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On the dimension of algebraic system of curves with nodes on a non-singular surface

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In this note we shall give an algebraic proof of the following

Theorem. *Let E be the linear system cut out on a non-singular surface by the hypersurfaces of order m. Then those curves of E which carry d or more nodes and no other kinds of singularities, together with their specializations, form an algebraic subsystem of E, and (if it is not vacuous) each of its components has dimension* $\geq \dim E - d$.

This theorem, as Severi showed already, plays an important rôle for a proof of the completeness of characteristic linear series in the classical case. The problem on the completeness of characteristic series in abstract case was taken up in the lectures of Zariski on minimal model of algebraic surfaces held in our department (October 1956),[1] and he conjectured that the problem would be true when the geometric genus is zero. Nakai proved at first the linearity of the characteristic series, but the question of the completeness remained unsolved until Zariski left here. The only missing point was the above theorem. Then we discussed on the problem in our seminar, thus we arrived at several proofs. So we wish to publish in the following papers, to which this paper also belongs, different proofs by several authors as a dear memory of Zariski and his lectures in Kyoto.

The principle of this first paper is very simple and elementary. So before stating the exact proof we shall explain the main idea by taking up the case of plane curves.

Let $\mathfrak{F}$ be an irreducible component of the algebraic system of

1) The contents of his lectures will be published soon as a series of his papers on minimal model and rationality of algebraic surfaces.

plane curves of m^{th} degree generated by the curves carrying d or more nodes and let C_λ be a generic curve of $\mathfrak{F}$ over an algebraically closed ground field k which has exactly d distinct nodes.

Further let $C_{\lambda'}$ be a special curve belonging to $\mathfrak{F}$ which has $d'(>d)$ distinct nodes and no other singularities. Let $\varphi(\lambda; x_0, x_1, x_2)=0$ be the equation of C_λ, and $\varphi(\lambda'; x)=0$ be that of $C_{\lambda'}$. The double points $P_i(1\leq i\leq d)$ of C_λ must satisfy

$$\varphi(\lambda, x)=0, \quad \partial\varphi/\partial x_1=0, \quad \partial\varphi/\partial x_2=0.$$

Let us now consider the intersection of two curves $\partial\varphi/\partial x_1=\partial\varphi/\partial x_2=0$. At a node P they intersect simply, because the equation $\varphi=0$ may be expressed locally at P as

$$\varphi=ax^2+hxy+\text{by}^2+\text{higher degree}=0, \text{ where } \begin{vmatrix} 2a & h \\ h & 2b \end{vmatrix}\neq 0.$$

From the $(m-1)^2$ intersecting points of $\partial\varphi/\partial x_1=\partial\varphi/\partial x_2=0$, we exclude the nodes $P_1, \cdots, P_d$ of C_λ and the points which are not simple intersecting points. We denote the remaining points by $Q_1, \cdots, Q_s$, and product up

$$R(\lambda)=\Pi\varphi(\lambda, Q_j).$$

The points $Q_1, \cdots, Q_s$ form evidently a rational cycle over $k(\lambda)$, hence $R(\lambda)$ is in $k(\lambda)$. Moreover $R(\lambda)\neq 0$. Thus the equation $R=0$ defines an algebraic subset $\mathfrak{B}$ of $\mathfrak{F}$, which is unmixed and of codimension 1 in $\mathfrak{F}$.[2] Let $\bar{\lambda}$ be a generic point of a component C of $\mathfrak{B}$ containing λ', and let $(\lambda, R(\lambda), P_1, \cdots, P_d, Q_1, \cdots, Q_s)\rightarrow(\bar{\lambda}, 0, \bar{P}_1, \cdots, \bar{P}_d, \bar{Q}_1, \cdots, \bar{Q}_s)\rightarrow(\lambda', 0, P_1', \cdots, P_d' Q_1', \cdots, Q_s')$ be specializations over k. Then

$$\Pi\varphi(\bar{\lambda}, \bar{Q}_j)=R(\bar{\lambda})=0.$$

Therefore there must exist at least one $\bar{Q}_j$ satisfying $\varphi(\bar{\lambda}, \bar{Q}_j)=0$. This point is either a multiple point of $C_{\bar{\lambda}}$ or a point at infinity. If the coordinate system was taken at the begining so general that $\partial\varphi/\partial x_1\neq 0$ at the points at infinity of $C_{\lambda'}$, the second case is impossible. On the other hand, Q_j' cannot coincide with any of the points P_i' since they are simple intersecting points of $\partial\varphi/\partial x_1=\partial\varphi/\partial x_2=0$. Hence $\bar{Q}_j$ cannot coincide with any of the $\bar{P}_i$. There-

2) $\mathfrak{B}$ is not empty as there exists Q' and $R(\lambda')=0$.

fore $C_{\bar{\lambda}}$ has $d+1$ or more nodes. So we get the theorem (at least locally) in the case of plane curves.

Let F be a nonsingular surface defined over an algebraically closed field k in a projective space L^n, and let $E=E_m$ be the linear system cut out on F by the hypersurfaces of order m. We are interested in those curves of E which have nodes and no other kinds of multiple points. We set $\dim E=N$, and denote by L'^N the parameter space of E. Let $\varphi_0, \cdots, \varphi_N$ be forms of order m which are linearly independent on F. We denote by $D(\lambda)$ the divisor on F corresponding to the parameter $\lambda \in L'$, i. e. the intersection of F with the hypersurface $\varphi_\lambda=\sum_0^N \lambda_i \varphi_i(x)=0$.

Lemma 1. *Let Q be a multiple point of a divisor $D(\lambda)$ of E. Assume that $x_0 \neq 0$ and $\Delta=\partial(f_1, \cdots, f_{n-1})/\partial(x_3, \cdots, x_n) \neq 0$ at Q (where $f_i \in \mathfrak{P}(F)$ =the homogeneous prime ideal of F). We set*

$$\Delta_j(\varphi_\lambda)=\partial(\varphi_\lambda, f_1, \cdots, f_{n-2})/\partial(x_j, x_3, \cdots, x_n) \quad (j=1,\ 2).$$

Then Q is a node of $D(\lambda)$ if and only if the n hypersurfaces $f_i=0$ $(i=1, \cdots, n-2)$, $\Delta_j(\varphi_\lambda)=0$ $(j=1, 2)$ are transversal at Q, i.e. if and only if

$$\Delta'(\varphi_\lambda)=\partial(\Delta_1(\varphi_\lambda), \Delta_2(\varphi_\lambda), f_1, \cdots, f_{n-2})/\partial(x_1, x_2, \cdots, x_n) \neq 0$$

at Q.

Proof. Let $K \supseteq k$ be an algebraically closed field over which Q and λ are rational, let $\mathfrak{o}$ be the local ring of Q on the space L^n with respect to K, and let $\mathfrak{M}$ be the maximal ideal of $\mathfrak{o}$. As $\Delta \neq 0$ at Q, $\mathfrak{p}=(f_1, \cdots, f_{n-2})\mathfrak{o}$ is the prime ideal of the surface F in $\mathfrak{o}$. (We use the inhomogeneous coordinates, setting $x_0=1$.) $\bar{\mathfrak{o}}=\mathfrak{o}/\mathfrak{p}$ is the local ring of Q on F, and $\{\bar{x}_1, \bar{x}_2\}$ is a regular syslem of parameters of $\bar{\mathfrak{o}}$. We denote the maximal ideal $(\bar{x}_1, \bar{x}_2)\bar{\mathfrak{o}}$ of $\bar{\mathfrak{o}}$ by $\mathfrak{m}$. Our assumption that Q is a multiple point of $D(\lambda)$ means that $(\varphi_\lambda \bmod \mathfrak{p}) \equiv 0\ (\mathfrak{m}^2)$. Now each element of $\mathfrak{m}^2/\mathfrak{m}^3$ is a quadratic form (hence the product of two linear forms) in $\bar{x}_1$ and $\bar{x}_2$ with coefficients in $K=\bar{\mathfrak{o}}/\mathfrak{m}$. Let l_1 and l_2 be the linear factors into which decomposes the image of φ_λ in $\mathfrak{m}^2/\mathfrak{m}^3$. Then Q is a node of $D(\lambda)$ if and only if l_1 and l_2 are not proportional in the Zariski tangent space $\mathfrak{m}/\mathfrak{m}^2$, or what is the same, if and only if $\Delta''=\partial(l_1, l_2, f_1, \cdots, f_{n-2})/\partial(x_1, \cdots, x_n) \neq 0$ at Q.

We have

$$\varphi_\lambda \equiv l_1 l_2 + \sum_{\nu=1}^{n-2} A_\nu f_\nu \pmod{\mathfrak{M}^3}, \quad A_\nu \in \mathfrak{o}.$$

It follows that, for $\alpha = 1, 2, \cdots, n$,

$$\partial\varphi_\lambda/\partial x_\alpha \equiv \partial(l_1 l_2)/\partial x_\alpha + \sum A_\nu \cdot \partial f_\nu/\partial x_\alpha \pmod{(f_1, \cdots, f_{n-2}, \mathfrak{M}^2)\mathfrak{o}},$$

hence

$$\Delta_i(\varphi_\lambda) \equiv \Delta_i(l_1 l_2) \pmod{(f_1, \cdots, f_{n-2}, \mathfrak{M}^2)\mathfrak{o}} \quad \text{for } i = 1, 2.$$

Repeating similar calculations once more, we have

$$\Delta'(\varphi_\lambda) \equiv \Delta'(l_1 l_2) \pmod{\mathfrak{M}}.$$

Since $\Delta_i(l_1 l_2) = l_1 \Delta_i(l_2) + l_2 \Delta_i(l_1)$ $(i = 1, 2)$ and since $l_1 \equiv 0$, $l_2 \equiv 0$ $(\text{mod } \mathfrak{M})$, we readily find by virtue of Jacobi's theorem on determinants that

$$\Delta'(\varphi_\lambda) \equiv -\Delta''^2 \cdot \Delta \pmod{\mathfrak{M}}.$$

This proves our lemma.

Corollary. *Let Q' be a multiple point of $D(\lambda')$ and let $(\lambda', Q') \to (\lambda, Q)$ be a specialization over k. If Q is a node of $D(\lambda)$, then also Q' is a node of $D(\lambda')$.*

Lemma 2. *If Q_1' and Q_2' are two multiple points of a curve $D(\lambda') \in E$, and if they are specialized to one and the same point Q over a specialization $\lambda' \to \lambda$, then Q is a multiple point, but not a node, of $D(\lambda)$.*

Proof. Assume that Q is a node of $D(\lambda)$. Using the notations of Lemma 1, we consider the algebraic correspondence $\mathfrak{N}$ between L' and L defined by $f_1(x) = 0, \cdots, f_{n-2}(x) = 0$, $\Delta_1(\varphi_\lambda) = 0$, $\Delta_2(\varphi_\lambda) = 0$. $\mathfrak{N}$ may very well be reducible, but since these n hypersurfaces in $L' \times L$ are transversal by Lemma 1, $\mathfrak{N}$ has a unique component W^N which contains $\lambda \times Q$. Also it is clear (by writing down the Jacobian matrix) that W and $\lambda \times L$ are transversal at $\lambda \times Q$. Therefore $\lambda \times Q$ is a proper point of intersection of $\lambda \times L$ and W with multiplicity 1. Then we can easily derive a contradiction by a theorem of Weil to the effect that: *Let U and V be two varieties; let W be a simple subvariety of $U \times V$, with the projection U on U, and of the same dimension as U. Let k be a common field of definition for U, V and W, and let P be a generic point of U over k. Then $W(P \times V) = P \times W(P)$ is defined and $W(P) = \sum_1^s Q_j$ is a prime rational cycle over $k(P)$. Moreover, if $P' \times Q'$ is a point of W such that it is a proper intersection, of multiplicity μ, of W and*

$P'\times V$ on $U\times V$, then, in any specialization of $(Q_1, \cdots, Q_s)$ over $P\to P'$ with respect to k, the point Q' occurs exactly μ times. ([7] Chap. 6 Th. 12).

Theorem. *Let $\mathfrak{E}_d$ be the set of those members of E which carry d or more nodes and no other kind of singularities, and let $\overline{\mathfrak{E}}_d$ be the subset of E obtained by adjoining to $\mathfrak{E}_d$ those members which are specializations over k of members of $\mathfrak{E}_d$. Then $\overline{\mathfrak{E}}_d$ is an algebraic subset of L', and (if it is not vacuous) each of its components has dimension $\geq N-d$.*

Proof. 1) In order to prove the theorem, it is sufficient to show the following: *Assuming that $\overline{\mathfrak{E}}_d$ is an algebraic set, let $\mathfrak{U}$ be one of its components. Then there exists an algebraic subset $\mathfrak{B}=\mathfrak{B}(\mathfrak{U})$ of $\mathfrak{U}$ with the following properties.*

(1) $\mathfrak{U}\cap\mathfrak{E}_{d+1}\supseteq\mathfrak{B}\supseteq\overline{\mathfrak{E}}_{d+1}$,

(2) *each component of $\mathfrak{B}$ has dimension* $\geq \dim\mathfrak{U}-1$.

In fact, it is evident that we shall have $\cup_{\mathfrak{U}}\mathfrak{B}(\mathfrak{U})=\overline{\mathfrak{E}}_{d+1}$ if this statement is proved.

2) If $\mathfrak{U}\subseteq\overline{\mathfrak{E}}_{d+1}$, or if $\mathfrak{U}\cap\mathfrak{E}_{d+1}=\phi$, there is nothing to prove. (We take $\mathfrak{B}=\mathfrak{U}$ or $\mathfrak{B}=\phi$ respectively.) Therefore we assume that the generic member of $\mathfrak{U}$ over k carries just d nodes, and that there is a member $D(\lambda^*)$ in $\mathfrak{U}$ with $d'(>d)$ nodes and no other singularities.

3) We first solve the problem locally. Let λ^* be as above and let $Q_1^*, \cdots, Q_{d'}^*$ be the nodes of $D(\lambda^*)$. Taking a suitable coordinate system (defined over k) in L and selecting suitable forms $f_1, \cdots, f_{n-2}$ from $\mathfrak{P}(F)$ we can assume that

(a) $x_0\cdot \varDelta\neq 0$ at Q_i^* $(i=1, \cdots, d'$; $\varDelta=\partial(f_1, \cdots, f_{n-2})/\partial(x_3, \cdots, x_n))$

(b) $D(\lambda^*)$ has only a finite number of points at infinity, and $\varDelta_1(\varphi_{\lambda^*})$ does not vanish at these points.

We use the notations of lemma 1 and 2. Let $\bar{\lambda}$ be a generic point of L' over k and let

$$(\bar{\lambda}\times L)\cdot W=\bar{\lambda}\times\sum_{j=1}^{\delta}\bar{P}_j.$$

Then each $\bar{P}_j$ is on F, and rank $(\partial(\varphi_{\bar{\lambda}}, f_1, \cdots, f_{n-2})/\partial(x_1, \cdots, x_n))$ $=n-2$ at each $\bar{P}_j$. Let λ' be a point of L' which has λ^* as a specialization over k, and let $(\bar{P}_1, \cdots, \bar{P}_\delta)\to(P_1', \cdots, P_\delta')$ be a specialization over $\bar{\lambda}\to\lambda'$. Assume that $\varphi_{\lambda'}(P_1')=0$. Then P_1' is a multiple point of $D(\lambda')$, because P_1' is at finite distance by our assumption

(b) and because rank $(\partial(\varphi_{\lambda'}, f_1, \cdots, f_{n-2})/\partial(x_1, \cdots, x_n)) \leq n-2$ at P_1' as $\lambda' \times P_1'$ is a specialization of $\bar{\lambda} \times \bar{P}_1$.

Now, let λ' be a generic point of $\mathfrak{U}$ over k, and consider those P_j' at which $\varphi_{\lambda'}(x) \cdot J'(\varphi_{\lambda'})$ does not vanish. We denote them by $P_1', \cdots, P_s'$, and set $\Phi(\lambda') = \Pi_1^s \varphi_{\lambda'}(P_j')$. Since $P_1' + \cdots + P_s'$ is clearly rational over the field $k(\lambda')$, $\Phi(\lambda')$ belongs to $k(\lambda')$ and hence defines a rational function Φ on $\mathfrak{U}$. The equation $\Phi = 0$ defines an unmixed algebraic subset $\mathfrak{B}'(\mathfrak{U}, \lambda^*)$ of codimension 1 in $\mathfrak{U}$. Let $\mathfrak{B}(\mathfrak{U}, \lambda^*)$ be the sum of those components of $\mathfrak{B}'(\mathfrak{U}, \lambda^*)$ which contain λ^*. Let $\mathfrak{B}$ be a component of $\mathfrak{B}(\mathfrak{U}, \lambda^*)$ and let λ'' be a generic point of $\mathfrak{B}$ over k. Then there exists at least one specialization $(P_1', \cdots, P_s') \rightarrow (P_1'', \cdots, P_s'')$ over $\lambda' \rightarrow \lambda''$ such that $\varphi_{\lambda''}(P_i'') = 0$ for some i, say for $i = 1$. P_1'' must be a multiple point of $D(\lambda'')$ by what was said above. From this, and from the proof of Lemma 2, it follows readily that $D(\lambda'')$ has at least $d+1$ nodes. Thus $\mathfrak{B}(\mathfrak{U}, \lambda^*) \subseteq \bar{\mathfrak{E}}_{d+1}$. Conversely, any point λ of $\mathfrak{E}_{d+1} \cap \mathfrak{U}$ which has λ^* as a specialization is in $\mathfrak{B}(\mathfrak{U}, \lambda^*)$, as is easily to be seen.

4) Now we can solve the problem globally. Set

$$\mathfrak{B} = \mathfrak{B}(\mathfrak{U}) = \cup_{\lambda^* \in \mathfrak{U} \cap \mathfrak{E}_{d+1}} \mathfrak{B}(\mathfrak{U}, \lambda^*).$$

Then $\mathfrak{B}$ is a union of subvarieties of codimension 1 on $\mathfrak{U}$, and has the property (1) required in 1). Hence we have only to show that $\mathfrak{B}$ is contained in a proper algebraic subset of $\mathfrak{U}$. (From that would follow the finiteness of the number of the components of $\mathfrak{B}$.) This can be done by the same method as in 3), and is much easier. We sketch the peoof: Let $f_1, \cdots, f_t$ be a basis of $\mathfrak{P}(F)$. Since the number of the minors of order $n-2$ of the Jacobian matrix $(\partial(f_1, \cdots, f_t)/\partial(x_0, x_1, \cdots, x_n))$ is finite, we see from Lemma 1 that there are a finite number of systems of n hypersurfaces $\mathfrak{S}_i = \{g_{i1}(\lambda, x) = 0, \cdots, g_{in}(\lambda, x) = 0\}$ $(1 \leq i \leq h)$ with the following property: if Q is a node of $D(\lambda)$, then for some i the n hypersurfaces of $\mathfrak{S}_i$ and $\lambda \times L$ are transversal at $\lambda \times Q$. Let W_{ij} be those components of the algebraic set $\mathfrak{N}_i$ defined by $\mathfrak{S}_i$ which contain some points of $\mathfrak{U} \cap \mathfrak{E}_{d+1}$, and let $\mathfrak{B}_{ij}'$ be the algebraic subset of $\mathfrak{U}$ derived from W_{ij} in a similar manner as the $\mathfrak{B}'(\mathfrak{U}, \lambda^*)$ was derived from W in 3). Then it is easy to see that $\mathfrak{E}_{d+1} \cap \mathfrak{U} \subseteq \cup_{ij} \mathfrak{B}_{ij}'$. This completes the proof of our theorem.

We add a few elementary lemmas concerning nodes, which will be used for the third proof of our theorem in [2].

Lemma 3. *If Q is a node of $D(\lambda)$, then Q is separably algebraic over the field $k(\lambda)$.*

Proof. We have only to prove that there is no non-trivial derivation of the field $k(\lambda, Q)$ over $k(\lambda)$. But this is an immediate consequence of Lemma 1.

Now consider in $L'\times L$ the algebraic correspondence $\overline{\mathfrak{M}}$ by which a curve of E corresponds to its multiple points. For every point P of the surface F, those curves of E which have P as a multiple point make up a linear system E_P of like dimension $N-3$. Therefore, if $\bar{P}$ is a generic point of F over k and if $\bar{\lambda}$ is a generic point of $E_{\bar{P}}$ over $k(\bar{P})$, then $\bar{\lambda}\times\bar{P}$ is a generic point of $\overline{\mathfrak{M}}$ over k. It follows that $\overline{\mathfrak{M}}$ is irreducible and has dimension $N-1$. (Cf. [6] p. 143 Lemma.) (We exclude the trivial case $N=2$.) Let $\mathfrak{M}$ be the projection of $\overline{\mathfrak{M}}$ on L'. We assume in the sequel that there is a member of $\mathfrak{M}$ which carries isolated multiple points. Then we have $\dim \mathfrak{M}=\dim \overline{\mathfrak{M}}=N-1$ (by [7] Chap. 4, Prop. 25). If, moreover, $\mathfrak{M}$ has a member which has one and only one node and no other multiple points, then we see by Lemma 3 that $\mathfrak{M}$ and $\overline{\mathfrak{M}}$ are birationally equivalent.

Lemma 4. *If Q is a node of $D(\lambda)$, then $\lambda\times Q$ is a simple point of $\overline{\mathfrak{M}}$.*

Proof. We may assume that Q satisfies the condition $x_0\cdot\Delta\neq 0$ of Lemma 1. Then by Lemma 1 we have that $\partial(\Delta_1(\varphi_\lambda), \Delta_2(\varphi_\lambda), f_1, \cdots, f_{n-2})/\partial(x_1, \cdots, x_n)\neq 0$ at Q. We may assume also that $\varphi_0=x_0^m$. Then λ_0 does not appear in $\Delta_i(\varphi_\lambda)$ for $i=1, 2$. Hence we have

$$\begin{aligned}&\partial(\varphi_\lambda, \Delta_1(\varphi_\lambda), \Delta_2(\varphi_\lambda), f_1, \cdots, f_{n-2})/\partial(\lambda_0, x_1, \cdots, x_n)\\&\qquad=x_0^m\cdot\partial(\Delta_1(\varphi_\lambda), \Delta_2(\varphi_\lambda), f_1, \cdots, f_{n-2})/\partial(x_1, \cdots, x_n)\neq 0\end{aligned}$$

at Q. This proves our assertion.

Lemma 5. *Assume that $E=E_m$, $m\geq 2$. Then $\mathfrak{M}$ and $\overline{\mathfrak{M}}$ are birationally equivalent.*

Proof. We have only to show that there is a member of E carrying only one node and no other multiple point. Let $\bar{P}$ be a generic point of F and let $\bar{\lambda}$ be a generic point of $E_{\bar{P}}$ over $k(\bar{P})$. First, $\bar{P}$ is a node of $D(\bar{\lambda})$ and $E_{\bar{P}}$ has no base points other than $\bar{P}$, since for any given point $Q(\neq\bar{P})$ we can find hypersurface sections D_1 and D_2, intersecting transversally at $\bar{P}$ and not passing through Q, such that $D_1+D_2\in E$. On the other hand, all multiple points of $D(\bar{\lambda})$ are conjugates of $\bar{P}$ over $k(\bar{\lambda})$ (because $\bar{\lambda}$ is a generic point of $\mathfrak{M}$ over k) and hence nodes. Therefore by Lemma 3, the theorem of Bertini on variable singular points is applicable for $E_{\bar{P}}$. (Cf. [1]). Hence $D(\bar{\lambda})$ has no multiple points other than the base point $\bar{P}$ itself.

Remark. From the proof of Lemma 5 we can easily derive the birationality of F and F^*, where F^* denotes the dual of F defined in [3].

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