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On the induced connexions

By

Seizi TAKIZAWA

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Dedicated to Prof. Jôyô Kanitani on His Sixty-third Birthday

Introduction

In the theory of connexions on differentiable principal bundles, the term "Induced Connexion" may be used in several different significances. What we shall introduce in the present paper is, however, one of the essential generalizations of the classical process of induction in the theory of submanifolds. It can be shown that a connexion is naturally induced on a certain principal bundle, whose bundle space and base space are both associated bundles of a given principal bundle with connexion.

In § 2, we give the definition and expositions of induced connexion, and derive generalizations of the equations of Gauss-Codazzi-Ricci. The structure of induced connexion appears not only in the theory of submanifolds, but also in the theories of various categories, for instance, canonical connexions on universal bundles, the Stiefel-Whitney characteristic classes, and reductive Cartan connexions. In the last four sections, we describe their applications.

§ 1. Survey of tensor calculus

We denote by $T(M)$ the tangent vector bundle over any C^∞-manifold M, and by $T_x(M)$ the tangent vector space at $x \in M$. Let $T^k(M)$ denote the space of all ordered sets $(t_1, \cdots, t_k)$ of tangent vectors such that $t_1, \cdots, t_k \in T_x(M)$, $x \in M$. Then $T^k(M)$ can be regarded as an associated bundle of $T(M)$. Let V be a vector space over the field of real numbers R. A *V-valued k-form* on M is, by definition, a C^∞-map

$$\theta : T^k(M) \to V,$$

such that $\theta(t_1, \cdots, t_k) \in V$ is multilinear and skew-symmetric with respect to the variables $t_1, \cdots, t_k \in T_x(M)$.

Let U, V and W be vector spaces. We consider a W-valued bilinear function $F(u, v)$, $u \in U$, $v \in V$, and take two forms on M

$$\theta : T^k(M) \rightarrow U \quad \text{and} \quad \varphi : T^l(M) \rightarrow V.$$

Then, θ and φ may be substituted into F, and we have a form

$$F(\theta, \varphi) : T^{k+l}(M) \rightarrow W,$$

which is defined as follows: for $t_1, \cdots, t_{k+l} \in T_x(M)$, $x \in M$,

$$F(\theta, \varphi)(t_1, \cdots, t_{k+l})$$
$$= \frac{1}{(k+l)!} \sum_{\sigma} \varepsilon(\sigma) F(\theta(t_{\sigma(1)}, \cdots, t_{\sigma(k)}), \varphi(t_{\sigma(k+1)}, \cdots, t_{\sigma(k+l)})),$$

where the summation is extended over all permutations σ of the set $\{1, \cdots, k+l\}$, and $\varepsilon(\sigma)$ denotes the sign of σ. Such substitution can be generalized for any multilinear function of vector spaces. In this paper, any concrete s-linear function $F(\theta_1, \cdots, \theta_s)$, $1 \leqq s$, into which forms θ_i are substituted will be understood in the above sense, without any mention. For instance, if $F(x, y) = xy$, $x, y \in R$, and if θ, φ are real-valued forms on M, $\theta\varphi$ expresses the exterior product of θ and φ.

We suppose that the exterior differentiation for real-valued forms is known in usual way. Let V^* be the dual space of V, and let θ be a V-valued k-form on M. The inner product $\langle v^*, \theta \rangle$, $v^* \in V^*$, gives a real-valued k-form, and there exists a unique V-valued $(k+1)$-form $d\theta$, called the exterior derivative of θ, such that

$$d\langle v^*, \theta \rangle = \langle v^*, d\theta \rangle, \text{ for any } v^* \in V^*.$$

The operation d is an *antiderivation*, that is, for a bilinear function F, it holds that

$$dF(\theta, \varphi) = F(d\theta, \varphi) + (-1)^k F(\theta, d\varphi).$$

We consider now a differentiable principal bundle $P(M, G)$ over M with group G. The Lie algebra of G and the linear adjoint group of G are denoted by $\mathfrak{g}$ and $ad(G)$ respectively.

Let $r : G \rightarrow GL(V)$ be a representation of G, where $GL(V)$ denotes the group of all automorphisms of a vector space V. A

V-valued k-form θ on the bundle space P is called a *pseudo-tensorial k-form of type* (r, V), if $\theta\rho(g)=r(g^{-1})\theta$ for any $g\in G$, where $\rho(g)$ is the right translation of the bundle $P(M, G)$ and is considered to operate on $T^k(P)$. A k-form θ on P is said to be *horizontal* provided that if t_1 is a vertical vector of $P(M, G)$ then $\theta(t_1, \cdots, t_k)=0$. A horizontal pseudo-tensorial form is called a *tensorial form.* A *connexion form* ω on $P(M, G)$ is a pseudo-tensorial 1-form of type $(ad, \mathfrak{g})$ and for every fundamental vector field A^*, $\omega(A^*)=A(=const.)\in\mathfrak{g}$. Cf. [7], p. 26.

Taking a k-form φ on the base space M, we have a k-form $\varphi\pi$ on the bundle space P, where π denotes the projection of the bundle structure and is considered to operate on $T^k(P)$. For simplicity, we express the form $\varphi\pi$ on P also as φ, and call it merely a *form on the base space M.*

PROPOSITION 1. *Let $P(M, G)$ be a differentiable principal bundle and let (r_0, V) be a trivial representation of G, i.e. $r_0(G)=(\varepsilon)$: the unit element. A form θ on P reduces to a form on the base space M, if and only if θ is a tensorial form of type (r_0, V); in other words, θ is a horizontal form which is invariant under the right translations of $P(M, G)$.*

This proposition can be easily proved, according to the following lemma: *if $t_1, t_2\in T(P)$ are tangent vectors such that $\pi t_1=\pi t_2\in T_x(M)$, then there exist uniquely an element $g\in G$ and a vertical vector t_0 of $P(M, G)$ such that*

$$t_2=\rho(g)t_1+t_0\,.$$

Assume that a connexion ω is defined on $P(M, G)$. Then, each tangent space $T_p(P)$, $p\in P$, is decomposed in a direct sum:

$$T_p(P)=\mathfrak{H}_p+\mathfrak{V}_p\,,$$

where $\mathfrak{H}_p$ and $\mathfrak{V}_p$ denote respectively the horziontal space and the vertical space at p. Cf. [7], p. 25. Let us denote by

$$\boldsymbol{h}: T_p(P)\to\mathfrak{H}_p,\ p\in P,$$

the projection with respect to the decomposition. The *covariant derivative* $D\theta$ of a form θ on P is defined by $D\theta=(d\theta)\boldsymbol{h}$.

PROPOSITION 2. *For the connexion form ω, it holds that $\omega\boldsymbol{h}=0$; and a form θ on P is horizontal if and only if $\theta=\theta\boldsymbol{h}$.*

This proposition is obvious by definition. Furthermore we have:

PROPOSITION 3. *If θ is a pseudo-tensorial k-form of type (r, V), then*

(i) *θh is a tensorial k-form of type (r, V),*

(ii) *$d\theta$ is a pseudo-tensorial $(k+1)$-form of type (r, V),*

(iii) *$D\theta$ is a tensorial $(k+1)$-form of type (r, V).*

Proof. The relation $\rho(g)h=h\rho(g)$ implies (i). Taking account that a right translation is a differentiable homeomorphism of P and that d is a linear operator, we have (ii). Moreover, (iii) follows from (i) and (ii).

PROPOSITION 4. *For the connexion form ω,*

$$D\omega=\Omega=d\omega+\frac{1}{2}[\omega,\omega] \quad (\textit{the equation of structure});$$

and if θ is a tensorial form of type (r, V),

$$D\theta=d\theta+\bar{r}(\omega)\theta,$$

where $\bar{r}: \mathfrak{g}\to\mathfrak{gl}(V)$ is the induced representation of r, and $\mathfrak{gl}(V)$ denotes the Lie algebra of all endomorphisms of V.

The form Ω is called the *curvature form* of the connexion ω, and is a tensorial 2-form of type $(ad, \mathfrak{g})$. To prove these formulas, it is sufficient to show that their right hand sides are horizontal forms. By direct calculations we obtain them. Cf. [1], [12]. In particular, we have:

PROPOSITION 5. *If θ is a form on the base space M,*

$$D\theta=d\theta;$$

and if θ is a tensorial form of type $(ad, \mathfrak{g})$,

$$D\theta=d\theta+[\omega,\theta].$$

These relations follows from $\bar{r}_0(\mathfrak{g})=(0)$ and $\overline{ad}(x)y=[x,y]$, $x, y\in\mathfrak{g}$.

PROPOSITION 6. *For the connexion form ω,*

$$D^2\omega=D\Omega=0 \quad (\textit{the Bianchi identity});$$

and if θ is a tensorial form of type (r, V),

$$D^2\theta=\bar{r}(\Omega)\theta \quad (\textit{the Ricci identity}). \text{ Cf. [5], p. 30.}$$

Proof. $(dD\omega)h=[d\omega,\omega]h=0$, since $\omega h=0$; and $(dD\theta)h=(\bar{r}(d\omega)\theta-\bar{r}(\omega)d\theta)h=\bar{r}(\Omega)\theta$, since $(d\omega)h=\Omega$ and $\theta h=\theta$.

PROPOSITION 7. *For horizontal forms, D is an antiderivation.*

Since d is an antiderivation and $\theta \boldsymbol{h}=\theta$ for a horizontal form θ, this proposition holds obvionsly.

A finite sum of W-valued multilinear functions

$$\sum F(\theta_1, \cdots, \theta_1, \theta_2, \cdots, \theta_2, \cdots, \theta_s, \cdots \theta_s)$$

into which forms $\theta_1, \cdots, \theta_s$ are substituted is called a W-valued *polynomial* of $\theta_1, \cdots, \theta_s$.

PROPOSITION 8. *Let ω and Ω be a connexion form and its curvature form respectively, and let $\theta_i (i=1, \cdots, s)$ be tensorial forms. The covariant derivative of any polynomial of ω, Ω, θ_i, $D\theta_i$ is a polynomial of Ω, θ_i, $D\theta_i$.*

Taking account of Proposition 6 and the relation $\omega \boldsymbol{h}=0$, we have Proposition 8.

§ 2. The induced connexions

Let $P(M, G)$ be a differentiable principal bundle. We take a closed subgroup $H \subset G$, and assume that the homogeneous space G/H is *reductive*, that is, there exists a canonical decomposition of the Lie algebra $\mathfrak{g}$: $\mathfrak{g}=\mathfrak{h}+\mathfrak{k}$ such that $ad(H)\mathfrak{k} \subset \mathfrak{k}$, where $\mathfrak{h}$ is the Lie algebra of H. The representation $(ad, \mathfrak{k})$ of H is called the *linear isotropy representation.* Let us denote by $a_{\mathfrak{h}}$ and $a_{\mathfrak{k}}$ respectively the $\mathfrak{h}$- and $\mathfrak{k}$-components of an element $a \in \mathfrak{g}$. Now we decompose H by a direct product $H=H_1 \times H_2$. In this case, H_2 may reduce to the unit element (ε) alone, and so the assumption is quite general. We have then a decomposition of $\mathfrak{g}$:

$$\mathfrak{g}=\mathfrak{h}_1+\mathfrak{k}+\mathfrak{h}_2,$$

where $\mathfrak{h}_1$, $\mathfrak{h}_2$ are the Lie algebras of H_1, H_2 respectively.

By the natural projection $\nu : G/H_2 \rightarrow G/H$, we get a principal bundle $G/H_2(G/H, H_1)$. Let E_1, E be the associated bundles of $P(M, G)$ with fibres G/H_2, G/H respectively. Then ν induces a projection

$$\nu^* : E_1 \rightarrow E,$$

$$\begin{tikzcd} G \arrow[r, dash, dashed] \arrow[d, dash] & P \arrow[d, dash] \arrow[dr, dash] & \\ G/H_2 \arrow[r, dash, dashed] \arrow[d, dash] & E_1 \arrow[d, dash] & E_2 \arrow[dl, dash] \\ G/H \arrow[r, dash, dashed] & E \arrow[d, dash] & \\ & M & \end{tikzcd}$$

and we get a principal bundle $E_1(E, H_1)$. When $H_2=(\varepsilon)$, $E_1(E, H_1)$ coincides with

$P(E, H)$. In the above considerations, if we exchange the groups H_1 and H_2 mutually, we get a bundle $E_2(E, H_2)$.

Consider now a connexion $\tilde{\omega}$ on $P(M, G)$, and denote by $\tilde{\Omega}$ its curvature form.

PROPOSITION 9. *Let $\tilde{\omega}$ be a connexion on $P(M, G)$. Then the form*

$$\omega_1 = \tilde{\omega}_{\mathfrak{h}_1}$$

reduces to a form on E_1, and becomes a connexion on $E_1(E, H_1)$.

We call ω_1 the *induced connexion* on $E_1(E, H_1)$ of the connexion $\tilde{\omega}$ on $P(M, G)$.

Proof. Since any right translation $\rho(h)$, $h \in H$, of $P(E, H)$ can be also regarded as a right translation of $P(M, G)$, we have

$$\tilde{\omega}_{\mathfrak{h}}\rho(h) = (\tilde{\omega}\rho(h))_{\mathfrak{h}} = (ad(h^{-1})\tilde{\omega})_{\mathfrak{h}} = ad(h^{-1})\tilde{\omega}_{\mathfrak{h}} \, ;$$

and, since a fundamental vector field A^*, $A \in \mathfrak{h}$, of $P(E, H)$ can be also regarded as a fundamental vector field of $P(M, G)$, we have

$$\tilde{\omega}_{\mathfrak{h}}(A^*) = (\tilde{\omega}(A^*))_{\mathfrak{h}} = A_{\mathfrak{h}} = A \, .$$

Hence, $\tilde{\omega}_{\mathfrak{h}}$ is a connexion on $P(E, H)$. The natural projection $p: H \to H_1$ given by $H = H_1 \times H_2$ is an onto homomorphism, and so it induces a bundle homomorphism

$$p^* : P(E, H) \to E_1(E, H_1).$$

Consequently, the connexion $\tilde{\omega}_{\mathfrak{h}}$ is mapped by p^* to a connexion ω^* on $E_1(E, H_1)$, whose horizontal vectors are given by the images of horizontal vectors of $P(E, H)$. Since $\tilde{\omega}_{\mathfrak{h}_1} = \omega^*$, $\tilde{\omega}_{\mathfrak{h}_1}$ is a connexion on $E_1(E, H_1)$. This completes the proof.

The induced connexion depends on the choice of canonical decomposition of the Lie algebra $\mathfrak{g}$.

Setting $\psi = \tilde{\omega}_{\mathfrak{k}}$ and $\omega_2 = \tilde{\omega}_{\mathfrak{y}_2}$, we have:

PROPOSITION 10. *The connexion $\tilde{\omega}$ is decomposed as follows:*

$$\tilde{\omega} = \omega_1 + \psi + \omega_2 \, ,$$

where

(i) ω_1 *is the induced connexion on* $E_1(E, H_1)$,

(ii) ψ *is a tensorial form on* $P(E, H)$ *of type* $(ad, \mathfrak{k})$,

(iii) ω_2 *is the induced connexion on* $E_2(E, H_2)$,

(iv) $\omega = \omega_2 + \omega_2$ *is the induced connexion on* $P(E, H)$.

Substituting this decomposition into the equation of structure of $\tilde{\omega}$:

$$d\tilde{\omega}=-\frac{1}{2}[\tilde{\omega},\ \tilde{\omega}]+\widetilde{\Omega}\ ,$$

we obtain the following formula.

PROPOSITION 11.

$$\Omega_1+D\psi+\Omega_2=-\frac{1}{2}[\psi,\ \psi]+\widetilde{\Omega}\ ,$$

where

(i) Ω_1 *is the curvature form of* ω_1,

(ii) $D\psi$ *is the covariant derivative of* ψ *with respect to* $\omega=\omega_1+\omega_2$,

(iii) Ω_2 *is the curvature form of* ω_2,

(iv) $\Omega=\Omega_1+\Omega_2$ *is the curvature form of* $\omega=\omega_1+\omega_2$.

The forms Ω_1, $D\psi$, Ω_2 have their values in $\mathfrak{h}_1$, $\mathfrak{k}$, $\mathfrak{h}_2$ respectively, and so we get the equations.

PROPOSITION 12.

$$\Omega_1=-\frac{1}{2}[\psi,\ \psi]_{\mathfrak{h}_1}+\widetilde{\Omega}_{\mathfrak{h}_1}\quad \textit{(the Gauss equation)},$$

$$D\psi=-\frac{1}{2}[\psi,\ \psi]_{\mathfrak{k}}+\widetilde{\Omega}_{\mathfrak{k}}\quad \textit{(the Codazzi equation)},$$

$$\Omega_2=-\frac{1}{2}[\psi,\ \psi]_{\mathfrak{h}_2}+\widetilde{\Omega}_{\mathfrak{h}_2}\quad \textit{(the Ricci equation)}.$$

Cf., for instance, [8].

When H_2 is discrete, the Ricci equation vanishes since $\omega_2=0$, and the Gauss-Codazzi equation assumes the form

$$\Omega+D\psi=-\frac{1}{2}[\psi,\ \psi]+\widetilde{\Omega}\ .$$

If the homogeneous space G/H is *symmetric, i.e.* $ad(H)\mathfrak{k}\subset\mathfrak{k}$ and $[\mathfrak{k},\ \mathfrak{k}]\subset\mathfrak{h}$, the Codazzi equation reduces to

$$D\psi=\widetilde{\Omega}_{\mathfrak{k}}\ .$$

We take a local cross-section α of the bundle $P(E_1,\ H_2)$ and denote by θ_α the dual image of any form θ on P by the map α. Then ψ_α is a local tensorial form on $E_1(E,\ H_1)$ of type $(ad,\ \mathfrak{k})$ and $\omega_{2,\alpha}$ reduce to a local $\mathfrak{h}_2$-valued form on E. The Codazzi equation takes now the form

$$D_1\varphi_\alpha+[\omega_{2,\alpha},\varphi_\alpha]=-\frac{1}{2}[\varphi_\alpha,\varphi_\alpha]_{\mathfrak{f}}+\widetilde{\Omega}_{\alpha\mathfrak{f}},$$

where D_1 denotes the covariant differentiation with respect to ω_1. In the classical theory, the Codazzi equation was written as the above form. Cf. [8], p. 278, or [5], p. 162.

§ 3. The deformation of a connexion

By a similar consideration to the preceding section, we can introduce the deformation of a connexion. Let ω be a connexion on a principal bundle $P(M, G)$ and φ be a tensorial 1-form on $P(M, G)$ of type $(ad, \mathfrak{g})$. A deformation of the connexion ω is defined by the 1-form

$$\omega_t=\omega+t\varphi,\ t\in[0,1]: \textit{parameter}.$$

Then ω_t is a connexion on $P(M, G)$ for each $t\in[0, 1]$. For two connexion ω, ω_1 on $P(M, G)$, there exists a deformation ω_t which joins ω to ω_1; precisely, it is furnished by $\omega_t=\omega+t(\omega_1-\omega)$.

Let $\omega_t=\omega+t\varphi$ be a deformation of ω. The curvature form Ω_t of ω_t, $t\in[0, 1]$, is given by

$$\Omega_t=\Omega+tD\varphi+\frac{t^2}{2}[\varphi,\varphi],$$

where Ω is the curvature form of $\omega=\omega_0$ and D denotes the covariant differentiation with respect to ω. Denoting by D_t the covariant differentiation with respect to ω_t and by "dot" the differentiation by the parameter t, we have the formulas:

$$D_t\varphi=D\varphi+t[\varphi,\varphi]=\dot{\Omega}_t,\quad i.e.\quad D_t\dot{\omega}_t=(D_t\omega_t)^{\cdot},$$

$$D_t\Omega_t=0\quad \text{(the Bianchi identity)}.$$

Now we take a real-valued k-linear function $F(x_1, \cdots, x_k)$, $x_1, \cdots, x_k\in\mathfrak{g}$, which is invariant under $ad(G)$. Since $F(\varphi, \Omega_t, \cdots, \Omega_t)$ reduces to a form on the base space M, we have

$$\begin{aligned} dF(\varphi,\Omega_t,\cdots,\Omega_t)&=D_tF(\varphi,\Omega_t,\cdots,\Omega_t)\\ &=F(\dot{\Omega}_t,\Omega_t,\cdots,\Omega_t). \end{aligned}$$

It is known that the form $F(\Omega, \cdots, \Omega)$ on M represents a characteristic class of the bundle structure $P(M, G)$. Cf. [2], p. 57.

§ 4. The induced Riemann connexion

Let M^m be an m-dimensional differentiable submanifold of a given

Riemann manifold $(M^{m+n}, \boldsymbol{g})$, where $\boldsymbol{g}$ denotes the metric tensor of M^{m+n}. The injection: $M^m \rightarrow M^{m+n}$ induces a unique Riemann metric $\boldsymbol{g}^*$ of M^m.

Let $\tilde{\omega}$ be the Riemann connexion on R^{m+n} determined by the metric $\boldsymbol{g}$. Then $\tilde{\omega}$ is a connexion on the bundle $P(M^{m+n}, O(m+n))$ of all orthonormal tangent $(m+n)$-frames on M^{m+n}, where $O(m+n)$ denotes the orthogonal group of degree $m+n$.

Let E_1 and E be the associated bundle of P with fibres $O(m+n)/O(n)$ and $O(m+n)/O(m)\times O(n)$ respectively. Then E_1 is the bundle of all orthonormal tangent m-frames on M^{m+n}, and E is the bundle of all tangent m-planes on M^{m+n}. Since the homogeneous space $O(m+n)/O(m)\times O(n)$ is symmetric, there exists a canonical decomposition of the Lie algebra of $O(m+n)$:

$$\mathfrak{so}(m+n)=\mathfrak{so}(m)+\mathfrak{k}+\mathfrak{so}(n).$$

Therefore, by the natural projections

$$P\rightarrow E_1\rightarrow E\rightarrow M,$$

we get an induced connexion $\omega=\tilde{\omega}_{\mathfrak{so}(m)}$ on the bundle $E_1(E, O(m))$. To each point $p \in M^m$, assigning the tangent space T_p of M^m at p, we have a map $T: M^m \rightarrow E$. Then a bundle structure $P^*(M^m, O(m))$ and a connexion ω^* on it are induced by T, from the bundle $E_1(E, O(m))$ and the connexion ω on it. It is notable that $P^*(M^m, O(m))$ becomes the bundle of all orthonormal tangent m-frames on M^m and ω^* gives the Riemann connexion determined by the induced metric $\boldsymbol{g}^*$.

In this case, the Gauss-Codazzi-Ricci equations assume the well-known forms. Cf., for instance, [5], p. 162.

§ 5. The canonical connexions on universal bundles

Let G be a compact Lie group. We can suppose that G is a subgroup of a special orthogonal group $SO(m)$ for m sufficient large. Setting

$$P_0=SO(m+n)/SO(n),\quad M_0=SO(m+n)/G\times SO(n),\quad G\subset SO(m),$$

by the natural projection: $P_0\rightarrow M_0$, we get a principal bundle $P_0(M_0, G)$, which is n-universal. Cf. [9].

The Grassmann manifold $\widetilde{M}=SO(m+n)/SO(m)\times SO(n)$ is a symmetric homogeneous space, and so we have a canonical decomposition of the Lie algebra of $SO(m+n)$:

$$\mathfrak{so}(m+n)=\mathfrak{so}(m)+\mathfrak{m}+\mathfrak{so}(n).$$

By the natural projection: $P_0 \to \widetilde{M}$, we get an n-universal bundle $P_0(\widetilde{M}, SO(m))$.

Let $\tilde{\omega}: T(SO(m+n)) \to \mathfrak{so}(m+n)$ be the Maurer-Cartan form of the group $SO(m+n)$, that is, by regarding any element $A \in \mathfrak{so}(m+n)$ as a left invariant vector field on $SO(m+n)$, $\omega(A)=A$ $(=const) \in \mathfrak{so}(m+n)$. Then $\tilde{\omega}$ becomes a connexion on the trivial bundle $SO(m+n) \times (x_0)$, and its equation of structure is given by

$$d\tilde{\omega}=-\frac{1}{2}[\tilde{\omega}, \tilde{\omega}].$$

Cf., for instance, [3], pp. 152 ~155. Accordingly, by the natural projections

$$SO(m+n) \to P_0 \to \widetilde{M} \to (x_0),$$

we obtain an induced connexion $\omega_0=\tilde{\omega}_{\mathfrak{so}(m)}$ on $P_0(\widetilde{M}, SO(m))$, and have the decomposition

$$\tilde{\omega}=\omega_0+\psi+\omega',$$

where $\psi=\tilde{\omega}_{\mathfrak{m}}$ and $\omega'=\tilde{\omega}_{\mathfrak{so}(n)}$. We call ω_0 the *canonical connexion* on the universal bundle $P_0(\widetilde{M}, SO(m))$. The curvature form Ω_0 of ω_0 is given by the Gauss equation

$$\Omega_0=-\frac{1}{2}[\psi, \psi]_{\mathfrak{so}(m)}.$$

Let $P_0 \to M_0 \to \widetilde{M}$ be the natural projections. Since the homogeneous space $SO(m)/G$ is reductive, we have a canonical decomposition of the Lie algebra $\mathfrak{so}(m)$:

$$\mathfrak{so}(m)=\mathfrak{g}+\mathfrak{n}.$$

Furthermore, a connexion $\omega=\omega_{0\mathfrak{g}}$ on $P_0(M_0, G)$ is induced, and is called the *canonical connexion* on the universal bundle $P_0(M_0, G)$.

Setting $\tau=\omega_{0\mathfrak{n}}$, we have the decomposition

$$\omega_0=\omega+\tau,$$

and get the Gauss-Codazzi equation

$$\Omega_0=\Omega+D\tau+\frac{1}{2}[\tau, \tau],$$

where Ω is the curvature form of ω and D denotes the covariant differentiation with respect to ω. The form τ is a tensorial form on $P_0(M_0, G)$ of type $(ad, \mathfrak{n})$. Setting $D\tau = T$, we have

$$DT = [\Omega, \tau] \quad \text{and} \quad D\Omega = 0,$$

which are respectively Ricci's and Bianchi's identities.

The canonical connexions on universal bundles may be employed in studies of characteristic classes. Cf. [2], pp. 59~64.

§ 6. The Stiefel-Whitney characteristic classes

Let $P(M, SO(m))$ be a differentiable principal bundle over a compact manifold M with structure group $SO(m)$. We take its associated bundle $E^k(M, Y^k, SO(m))$, whose fibre is the Stiefel manifold

$$Y^k = SO(m)/SO(k), \quad 1 \leq k < m.$$

The obstruction class W^{k+1} of the bundle E^k is a $(k+1)$-dimensional cohomology class of M with coefficients in the homotopy group $\pi_k(Y^k)$, and is called the $(k+1)$-th Stiefel-Whitney characteristic class of $P(M, SO(m))$.

We suppose that the group $SO(q)$ operates on a q-dimensional vector space V^q. Denoting by

$$\Lambda = \sum_{i=0}^{q} \Lambda^i$$

the exterior algebra generated by V^q, we have the identifications

$$\Lambda^2 = \mathfrak{so}(q) \quad \text{and} \quad \Lambda^q = R: \text{ the real numbers.}$$

We take a Λ-valued k-linear function F defined by

$$F(x_1, \cdots, x_k) = x_1 \wedge \cdots \wedge x_k, \quad x_1, \cdots, x_k \in \Lambda,$$

and set

$$\theta^k = F(\theta, \cdots, \theta),$$

for any Λ-valued form θ on a manifold.

Assume that a connexion ω is given on $P(M, SO(m))$. By the natural projections

$$P \rightarrow E^q \rightarrow M,$$

we get a bundle $P(E^q, SO(q))$. Since Y^q is reductive, there exists a canonical decomposition

$$\mathfrak{so}(m)=\mathfrak{so}(q)+\mathfrak{y},$$

and we have an induced connexion $\omega^{(q)}=\omega_{\mathfrak{so}(q)}$ on $P(E^q, SO(q))$. Let $\Omega^{(q)}$ denote the curvature form of $\omega^{(q)}$, and set

$$\Theta^q=c_q(\Omega^{(q)})^{q/2}, \quad \text{if } q \text{ is even,}$$
$$=0, \quad \text{if } q \text{ is odd,}$$

where $c_q=(-1)^{q/2}/2^q\pi^{q/2}(q/2)!$. Then Θ^q reduces to a real-valued q-form on E^q. Moreover, we consider the natural projections

$$P\to E^{q-1}\to E^q.$$

Since $(q-1)$-sphere $SO(q)/SO(q-1)$ is symmetric, there exists a canonical decomposition

$$\mathfrak{so}(q)=\mathfrak{so}(q-1)+\mathfrak{v},$$

where $\mathfrak{v}$ is a $(q-1)$-dimensional vector space. Taking the decomposition

$$\omega^{(q)}=\omega^{(q-1)}+\psi, \quad \psi=\omega^{(q)}_{\mathfrak{v}},$$

we obtain the Gauss-Codazzi equations:

$$\Omega^{(q)}_{\mathfrak{so}(q-1)}=\Omega^{(q-1)}+\psi\wedge\psi,$$
$$D\psi=\Omega^{(q)}_{\mathfrak{v}},$$

where D denotes the covariant differentiation with respect to the connexion $\omega^{(q-1)}$ on $P(E^{q-1}, SO(q-1))$. Now, we set

$$\Pi^{q-1}=a_q\sum_{k\geq 0} b_k(\Omega^{(q)}_{\mathfrak{so}(q-1)})^k\wedge\psi^{q-2k-1}$$

where $a_q=(-1)^q/2^q\pi^{(q-1)/2}$, $b_k=(-1)^k/k!\ \Gamma((q-2k+1)/2)$. Then Π^{q-1} reduces to a real-valued $(q-1)$-form on E^{q-1}, and we can easily show that

$$-d\Pi^{q-1}=-D\Pi^{q-1}=\Theta^q,$$

taking account of the Bianchi identity for the connexion $\omega^{(q-1)}$. The obstruction cocycles of the bundles E^{q-1} can be expressed in terms of the forms Θ^q and Π^{q-1}. Cf. [10], [11].

§ 7. The reductive Cartan connexion

We consider now a differentiable fibre bundle $E(M, F, G^*)$ satisfying the following conditions.

(i) *The fibre F is a reductive homogeneous space $F=G^*/G$, $G\subset G^*$, and* $\dim F=\dim M$.

(ii) *There exists a differentiable cross-section $\alpha: M\to E$.*

We take a canonical decomposition of the Lie algebra of G^*:

$$\mathfrak{g}^*=\mathfrak{g}+\mathfrak{k}.$$

Let ω^* be a connexion on the associated principal bundle $P^*(M, G^*)$ of $E(M, F, G^*)$. Then, by the natural projections

$$P^*\to E\to M,$$

we have an induced connexion $\omega_1{}^*=\omega_{\mathfrak{g}}{}^*$ on $P^*(E, G)$ and the decomposition

$$\omega^*=\omega_1{}^*+\psi^*, \quad \psi^*=\omega_{\mathfrak{k}}{}^*.$$

Moreover, we have the Gauss-Codazzi equation

$$\Omega_1{}^*+D_1{}^*\psi^*=-\frac{1}{2}[\psi^*, \psi^*]+\Omega^*,$$

where Ω^*, $\Omega_1{}^*$ are curvature forms of ω^*, $\omega_1{}^*$ respectively, and $D_1{}^*$ denotes the covariant differentiation with respect to $\omega_1{}^*$. The injection $\alpha: M\to E$ induces a bundle structure $P(M, G)$ over M from the bundle $P^*(E, G)$. Denoting by ω, ω_1, ψ the restrictions on $P(M, G)$ of ω^*, $\omega_1{}^*$, ψ^* respectively, we have

$$\omega=\omega_1+\psi.$$

Properly, ω_1 becomes a connexion on $P(M, G)$, and ψ is a tensorial form on $P(M, G)$ of type $(ad, \mathfrak{k})$. In the case that ψ maps $T_p(P)$ onto $\mathfrak{k}$ for each $p\in P$, ψ defines a soldered structure of $E(M, F, G^*)$ and ω^* becomes a *reductive Cartan connexion*. The form ψ is called the *basic form* of soldered structure. The *torsion form* Θ is given by the Codazzi equation:

$$\Theta=D_1\psi=-\frac{1}{2}[\psi, \psi]_{\mathfrak{k}}+\Omega^{\mathfrak{k}},$$

where Ω is the restricted curvature form of ω^* and D_1 denotes the covariant differentiation with respect to ω_1. If F is a symmetric homogeneous space, Θ coincides with $\Omega_{\mathfrak{k}}$. Cf. [4], [6], [12].

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Added in Proof.

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