MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES A
Vol. XXXI, Mathematics No. 1, 1958.

On exact sequences in Steenrod algebra mod. 2

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(Received December 10, 1957)

A Steenrod algebra A^* will mean a stable Eilenberg-MacLane cohomology group $A^*(Z_2, Z_2) = \lim H^*(Z_2, n; Z_2)$ in which the multiplication is defined by the composition of the squaring operations Sq^t. The formula $\varphi_a(b) = ba$ associates for each element a of A^* an (additive) homomorphism $\varphi_a : A^* \to A^*$. We write $\varphi_a = \varphi_t$ if $a = Sq^t$, then $A^*(Z, Z_2) = A^*/\varphi_1 A^*$. We shall give an elementary proof of the following

Theorem I. *The following two sequences of homomorphisms are exact.*

$$\begin{array}{ccc} A^* & \xrightarrow{\varphi_2} & A^* \\ \uparrow{\scriptstyle \varphi_3} & & \downarrow{\scriptstyle \varphi_2} \\ A^*/\varphi_1 A^* & \xleftarrow{\varphi_5} & A^*/\varphi_1 A^* , \end{array}$$

$$A^*/\varphi_1 A^* \xrightarrow{\varphi_3} A^*/\varphi_1 A^* \xrightarrow{\varphi_3} A^*/\varphi_1 A^* .$$

Several exact sequences are known experimentally for lower dimensions. For example, it seems that the sequence

$$A^* \xrightarrow{\varphi_{2^r}} A^*/(\sum_{i=0}^{r-2} \varphi_{2^i} A^*) \xrightarrow{\varphi_{2^r}} A^*/(\sum_{i=0}^{r-1} \varphi_{2^i} A^*)$$

is exact. More generally we propose

Problem. *Let* $a, b_1, \cdots, b_r \in A^*$. *Is the kernel of* $\varphi_a : A^* \to A^*/(\sum_{i=1}^{r} \varphi_{b_i} A^*)$ *finitely generated (as a left ideal)?*

In place of φ_a, take a homomorphism φ_a^* defined by the formula $\varphi_a^*(b) = ab$, then the exactness of analogous sequences is proved by T. Yamanoshita and A. Negishi (*cf.* [5]).

Theorem II. *Let $B^*=\sum B^i$ be one of the five kernel-images in the exact sequences of Theorem I, then in the sequence*

$$B^{i-1}\xrightarrow{\varphi_1^*}B^i\xrightarrow{\varphi_1^*}B^{i+1}$$

we have $$(\varphi_1^*)^{-1}(0)/\varphi_1^*(B^{i-1})\approx\begin{cases}Z_2 & \text{for } i\equiv\lambda \pmod 4),\ i\geqq 2,\\ 0 & \text{otherwise,}\end{cases}$$

where λ takes the following values:

when	$B^*=$ *image of*	φ_2	φ_2	φ_5	φ_3	φ_3
	$=$ *kernel of*	φ_2	φ_5	φ_3	φ_2	φ_3
then	$\lambda=$	0	1	1	3	1 or 3.

The above two theorems are proved in §2 under some preparations in §1. In §3, we see some partial exact sequences, which are applied in §4 to study the cohomology of fibre spaces over a sphere and to calculate the following values of 2-components of the stable homotopy groups $\pi_k=\lim\pi_{k+n}(S^n)$ of the sphere:

$k=$	1	2	3	4	5	6	7	8	9	10	11	12	13
2-comp. of π_k	Z_2	Z_2	Z_8	0	0	Z_2	Z_{16}	Z_2+Z_2	$Z_2+Z_2+Z_2$	Z_2	Z_8	0	0.

We have also a partial result on π_{14} which will be useful for determining the groups π_{14} and π_{15}.

§1. Steenrod algebra $A^*=A^*(Z_2, Z_2)$.

Consider a sequence $\mathfrak{X}=\{X_k, f_k;\ k=N, N+1, N+2, \cdots\}$ which satisfies the conditions

(1.1). i) *X_k are $(k-1)$-connected spaces.*

ii) *f_k are mappings of the suspensions $S(X_k)$ of X_k in X_{k+1}.*

iii) *For each integer i, there exists an integer $\lambda(i)$ such that $f_{k*}:\pi_{i+k+1}(S(X_k))\to\pi_{i+k+1}(X_{k+1})$ are isomorphisms for $k\geqq\lambda(i)$.*

Then it is verified that the condition iii) may be replaced by the same condition for homology groups. Denote that

$$G_i(\mathfrak{X})=\text{Dir. lim}\ \{\pi_{i+k}(X_k),\ f_{k*}\circ S\}\ ,$$
$$A_i(\mathfrak{X})=\text{Dir. lim}\ \{H_{i+k}(X_k),\ f_{k*}\circ S_*\}\ ,$$
$$A^i(\mathfrak{X})=\text{Inv. lim}\ \{H^{i+k}(X_k),\ S^*\ f_k^*\}\ ,$$

where S, S_* and S^* denote the suspension homomorphisms. Remark that these groups can be defined without the condition iii). By the condition iii), we may regard that

$$(1.2)\qquad \begin{aligned} G_i(\mathfrak{X}) &= \pi_{i+k}(X_k)\,, \\ A_i(\mathfrak{X}) &= H_{i+k}(X_k)\,, \\ A^i(\mathfrak{X}) &= H^{i+k}(X_k)\,, \end{aligned}$$

for sufficiently large k. Cohomological operations which commute with f_k^* and S^* are naturally defined in $A^i(\mathfrak{X})$. For example, the squaring operation $Sq^t: A^i(\mathfrak{X}, Z_2) \to A^{i+t}(\mathfrak{X}, Z_2)$ is defined. The groups $G_i(\mathfrak{X})$, $A_i(\mathfrak{X})$ and $A^i(\mathfrak{X})$ are called *the stable homotopy, homology* and *cohomology groups of* $\mathfrak{X}$ respectively.

The *i-th stable homotopy group* π_i *of the sphere* is defined by

$$\pi_i = G_i(\mathfrak{S})$$

where $\mathfrak{S} = \{S^k, i_k\}$ is a sequence consists of the k-spheres S^k and the identities of $S^{k+1} = S(S^k)$. It is well known that $\pi_i = \pi_{i+N}(S^N)$ for $N > i+1$ under the convension (1.2).

The *i-th stable Eilenbrg-MacLane homology group* $A_i(\pi)$ and *cohomology group* $A^i(\pi, Z_2)$ *of an abelian group* π are defined by

$$A_i(\pi) = A_i(\mathfrak{K}(\pi)) \quad \textit{and} \quad A^i(\pi, Z_2) = A^i(\mathfrak{K}(\pi), Z_2)\,,$$

where $\mathfrak{K}(\pi)$ consists of Eilenberg-MacLane spaces $K(\pi, k)$ and mappings $f_k: S(K(\pi, k)) \to K(\pi, k+1)$ which induce isomorphisms of $(k+1)$-th homotopy groups. It is well known that $A_i(\pi) = H_{i+N}(\pi, N)$ and $A^i(\pi, Z_2) = H^{i+N}(\pi, N; Z_2)$ for $N \geqq i+1$ under the convension (1.2).

A symbol I will denote a finite sequence $I = (i_1, \cdots, i_r)$ of positive integers. It is convenient to introduce the empty sequence $I = (\phi)$. We use the following notations:

$$\begin{aligned} &\deg I = i_1 + \cdots + i_r \ \textit{(degree of } I\textit{)}\,, \qquad \deg(\phi) = 0\,, \\ &l(I) = r \ \textit{(length of } I\textit{)}\,, \qquad l(\phi) = 0\,, \\ &t_j(I) = i_j \ \textit{(j-th element)}\,, \\ &t(I) = i_r = t_{l(I)} \ \textit{(last element)}\,. \end{aligned}$$

A sequence $I = (i_1, \cdots, i_r)$ is called to be *admissible* if $i_j \geqq 2i_{j+1}$ for $j = 1, \cdots, r-1$.

By Serre's work [4], the stable Eilenberg-MacLane cohomology group $A^i(Z_2, Z_2)$ has its Z_2-base $\{Sq^I u\}$, where $Sq^I = Sq^{i_1} \circ \cdots \circ Sq^{i_r}$, I is admissible, $\deg I = i$ and u is the fundamental class of $A^0(Z_2, Z_2)$. For an arbitrary sequence I, $Sq^I u$ belongs to $A^i(Z_2, Z_2)$, $i = \deg I$. Thus $Sq^I u$ is a sum of admissible squares $Sq^{I_k} u$. The result $Sq^I u = \sum Sq^{I_k} u$ is obviously unique and is called the *normalization* of $Sq^I u$.

For the simplicity, we set

$$(1.3) \qquad Sq^I u = I, \quad A^i(Z_2, Z_2) = A^i \quad \textit{and} \quad A^* = \sum A^i .$$

Then A^* is a graded Z_2-module generated by the sequences I with the relation determined by the normalization $I = \sum I_k$. Set

$$IJ = (i_1, \cdots, i_r, j_1, \cdots, j_s)$$

for $I = (i_1, \cdots, i_r)$ and $J = (j_1, \cdots, j_s)$, then a multiplication is defined in A^*, since the product IJ corresponds to the composition $Sq^I \circ Sq^J$ of the squaring operations. Now A^* becomes a graded algebra over Z_2, namely *Steenrod algebra* mod 2.

When $l(I) = 2$, the normalization process is given precisely by the Adem's relations [1], [2]:

$$(1.4) \qquad (2h-m, h) = \sum_{h-m+t \geq 0} \binom{m-t-1}{t-1}_2 (2h-t, h-m+t),$$

where $m > 0$, $\binom{a}{b}_2 = \binom{a}{a-b}_2$ is the binomial coefficient mod 2 with the convension $\binom{a}{b}_2 = 0$ if $b < 0$ and we omit the term $h-m+t$ if $h-m+t=0$. The coefficient $\binom{m-t-1}{t-1}_2 = \binom{m-t-1}{m-2t}_2$ vanishes if $t-1 < 0$ or $m-2t < 0$. Therefore the summation of (1.4) is valid for the following values of t:

$$(1.5) \qquad \text{Max.}\,(1, m-h) \leq t \leq m/2 .$$

Since $t \leq m/2 < 2m/3$, we have $2h-t > 2(h-m+t)$. Thus the relation (1.4) gives the normalization of $(2h-m, h)$.

Lemma 1.1. *Each sequence I is normalized by use of the Adem's relations. The normalization preserves the degree and does not augument the length.*

Proof. By (1.4), the lemma is true for $l(I) = 2$. Put $I = (i_1, \cdots, i_r)$, $r > 2$ and assume that the lemma is true for $l(I) < r$ and for $t_1(I) > i_1$ (and $l(I) = r$). Then $(i_2, \cdots, i_r)$ is normalized.

Thus we may assume that $(i_2, \cdots, i_r)$ is admissible. If $i_1 \geqq 2i_2$, I is already admissible. If $i_1 < 2i_2$, (i_1, i_2) is normalized to $\sum (a_k, b_k)$ where $a_k + b_k = i_1 + i_2$ and $a_k \geqq 2b_k$. Then $a_k > i_1$ and $I = \sum (a_k, b_k, i_3, \cdots, i_r)$, and each term of the summation is normalized by the assumption. Obviously these processes preserve the degree and do not augument the length. The lemma is proved inductively since $t_1(I) \leqq \deg I$.

Lemma 1.2. *Let $I = (i_1, \cdots, i_r)$ be an admissible sequence and let i be a positive integer less than $2i_1$. Let $\sum I_j$ be the normalization of $(i)I = (i, i_1, \cdots, i_r)$. Then $t_1(I_j) \leqq 2i_1 - 1$ for all j. If $t_1(I_j) = 2i_1 - 1$, then $I_j = (2i_1 - 1, i - i_1 + 1, i_2, \cdots, i_r)$ or $I_j = (2i_1 - 1, i_2, \cdots, i_r)$. The term $I_j = (2i_1 - 1, i - i_1 + 1, i_2, \cdots, i_r)$ exists if and only if $2(i_1 - 1) \geqq i \geqq i_1 - 1$ and $i - i_1 + 1 \geqq 2i_2$. The term $I_j = (2i_1 - 1, i_2, \cdots, i_r)$ exists if and only if $i = i_1 - 1$.*

Proof. First consider the case $r = 1$. By (1.4) and (1.5), each term I_j has a form $(2i_1 - t, i - i_1 + t)$ for $\mathrm{Max.}(1, i_1 - i) \leqq t \leqq i_1 - i/2$. Then $t_1(I_j) = 2i_1 - t \leqq 2i_1 - 1$. If $t_1(I_j) = 2i_1 - 1$ then $t = 1$ and whence the condition $i_1 - i \leqq t \leqq i_1 - i/2$ implies that $2(i_1 - 1) \geqq i \geqq i_1 - 1$. Conversely, if $2(i_1 - 1) \geqq i \geqq i_1 - 1$ then the coefficient $\binom{2i_1 - i - 2}{0}_2$ of $(2i_1 - 1, i - i_1 + 1)$ equals to 1. Therefore the lemma is true for $r = 1$.

Now let $r > 1$ and assume that the lemma is true for $l(I) < r$ and for $l(I) = r$ and $t_1(I) < i_1$. Applying the lemma of the case $r = 1$, we have that $(i, i_1, \cdots, i_r)$ is a sum of some $J_t = (2i_1 - t, i - i_1 + t, i_2, \cdots, i_r)$ and the term J_1 appears if and only if $2(i_1 - 1) \geqq i \geqq i_1 - 1$. The term J_1 is admissible if $i - i_1 + 1 \geqq 2i_2$ and also if $i - i_1 + 1 = 0$ since $2i_1 - 1 \geqq i_1 \geqq 2i_2$. In the case $0 < i - i_1 + 1 < 2i_2$, applying the lemma to $(i - i_1 + 1)(i_2, \cdots, i_r)$, $(i - i_1 + 1, i_2, \cdots, i_r)$ is normalized to $\sum (a_k, b_k, \cdots)$ such as $a_k \leqq 2i_2 - 1$. Since $a_k \leqq 2i_2 - 1 \leqq i_1 - 1 < i_1$, we may apply the lemma to $(2i_1 - 1)(a_k, b_k, \cdots)$ by the assumption. Then $J_1 = (2i_1 - 1, i - i_1 + 1, i_2, \cdots, i_r)$ is normalized to $\sum I_j$ such as $t_1(I_j) \leqq 2a_j - 1 < 2i_1 - 1$. Therefore, when $0 < i - i_1 + 1 < 2i_2$, the normalization of J_1 has no term I_m of $t_1(I_m) = 2i_1 - 1$. Next consider the term J_t for $t < 1$. If J_t is not admissible, by similar arguments to the case $t = 1$, we have that the normalization of J_t consists of I_j such as $t_1(I_j) < 2i_1 - 1$. If J_t is admissible, $t_1(J_t) = 2i_1 - t < 2i_1 - 1$. Consequently we see that the lemma is proved by the induction since $t_1(I) \geqq 2^{l(I)-1}$.

Lemma 1.3. *Let $I=(i_1, \cdots, i_r)$ be admissible and $s\geqq 0$. If $2^{s+r-j}\leqq i_j\leqq i^{s+r-j+1}$ for $j=1, \cdots, r$, then the normalization of $\varphi_{2^s+1}I=(i_1, \cdots, i_r, 2^s+1)$ is $(2^{s+r}+1, i_1-2^{s+r-1}, i_2-2^{s+r-2}, \cdots, i_r-2^s)+\sum(a_k, b_k, \cdots)$ where $a_k\leqq 2^{s+r}$ and i_j-2^{s+r-j} are omitted if $i_j=2^{s+r-j}$.*

Proof. Then lemma is obvious for $r=0$. Suppose that the lemma is true for $l(I)=r-1$. Then $\varphi_{2^s+1}I=(i_1)(i_2, \cdots, i_r, 2^s+1)=(i_1, 2^{s+r-1}+1, i_2-2^{s+r-2}, \cdots, i_r-2^s)+\sum(i_1, a_k', b_k', \cdots)$ and $a'_k\leqq 2^{s+r-1}$. By Lemma 1.2, each term I_m' of the normalization $\sum I_m'$ of $(i_1, a_k', b_k', \cdots)$ satisfies $t_1(I_m')\leqq 2^{s+r}-1$. Next the term $(i_1)(2^{s+r-1}+1, i_2-2^{s+r-2}, \cdots, i_r-2^s)$ satisfies the conditions $2(2^{s+r-1}+1-1)\geqq i_1\geqq 2^{s+r-1}+1-1$ and $i_1-(2^{s+r-1}+1)+1\geqq 2(i_2-2^{s+r-2})=2i_2-2^{s+r-1}$ of Lemma 1.2, by the assumption of this lemma. Then the normalization of $(i_1, 2^{s+r-1}+1, i_2-2^{s+r-2}, \cdots, i_r-2^s)$ is $(2^{s+r}+1, i_1-2^{s+r-1}, i_2-2^{s+r-2}, \cdots, i_r-2^s)+\sum(a_l, b_l, \cdots)$ where $a_l\leqq 2^{s+r}$. Therefore the lemma is proved by the induction on $l(I)$.

For the convenience, we note some relations obtained directly from (1.4).

$$
\begin{aligned}
&(1, 2i)=(2i+1), \quad (1, 2i-1)=0,\\
&(2, 2)=(3, 1), \quad (3, 2)=0,\\
(1.6)\quad &(2, 3)=(5)+(4, 1), \quad (3, 3)=(5, 1), \quad (4, 3)=(5, 2), \quad (5, 3)=0,\\
&(2, 4)=(6)+(5, 1), \quad (3, 4)=(7), \quad (4, 4)=(7, 1)+(6, 2), \cdots,\\
&(2, 5)=(6, 1), \quad (3, 5)=0, \quad (4, 5)=(9)+(8, 1)+(7, 2), \cdots.
\end{aligned}
$$

§ 2. Proof of Theorems.

The formula $\varphi_a(b)=ba$ defines a homomorphism φ_a of the left A^*-modules. In particular, for an integer t the homomorphism $\varphi_{(t)}$, denoted by φ_t, is defined by $\varphi_t(i_1, \cdots, i_r)=(i_1, \cdots, i_r, t)$.

By (1.6), $\varphi_1(i_1, \cdots, i_r)=0$ if $i_r=1$. If $i_r>1$, then $\varphi_1(i_1, \cdots, i_r)=(i_1, \cdots, i_r, 1)$ is admissible. Thus the sequence

$$(2.1)\qquad A^*\xrightarrow{\varphi_1}A^*\xrightarrow{\varphi_1}A^*$$

is exact. The kernel-image $\varphi_1(A^*)=\varphi_1^{-1}(0)$ of the sequence has the admissible sequences I of the last element $t(I)=1$ as its Z_2-base. The factor group A^*/φ_1A^* has a Z_2-base $\{I \mid \text{admissible}, t(I)\geqq 2\}$.

For an odd t, $\varphi_t\circ\varphi_1=\varphi_{(1,t)}=0$ by (1.6). Then φ_t defines an A^*-homomorphism of A^*/φ_1A^* into A^* which will be denoted by the same symbol

$$\varphi_t : A^*/\varphi_1 A^* \to A^* .$$

We denote the composition of φ_t and the natural homomorphism of A^* onto $A^*/\varphi_1 A^*$ by

$$\bar{\varphi}_t : A^* \to A^*/\varphi_1 A^* ,$$
$$\bar{\varphi}_t : A^*/\varphi_1 A^* \to A^*/\varphi_1 A^*, \qquad t: \textit{odd}.$$

Now the first theorem is stated as follows.

Theorem 1. *The following sequences are exact.*

i) $A^* \xrightarrow{\varphi_2} A^* \xrightarrow{\bar{\varphi}_2} A^*/\varphi_1 A_* ,$

ii) $A^* \xrightarrow{\bar{\varphi}_2} A^*/\varphi_1 A^* \xrightarrow{\bar{\varphi}_5} A^*/\varphi_1 A^* ,$

iii) $A^*/\varphi_1 A^* \xrightarrow{\bar{\varphi}_5} A^*/\varphi_1 A^* \xrightarrow{\varphi_3} A^* ,$

iv) $A^*/\varphi_1 A^* \xrightarrow{\varphi_3} A^* \xrightarrow{\varphi_2} A^* ,$

v) $A^*/\varphi_1 A^* \xrightarrow{\bar{\varphi}_3} A^*/\varphi_1 A^* \xrightarrow{\bar{\varphi}_3} A^*/\varphi_1 A^* .$

We introduce the following notations:

$\alpha_i =$ (*the rank of* A^i) $=$ (*the number of the admissible sequences of a degree* i),

$\bar{\alpha}_i =$ (*the rank of* $A^i/\varphi_1 A^{i-1}$) $=$ (*the number of the admissible sequences* I *of a degree* i *such that* $t(I) \geqq 2$),[1]

$\beta_i(t) =$ (*the rank of the image* $\varphi_t(A^{i-t})$ *in* A^i),

$\bar{\beta}_i(t) =$ (*the rank of the image* $\bar{\varphi}_t(A^{i-t})$ *in* $A^i/\varphi_1 A^{i-1}$).

An admissible sequence $I=(i_1, \cdots, i_r)$ is called to be of *a type*[1] (i, s) if $\deg I = i$ and if there exist integers j and t such that $i_j = 2^t+1$, $1 \leqq j \leqq r$ and $t \geqq s+(r-j)$. Obviously an admissible sequences of a type (i, s) is of a type (i, s') for $s' \leqq s$. Denote that

$\gamma_i(2^s+1) =$ (*the number of the admissible sequnces of a type* (i, s)),

$\bar{\gamma}_i(2^s+1) =$ (*the number of the admissible sequences* I *of a type* (i, s) *such that* $t(I) \geqq 2$).[1]

1) We consider that the empty sequence (ϕ) satisties the condition $t(I) \geqq 2$ and has a type $(0, s)$ for arbitrary s.

For an admissible sequence $I=(i_1, \cdots, i_r)$ of a type (i, s), we define an admissible sequence $\sigma_{2^s+1}I$ as follows. Let j be the least integer such that i_j has a form 2^t+1. Then $t-(r-j)\geqq s$. For, there are t' and $j'\geqq j$ such that $i_{j'}=2^{t'}+1$ and $t'-(r-j')\geqq s$, then $2^t+1=i_j\geqq 2^{j'-j}i_{j'}\geqq 2^{t'+j'-j}+1$ implies $t-(r-j)\geqq t'-(r-j')\geqq s$. Then we set

$$(2.2)\quad \sigma_{2^s+1}I=(i_1, \cdots, i_{j-1}, i_{j+1}+2^{t-1}, \cdots, i_r+2^{t-(r-j)}, 2^{t-(r-j)-1}, \cdots, 2^s).$$

It is easily verified that the sequence $\sigma_{2^s+1}I$ is admissible.

For an admissible sequence $I=(i_1, \cdots, i_r)$ such that $i_r\geqq 2^s$, we define a sequence $\tau_{2^s+1}I$ as follows. Let k be the largest integer such that $i_k>2^{s+r-k+1}$. We set $k=0$ if $i_j\leqq 2^{s+r-j+1}$ for $1\leqq j\leqq r$. Now we set

$$(2.3)\quad \tau_{2^s+1}I=(i_1, \cdots, i_k, 2^{s+r-k}+1, i_{k+1}-2^{s+r-(k+1)}, \cdots, i_r-2^s),$$

where we omit $i_{k+n}-2^{s+r-(k+n)}$ if $i_{k+n}=2^{s+r-(k+n)}$. It is easily seen that $\tau_{2^s+1}I$ is admissible if and only if $i_k\neq 2^{s+r-k+1}+1$.

Lemma 2.1. i) *Let I be an admissible sequence of a type $(i+2^s+1, s)$. Then $t(\sigma_{2^s+1}I)\geqq 2^s$, $\tau_{2^s+1}(\sigma_{2^s+1}I)=I$ and $\sigma_{2^s+1}I$ is not a type $(i, s+1)$. If $t(I)\geqq 2$, then $\sigma_{2^s+1}I$ is not a type $(i, 1)$. If $t(I)\geqq 2$ and $s\neq 1$, then $\sigma_{2^s+1}I$ is not a type $(i, 0)$.*

ii) *Let I be an admissible sequence of a degree i which is not a type $(i, s+1)$ and which has a last element $t(I)\geqq 2^s$. Then $\tau_{2^s+1}I$ is an admissible sequence of a type $(i+2^s+1, s)$ and we have $\sigma_{2^s+1}(\tau_{2^s+1}I)=I$. Furthermore $t(\tau_{2^s+1}I)\geqq 2$ if I is not a type (i, s).*

Proof. i) Let $\sigma_{2^s+1}I$ be defined by (2.2). Obviously $t(\sigma_{2^s+1}I)\geqq 2^s$. We set $l(\sigma_{2^s+1}I)=t-s+j-1=r'$. Since $i_{j+n}\leqq(2^t+1)/2^n=2^{t-n}+2^{-n}$, $n=1, \cdots, r-j$, we have $t_{j+n-1}(\sigma_{2^s+1}I)=i_{j+n}+2^{t-n}\leqq 2^{t-n+1}=2^{s+r'-(j+n-1)+1}$. Also $t_{j-1}(\sigma_{2^s+1}I)=i_{j-1}\geqq 2(2^t+1)>2^{t+1}=2^{s+r'-(j-1)+1}$. Then it is verified directly from (2.3), where $k=j-1$, that $\tau_{2^s+1}(\sigma_{2^s+1}I)=I$. Next consider the type of $\sigma_{2^s+1}I$. For $1\leqq n\leqq j-1$, i_n is not a form 2^p+1. Since $i_{j+n}\leqq 2^{t-n}+2^{-n}$, $n=1, \cdots, r-j$, if $i_{j+n}+2^{t-n}=2^p+1$, then $i_{j+n}=i_r=1$ and $p=t-n=t-(r-j)$. In this case, however, the condition of the type $(i, s+1)$ is not satisfied, since $p=s+(r'-(r-1))<s+1+(r'-(r-1))$. The elements $2^{t-(r-j)-1}, \cdots, 2^s$ are not forms 2^p+1 except for $2^1=2^0+1$ whence $s=0$ or 1. When $s=0$, we have $2^0+1=t_{r'-1}(\sigma_{2^s+1}I)$ and this does not satisfy the condition of the type $(i, 0)$ since $0<0+(r'-(r'-1))=1$. When $s=1$, we have $2^0+1=t_{r'}(\sigma_{2^s+1}I)$ and this does not satisfy the condition of the type $(i, 1)$ since $0<1+(r'-r')=1$. Consequently $\sigma_{2^s+1}I$ is not a type $(i, s+1)$. In the case $t(I)=i_r\geqq 2$, the only element of a form 2^p+1 is $2^1=2^0+1$. Then $\sigma_{2^s+1}I$ is not a type $(i, 1)$ and further not a type $(i, 0)$ if $s\neq 1$.

ii) Let $\tau_{2^s+1}I$ be defined by (2.3). Since I is not a type $(i, s+1)$, $i_n=2^p+1$ implies $p<s+1+(r-n)=s+r-n+1$ and $i_n\leqq 2^{s+r-n+1}$. From $i_k>2^{s+r-k+1}$ we have $i_n\geqq 2^{k-n}i_k>2^{s+r-n+1}$ for $n\leqq k$. Therefore i_n is not a form 2^p+1 for $1\leqq n\leqq k$. In particular, $i_k\neq 2^{s+r-k+1}+1$ and this shows that $\tau_{2^s+1}I$ is admissible. Since $l(\tau_{2^s+1}I)\leqq r+1$, we have $t_{k+1}(\tau_{2^s+1}I)=2^{s+r-k}+1$ and $s+r-k\geqq s+(l(\tau_{2^s+1}I)-(k+1))$. Thus $\tau_{2^s+1}I$ has a type $(i+2^s+1, s)$. Since $k+1$ is the least integer such that $t_{k+1}(\tau_{2^s+1}I)$ is of a form 2^p+1, it is verified directly from (2.2) that $\sigma_{2^s+1}(\tau_{2^s+1}I)=I$. Next suppose that $t(\tau_{2^s+1}I)=1$, then $i_{k+n}=2^{s+r-(k+n)}+1$, $i_{k+n+1}=2^{s+r-(k+n+1)}$, $\cdots$, $i_r=2^s$ for some n, and this indicates that I has a type (i, s). Therefore $t(\tau_{2^s+1}I)\geqq 2$ if I is not a type (i, s). q. e. d.

Lemma 2. 2. i) $\gamma_i(2)+\bar{\gamma}_{i+2}(2)=\alpha_i$,
ii) $\bar{\gamma}_i(2)+\bar{\gamma}_{i+5}(5)=\bar{\alpha}_i$,
iii) $\bar{\gamma}_i(5)+\gamma_{i+3}(3)=\bar{\alpha}_i$,
iv) $\gamma_i(3)+\gamma_{i+2}(2)=\alpha_i$,
v) $\bar{\gamma}_i(3)+\bar{\gamma}_{i+3}(3)=\bar{\alpha}_i$.

Proof. i) $\bar{\gamma}_{i+2}(2)$ is the number of the admissible sequences I of a type $(i+2, 0)$ such that $t(I)\geqq 2$. $\alpha_i-\gamma_i(2)$ is the number of the admissible sequences J of the degree i which is not a type $(i, 0)$. By Lemma 2. 1, i), $\sigma_2 I$ is not a type $(i, 0)$ and $\tau_2(\sigma_2 I)=I$. By Lemma 2. 1, ii), $\tau_2 J$ is an admissible sequence of a type $(i+2, 0)$ such that $\sigma_2(\tau_2 J)=J$ and $t(\tau_2 J)\geqq 2$. Therefore σ_2 and τ_2 are the inverses of the others, and we have $\bar{\gamma}_{i+2}(2)=\alpha_i-\gamma_i(2)$.

ii) Let I be an admissible sequence of a type $(i+5, 2)$ such that $t(I)\geqq 2$. Let J be an admissible sequence of the degree i which is not a type $(i, 0)$ and which satisfies $t(J)\geqq 2$. By Lemma 2. 1, i), $\sigma_5 I$ is not a type $(i, 0)$, $\tau_5(\sigma_5 I)=I$ and $t(\sigma_5 I)\geqq 2^2\geqq 2$. Since J is not a type $(i, 0)$, we have $t(J)\neq 2=2^0+1$ and $t(J)\neq 3=2^1+1$. Thus J is not a type $(i, 2)$ and $t(J)\geqq 4=2^2$. Then, by Lemma 2. 1, ii), $\tau_5 J$ is an admissible sequence of a type $(i+5, 2)$, $\sigma_5(\tau_5 J)=J$ and $t(\tau_5 J)\geqq 2$. σ_5 and τ_5 shows the equality $\bar{\gamma}_{i+5}(5)=\bar{\alpha}_i-\bar{\gamma}_i(2)$.

iii) Let I be an admissible sequence of a type $(i+3, 1)$. Let J be an admissible sequence of a degree i which is not a type

$(i, 2)$ and which satisfies $t(J)\geqq 2$. By Lemma 2.1, we have that $\sigma_3 I$ is not a type $(i, 2)$, $\tau_3(\sigma_3 I)=I$ and $t(\sigma_3 I)\geqq 2$ and that $\tau_3 J$ is an admissible sequence of a type $(i+3, 1)$ and $\sigma_3(\tau_3 J)=J$. Then $\gamma_{i+3}(3)=\bar{\alpha}_i-\bar{\gamma}_i(5)$.

The proofs of iv) and v) are similar to the above one and omitted. q. e. d.

Lemma 2.3. $\gamma_i(2^s+1)\leqq\beta_i(2^s+1)$ *and* $\bar{\gamma}_i(2^s+1)\leqq\bar{\beta}_i(2^s+1)$.

Proof. We order the sequences of A^i by the following rule. $I=(i_1, \cdots, i_r)>J=(j_1, \cdots, j_s)$ if $i_1=j_1, \cdots, i_{p-1}=j_{p-1}$ and $i_p>j_p$ for some p. First we prove that for an admissible sequence of a type (i, s) the following formula holds:

(2.4) $\qquad \varphi_{2^s+1}(\sigma_{2^s+1}I) = I+\sum I_k \quad$ *for some* $I_k<I$.

Let $\sigma_{2^s+1}I$ be given by (2.2), then its subsequence $(i_{j+1}+2^{t-1}, \cdots, 2^s)$ satisfies the condition of Lemma 1.3. By Lemma 1.3,

$$\begin{aligned}\varphi_{2^s+1}(\sigma_{2^s+1}I) &= (i_1, \cdots, i_{j-1})\,\varphi_{2^s+1}(i_{j+1}+2^{t-1}, \cdots, 2^s)\\ &=I+\sum (i_1, \cdots, i_{j-1}, a_k, b_k, \cdots)\,, \quad a_k\leqq 2^t\,,\\ &=I+\sum I_k\end{aligned}$$

for some $I_k=(i_1, \cdots, i_{j-1}, a_k, b_k, \cdots)<I$. Now assume that there is a relation $\varphi_{2^s+1}(\sigma_{2^s+1}I_1+\cdots+\sigma_{2^s+1}I_n)=0$ for some $I_1>I_2>\cdots>I_n$. Then by (2.4), $I_1+\sum J_m=0$ for some $J_m<I_1$ and this implies a contradiction $I_1=0$. Therefore $\varphi_{2^s+1}(\sigma_{2^s+1}I)$ are linearly independent for all sequences I of the type (i, s). Thus $\gamma_i(2^s+1)\leqq\beta_i(2^s+1)$. Another inequality $\bar{\gamma}_i(2^s+1)\leqq\bar{\beta}_i(2^s+1)$ is proved similarly.

Proof. of Theorem I. By (1.6), we have that $\bar{\varphi}_2\circ\varphi_2=\bar{\varphi}_5\circ\bar{\varphi}_2=\varphi_3\circ\bar{\varphi}_5=\varphi_2\circ\varphi_3=\bar{\varphi}_3\circ\bar{\varphi}_3=0$. From $\bar{\varphi}_2\circ\varphi_2=0$, we have $\varphi_2(A^{i-2})\subset\bar{\varphi}_2^{-1}(0)$. Thus $\beta_i(2)\leqq\alpha_i-\bar{\beta}_{i+2}(2)$. By Lemma 2.3 and 2.2, $\beta_i(2)\geqq\gamma_i(2)=\alpha_i-\bar{\gamma}_{i+2}(2)\geqq\alpha_i-\bar{\beta}_{i+2}(2)$. Therefore $\beta_i(2)=\alpha_i-\bar{\beta}_{i+2}(2)$ and this implies that $\varphi_2(A^{i-2})=\bar{\varphi}_2^{-1}(0)$. Then the exactness of the sequence i) of Theorem I is proved. The exactness of the other sequences ii)-v) is proved similarly. q. e. d.

Corollary. $\gamma_i(2)=\beta_i(2)$, $\bar{\gamma}_i(2)=\bar{\beta}_i(2)$, $\gamma_i(3)=\beta_i(3)$, $\bar{\gamma}_i(3)=\bar{\beta}_i(3)$ *and* $\bar{\gamma}_i(5)=\bar{\beta}_i(5)$.

Define a homomorphism $\varphi_u^*: A^*\to A^*$ by the formula $\varphi_u^*(i_1, \cdots, i_r)=(u, i_1, \cdots, i_r)$. Then

(2.5) $\qquad \varphi_u^*\circ\varphi_t=\varphi_t\circ\varphi_u^*\,.$

By (1.6), we have $\varphi_1^*(i_1, \cdots, i_r)=0$ for odd i_1 and $\varphi_1^*(i_1, \cdots, i_r) =(i_1+1, i_2, \cdots, i_r)$ for even i_1. Then it is easy to see that the sequence

$$A^{i-1} \xrightarrow{\varphi_1^*} A^i \xrightarrow{\varphi_1^*} A^{i+1}$$

in exact. Also we have an exact sequence

$$A^{i-1}/\varphi_1 A^{i-2} \xrightarrow{\varphi_1^*} A^i/\varphi_1 A^{i-1} \xrightarrow{\varphi_1^*} A^{i+1}/\varphi_1 A^i$$

for $i \geqq 1$. Define subgroups B_t^i and $\bar{B}_t^i$ by setting

$$B_t^i = \varphi_t(A^{i-t}) \subset A^i \quad \text{and} \quad \bar{B}_t^i = \bar{\varphi}_t(A^{i-t}) \subset A^i/\varphi_1 A^{i-1}.$$

By (2.5), $\varphi_1^*(B_t^i) \subset B_t^{i+1}$ and $\varphi_1^*(\bar{B}_t^i) \subset \bar{B}_t^{i+1}$. Since $\varphi_1^* \circ \varphi_1^* = 0$, A^*, $A^*/\varphi_1 A^*$, $B_t^* = \sum B_t^i$ and $\bar{B}_t^* = \sum \bar{B}_t^i$ are cochain complexes with respect to the coboundary operator $\delta = \varphi_1^*$. From the exactness of the above two sequences, we have

$$\begin{aligned} (2.6) \qquad H(A^i) &= 0 \qquad \textit{for} \quad i \geqq 0, \\ H(A^i/\varphi_1 A^{i-1}) &= 0 \qquad \textit{for} \quad i \geqq 1. \end{aligned}$$

From Theorem I and (2.5), we have an exact sequence

$$0 \to B_2^i \to A^i \to \bar{B}_2^{i+2} \to 0$$

which is compatible with φ_1^*. This induces the following cohomology exact sequence:

$$\cdots \to H(A^i) \to H(\bar{B}_2^{i+2}) \xrightarrow{\delta^*} H(B_2^{i+1}) \to H(A^{i+1}) \to \cdots.$$

Then, from (2.6), we have an isomorphism

(2.7), i) $\quad \delta^* : H(\bar{B}_2^{i+1}) \approx H(B_2^i) \quad$ *for all* i.

Similarly we have the following isomorphisms:

ii) $\quad \delta^* : H(\bar{B}_5^{i+4}) \approx H(\bar{B}_2^i) \quad$ *for* $i \geqq 2$,

iii) $\quad \delta^* : H(B_3^{i+2}) \approx H(\bar{B}_5^i) \quad$ *for* $i \geqq 2$,

iv) $\quad \delta^* : H(B_2^{i+1}) \approx H(B_3^i) \quad$ *for all* i,

v) $\quad \delta^* : H(\bar{B}_3^{i+2}) \approx H(\bar{B}_3^i) \quad$ *for* $i \geqq 2$.

From (1.6), we calculate easily that $\bar{B}_3^3 \approx B_3^3 \approx \bar{B}_5^5 \approx Z_2$ and $B_3^2 = \bar{B}_3^2 = B_3^4 = \bar{B}_3^4 = \bar{B}_5^2 = \bar{B}_5^3 = \bar{B}_5^4 = \bar{B}_5^6 = \bar{B}_5^7 = \bar{B}_5^8 = 0$. Then we obtain the following theorem by the isomorphisms of (2.7).

Theorem II. *Let* B^i *be one of* B_2^i, $\bar{B}_2^i$, $\bar{B}_5^i$, B_3^i *and* $\bar{B}_3^i$. *Then*

$$H(B^i) \approx \begin{cases} Z_2 & \textit{for } i \equiv \lambda \pmod 4 \textit{ and } i \geqq 2, \\ 0 & \textit{otherwise}, \end{cases}$$

where λ *takes the following values*:

$B^i =$	B_2^i	$\bar{B}_2^i$	$\bar{B}_5^i$	B_3^i	$\bar{B}_3^i$
$\lambda =$	0	1	1	3	1 or 3 .

We note here the following representatives of the generator of $H(B^i)$.

	$H(B_1^{4k})$	$H(\bar{B}_2^{4k+1})$ and $H(\bar{B}_5^{4k+1})$	$H(B_3^{4k-1})$	$H(\bar{B}_3^{2k+1})$
$k=1$	(3, 1)	(5)	(3)	(3)
$k=2$	(5, 2, 1)	(9) + (7, 2)	(5, 2)	(5)
$k=3$	(9, 2, 1)	(9, 4)	(9, 2) + (7, 3, 1)	(5, 2)
$k=4$	(9, 4, 2, 1)	(17) + (15, 2) + (13, 4) + (11, 4, 2).	(9, 4, 2)	(9) + (7, 2)

§ 3. Some tables and lemmas.

In the following, several practical values of φ_a-images are calculated by (1. 4).

The table indicated at the end of the previous § follows from the following diagram:

(3. 1)

$$\begin{array}{ccc}
(2) & \xrightarrow{\varphi_1^*} & (3)\\
 & \searrow \varphi_2 & \\
(2,1) & \longrightarrow & (3,1)\\
 & \searrow \bar{\varphi}_2 & \\
(4) & \longrightarrow & (5)\\
 & \searrow \bar{\varphi}_5 & \\
(8)+(6,2) & \longrightarrow & (9)+(7,2)\\
 & \searrow \varphi_3 & \\
(8,2)+(6,3,1) & \longrightarrow & (9,2)+(7,3,1)\\
 & \searrow \varphi_2 & \\
(8,2,1) & \longrightarrow & (9,2,1)\\
 & \searrow \bar{\varphi}_2 & \\
(8,4) & \longrightarrow & (9,4)\\
 & \searrow \bar{\varphi}_5 & \\
(16)+\cdots & \longrightarrow & (17)+\cdots\\
 & \vdots &
\end{array}
\qquad
\begin{array}{ccc}
(4) & \xrightarrow{\varphi_1^*} & (5)\\
 & \searrow \varphi_3 & \\
(4,2) & \longrightarrow & (5,2)\\
 & \searrow \varphi_2 & \\
(4,2,1) & \longrightarrow & (5,2,1)\\
 & \searrow \bar{\varphi}_2 & \\
(8)+(6,2) & \longrightarrow & (9)+(7,2)\\
 & \searrow \bar{\varphi}_5 & \\
(8,4) & \longrightarrow & (9,4)\\
 & \searrow \varphi_3 & \\
(8,4,2) & \longrightarrow & (9,4,2)\\
 & \searrow \varphi_2 & \\
(8,4,2,1) & \longrightarrow & (9,4,2,1)\\
 & \searrow \bar{\varphi}_2 & \\
(16)+\cdots & \longrightarrow & (17)+\cdots\\
 & \vdots &
\end{array} .$$

The image of the homomorphism $\bar{\varphi}_4: A^* \to A^*/\varphi_1 A^*$ contains the following linearly independent elements:

(4), $(i, 4)$ for $i \geqq 8$; (5), $(i, 5)$ for $i \geqq 10$; (6), $(i, 6)$ for $i \geqq 12$;
(7), $(i, 7)$ for $i \geqq 14$; $(6, 2)$, $(i, 6, 2)$ for $i \geqq 12$;
(9), $(7, 2)$; $(10)+(8, 2)$, $(7, 3)$; $(11)+(9, 2)$, $(9, 2)+(8, 3)$;
$(10, 2)$, $(9, 3)$; $(13)+(10, 3)$, $(11, 2)$; $(13, 2)+(12, 3)$;
$(13, 3)$, $(10, 4, 2)$; $(17)+(15, 2)$, $(11, 4, 2)$;
$(18)+(16, 2)+(12, 4, 2)$, $(11, 5, 2)$;
$(19)+(16, 3)$, $(17, 2)+(16, 3)+(12, 5, 2)$, $(13, 4, 2)+(12, 5, 2)$;
$(18, 2)+(14, 4, 2)$, $(17, 3)$, $(13, 5, 2)$;
$(21)+(18, 3)+(14, 5, 2)$, $(19, 2)+(15, 4, 2)$; $\cdots$.

Consider a homomorphism $\tilde{\varphi}_4: A^* \to A^*/(\varphi_1 A^* + \varphi_2 A^*)$ defined by φ_4. For the degrees less than 22, $\tilde{\varphi}_4$ is given from $\bar{\varphi}_4$ by adding the following relations generated by $(2)=0$; $(*, *, 2) = (*, 2) = (2) = 0$, $(*, 3) = (3) = 0$, $(*, 5) = (5) = 0$, $(9) = 0$, $(9, 4) = 0$, $(17)+(13, 4) = 0$, $(17, 4)+(15, 6) = 0$. Therefore the image of $\tilde{\varphi}_4$ contains the following linearly independent elements (representatives):

(4), $(8, 4)$, $(i, 4)$ for $10 \leqq i \leqq 16$; (6), $(i, 6)$ for $12 \leqq i \leqq 15$;
(7), $(14, 7)$; (10), (11), (13), (18), (19), (21).

Next consider the kernel of $\bar{\varphi}_4: A^* \to A^*/\varphi_1 A^*$. Since $\bar{\varphi}_4(2, 1) = (2, 1, 4) = (2, 5) = 0$, $\bar{\varphi}_4(7) = (7, 4) = 0$ and $\bar{\varphi}_4((10)+(8, 2)+(7, 3)) = (10, 4)+(8, 6)+(7, 7) = 0$, the kernel contains $\varphi_{(2,1)} A^* + \varphi_7 A^* + \varphi_{(10)+(8,2)+(7,3)} A^*$. Since $\varphi_1: A^*/\varphi_1 A^* \to A^*$ is an isomorphism into and since $\varphi_{(t,1)} = \varphi_1 \circ \bar{\varphi}_t : A^* \to A^*$, we have from Theorem I

(3.2). *The sequences*

$$A^* \xrightarrow{\varphi_2} A^* \xrightarrow{\varphi_{(2,1)}} A^*,$$

$$A^* \xrightarrow{\bar{\varphi}_2} A^*/\varphi_1 A^* \xrightarrow{\varphi_{(5,1)}} A^*,$$

and
$$A^* \xrightarrow{\bar{\varphi}_3} A^*/\varphi_1 A^* \xrightarrow{\varphi_{(3,1)}} A^*$$

are exact. The rank of the image $\varphi_{(t,1)}(A^{i-t})$ *eqals to* $\bar{\beta}_i(t)$.

In $A^*/\varphi_1 A^*$ we have the following linearly independent elements:

$$\varphi_7(2)=(9),\quad \varphi_7(4)=(11)+(9,2),\quad \varphi_7(6)=(13)+(10,3),$$
$$\varphi_7(4,2)=(11,2)\ ;\quad \varphi_{(10)+(8,2)+(7,3)}(2)=(10,2).$$

Since $\varphi_{(2,1)}A^*\subset\varphi_1A^*$, the above images of φ_7 and $\varphi_{(10)+(8,2)+(7,3)}$ are independent of $\varphi_{(2,1)}A^*$. Let $\tilde{\beta}_i(4)$ and ε_i be the ranks of the image $\tilde{\varphi}_4A^{i-4}$ and the kernel of $\bar{\varphi}_4$ respectively. Then the following table follows from the above results.

i	$=4$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$\tilde{\beta}_i(4)\geqq$	1	0	1	1	0	0	1	1	1	1	1	1	1	1	3	3	2	3
$\bar{\alpha}_{i-4}=$	1	0	1	1	1	1	2	2	2	3	3	3	4	4	5	6	6	7
$\bar{\beta}_{i-4}(4)\geqq$					1	1	1	1	1	2	2	2	3	3	2	3	4	4
$\alpha_{i-8}=$					1	1	1	2	2	2	3	4	4	5	6	6	7	8
$\bar{\beta}_{i-9}(2)=$								1	1	0	1	1	1	1	3	2	2	2
$\varepsilon_{i-8}-\bar{\beta}_{i-9}(2)\geqq$												1	0	1	1	1	1	2.

Since $(4,4)=(6,2)+(7,1)$, we have $\tilde{\varphi}_4\circ\bar{\varphi}_4=0$. By a similar argument to the proof of Theorem I, we have

Lemma 3.1. *The sequence*

$$A^{i-8}\xrightarrow{\bar{\varphi}_4}A^{i-4}/\varphi_1A^{i-5}\xrightarrow{\tilde{\varphi}_4}A^i/(\varphi_1A^{i-1}+\varphi_2A^{i-2})$$

is exact for $i<22$ *and the kernel of* $\bar{\varphi}_4$ *is generated by* $(2,1)$, (7), $(10)+(8,2)+(7,3)$ *for* $i<22$. *In the above table the equalities hold.*

It seems that this lemma is true for all i.

The image of a homomorphism $\tilde{\varphi}_8: A^*\rightarrow A^*/(\varphi_1A^*+\varphi_2A^*)$, defined by φ_8, contains the following linearly independent elements:

$$(8),\ (10),\ (11),\ (12),\ (13),\ (14),\ (15),\ (12,4),\ (13,4),\ (18),$$
$$(14,4),\ (19),\ (15,4),\ (20)+(16,4),\ (14,6),\ (21),\ (15,6).$$

By adding relations generated by $(4)=0$, we see that the image of a homomorphism $\hat{\varphi}_8: A^*\rightarrow A^*/(\varphi_1A^*+\varphi_2A^*+\varphi_4A^*)$, defined by φ_8, contains the following linearly independent elements:

$$(8),\ (12),\ (14),\ (15),\ (20),$$

Let $\tilde{\beta}_i(8)$ and $\hat{\beta}_i(8)$ be the ranks of the images $\tilde{\varphi}_8(A^{i-8})$ and $\hat{\varphi}_8(A^{i-8})$ respectively, then we have the following table:

$i =$	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$\hat{\beta}_i(8) \geqq$	1	0	0	0	1	0	1	1	0	0	0	0	1	0
$\bar{\alpha}_{i-8} - \bar{\beta}_{i-8}(2) = \beta_{i-3}(5) =$	1	0	0	0	1	0	1	1	1	0	1	1	2	1
$\bar{\alpha}_{i-16} = \tilde{\beta}_{i-8}(8) =$									1	0	1	1	1	1.

Since $\varphi_8(1) = (9) = \varphi_2((4, 2, 1) + (7)) + \varphi_1((8) + (6, 2))$, $\varphi_8(2) = (10) + (9, 1) = \varphi_4(4, 2) + \varphi_2(8) + \varphi_1(7, 2)$ and since $\varphi_8(8) = (15, 1) + (14, 2) + (12, 4) = \varphi_4(12) + \varphi_2(14) + \varphi_1(15)$, we have

$$\tilde{\varphi}_8(\varphi_1 A^*) = \hat{\varphi}_8(\varphi_1 A^* + \varphi_2 A^*) = \hat{\varphi}_8(\tilde{\varphi}_8(A^*)) = 0\,.$$

Then we have the following lemma by a similar argument to the proof of Theorem I.

Lemma 3. 2. *The sequence*

$$0 \to A^{i-16}/\varphi_1 A^{i-17} \xrightarrow{\tilde{\varphi}_8} A^{i-8}/(\varphi_1 A^{i-9} + \varphi_2 A^{i-10}) \xrightarrow{\hat{\varphi}_8} A^i/(\varphi_1 A^{i-1} + \varphi_2 A^{i-2} + \varphi_4 A^{i-4})$$

is exact for $i < 22$. *In the above table the equality holds.*

Remark that the kernal of $\tilde{\varphi}_8$ contains non-zero elements (4, 2), (15), etc..

We introduce *a Bockstein homomorphism*

$$\frac{\delta}{2^r} : \frac{\delta}{2^{r-1}}\text{-}kernel \to \frac{\delta}{2^{r-1}}\text{-}cokernel, \qquad r \geqq 1\,,$$

as follows. A cohomology class $\alpha \in H^i(X, A, Z_2)$ is in the $\frac{\delta}{2^{r-1}}$-*kernel* if there exist integral cochains $a \in C^i(X, A)$ and $a' \in C^{i+1}(X, A)$ such that $\delta a = 2^r a'$ and a represents α. A cohomology class $\beta \in H^{i+1}(X, A, Z_2)$ is in the $\frac{\delta}{2^{r-1}}$-*image* if there exist integral cochains $b \in C^i(X, A)$ and $b' \in C^{i+1}(X, A)$ such that $\delta b = 2^{r-1} b'$ and b' represents β. The $\frac{\delta}{2^{r-1}}$-*cokernel* is the factor group $H^{i+1}(X, A, Z_2)/\left(\frac{\delta}{2^{r-1}}\text{-}image\right)$. Let a and a' be integral cochains as above, then $\frac{\delta}{2^r}\alpha$ is defined as the class represented by a'. Let a_1 be another integral cochain such that $\delta a_1 = 2^r a_1'$ for some a_1' and a_1 represents α. Then $a - a_1 = 2b + \delta c$ for some integral cochains b and c. $2^r(a' - a'_1) = \delta(a - a_1) = 2\delta b$ implies that $2^{r-1}(a' - a_1') = \delta b$. Thus a' and a_1' represent the same class of $\frac{\delta}{2^{r-1}}$-*cokernel*, and a Bockstein homomorphism $\frac{\delta}{2^r}$ is defined uniquely. The following properties are well known.

(3. 3) i) $\frac{\delta}{2^r}$*-kernel = the kernel of* $\frac{\delta}{2^r}$.

ii) $\frac{\delta}{2^r}$*-image*$/\left(\frac{\delta}{2^{r-1}}\right.$*-image*$\left.\right)$ *= the image of* $\frac{\delta}{2^r}$.

iii) $\frac{\delta}{2}=Sq^1: H^i(X, A, Z_2)\to H^{i+1}(X, A, Z_2)$.

iv) *The naturality* $f^*\circ\frac{\delta}{2^r}=\frac{\delta}{2^r}\circ f^*$ *holds for homomorphisms* f^* *of cohomology groups induced by a mapping* $f: (X, A)\to(Y, B)$.

v) $\delta^*\circ\frac{\delta}{2^r}=\frac{\delta}{2^r}\circ\delta^*$ *for coboundary homomorphisms* $\delta^*: H^i(A, Z_2)\to H^{i+1}(X, A, Z_2)$.

vi) $\frac{\delta}{2^r}\circ\frac{\delta}{2^s}=0$.

vii) *Let* $H_i(X)$ *be finitely generated. Then the rank of the image of* $\frac{\delta}{2^r}$ *is the number of direct factors of* $H_i(X)$ *which are isomorphic to the cyclic group* Z_{2^r} *of the order* 2^r.

Denote by $H^*_{(r)}(X, A, Z_2)$ the factor group $\frac{\delta}{2^r}$*-kernel*$/\left(\frac{\delta}{2^r}\right.$*-image*$\left.\right)$. By (3. 3), $\frac{\delta}{2^r}$ defines a homomorphism of $H^*_{(r-1)}(X, A, Z_2)$ which will be denoted by the same symbol ($H^*_{(0)}=H^*$)

$$\frac{\delta}{2^r}: H^i_{(r-1)}(X, A, Z_2)\to H^{i+1}_{(r-1)}(X, A, Z_2)\subset\frac{\delta}{2^{r-1}}\text{-cokenral.} \tag{3. 4}$$

By regarding $\frac{\delta}{2^{r+1}}$ as a cobundary operator in $H^*_{(r)}(X, A, Z_2)$, we see that $H^*_{(r+1)}(X, A, Z_2)$ is the cohomology group of $H^*_{(r)}(X, A, Z_2)$.

Consider the cohomology exact sequence for a pair (X, A):

$$\cdots\to H^i(X, A, Z_2)\xrightarrow{j^*}H^i(X, Z_2)\xrightarrow{i^*}H^i(A, Z_2)\xrightarrow{\delta^*}H^{i+1}(X, A, Z_2)\to\cdots.$$

The following lemma is a modification of theorems in [6], §3.

Lemma 3. 3. i) *For* $\alpha\in H^i(A, Z_2)$ *and* $\beta\in H^i(X, A, Z_2)$, *assume that* $\frac{\delta}{2^r}\beta=\{\delta^*\alpha\}$. *Then there is an element* $\tilde{\alpha}\in H^{i+1}(X, Z_2)$ *such that* $i^*\tilde{\alpha}=Sq^1\alpha$ *and* $\frac{\delta}{2^{r+1}}(j^*\beta)=\{\tilde{\alpha}\}$ $(r\geqq 1)$.

ii) *For* $\alpha\in H^i(A, Z_2)$ *and* $\beta\in H^{i+1}(X, A, Z_2)$, *assume that* $\delta^*\alpha=\beta$ *and* $\beta\in\frac{\delta}{2^{r-1}}$*-kernel. Then there are elements* $\tilde{\alpha}\in H^{i+1}(X, Z_2)$ *and* $\gamma\in H^{i+2}(X, A, Z_2)$ *such that* $i^*\tilde{\alpha}=Sq^1\alpha$, $\frac{\delta}{2^r}\beta=\{\gamma\}$ *and* $\frac{\delta}{2^{r-1}}\tilde{\alpha}=\{j^*\gamma\}$ $(r\geqq 2)$.

iii) *For* $\alpha \in H^i(A, Z_2)$ *and* $\beta \in H^{i+1}(A, Z_2)$, *assume that* $\frac{\delta}{2^r}(\delta^*\alpha) = \{\delta^*\alpha\}$. *Then there are elements* $\tilde{\alpha} \in H^{i+1}(X, Z_2)$ *and* $\tilde{\beta} \in H^{i+2}(X, Z_2)$ *such that* $i^*\tilde{\alpha} = Sq^1\alpha + 2^{r-1}\beta$, $i^*\tilde{\beta} = Sq^1\beta$ *and* $\frac{\delta}{2^r}\tilde{\alpha} = \{\tilde{\beta}\}$ $(r \geqq 1)$.

Proof. i) Let $a \in C^i(A)$ and $b \in C^i(X, A)$ be representatives of α and β respectively such that $\delta a = 2a' + b_1$ and $\delta b = 2^r b'$ for some $a' \in C^{i+1}(A)$ and $b_1, b' \in C^{i+1}(X, A)$, then a', b_1 and b' represent $Sq^1\alpha$, $\delta^*\alpha$ and $\frac{\delta}{2^r}\beta$ respectively. From the assumption $\frac{\delta}{2^r}\beta = \{\delta^*\alpha\}$, we have $b_1 - b' = 2b_2 + c' + \delta c_1$ and $\delta c = 2^{r-1}c'$ for some $c, c_1 \in C^i(X, A)$ and $b_2, c' \in C^{i+1}(X, A)$. The element $b + 2(c - 2^{r-1}a + 2^{r-1}c_1)$ represents $j^*\beta$. From $\delta(b + 2(c - 2^{r-1}a + 2^{r-1}c_1)) = 2^r b' + 2\delta c - 2^r\delta a + 2^r\delta c_1 = 2^r(b_1 - 2b_2 - c' - \delta c_1) + 2^r c' - 2^r(2a' + b_1) + 2^r\delta c_1 = -2^{r+1}(b_2 + a')$, we see that $\frac{\delta}{2^{r+1}}(j^*\beta) = \{\tilde{\alpha}\}$ for an element $\tilde{\alpha}$ represented by $-(b_2 + a')$. Obviously $i^*\tilde{\alpha} = Sq^1(-\alpha) = Sq^1\alpha$.

ii) Let $a \in C^i(A)$ and $b \in C^{i+1}(X, A)$ be representatives of α and β respectively such that $\delta a = 2a' + b_1$ and $\delta b = 2^r b'$ for some $a' \in C^{i+1}(A)$, $b_1 \in C^{i+1}(X, A)$ and $b' \in C^{i+2}(X, A)$, then a', b_1 and b' represent $Sq^1\alpha$, $\delta^*\alpha$ and $\frac{\delta}{2^r}\beta$ respectively. From the assumption $\delta^*\alpha = \beta$, we have $b_1 - b = 2b_2 + \delta c$ for some $b_2 \in C^{i+1}(X, A)$ and $c \in C^i(X, A)$. From $2\delta(b_2 + a') = \delta(b_1 - b - \delta c) + \delta(\delta a - b_1) = -\delta b = 2^r(-b')$, we have $\delta(b_2 + a') = 2^{r-1}(-b')$. Let $\tilde{\alpha}$ and γ be represented by $b_2 + a'$ and b' respectively, then we see that $i^*\tilde{\alpha} = Sq^1\alpha$, $\frac{\delta}{2^r}\beta = \{\gamma\}$ and $\frac{\delta}{2^{r-1}}\tilde{\alpha} = \{-j^*\gamma\} = \{j^*\gamma\}$.

iii) Let $a \in C^i(A)$ and $b \in C^{i+1}(A)$ be representatives of α and β respectively such that $\delta a = 2a' + a_1$ and $\delta b = 2b' + b_1$ for some $a' \in C^{i+1}(A)$, $b' \in C^{i+2}(A)$, $a_1 \in C^{i+1}(X, A)$ and $b_1 \in C^{i+2}(X, A)$. Then a', b', a_1 and b_1 represent $Sq^1\alpha$, $Sq^1\beta$, $\delta^*\alpha$ and $\delta^*\beta$ respectively. From the assumption $\frac{\delta}{2^r}(\delta^*\alpha) = \{\delta^*\beta\}$, we have $\delta a_2 = 2^r b_2$, $a_2 - a_1 = 2c + \delta c_1$, $b_2 - b_1 = 2d_1 + d' + \delta d_2$ and $2^{r-1}d' = \delta d$ for some $a_2, c, d_2, d \in C^{i+1}(X, A)$, $c_1 \in C^i(X, A)$ and $d', d_1 \in C^{i+2}(X, A)$. From $2\delta(a' + 2^{r-1}b + d + 2^{r-1}d_2 - c) = \delta(\delta a - a_1) + 2^r(2b' + b_1) + 2^r d' - 2^r(2d_1 + d' + b_1 - b_2) - \delta(a_2 - a_1 - \delta c_1) = 2^{r+1}(b' - d_1) + (2^r b_2 - \delta a_2) = 2^{r+1}(b' - d_1)$, we have $\delta(a' + 2^{r-1}b + (d + 2^{r+1}d_2 - c)) = 2^r(b' - d_1)$. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be represented by $a' + 2^{r-1}b + (d + 2^{r-1}d_2 - c)$ and $b' - d_1$ respectively, then we see that $i^*\tilde{\alpha} = Sq^1\alpha + 2^{r-1}\beta$, $i^*\tilde{\beta} = Sq^1\beta$ and $\frac{\delta}{2^r}\tilde{\alpha} = \{\tilde{\beta}\}$. q. e. d.

Remark that the above lemma is valid for a fibre space in the following manner. Let X be a fibre space over an m-connected space B having an n-connected fibre F. Then, for $i \leqq m+n+1$, we have isomorphisms

$$p^*: H^i(B, Z_2) \approx H^i(X, F, Z_2).$$

By the above isomorphisms, Lemma 3.1. is valid for the exact sequence of the fibering:

$$\cdots \rightarrow H^{i-1}(F, Z_2) \xrightarrow{\Delta^*} H^i(B, Z_2) \xrightarrow{p^*} H^i(X, Z_2) \xrightarrow{i^*} H^i(F, Z_2) \rightarrow \cdots,$$

replacing $H^i(X, A, Z_2)$ by $H^i(B, Z_2)$, j^* by p^* and δ^* by Δ^*.

§ 4. Application to the stable homotopy groups of the sphere.

Let S^N be an N-sphere. Consider a CW-complex K_k, $k \geqq 2$, whose $(N+k)$-skeleton K_k^{N+k} is S^N. By attaching cells of dim. $\geqq N+k$ to K_k, we can construct a CW-complex K_{k-1} such that $K_{k-1} \supset K_k$, $K_{k-1}^{N+k-1} = S^N$ and $\pi_i(K_{k-1}) = 0$ for $i \geqq N+k-1$. Repeating this construction from $K_N = S^N$, we have a sequence of complexes

$$K_1 \supset K_2 \supset \cdots \supset K_{k-1} \supset K_k \supset \cdots \supset K_{N-1} \supset S^N$$

such that $K_k^{N+k} = S^N$ and $\pi_i(K_k) = 0$ for $i \geqq N+k$. It is easy to see that the injection $i: S^N \subset K_k$ induces isomorphisms

$$i_*: \pi_i(S^N) \approx \pi_i(K_k) \qquad \text{for} \quad i < N+k.$$

Let Y_k be a space of the paths in K_k starting in S^N. S^N is naturally imbedded in Y_k as its deformation retract. We have a retraction (fibering)

$$p_0: Y_k \rightarrow S^N$$

by associating to each path the starting point. Also associating the end point, we have a fibering

$$p_1: Y_k \rightarrow K_k,$$

in the sense of Serre [3], a fibre X_k of which is a space of the paths in K_k starting in S^N and ending at a point. The restriction

$$p': X_k \rightarrow S^N$$

of p_0 on X_k is also a fibering. Consider a diagram

$$\begin{array}{ccccccc} & & & \nearrow \pi_i(Y_k) \searrow^{p_{1*}} & & & \\ \cdots \to \pi_{i+1}(K_k) \to & \pi_i(X_k) & & \wr\!\wr \downarrow p_{0*} & & \pi_i(K_k) & \xrightarrow{\Delta} \cdots \\ & & & {}_{p'_*}\searrow \pi_i(S^N) \nearrow_{i_*} & & & \end{array}$$

then it is easily verified from the conditions on $\pi_i(K_k)$ and i_* that

$$\pi_i(X_k) \begin{cases} = 0 & \text{for } \ i < N+k\,, \\ \overset{p'_*}{\approx} \pi_i(S^N) & \text{for } \ i \geqq N+k\,. \end{cases}$$

This indicates that X_k is *an* $(N+k-1)$*-connective fibre space over* S^N.

Since X_k and K_k are $(N+k-1)$- and $(N-1)$-connected respectively, we have the following homology exact sequence for $i \leqq 2N+k-1$:

$$\cdots \to H_i(X_k) \to H_i(Y_k) \to H_i(K_k) \xrightarrow{\partial_*} H_{i-1}(X_k) \to \cdots .$$

Since $H_i(Y_k) = H_i(S^N) = 0$ for $i \neq 0$, N, we have isomorphisms

(4. 1) $\quad \partial_* : H_i(K_k) \approx H_{i-1}(X_k) \qquad$ *for* $\ N \neq i \leqq 2N+k-1\,.$

Similarly we have isomorphisms

(4. 1)′ $\quad \delta^* : H^{i-1}(X_k, Z_2) \approx H^i(K_k, Z_2) \qquad$ *for* $\ N \neq i \leqq 2N+k-1\,.$

Combining (4. 1) to the Hurewicz isomorphism, we have

(4. 2) $\quad \pi_{N+k}(S^N) \approx \pi_{N+k}(X_k) \approx H_{N+k}(X_k) \approx H_{N+k+1}(K_k)$

for $\ 1 \leqq k < N-1\,.$

Remark that (4.2) is proved directly as follows:

$$H_{N+k+1}(K_k) \overset{j_*}{\approx} H_{N+k+1}(K_k, S^N) \approx \pi_{N+k+1}(K_k, S^N) \overset{\partial}{\approx} \pi_{N+k}(S^N)\,.$$

Let $\tilde{K}_{k+1}$ be a space of the paths in K_k which start in K_{k+1}. Then $\tilde{K}_{k+1}$ is a fibre space over K_k containning K_{k+1} as its deformation retract. Let F_k be a fibre of this fibering and consider a diagram

$$\begin{array}{ccccccc} \cdots \to \pi_{i+1}(K_k) \to \pi_i(F_k) \to & \pi_i(\tilde{K}_{k+1}) & \longrightarrow & \pi_i(K_k) \to \cdots \\ & \wr\!\wr & \nearrow & \uparrow i_* \\ & \pi_i(K_{k+1}) & \xleftarrow{\ i_*\ } & \pi_i(S^N) \end{array} .$$

Then it is verified easily from the conditions of $\pi_i(K_k)$, $\pi_i(K_{k+1})$ and i_* that

$$\pi_i(F_k) \approx \begin{cases} \pi_{N+k}(S^N) & \text{for } i=N+k, \\ 0 & \text{for } i\neq N+k. \end{cases}$$

Therefore F_k is an Eilenberg-MacLane space of the type $(\pi_{N+k}(S^N), N+k)$ and $H^i(F_k, Z_2) \approx H^i(\pi_{N+k}(S^N), N+k, Z_2)$. Since K_k and F_k are $(N-1)$- and $(N+k-1)$-connected respectively, we have the following exact sequence for $i \leqq 2N+k-1$:

(4.3) $\cdots \to H^i(K_k, Z_2) \to H^i(K_{k+1}, Z_2) \to H^i(\pi_{N+k}(S^N), N+k, Z_2) \to \cdots$.

Now we write $K_k = K_k(N)$ and consider $K_k(N+1)$. The suspension $S(K_k(N))$ of $K_k(N)$ is a CW-complex whose $(N+k+1)$-skeleton is S^{N+1}. Since $\pi_i(K_k(N+1))=0$ for $i \geqq N+k+1$, we can construct a mapping

$$f_k^{(N)}: S(K_k(N)) \to K_k(N+1)$$

such that $f_k^{(N)}$ is identical on the $(N+k+1)$-skeletons. It is easy to see that the sequence

$$\mathfrak{K}_k = \{K_k(N), f_k^{(N)}\}$$

satisfies the conditions of (1.1). Then the stable groups

$$A_i(\mathfrak{K}_k) \quad \textit{and} \quad A^i(\mathfrak{K}_k, Z_2)$$

are defined. By the convension (1.2), we may regard that for sufficiently large N,

$$A_i(\mathfrak{K}_k) = H_{i+N}(K_k(N)) = H_{i+N}(K_k),$$
$$A^i(\mathfrak{K}_k, Z_2) = H^{i+N}(K_k(N), Z_2) = H^{i+N}(K_k, Z_2).$$

Then from (4.2) and (4.3), we have

(4.4) i) $\pi_k \approx A_{k+1}(\mathfrak{K}_k)$.

ii) *The following sequence is exact.*

$$\cdots \to A^i(\mathfrak{K}_k, Z_2) \xrightarrow{p^*} A^i(\mathfrak{K}_{k+1}, Z_2) \xrightarrow{i^*} A^{i-k}(\pi_k, Z_2) \xrightarrow{\Delta^*} A^{i+1}(\mathfrak{K}_k, Z_2) \to \cdots.$$

The squaring operations in $A^*(\mathfrak{K}_k, Z_2) = \sum A^i(\mathfrak{K}_k, Z_2)$ and $A^*(\pi_k, Z_2) = \sum A^i(\pi_k, Z_2)$ define naturally a (left) A^*-module structure of $A^*(\mathfrak{K}_k, Z_2)$ and $A^*(\pi_k, Z_2)$. Then the above exact sequence is one of A^*-homomorphisms, since the squaring operation commutes with the homomorphisms of the sequence.

The Bockstein homomorphism $\frac{\delta}{2^r}$ is also defined naturally in $A^*(\mathfrak{K}_k, Z_2)$ and $A^*(\pi_k, Z_2)$ and it satisfies the properties of (3. 3), i)–vii) by replacing H^i by A^i, H_i by A_i and δ^* by Δ^*. Then the following lemma follows from Lemma 3. 3.

Lemma 4. 1. i) *For $\alpha \in A^{i-k}(\pi_k, Z_2)$ and $\beta \in A^i(\mathfrak{K}_k, Z_2)$, assume that $\frac{\delta}{2^r}\beta = \{\Delta^*\alpha\}$. Then there is an element $\tilde{\alpha}$ of $A^{i+1}(\mathfrak{K}_{k+1}, Z_2)$ such that $i^*\tilde{\alpha} = Sq^1\alpha$ and $\frac{\delta}{2^{r+1}}(p^*\beta) = \{\tilde{\alpha}\}$.*

ii) *For $\alpha \in A^{i-k}(\pi_k, Z_2)$ and $\beta \in A^{i+1}(\mathfrak{K}_k, Z_2)$, assume that $\Delta^*\alpha = \beta$, $r > 1$ and $\beta \in \frac{\delta}{2^{r-1}}$-kernel. Then there are elements $\tilde{\alpha} \in A^{i+1}(\mathfrak{K}_{k+1}, Z_2)$ and $\gamma \in A^{i+2}(\mathfrak{K}_k, Z_2)$ such that $i^*\tilde{\alpha} = Sq^1\alpha$, $\frac{\delta}{2^r}\beta = \{\gamma\}$ and $\frac{\delta}{2^{r-1}}\tilde{\alpha} = \{p^*\gamma\}$.*

iii) *For $\alpha \in A^{i-k}(\pi_k, Z_2)$ and $\beta \in A^{i-k+1}(\pi_k, Z_2)$, assume that $\frac{\delta}{2^r}(\Delta^*\alpha) = \{\Delta^*\beta\}$. Then there are elements $\tilde{\alpha} \in A^{i+1}(\mathfrak{K}_{k+1}, Z_2)$ and $\tilde{\beta} \in A^{i+2}(\mathfrak{K}_{k+1}, Z_2)$ such that $i^*\tilde{\alpha} = Sq^1\alpha + 2^{r-1}\beta$, $i^*\tilde{\beta} = Sq^1\beta$ and $\frac{\delta}{2^r}\tilde{\alpha} = \{\tilde{\beta}\}$.*

By [3], π_k is a finite group for $k \neq 0$. Then by [4], $A^i(\pi_k, Z_2)$ is isomorphic to the sum of some A^i and $A^i/\varphi_1 A^{i-1} + A^{i-1}/\varphi_1 A^{i-2}$.

In the following lemma, we denote by u, $u_0 \in A^0(\pi_k, Z_2)$ and $u_1 \in A^1(\pi_k, Z_2)$ the fundamental elements which generate direct summands A^* and $A^*/\varphi_1 A^*$. Note that $Sq^1 u_0 = Sq^1 u_1 = 0$. Consider the exact sequence of (4. 4), ii).

Lemma 4. 2. i) *Assume that $\Delta^* Sq^2 u = 0$ and that $Sq^1 : p^*A^{k+5}(\mathfrak{K}_k, Z_2) \to p^*A^{k+6}(\mathfrak{K}_k, Z_2)$ is an isomorphism into. Then there is an element v of $A^{k+2}(\mathfrak{K}_{k+1}, Z_2)$ such that $i^*v = Sq^2 u$, $Sq^3 v = 0$ and that the A^*-submodule generated by v and by the image of p^* is isomorphic to $A^*/\varphi_3 A^* + p^*A^*(\mathfrak{K}_k, Z_2)$ (direct sum of A^*-modules).*

ii) *Assume that $\Delta^* Sq^3 u = 0$ and that $Sq^1 : p^*A^{k+i}(\mathfrak{K}_k, Z_2) \to p^*A^{k+i+1}(\mathfrak{K}_k, Z_2)$, $i = 4, 8$, are isomorphisms into. Then there is an element v of $A^{k+3}(\mathfrak{K}_{k+1}, Z_2)$ such that $i^*v = Sq^3 u$, $Sq^1 v = Sq^5 v = 0$ and that the A^*-submodule generated by v and by the image of p^* is isomorphic to $A^*/(\varphi_1 A^* + \varphi_5 A^*) + p^*A^*(\mathfrak{K}_k, Z_2)$.*

iii) *Assume that $\Delta^* Sq^5 u_0 = 0$ (resp. $\Delta^* Sq^5 u_1 = 0$) and $p^*A^{k+6}(\mathfrak{K}_k, Z_2) = p^*A^{k+7}(\mathfrak{K}_k, Z_2) = 0$. (resp. $p^*A^{k+7}(\mathfrak{K}_k, Z_2) = p^*A^{k+8}(\mathfrak{K}_k, Z_2) = 0$). Then there is an element v of $A^{k+5}(\mathfrak{K}_{k+1}, Z_2)$ (resp. $A^{k+6}(\mathfrak{K}_{k+1}, Z_2)$) such that $i^*v = u_0$ (resp. u_1), $Sq^1 v = Sq^2 v = 0$ and that the A^*-*

submodule generated by v and by the image of p^* is isomorphic to $A^*/(\varphi_1 A^*+\varphi_2 A^*)+p^*A^*(\mathfrak{R}_k, Z_2)$.

iv) *Assume that* $\Delta^* Sq^2 u_0=0$ (*resp.* $\Delta^* Sq^2 u_1=0$ *or* $\Delta^* Sq^2 Sq^1 u=0$) *and that* $p^*A^{k+4}(\mathfrak{R}_k, Z_2)=0$ (*resp.* $p^*A^{k+5}(\mathfrak{R}_k, Z_2)=0$). *Then there is an element v of* $A^{k+2}(\mathfrak{R}_{k+1}, Z_2)$ (*resp.* $A^{k+3}(\mathfrak{R}_{k+1}, Z_2)$) *such that* $i^*v=Sq^2u_0$ (*resp.* Sq^2u_1 *or* Sq^2Sq^1u), $Sq^2v=0$ *and that the A^*-submodule generated by v and by the imag of p^* is isomorphic to* $A^*/\varphi_2 A^*+p^*A^*(\mathfrak{R}_k, Z_2)$.

v) *Assume that* $\Delta^* Sq^4 u_0=0$, $p^*A^{k+7}(\mathfrak{R}_k, Z_2)=p^*A^{k+14}(\mathfrak{R}_k, Z_2)=0$ *and that* $Sq^1: p^*A^{k+11}(\mathfrak{R}_k, Z_2)\to p^*A^{k+12}(\mathfrak{R}_k, Z_2)$ *is an isomorphism into. Then there is an element v of* $A^{k+4}(\mathfrak{R}_{k+1}, Z_2)$ *such that* $i^*v=Sq^4u_0$, $Sq^2Sq^1v=Sq^7v=(Sq^{10}+Sq^8Sq^2+Sq^7Sq^3)v=0$ *and that the A^*-submodule generated by v and by the imag of p^* is isomorphic to* $A^*/(\varphi_{(2,1)}A^*+\varphi_7 A^*+\varphi_{(10)+(8,2)+(7,3)}A^*)+p^*A^*(\mathfrak{R}_k, Z_2)$ *for dimensions less than* 21.

Proof. i) From the exactness of the sequence (4.4), ii), $\Delta^* Sq^2 u=0$ implies the existence of an element v such that $i^*v=Sq^2u$. Also from $i^*(Sq^3v)=Sq^3Sq^2u=0$, we have that there is an element w of $A^{k+5}(\mathfrak{R}_k, Z_2)$ such that $p^*w=Sq^3v$. Since $0=Sq^1Sq^3v=Sq^1p^*w$, p^*w is in the kernel of $Sq^1: p^*A^{k+5}(\mathfrak{R}_k, Z_2)\to p^*A^{k+6}(\mathfrak{R}_k, Z_2)$. Thus $Sq^3v=p^*w=0$. Let A_0^* be the A^*-submodule generated by v and the image of p^*. The formula $f'(\alpha u)=\alpha v$, $\alpha\in A^*$, defines an A^*-homomorphism f' of A^* into A_0^*. Since $f'(\varphi_3(\alpha u))=\alpha Sq^3 v=0$, f' defines an A^*-homomorphism f of $A^*/\varphi_3 A^*$ into A_0^*. Obviously the composition $i^*\circ f: A^*/\varphi_3 A^*\to A^*$ equals to φ_2. By Theorem I, $i^*\circ f$ is an isomorphism into. Therefore $A^*/\varphi_3 A^*$ is isomorphic to $f(A^*/\varphi_3 A^*)$ which is a direct summand of A_0^*. Since $A_0^*/p^*A^*(\mathfrak{R}_k, Z_2)\approx i^*(f(A^*/\varphi_3 A^*))$, we have $A_0^*=f(A^*/\varphi_3 A^*)+p^*A^*(\mathfrak{R}_k, Z_2)$.

The proofs of ii)-v) are similar by use of Theorem I, (3.2) or Lemma 3.1. q.e.d.

In the following, we treat A^*-module structures of $A^*(\mathfrak{R}_k, Z_2)$ and some Bockstein operators in it. Then several results on the stable homotopy groups π_k of the sphere are clarified.

Since $K_k^{N+k}=S^N$, we have easily

$$A^i(\mathfrak{R}_k, Z_2)=0 \quad \text{for} \quad 0<i\leq k. \tag{4.5}$$

The complex K_1 has the only non-trivial homotopy group $\pi_N(K_1)\approx\pi_N(S^N)\approx Z$. Then K_1 is an Eilenberg-MacLane space of a type (Z, N), and we have

Proposition 4. 3. *$A^*(\mathfrak{K}_1, Z_2)$ is an A^*-module generated by an element $a_1 \in A^0(\mathfrak{K}_1, Z_2)$. We have a relation $Sq^1 a_1 = 0$ and an isomorphism $A^*(\mathfrak{K}_1, Z_2) \approx A^*/\varphi_1 A^*$. The Bockstein operators $\frac{\delta}{2^r}$ are trivial for $r > 1$.*

The triviality follows from (2. 6): $H(A^i/\varphi_1 A^{i-1}) = A^i_{(1)}(\mathfrak{K}_1, Z_2) = 0$ for $i > 1$. $A^2(\mathfrak{K}_1, Z_2) = \{Sq^2 a_1\}$, $A^3(\mathfrak{K}_1, Z_2) = \{Sq^3 a_1\}$ and $Sq^1 Sq^2 a_1 = Sq^3 a_1$. Then from (3. 3), vii) and (4. 4), i), we have

Corollary. *2-component of $\pi_1 = Z_2$.*

From the corollary, $A^*(\pi_1, Z_2)$ is isomorphic to A^* and is generated by an element $u \in A^0(\pi_1, Z_2)$. Consider the exact sequence of (4. 4), ii) for $k = 1$.

Proposition 4. 4. *There exists an element b_2 of $A^3(\mathfrak{K}_2, Z_2)$ such that $i^* b_2 = Sq^2 u$. $A^*(\mathfrak{K}_2, Z_2)$ is an A^*-module generated by $a_2 = p^* a_1$ and b_2. We have relations $Sq^1 a_1 = Sq^2 a_2 = Sq^3 b_2 = 0$ and an isomorphism $A^i(\mathfrak{K}_2, Z_2) \approx A^i/(\varphi_1 A^{i-1} + \varphi_2 A^{i-2}) \oplus A^{i-3}/\varphi_3 A^{i-6}$.[1] The Bockstein homomorphisms $\frac{\delta}{2^r}$, $r > 1$, are trivial except for the case $r = 2$ and $\deg \equiv 0 \pmod{4}$, and in the case the rank of the image of $\frac{\delta}{4}$ is 1. In particular, $\frac{\delta}{4} Sq^4 a_2 = \{Sq^2 b_2\}$, $\frac{\delta}{4} Sq^8 a_2 = \{Sq^4 Sq^2 b_2\}$ and $\frac{\delta}{4} Sq^8 Sq^4 a_2 = \{(Sq^8 Sq^2 + Sq^6 Sq^3 Sq^1) b_2\}$.*

Proof. By (4. 5), $A^2(\mathfrak{K}_2, Z_2) = 0$. Then $\Delta^*: A^0(\pi_1, Z_2) \to A^2(\mathfrak{K}_1, Z_2) = \{Sq^2 a_1\}$ is onto and $\Delta^* u = Sq^2 a_1$. Since $\Delta^* \alpha u = \alpha Sq^2 a_1$, Δ_* is equivalent to $\bar{\varphi}_2: A^* \to A^*/\varphi_1 A^*$. By Theorem I, the kernel of Δ^* is generated by $Sq^2 u$. From the exactness of the sequence (4. 4), ii), we have that $A^*(\mathfrak{K}_2, Z_2)$ is generated by $a_2 = p^* a_1$ and an element b_2 such that $i^* b_2 = Sq^2 u$. We see that $p^* A^6(\mathfrak{K}_1, Z_2) = \{Sq^6 a_2\}$ and $p^* A^7(\mathfrak{K}_1, Z_2) = \{Sq^7 a_2\}$. Then $Sq^1: p^* A^6(\mathfrak{K}_1, Z_2) \to p^* A^7(\mathfrak{K}_1, Z_2)$ is an isomorphism. By Lemma 4. 2, i) we have an isomorphism $A^i(\mathfrak{K}_2, Z_2) \approx A^{i-3}/\varphi_3 A^{i-6} \oplus p^* A^i(\mathfrak{K}_1, Z_2)$ and a relation $Sq^3 b = 0$. Obviously $p^* A^i(\mathfrak{K}_1, Z_2) \approx A^i/(\varphi_1 A^{i-1} + \varphi_2 A^{i-2})$, $Sq^1 a_2 = p^* Sq^1 a_1 = 0$ and $Sq^2 a_2 = p^* Sq^2 a_1 = p^* \Delta^* u = 0$.

By Theorem I, $A^*/\varphi_3 A^*$ and $A^*/(\varphi_1 A^* + \varphi_2 A^*)$ are $(A^*$-)isomorphic to $\varphi_2 A^* = B_2^*$ and $\bar{\varphi}_5 A^* = \bar{B}_5^*$. Since $\frac{\delta}{2} = Sq^1 = \varphi_1^*$, we have from Theorem II that

1) $B^i = C^j \oplus D^k$ means that $B^* = \sum B^i$ is a direct sum of A^*-moducles $C^* = \sum C^j$ and $D^* = \sum D^k$.

$$A^i_{(1)}(\mathfrak{K}_2, Z_2) \approx \begin{cases} Z_2 & i \equiv 0, 1 \pmod 4),\ i \geqq 4, \\ 0 & i \equiv 2, 3 \pmod 4). \end{cases}$$

By Lemma 4.1, i), we see that $\frac{\delta}{4}: A^{4k}_{(1)}(\mathfrak{K}_2, Z_2) \to A^{4k+1}_{(1)}(\mathfrak{K}_2, Z_2)$ is not trivial and hence an isomorphism. Then $A^*_{(r)}(\mathfrak{K}_2, Z_2) = 0$ for $r \geqq 2$. The last assertion of the lemma follows from the diagram (3.1). q.e.d.

$A^3(\mathfrak{K}_2, Z_2) = \{b_2\}$ and $A^4(\mathfrak{K}_2, Z_2) = \{Sq^1 b_2, Sq^4 a_2\}$, then from (3.3), vii) and (4.4), i), we have

Corollary. *2-component of $\pi_2 = Z_2$.*

From the corollary, $A^*(\pi_2, Z_2)$ is isomorphic to A^* and generated by an element u of $A^0(\pi_2, Z_2)$. Consider the exact sequence of (4.4), ii) for $k=2$.

Proposition 4.5. *There exists an element c_3 of $A^5(\mathfrak{K}_3, Z_2)$ such that $i^* c_3 = Sq^3 u$. $A^*(\mathfrak{K}_3, Z_2)$ is generated by $a_3 = p^* a_2$ and c_3. We have relations $Sq^1 a_3 = Sq^2 a_3 = Sq^1 c_3 = Sq^5 c_3 = 0$ and an isomorphism $A^i(\mathfrak{K}_3, Z_2) \approx A^i/(\varphi_1 A^{i-1} + \varphi_2 A^{i-2}) \oplus A^{i-5}/(\varphi_1 A^{i-6} + \varphi_5 A^{i-10})$. The Bockstein homomorphisms $\frac{\delta}{2^r}$, $r > 1$, are trivial except for the case $r=3$ and* deg$\equiv 0$ (mod 4), *and in the case the rank of the image of $\frac{\delta}{8}$ is* 1. *In particular, $\frac{\delta}{8} Sq^4 a_3 = \{c_3\}$, $\frac{\delta}{8} Sq^4 a_3 = \{Sq^4 c_3\}$, and $\frac{\delta}{8} Sq^8 Sq^4 a_3 = \{(Sq^8 + Sq^6 Sq^2) c_3\}$.*

Proof. $A^3(\mathfrak{K}_3, Z_2) = 0$ by (4.5), then $\Delta^*: A^0(\pi_2, Z_2) \to A^3(\mathfrak{K}_2, Z_2) = \{b_2\}$ is onto and $\Delta^*(\alpha u) = \alpha b_2$. From the exactness of the sequence (4.4), ii) and from Proposition 4.4, we have that $p^* A^*(\mathfrak{K}_2, Z_2)$ is generated by $a_3 = p^* a_2$ and isomorphic to $A^*/(\varphi_1 A^* + \varphi_2 A^*)$ and that the kernal of Δ^*, i.e. the image of i^*, is generated by $Sq^3 u$. Therefore $A^*(\mathfrak{K}_3, Z_2)$ is generated by a_3 and an element c_3 such that $i^* c_3 = Sq^3 u$. We see that $p^* A^i(\mathfrak{K}_2, Z_2) = \{Sq^i a_2\}$ for $i=6$, 7, 10 and 11. Then $Sq^1: p^* A^i(\mathfrak{K}_2, Z_2) \to p^* A^{i+1}(\mathfrak{K}_2, Z_2)$ is an isomorphism if $i=6$ or 10. Applying Lemma 4.2, ii), we have relations $Sq^1 c_3 = Sq^5 c_3 = 0$ and an isomorphism: $A^i(\mathfrak{K}_3, Z_3) \approx A^{i-5}/(\varphi_1 A^{i-6} + \varphi_5 A^{i-10}) \oplus p^* A^i(\mathfrak{K}_2, Z_2) \approx A^{i-5}/(\varphi_1 A^{i-6} + \varphi_5 A^{i-10}) \oplus A^i/(\varphi_1 A^{i-1} + \varphi_2 A^{i-2})$. For Bockstein operators, the proof is similar to the previous proposition. q.e.d.

$A^4(\mathfrak{K}_3, Z_2) = \{Sq^4 a_3\}$, $A^5(\mathfrak{K}_3, Z_2) = \{c_3\}$ and $\frac{\delta}{8} Sq^4 a_3 = c_3$, then by (3.3), vii) and (4.4), i),

Corollary. *2-component of* $\pi_3 = Z_8$.

From the corollary, $A^*(\pi_3, Z_2)$ is isomorphic to $A^*/\varphi_1 A^* + A^*/\varphi_1 A^*$ generated by elements $u_0 \in A^0(\pi_3, Z_2)$ and $u_1 \in A^1(\pi_3, Z_2)$ such that $\frac{\delta}{8} u_0 = u_1$. Consider the exact sequence (4.4), ii) for $k=3$. Denote that $p^* a_3 = a_4$.

Proposition 4.6. *There exist elements* $d_4 \in A^7(\mathfrak{K}_4, Z_2)$ *and* $e_4 \in A^9(\mathfrak{K}_4, Z_2)$ *such that* $i^* d_4 = Sq^4 u_0$ *and* $i^* e_4 = Sq^5 u_1$. *We have relations* $Sq^1 a_4 = Sq^2 a_4 = Sq^4 a_4 = Sq^1 e_4 = Sq^2 e_4 = Sq^2 Sq^1 d_4 = Sq^7 d_4 = (Sq^{10} + Sq^8 Sq^2 + Sq^7 Sq^3) d_4 = 0$, $\frac{\delta}{16} Sq^8 a_4 = \{e_4\}$, $\frac{\delta}{4} Sq^{12} a_4 = \{Sq^6 d_4\}$ *and* $\frac{\delta}{8}((Sq^5 + Sq^4 Sq^1) d_4 + \varepsilon Sq^{12} a_4) = \{Sq^4 e_4\}$ *for some* $\varepsilon = 0$ *or* 1. *Let* $A_0^* = \sum A_0^i$ *be an* A^**-submodule generated by* a_4 *and* e_4, *then* $A_0^i \approx A^i/(\varphi_1 A^{i-1} + \varphi_2 A^{i-2} + \varphi_4 A^{i-4}) \oplus A^{i-9}/(\varphi_1 A^{i-10} + \varphi_2 A^{i-11})$. *For* $i < 21$, $A^i(\mathfrak{K}_4, Z_2)$ *is generated by* a_4, d_4 *and* e_4, *and* $A^i(\mathfrak{K}_4, Z_2) \approx A_0^i \oplus A^{i-7}/(\varphi_{(2,1)} A^{i-10} + \varphi_7 A^{i-14} + \varphi_{(10)+(8,2)+(7,3)} A^{i-17})$.

Proof. $A^4(\mathfrak{K}_4, Z_2) = 0$ by (4.5), then $\Delta^* : A^0(\pi_3, Z_2) \to A^4(\mathfrak{K}_3, Z_2) = \{Sq^4 a_3\}$ is onto. Thus $\Delta^* u_0 = Sq^4 a_3$ and $\Delta^* u_1 = c_3$ by (3.3), v). It follows from (4.4), ii) that $p^* A^*(\mathfrak{K}_3, Z_2)$ is generated by $a_4 = p^* a_3$ and isomorphic to $A^*/(\varphi_1 A^* + \varphi_2 A^* + \varphi_4 A^*)$. Since $\Delta^* Sq^5 u_1 = Sq^5 c_3 = 0$ and $\Delta^* Sq^4 u_0 = Sq^4 Sq^4 a_3 = Sq^6 Sq^2 a_3 + Sq^7 Sq^1 a_3 = 0$, there are elements e_4 and d_4 such that $i^* e_4 = Sq^5 u_1$ and $i^* d_4 = Sq^4 u_0$. Let A_e^* and A_d^* be A^*-submodules generated by e_4 and d_4 respectively. Since $p^* A^{10}(\mathfrak{K}_3, Z_2) = p^* A^{11}(\mathfrak{K}_3, Z_2) = 0$, we have from Lemma 4.2, iii) that $A^* = A_e^* + p^* A^*(\mathfrak{K}_3, Z_2)$ and $A_e^* \approx A^*/(\varphi_1 A^* + \varphi_2 A^*)$. Since $p^* A^{10}(\mathfrak{K}_3, Z_2) = p^* A^{17}(\mathfrak{K}_3, Z_2) = 0$ and $Sq^1 : p^* A^{14}(\mathfrak{K}_3, Z_2) = \{Sq^{14} a_4\} \approx p^* A^{15}(\mathfrak{K}_3, Z_2) = \{Sq^{15} a_4\}$, we have from Lemma 4.2, v) that $p^* A^*(\mathfrak{K}_3, Z_2) \cup A_d^*$[1] $= p^* A^*(\mathfrak{K}_3, Z_2) + A_d^*$ and $A_d^* \approx A^*/(\varphi_{(2,1)} A^* + \varphi_7 A^* + \varphi_{(10)+(8,2)+(7,3)} A^*)$ for dimensions less than 21. From Lemma 3.1 and from (4.4), ii), we have $A^*(\mathfrak{K}_4, Z_2) = p^* A^*(\mathfrak{K}_3, Z_2) \cup A_d^* \cup A_e^*$ for dimensions less than 21. Since A_d^* and A_e^* are imbedded by i^* into direct factors, we have $A^*(\mathfrak{K}_4, Z_2) = p^* A^*(\mathfrak{K}_3, Z_2) + A_d^* + A_e^*$ for dimensions less than 21.

Since $\frac{\delta}{8} Sq^8 a_3 = \{Sq^4 c_3\} = \Delta^* Sq^4 u_1$, we have from Lemma 4.1, i), an element $\tilde{\alpha} \in A^9(\mathfrak{K}_4, Z_2)$ such that $\frac{\delta}{16}(p^* Sq^8 a_3) = \frac{\delta}{16} Sq^8 a_4 = \{\tilde{\alpha}\}$ and $i^* \tilde{\alpha} = Sq^1 Sq^4 u_1 = Sq^5 u_1 = i^* e_4$. Since $p^* A^9(\mathfrak{K}_3, Z_2) = 0$, $i^* \tilde{\alpha} = i^* e_4$

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1) $B^* \cup C^*$ means the minimal A^*-submodule containning B^* and C^*.

implies $\tilde{\alpha}=e_4$ and $\frac{\delta}{16}Sq^8a_4=\{e_4\}$. Similarly from $\frac{\delta}{2}Sq^{12}a_3=Sq^{13}a_3$ $=Sq^6Sq^3Sq^4a_3=\Delta^*Sq^6Sq^3u_0$, $Sq^1Sq^6Sq^3u_0=Sq^6Sq^4u_0=i^*Sq^6d_4$ and from $p^*A^{13}(\mathfrak{K}_3, Z_2)=0$, we have $\frac{\delta}{4}Sq^{12}a_4=\{Sq^6a_4\}$.

Since $\frac{\delta}{8}Sq^8Sq^4a_3=\frac{\delta}{8}\Delta^*Sq^8u_0=(Sq^8+Sq^6Sq^2)c_3=\Delta^*(Sq^8+Sq^6Sq^2)u_1$, we have, from Lemma 4.1, iii), elements $\tilde{\alpha}\in A^{12}(\mathfrak{K}_4, Z_2)$ and $\tilde{\beta}\in A^{13}(\mathfrak{K}_4, Z_2)$ such that $\frac{\delta}{8}\tilde{\alpha}=\{\tilde{\beta}\}$, $i^*\tilde{\alpha}=Sq^1Sq^8u_0=(Sq^5+Sq^4Sq^1)Sq^4u_0$ $=i^*(Sq^5+Sq^4Sq^1)d_4$ and $i^*\tilde{\beta}=Sq^1(Sq^8+Sq^6Sq^2)u_1=Sq^4Sq^5u_1=i^*Sq^4e_4$. From $p^*A^{13}(\mathfrak{K}_3, Z_2)=0$ and $p^*A^{12}(\mathfrak{K}_3, Z_2)=\{Sq^{12}a_4\}$, we have that $\tilde{\beta}=Sq^4e_4$ and $\tilde{\alpha}=(Sq^5+Sq^4Sq^1)d_4+\varepsilon Sq^{12}a_4$ for some $\varepsilon=0$ or 1. Then $\frac{\delta}{8}((Sq^5+Sq^4Sq^1)d_4+\varepsilon Sq^{12}a_4)=\{Sq^4e_4\}$. q. e. d.

$A^5(\mathfrak{K}_4, Z_2)=A^6(\mathfrak{K}_4, Z_2)=0$, $A^7(\mathfrak{K}_4, Z_2)=\{d_4\}$ and $A^8(\mathfrak{K}_4, Z_2)$ $=\{Sq^1d_4, Sq^8a_4\}$. By (3.3), vii), and (4.4), i), the 2-component of π_4 vanishes. Then $A^*(\pi_4, Z_2)=0$. From the exact sequence (4.4), ii), we have an isomorphism

$$p^*: A^*(\mathfrak{K}_4, Z_2)\approx A^*(\mathfrak{K}_5, Z_2).$$

Similarly we have an isomorphism

$$p^*: A^*(\mathfrak{K}_5, Z_2)\approx A^*(\mathfrak{K}_6, Z_2).$$

Again from (3.3), vii) and (4.4), i),

Corollary. *2-component of* $\pi_4=$ *2-component of* $\pi_5=0$,
2-component of $\pi_6=Z_2$.

From the corollary, $A^*(\pi_6, Z_2)$ is isomorphic to A^* and is generated by an element u of $A^0(\pi_6, Z_2)$. Consider the exact sequence of (4.4), ii) for $k=6$, where we identify $A^*(\mathfrak{K}_6, Z_2)$ with $A^*(\mathfrak{K}_4, Z_2)$ by the above two isomorphisms p^*. Denote that $a_7=p^*a_4\in A^0(\mathfrak{K}_7, Z_2)$ and $e_7=p^*e_4\in A^9(\mathfrak{K}_7, Z_2)$.

Proposition 4.7. *There exists elements* $f_7\in A^9(\mathfrak{K}_7, Z_2)$, $f_7'\in A^{13}(\mathfrak{K}_7, Z_2)$ *and* $f_7''\in A^{16}(\mathfrak{K}_7, Z_2)$ *such that* $i^*f_7=Sq^2Sq^1u$, $i^*f_7'=Sq^7u$ *and* $i^*f_7''=(Sq^{10}+Sq^8Sq^2+Sq^7Sq^3)u$. *Let* A_0^* *be an* A^**-submodule generated by* a_7, e_7 *and* f_7. *We have relations* $Sq^1a_7=Sq^2a_7=Sq^4a_7$ $=Sq^1e_7=Sq^2e_7=Sq^2f_7=0$ *and an isomorphism* $A_0^i\approx A^i/(\varphi_1A^{i-1}+\varphi_2A^{i-2}+\varphi_4A^{i-4})\oplus A^{i-9}/(\varphi_1A^{i-10}+\varphi_2A^{i-11})\oplus A^{i-9}/\varphi_2A^{i-11}$. $A^*(\mathfrak{K}_7, Z_2)/A_0^*$ *has a linearly independent base* $\{f_7', Sq^2f_7', f_7'', Sq^4f_7', Sq^2f_7''; Sq^6f_7', Sq^4Sq^2f_7'; \cdots\}$.

Proof. The existence of f_7, f_7' and f_7'' follows from (4.4), ii) and the previous proposition. The second assertion follows from Lemma 4.2, iv) since $p^*A^{11}(\mathfrak{K}_4, Z_2)=0$. The last assertion follows from (4.4), ii) and from the calculation in the proof of Lemma 3.1. q. e. d.

We see $A^8(\mathfrak{K}_7, Z_2)=\{Sq^8a_7\}$ and $A^9(\mathfrak{K}_7, Z_2)=\{e_7, f_7\}$. By proposition 4.6 and (3.3), iv), $\frac{\delta}{16}Sq^8a_7=p^*\frac{\delta}{16}Sq^8a_4=\{p^*e_4\}=\{e_7\}$. Then by (3.3), vii) and (4.4), i),

Corollary. *2-component of* $\pi_7=Z_{16}$.

$A^*(\pi_7, Z_2)$ is isomorphic to $A^*/\varphi_1A^*+A^*/\varphi_1A^*$ and generated $u_0\in A^0(\pi_7, Z_2)$ and $u_1\in A^1(\pi_7, Z_2)$ such that $\frac{\delta}{16}u_0=u_1$. Consider the exact sequence (4.4), ii) for $k=7$. Denote that $p^*a_7=a_8$, $p^*f_7=f_8$, $p^*f_7'=f_8'$ and $p^*f_7''=f_8''$.

Porposition 4.8. *There exist elements* $g_8\in A^9(\mathfrak{K}_8, Z_2)$, $g_8'\in A^{15}(\mathfrak{K}_8, Z_2)$ *and* $h_8\in A^{10}(\mathfrak{K}_8, Z_2)$ *such that* $i^*g_8=Sq^2u_0$, $i^*g_8'=Sq^8u_0$, $i^*h_8=Sq^2u_1$ *and* $Sq^2h_8=0$. *Let* A_0^* *be an* A^**-submodule generated by* a_8, f_8, g_8 *and* h_8, *then we have relations* $Sq^1a_8=Sq^2a_8=Sq^4a_8=Sq^8a_8=Sq^2f_8=Sq^2g_8=0$ *and an isomorphism* $A_0^i\approx A^i/(\varphi_1A^{i-1}+\varphi_2A^{i-2}+\varphi_4A^{i-4}+\varphi_8A^{i-8})\oplus A^{i-9}/\varphi_2A^{i-11}\oplus A^{i-9}/\varphi_2A^{i-11}\oplus A^{i-10}/\varphi_2A^{i-12}$. $A^*(\mathfrak{K}_8, Z_2)/A_0^*$ *has a linearly independent base* $\{f_8'; Sq^2f_8', g_8'; f_8'': Sq^4f_8', Sq^2g_8'; Sq^2f_8'', Sq^3g_8'; Sq^6f_8', Sq^4Sq^2f_8', Sq^4g_8'; \cdots\}$.

Proof. As is seen in the proof of Proposition 4.6, $\Delta^*u_0=Sq^8a_7$ and $\Delta^*u_1=e_7$. From Proposition 4.8, Lemma 3.2 and from (4.4), ii), there are elements g_8, g_8' and h_8' such that $i^*g_8=Sq^2u_0$, $i^*g_8'=Sq^8u_0$ and $i^*h_8'=Sq^2u_1$. Since $i^*Sq^2h_8'=Sq^2Sq^2u_1=0$ and since $p^*A^{12}(\mathfrak{K}_7, Z_2)=\{Sq^2Sq^1f_8\}$, we have $Sq^2h_8'=\varepsilon Sq^2Sq^1f_8$ for some $\varepsilon=0$ or 1. Setting $h_8=h_8'+\varepsilon Sq^1f_8$ we have that $i^*h_8=Sq^2u_1$ and $Sq^2h_8=0$. Remark that the condition $p^*A^{k+5}(\mathfrak{K}_k, Z_2)=0$ of Lemma 4.2, iv) may be replaced by the condition $Sq^2v=0$. Then the proposition is proved by Lemma 4.2, iv), the exact sequence (4.4), ii) and by Lemma 3.2. q. e. d.

We see $A^9(\mathfrak{K}_8, Z_2)=\{f_8, g_8\}$ and $A^{10}(\mathfrak{K}_8, Z_2)=\{Sq^1f_8, Sq^1g_8, h_8\}$. By (3.3), vii) and (4.4), i),

Corollary. *2-component of* $\pi_8=Z_2+Z_2$.

Then $A^*(\pi_8, Z_2)\approx A^*+A^*$. We may chose generators u and u' of $A^*(\pi_8, Z_2)$ such that, in the exact sequence (4.4), ii) for

$k=8$, the relations $\Delta^* u=f_8$ and $\Delta^* u'=g_8$ hold. Denote that $p^* a_8 = a_9$, $p^* f_8'=f_9'$, $p^* f_8''=f_9''$, $p^* g_8'=g_9'$ and $p^* h_8=h_9$.

Since $\Delta^* Sq^2 u=Sq^2 f_8=0$ and $\Delta^* Sq^2 u'=Sq^2 g_8=0$, there exist elements i_9' and j_9' of $A^{10}(\mathfrak{K}_9, Z_2)$ such that

$$i^* i_9' = Sq^2 u \quad \text{and} \quad i^* j_9' = Sq^2 u' .$$

To determine $Sq^3 i_9'$ and $Sq^3 j_9'$, we shall consider the Bockstein operators in $A^i(\mathfrak{K}_k, Z_2)$ for $i=12, 13$ and $k=7, 8, 9$.

$A^{12}(\mathfrak{K}_7, Z_2)=\{Sq^{12}a_7, Sq^2Sq^1f_7\}$ and $A^{13}(\mathfrak{K}_7, Z_2)=\{Sq^4e_7, Sq^4f_7, f_7'\}$. Then the following three possibilities are considered.

$$\text{(4. 6)} \quad \text{i)} \quad \frac{\delta}{8} Sq^{12}a_7 = \{f_7'\} \quad \textit{and} \quad \frac{\delta}{4} Sq^2Sq^1f_7 = \{Sq^4e_7\} ;$$

$$\text{ii)} \quad \frac{\delta}{8} Sq^{12}a_7 = \{Sq^4e_7\} \quad \textit{and} \quad \frac{\delta}{4} Sq^2Sq^1f_7 = \{f_7'\} ;$$

$$\text{iii)} \quad \frac{\delta}{8} Sq^{12}a_7 = \{Sq^4e_7\} \quad \textit{and} \quad \frac{\delta}{4} Sq^2Sq^1f_7 = \{f_7'+Sq^4e_7\} .$$

Proof. First we remark that $p^* A^{12}(\mathfrak{K}_4, Z_2)=\{Sq^{12}a_7\}$ and $p^* A^{13}(\mathfrak{K}_4, Z_2)=\{Sq^4e_7\}$. By Proposition 4. 6, $\frac{\delta}{4} Sq^{12}a_4=\{Sq^6d_4\}=\{\Delta^* Sq^6u\}$. Since $i^* f_7'=Sq^7u=Sq^1Sq^6u$, we have by Lemma 4. 1, i), $\frac{\delta}{8} Sq^{12}a_4 =\{f_7'+\lambda Sq^4e_7\}$ for some $\lambda=0$ or 1. By Proposition 4. 6, $\frac{\delta}{8}((Sq^5+Sq^4Sq^1)d_4+\varepsilon Sq^{12}a_4)=\{Sq^4e_4\}=\{Sq^4e_4+Sq^6d_4\}$. In the case $\varepsilon=0$, applying Lemma 4. 1, ii), we have from $Sq^2Sq^1Sq^2Sq^1u=Sq^1(Sq^5+Sq^4Sq^1)u$ that $\frac{\delta}{4}(Sq^2Sq^1f_7+\nu Sq^{12}a_7)=\{Sq^4e_7\}$ for some ν. Since $Sq^{12}a_7 \in \frac{\delta}{4}$-*kernel*, we have $\frac{\delta}{4}(Sq^2Sq^1f_7)=\{Sq^4e_7\}$. Since $Sq^4e_7 \in \frac{\delta}{4}$-*image*, $\frac{\delta}{8} Sq^{12}a_7=\{f_7'+\lambda Sq^4e_7\}=\{f_7'\}$. Then we have the case i). Next consider the case $\varepsilon=1$. By (3. 3), iv), $\frac{\delta}{8}((Sq^5+Sq^4Sq^1)d_4+Sq^{12}a_4)=\{Sq^4e_4\}$ implies $\frac{\delta}{8} Sq^{12}a_7=\{Sq^4e_7\}$. Since $(Sq^5+Sq^4Sq^1)d_4+Sq^{12}a_4 \in \frac{\delta}{4}$-*kernel*, we have $\frac{\delta}{4} Sq^{12}a_4=\frac{\delta}{4}(Sq^5+Sq^4Sq^1)d_4=\{Sq^6d_4\}$. Then we have from Lemma 4. 2, iii), $\frac{\delta}{4}(Sq^2Sq^1f_7+\lambda Sq^{12}a_7)=\{f_7'+\nu Sq^4e_7\}$ for some λ, $\nu=0$ or 1. Since $Sq^{12}a_7 \in \frac{\delta}{4}$-*kernel*, $\frac{\delta}{4}(Sq^2Sq^1f_7) =\{f_7'+\nu Sq^4e_7\}$. Then we have the cases ii) and iii). q. e. d.

$A^{12}(\mathfrak{K}_8, Z_2)=\{Sq^2Sq^1f_8, Sq^2Sq^1g_8\}$ and $A^{13}(\mathfrak{K}_8, Z_2)=\{f_8', Sq^2Sq^1h_8, Sq^4f_8, Sq^4g_8\}$. Then the following three possibilities are considered.

(4.7) i) $\frac{\delta}{4}Sq^2Sq^1g_8=\{f_8'\}$ *and* $\frac{\delta}{8}Sq^2Sq^1f_8=\{Sq^2Sq^1h_8\}$;

ii) $\frac{\delta}{4}Sq^2Sq^1f_8=\{f_8'\}$ *and* $\frac{\delta}{8}Sq^2Sq^1g_8=\{Sq^2Sq^1h_8\}$;

iii) $\frac{\delta}{4}Sq^2Sq^1f_8=\{f_8'\}$ *and* $\frac{\delta}{8}Sq^2Sq^1(f_8+g_8)=\{Sq^2Sq^1h_8\}$.

Proof. First we remark that the term Sq^4f_8 does not appear in any representatives of a $\frac{\delta}{2^r}$*-image,* because $Sq^1Sq^4f_8=Sq^5f_8\neq 0$. Applying the Lemma 4.1, i) and ii), we have from the case i) of (4.6) that $\frac{\delta}{4}Sq^2Sq^1(g_8+\lambda f_8)=\{f_8'\}$ and $\frac{\delta}{8}Sq^2Sq^1f_8=\{Sq^2Sq^1h_8+\nu f_8'\}$ for some λ and ν. Then $\frac{\delta}{4}Sq^2Sq^1f_8=0$ and $\{Sq^2Sq^1h_8+\nu f_8'\}=\{Sq^2Sq^1h_8\}$. Therefore (4.6), i) implies (4.7), i).

Next consider the cases ii) and iii) of (4.6). By (3.3), $\frac{\delta}{4}Sq^2Sq^1f_7=\{f_7'+\nu Sq^4e_7\}$ implies $\frac{\delta}{4}Sq^2Sq^1f_8=\{f_8'\}$. Applying Lemma 4.1, iii) to $\frac{\delta}{8}Sq^{12}a_7=\{Sq^4e_7\}$, we have that $\frac{\delta}{8}Sq^2Sq^1(g_8+\lambda f_8)=\{Sq^2Sq^1h_8+\nu f_8'\}=\{Sq^2Sq^1h_8\}$. Then we have the case ii) and iii). q. e. d.

From (4.4), ii) for $k=8$ and from Theorem I, we have that $A^*(\mathfrak{K}_9, Z_2)/p^*A^*(\mathfrak{K}_8, Z_2)$ is isomorphic to $A^*/\varphi_3A^*+A^*/\varphi_3A^*$ and generated by i_9' and j_9'. In particular $A^{12}(\mathfrak{K}_9, Z_2)=\{Sq^2i_9', Sq^2j_9'\}$ and $A^{13}(\mathfrak{K}_9, Z_2)=\{Sq^2Sq^1h_9, f_9', Sq^2Sq^1i_9', Sq^2Sq^1j_9'\}$. Then the following three possibilities are considered.

(4.8) i) $Sq^3j_9'=\{f_9'\}$ *and* $\frac{\delta}{4}Sq^2i_9'=\{Sq^2Sq^1h_9\}$;

ii) $Sq^3i_9'=\{f_9'\}$ *and* $\frac{\delta}{4}Sq^2j_9'=\{Sq^2Sq^1h_9\}$;

iii) $Sq^3i_9'=\{f_9'\}$ *and* $\frac{\delta}{4}Sq^2(i_9'+j_9')=\{Sq^2Sq^1h_9\}$.

Proof. Consider the case i) of (4.7). From Lemma 4.1, ii), we have elements $\tilde{\alpha}$ and γ such that $i^*\tilde{\alpha}=Sq^1Sq^2Sq^1u'=Sq^2Sq^2u'=i^*Sq^2j_9'$, $\{\gamma\}=\{f_8'\}$ and $\frac{\delta}{2}\tilde{\alpha}=p^*\gamma$. Since $p^*A^{12}(\mathfrak{K}_8, Z_2)=0$, $i^*\tilde{\alpha}=i^*Sq^2j_9'$ implies $\tilde{\alpha}=Sq^2j_9'$. Since $\frac{\delta}{2}$*-image*$=0$ in $A^{12}(\mathfrak{K}_8, Z_2)$, $\{\gamma\}=\{f_8'\}$ implies $\gamma=f_8'$ and $p^*\gamma=f_9'$. Therefore $Sq^3j_9'=\frac{\delta}{2}Sq^2j_9'=f_9'$. We have also, from Lemma 4.1, ii), $\frac{\delta}{4}Sq^2i_9'=\{Sq^2Sq^1h_9+\varepsilon f_9'\}=\{Sq^2Sq^1h_9\}$.

Similarly (4. 7), ii) implies (4. 8), ii) and (4. 7), iii) implies (4. 8), iii). q. e. d.

Now we define elements i_9 and j_9 of $A^{10}(\mathfrak{K}_9, Z_2)$ as follows corresponding for each cases of (4. 8);

i) $i_9 = i_9'$ *and* $j_9 = j_9'$,

ii) $i_9 = j_9'$ *and* $j_9 = i_9'$,

iii) $i_9 = i_9' + j_9'$ *and* $j_9 = i_9'$.

Then $Sq^3 j_9 = f_9'$, $\frac{\delta}{4} Sq^2 i_9 = \{Sq^2 Sq^1 h_9\}$ and $Sq^3 i_9 = \frac{\delta}{2} Sq^2 i_9 = 0$. Obviously i_9 and j_9 generate $A^*(\mathfrak{K}_9, Z_2)/p^*A^*(\mathfrak{K}_8, Z_2)$. By making use of the condition $Sq^3 i_9 = 0$, in place of the condition on Sq^1 in Lemma 4. 2, i), we have

Proposition 4. 9. *Let A_0^* be an A^*-submodule generated by h_9 and i_9, then we have relations $Sq^2 h_9 = Sq^3 i_9 = 0$ and an isomorphism $A_0^i \approx A^{i-10}/\varphi_2 A^{i-12} \oplus A^{i-10}/\varphi_3 A^{i-13}$. $A^*(\mathfrak{K}_9, Z_2)/A_0^*$ has a linearly independent base $\{Sq^{16} a_9;\ g_9',\ Sq^i g_9',\ i = 2, 3, 4;\ f_9'',\ Sq^2 f_9'';\ Sq^I j_9,\ I \neq (5, 1)\}$, for dimensions less than 20.*

Remark that $f_9' = Sq^3 j_9$, $Sq^2 f_9' = (Sq^5 + Sq^4 Sq^1) j_9$, $Sq^4 f_9' = Sq^5 Sq^2 j_9$, $Sq^6 f_9' = Sq^6 Sq^3 j_9$ and $Sq^4 Sq^2 f_9' = (Sq^9 + Sq^8 Sq^1 + Sq^7 Sq^2 + Sq^6 Sq^2 Sq^1) j_9$.

Since $i^* Sq^3 f_7' = Sq^3 Sq^7 u = (Sq^4 Sq^2 Sq^1 + Sq^7) Sq^2 Sq^1 u = i^*(Sq^4 Sq^2 Sq^1 + Sq^7) f_7$, we have $Sq^3 f_7' - (Sq^4 Sq^2 Sq^1 + Sq^7) f_7 \in p^* A^{16}(\mathfrak{K}_4, Z_2) = \{Sq^{16} a_7, Sq^7 e_7\}$. By operating p^*, we have that $Sq^3 f_9' = \varepsilon Sq^{16} a_9$ for some $\varepsilon = 0$ or 1. Thus we consider the following two cases:

A) $Sq^5 Sq^1 j_9 = Sq^3 f_9' = 0$,

B) $Sq^5 Sq^1 j_9 = Sq^3 f_9' = Sq^{16} a_9$.

By (3. 3), vii) and (4. 4), i), we have from $A^{10}(\mathfrak{K}_9, Z_2) = \{h_9, i_9, j_9\}$ and $A^{11}(\mathfrak{K}_9, Z_2) = \{Sq^1 h_9, Sq^1 i_9, Sq^1 j_9\}$,

Corollary. *2-component of* $\pi_9 = Z_2 + Z_2 + Z_2$.

$A^*(\pi_9, Z_2) \approx A^* + A^* + A^*$. We may chose generators u, u' and u'' such that the relations $\Delta^* u = h_9$, $\Delta^* u' = i_9$ and $\Delta^* u'' = j_9$ hold in the exact sequence (4. 4), ii), $k = 9$. Denote that $p^* a_9 = a_{10}$, $p^* f_9'' = f_{10}''$ and $p^* g_9' = g_{10}'$.

Proposition 4. 10. *There exist elements $k_{10} \in A^{11}(\mathfrak{K}_{10}, Z_2)$ and $l_{10} \in A^{12}(\mathfrak{K}_{10}, Z_2)$ such that $i^* k_{10} = Sq^2 u$ and $i^* l_{10} = Sq^3 u'$. Let A_0^* be an A^*-submodule generated by k_{10} and l_{10}, then we have relations $Sq^3 k_{10} = Sq^1 l_{10} = Sq^5 l_{10} = 0$ and $\frac{\delta}{4} l_{10} = \{Sq^2 k_{10}\}$ and an isomorphism*

$A_0^i \approx A^{i-10}/\varphi_3 A^{i-13} \oplus A^{i-11}/(\varphi_1 A^{i-12}+\varphi_5 A^{i-16})$. *For the case A), $A^*(\mathfrak{K}_{10}, Z_2)/A_0^* = \{g_{10}'', m_{10}; Sq^{16}a_{10}, f_{10}''; Sq^2 g_{10}'; Sq^2 f_{10}'', Sq^3 g_{10}'; \cdots\}$ where $i^* m_{10} = Sq^5 Sq^1 u''$. For the case B), $A^*(\mathfrak{K}_{10}, Z_2)/A_0^* = \{g_{10}'; f_{10}''; Sq^2 g_{10}'; Sq^2 f_{10}'', Sq^3 g_{10}'; \cdots\}$.*

Proof. From the previous proposition, $\Delta^* Sq^2 u = \Delta^* Sq^3 u' = 0$, $p^* A^i(\mathfrak{K}_9, Z_2) = 0$, $i = 12, 13$, and $Sq^1: p^* A^{17}(\mathfrak{K}_9, Z_2) = \{Sq^2 g_{10}'\} \to p^* A(\mathfrak{K}_9, Z_2) = \{Sq^3 g_{10}', Sq^2 f_{10}''\}$ is an isomorphism into. Then we have, from i) and ii) of Lemma 4. 2, the first two assertions of the proposition. Since $p^* A^{12}(\mathfrak{K}_9, Z_2) = p^* A^{13}(\mathfrak{K}_9, Z_2) = 0$, $\frac{\delta}{4} Sq^2 i_9 = \{Sq^2 Sq^1 h_9\}$ implies $\frac{\delta}{4} l_{10} = \{Sq^2 k_{10}\}$ by Lemma 4. 1, iii). The last two assertions are verified directly. q. e. d.

From $A^{11}(\mathfrak{K}_{10}, Z_2) = \{k_{10}\}$ and $A^{12}(\mathfrak{K}_{10}, Z_2) = \{Sq^1 k_{10}', l_{10}\}$,

Corollary. *2-component of $\pi_{10} = Z_2$.*

The $A^*(\pi_{10}, Z_2)$ is isomorphic to A^* and generated by an element u of $A^0(\pi_{10}, Z_2)$.

Continuing our calculation, we have the following results without difficulties:

$$A^*(\mathfrak{K}_{11}, Z_2) = \{a_{11}; l_{11}; n_{11}; Sq^2 l_{11}; g_{11}', Sq^3 l_{11}, m_{11}, Sq^2 n_{11}; Sq^{16} a_{11}, f_{11}'', Sq^4 l_{11}, Sq^3 n_{11}; Sq^2 g_{11}', Sq^4 n_{11}; \cdots\}$$

where $i^* n_{11} = Sq^2 u$ and the elements m_{11} and $Sq^{16} a_{11}$ are omitted for the case B). $\frac{\delta}{8} l_{11} = \{n_{11}\}$.

$$A^*(\mathfrak{K}_{12}, Z_2) = \{a_{12}; g_{12}', m_{12}; Sq^{16} a_{12}, f_{12}'', o_{12}; \cdots\}$$

where $i^* o_{12} = Sq^5 u_0$ and the elements m_{12} and $Sq^{16} a_{12}$ are omitted for the case B).

Therefore we have from (3. 3), vii) and (4. 4), i),

Proposition 4. 11. i) *2-component of $\pi_{11} = Z_8$,*

ii) *2-component of π_{12} = 2-component of $\pi_{13} = 0$,*

iii) *the 2-component of π_{14} has at most two generators.*

Remark. If $\pi_N(S^N)$, k_N, $\pi_{N+1}(S^N)$, k_{N+2}, $\cdots$ are Postnikov's invariant system of S^N, then K_k has an invariant system $\pi_N(S^N)$, k_N, $\cdots$, $\pi_{N+k-1}(S^N)$, 0, 0, $\cdots$.

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