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On a New Definition of Stochastic Integral by Random Riemann Sum

By

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§1. Let $\{Y(t, \omega); t \in T\}$ be a stochastic process and let $\Phi(t, \omega)$ be a function of the time parameter $t \in T$ and the probability parameter ω. The integral

$$\varphi(\omega) = \int_T \Phi(t, \omega)\, dY_t(\omega)$$

cannot be defined as an ordinary Stieltjes integral for each sample function, since the sample functions are not of bounded variation except in trivial cases.

In case $Y(t, \omega)$ is a Wiener process and $\Phi(t, \omega)$ satisfies the following conditions:

C 1. $\Phi(t, \omega)$ is measurable in (t, ω),

C 2. $\int_T E\Phi^2(t, \omega)\, dt < \infty$(*),

C 3. $\Phi(t, \cdot)$ is measurable in B_t for each t, where B_t is the Borel field determined by $\{Y(s, \omega); s \leq t\}$,

K. Itô defined the above integral and called it a stochastic integral [1]. His definition consists of two steps. First he defined the integral in usual way when the integrand Φ is a uniformly step-wise function, namely when there exist $t_1 < t_2 < t_3 < \cdots < t_n$ independent of ω such that

* In [1] he took the weaker hypothesis that $\int_T \Phi^2(t, \omega) dt < \infty$ for almost all ω, but this part has no connection to the purpose of this note.

$$\Phi(t, \omega) = 0 \qquad t < t_1$$
$$\Phi(t, \omega) = \Phi(t_\nu, \omega) \qquad t_\nu \leq t < t_{\nu+1}, \ \nu \leq n-1$$
$$\Phi(t, \omega) = 0 \qquad t_n \leq t,$$

and next he showed, for general Φ, the existence of a sequence of uniformly stepwise function $\{\Phi_n\}$ such that

$$\int_T E(\Phi_n - \Phi)^2 \, dt \to 0$$

and defined its integral as $\varphi(\omega) = \underset{n \to \infty}{\text{l.i.m.}} \int_T \Phi_n(t, \omega) \, dY_t(\omega)$.
J. L. Doob [2] showed that this definition is also available to the more general case in which $Y(t)$ is a martingale process.

In order to avoid the technical trouble of making two cases we shall use the Random Riemann Sum which we introduced in connection with Lebesgue integral in our previous paper [3].

We shall here discuss Itô's case, since Doob's general case can be treated with a slight modification.

§ 2. Let (Ω, B, P) be the probability space on which we have a Wiener process $\{Y(t, \omega) : t \in [0, \infty)\}$ and a function $\Phi(t, \omega)$ satisfing C1, C2 and C3. Let us introduce a probability space (Ω', B', P') on which we define Poisson processes $\{P_n(t, \omega') ; t \in [0, \infty)\}$, n being the mean value of $P_n(1, \omega')$; it makes no difference here whether these Poisson processes are independent or dependent of each other. We shall take the direct product probability space $(\bar{\Omega}, \bar{B}, \bar{P}) = (\Omega, B, P) \times (\Omega', B', P')$ as a basic one, so that ω and ω' are independent. Let $t_i^n(\omega')$ be the i-th jumping time of the Poisson process $P_n(t, \omega')$ and set $t_0^n(\omega') \equiv 0$ for convenience.

For brevity we use the following notations. Let $z \equiv z(\bar{\omega}) = z(\omega, \omega')$ be a random variable on $(\bar{\Omega}, \bar{B}, \bar{P})$. Then $\bar{E}$, E and E' are defined as follows

$$Ez = \int_\Omega z(\omega, \omega') \, P(d\omega)$$
$$E'z = \int_{\Omega'} z(\omega, \omega') \, P'(d\omega')$$
$$\bar{E}z = \int_{\bar{\Omega}} z(\bar{\omega}) \, P(d\bar{\omega}) = \int_{\Omega'} \int_\Omega z(\omega, \omega') \, P(d\omega) \, P'(d\omega')$$
$$= E'Ez = EE'z .$$

It is clear that both Ez and $E'z$ are random variables depending on ω' and ω respectively. By the same way we understand the notations $\bar{P}$, P and P'. It is clear that

$$\bar{P}(A) = E'P(A) = EP'(A)\,.$$

Consider the Random Riemann Sum for sample process

$$S_n(\bar{\omega}) = \sum_{i\geq 1} \Phi(t_i^n(\omega'), \omega)[Y(t_{i+1}^n(\omega'), \omega) - Y(t_i^n(\omega'), \omega)]\,.$$

We will define stochastic integral as follows.

Definition. $\varphi^*(\omega)$, which is independent of ω', is the stochastic integral of Φ with respect to dY_t, say $(*)\int_0^\infty \Phi(t, \omega)\, dY_t(\omega)$, if and only if

$$P'[E(S_n(\bar{\omega}) - \varphi^*(\omega))^2 > \varepsilon] \to 0 \quad \text{for} \quad n \to \infty\,.$$

(We put $*$ in order to distinguish this definition from Itô's.)

Now we shall prove the following

Theorem 1. $(*)\int_0^\infty \Phi(t, \omega)\, dY_t(\omega)$ exists and is uniquely determined except P-measure zero.

Proof. We will prove that $E'E(S_n(\bar{\omega}) - S_m(\bar{\omega}))^2 \to 0$ for $n, m \to \infty$.

For convenience's sake we introduce the following notations

$$X_n(t, \omega') = t_i^n(\omega') \quad \text{for} \quad t_i^n(\omega') \leq t < t_{i+1}^n(\omega') \qquad i = 0, 1, 2, \cdots$$
$$\Phi^*(t, \omega) = \Phi(t, \omega) \quad \text{for} \quad t > 0 \qquad = 0 \quad \text{for} \quad t = 0\,.$$

Arrange the elements of the set $\{t_i^n(\omega')\} \cup \{t_i^m(\omega')\}$ in the order of magnitude and denote the i-th term by $s_i^{n,m}(\omega')$.

$$\begin{aligned}
&\bar{E}(S_n(\bar{\omega}) - S_m(\bar{\omega}))^2 \\
&\quad = E' \sum_{i\geq 1} E[\Phi^*(X_n(s_i^{n,m}(\omega'), \omega'), \omega) - \Phi^*(X_m(s_i^{n,m}(\omega'), \omega'), \omega)]^2 \\
&\qquad \times [s_{i+1}^{n,m}(\omega') - s_i^{n,m}(\omega')] \\
&\quad = E' \int_0^\infty E[\Phi^*(X_n(t, \omega'), \omega) - \Phi^*(X_m(t, \omega'), \omega)]^2\, dt \\
&\quad = E' \int_0^\infty E[\Phi^*(X_n(t, \omega'), \omega) - \Phi(t, \omega) + \Phi(t, \omega) - \Phi^*(X_m(t, \omega'), \omega)]^2\, dt \\
&\quad \leq 2E' \int_0^\infty E[\Phi^*(X_n(t, \omega'), \omega) - \Phi(t, \omega)]^2\, dt \\
&\qquad + 2E' \int_0^\infty E[\Phi^*(X_m(t, \omega'), \omega) - \Phi(t, \omega)]^2\, dt\,.
\end{aligned}$$

$$E'\int_0^\infty E[\Phi^*(X_n(t,\omega'),\omega)-\Phi(t,\omega)]^2 dt$$
$$=E'\{E'\int_0^\infty E(\Phi^*(X_n(t,\omega),\omega)-\Phi(t,\omega))^2 dt/t_1^n(\omega')\}$$
$$=\int_0^\infty e^{-nh}ndh\,[\int_0^h E\Phi^2(t,\omega)\,dt+E'\int_h^\infty E(\Phi(X_n(t,\omega'),\omega)-\Phi(t,\omega))^2 dt]$$
$$\equiv A_n+B_n.$$
$$A_n=\int_{t=0}^\infty E\Phi^2(t,\omega)\,dt\int_{h=t}^\infty e^{-nh}ndh=\int_0^\infty E\Phi^2(t,\omega)\,e^{-nt}dt\to 0 \quad \text{for} \quad n\to\infty.$$

Since the probability that $P_n(t,\omega')$ has a jump in $(t-s,t-s+ds)$ and no jump in $(t-s,t)$ is $nds\cdot e^{-ns}$, we have

$$B_n=\int_{h=0}^\infty e^{-nh}ndh\int_{t=h}^\infty dt\int_{s=0}^{t-h}E(\Phi(t-s,\omega)-\Phi(t,\omega))^2 e^{-ns}nds$$
$$\leq\int_{h=0}^\infty e^{-nh}ndh\int_{t=0}^\infty dt\int_{s=0}^{t}E(\Phi(t-s,\omega)-\Phi(t,\omega))^2 e^{-ns}nds$$
$$=\int_{s=0}^\infty e^{-ns}nds\int_{t=s}^\infty E(\Phi(t-s,\omega)-\Phi(t,\omega))^2 dt$$
$$=\int_{s=0}^\infty e^{-ns}nds\int_{t=0}^\infty E(\Phi(t,\omega)-\Phi(t+s,\omega))^2 dt$$
$$=\int_{s=0}^\infty e^{-s}ds\int_{t=0}^\infty E\Big(\Phi(t,\omega)-\Phi\Big(t+\frac{s}{n},\omega\Big)\Big)^2 dt,$$

which tends to 0 by virtue of C2. as $n\to\infty$.
So we obtain $\bar{E}(S_n-S_m)^2\to 0$, so that there exists $\varphi^*(\omega,\omega')$ such that

$$P'[E(S_n(\bar{\omega})-\varphi^*(\omega,\omega'))^2>\varepsilon]\to 0 \quad \text{for} \quad n\to\infty.$$

Next we shall show that $\varphi^*(\omega,\omega')$ is independent of ω'.

$$\bar{E}(S_n(\bar{\omega})-E'S_n(\bar{\omega}))^2=\bar{E}S_n^2(\bar{\omega})-E(E'S_n(\bar{\omega}))^2$$
$$\bar{E}S_n^2(\bar{\omega})=E'\sum_{i\geq 1}E\Phi^2(t_i^n(\omega'),\omega)(t_{i+1}^n(\omega')-t_i^n(\omega'))$$
$$=\sum_{i\geq 1}\int_0^\infty E\Phi^2(t,\omega)\,\frac{\exp(-nt)(nt)^{i-1}}{(i-1)!}ndt\int_0^\infty he^{-nh}ndh$$
$$=\int_0^\infty E\Phi^2(t,\omega)\,dt<\infty.$$
$$E'S_n(\bar{\omega})=\sum_{i\geq 1}\int_0^\infty \Phi(t,\omega)\,\frac{\exp(-nt)(nt)^{i-1}}{(i-1)!}ndt$$
$$\times\int_0^\infty (Y(t+h,\omega)-Y(t,\omega))\,e^{-nh}ndh$$
$$=\int_0^\infty \Phi(t,\omega)\,ndt\int_0^\infty (Y(t+h,\omega)-Y(t,\omega))\,e^{-nh}ndh$$

$$E(E'S_n(\bar{\omega}))^2 = E[\int_0^\infty \Phi(t, \omega)\, ndt \int_0^\infty (Y(t+h, \omega) - Y(t, \omega))\, e^{-nh}\, ndh]^2$$

$$= 2E[\int_{t=0}^\infty \Phi(t, \omega)\, ndt \int_{h=0}^\infty (Y(t+h, \omega) - Y(t, \omega))\, e^{-nh}\, ndh$$

$$\times \int_{t'=0}^t \Phi(t', \omega)\, ndt' \int_{h'=0}^\infty (Y(t'+h', \omega) - Y(t, \omega))\, e^{-nh'}\, ndh']$$

$$= 2E[\int_{t=0}^\infty \Phi(t, \omega)\, ndt \int_{h=0}^\infty (Y(t+h, \omega) - Y(t, \omega))\, e^{-nh}\, ndh$$

$$\times \int_{t'=0}^t \Phi(t', \omega)\, ndt' \int_{h'=t-t'}^\infty (Y(t'+h', \omega) - Y(t', \omega))\, e^{-nh'}\, ndh']$$

$$= 2\int_{t=0}^\infty ndt \int_{t'=0}^t E(\Phi(t, \omega)\, \Phi(t', \omega'))\, ndt' \int_{h=0}^\infty e^{-nh}\, ndh$$

$$\times \int_{h'=t-t'}^\infty [\min (t'+h', t+h) - t]\, e^{-nh'}\, ndh'$$

$$= \int_0^\infty dt \int_0^t E(\Phi(t, \omega)\, \Phi(t', \omega))\, e^{-n(t-t')}\, ndt' \to \int_0^\infty E\Phi^2(t, \omega)\, dt\,.$$

Therefore $\bar{E}(S_n(\bar{\omega}) - E'S_n(\bar{\omega}))^2$ tends to zero when n tends to infinity. Now we have

$$E(E'(\varphi^*(\omega, \omega') - E'S_n(\bar{\omega}))^2) \leq \bar{E}(\varphi^*(\omega, \omega') - E'S_n(\bar{\omega}))^2 \to 0\,,$$

and therefore

$$E(\varphi^*(\omega, \omega') - E'\varphi^*(\omega, \omega'))^2 = 0\,,$$

namely

$$\varphi^*(\omega, \omega') = E'\varphi^*(\omega, \omega')\,.$$

This completes our proof.

From the above calculation we can easily obtain the following properties.

P 1. $E\varphi^*(\omega) = 0$,

P 2. $E\varphi_1^*(\omega)\varphi_2^*(\omega) = \int_0^\infty E\Phi_1(t, \omega)\, \Phi_2(t, \omega)\, dt$,

P 3. Let $\chi_t(\cdot)$ is the characteristic function of the set $[0, t]$. If we denote the separable modification of (*) $\int_0^\infty \chi_t(s)\, \Phi(s, \omega)\, dY_s(\omega)$ with the same expression, then we have

$$c^2P[\sup_{0\leq t<\infty} |\int_0^\infty \chi_t(s)\, \Phi(s, \omega)\, dY_s(\omega)| \geq c] \leq E\varphi^*(\omega)^2 \qquad (c>0)\,.$$

Proof. P1: It is clear that for almost all ω', $ES_n(\bar{\omega}) = 0$. Since there are subsequence $\{n_i\}$ such that for almost all ω'

$$E|S_{n_i}(\bar{\omega}) - \varphi^*(\omega)|^2 \to 0$$

we have $E\varphi^*(\omega) = 0$.

P 2: We use $S_n(\bar{\omega}\,;\,\Phi)$ for Random Riemann Sum to specify the integrand function Φ.

We have

$$ES_n(\omega\,;\,\Phi_1)\,S_n(\bar{\omega}\,;\,\Phi_2)=\int_0^\infty E\Phi_1^*(X_n(t,\,\omega'),\,\omega)\,\Phi_2^*(X_n(t,\,\omega'),\,\omega)\,dt$$

for almost all ω'. If we take subsequence $\{n_i\}$ such that

$$\int_0^\infty E[\Phi_i^*(X_{n_j}(t,\,\omega'),\,\omega)-\Phi_i(t,\,\omega)]^2\,dt\to 0\,,\quad E|S_{n_j}(\bar{\omega}\,;\,\Phi_i)-\varphi_i^*(\omega)|^2\to 0$$
$$i=1,\,2\,,\quad n_j\to\infty\,,$$

for almost all ω', we obtain P2 by the property of inner product.

P 3: Keeping in mind that $S_n(\bar{\omega}\,;\,\chi_t\Phi)$ is the only finite sum for almost all $\bar{\omega}$, we have for almost all ω'

$$c^2P(\sup_{0\leq t<\infty}|S_n(\bar{\omega}\,;\,\chi_t\Phi)|\geq c)\leq\int_0^\infty E\Phi^*(X_n(t,\,\omega'),\,\omega)^2\,dt$$

$$c^2P(\sup_{0\leq t<\infty}|S_n(\bar{\omega}\,;\,\chi_t\Phi)-S_m(\bar{\omega}\,;\,\chi_t\Phi)|\geq c)$$
$$\leq\int_0^\infty E(\Phi^*(X_n(t,\,\omega'),\,\omega)-\Phi^*(X_m(t,\,\omega'),\,\omega)^2\,dt$$

by Kolmogorav's inequality in infinite sequence.

Therefore for some subsequence $\{n_i\}$ we have

$$\sup_{0\leq t<\infty}|S_{n_i}(\bar{\omega}\,;\,\chi_t\Phi)-S_{n_j}(\bar{\omega}\,;\,\chi_t\Phi)|\to 0\quad\text{and}$$
$$\int_0^\infty E[\Phi^*(X_{n_i}(t,\,\omega'),\,\omega)-\Phi(t,\,\omega)]^2\,dt\to 0\quad\text{for}\quad n_i,\ n_j\to\infty$$

for almost all $\bar{\omega}$.

So if we take the sequence $\{S_{n_i}(\bar{\omega}\,;\,\chi_t\Phi)\}$, we can prove P3.

Remark. For almost all ω, (*) $\int_0^\infty \chi_t(s)\,\Phi(s,\,\omega)\,dY_s(\omega)$ is continuous.

Consider the modified Random Riemann Sum;

$$\bar{S}_n(\omega\,;\,\chi_t\Phi)=\sum_{i=1}^{N-1}\Phi(t_i^n(\omega'),\,\omega)\,(Y(t_{i+1}^n(\omega'),\,\omega)-Y(t_i^n(\omega'),\,\omega))$$
$$+\Phi(t_N^n(\omega'),\,\omega)\,(Y(t,\,\omega)-Y(t_N^n(\omega'),\,\omega))$$
$$\text{for}\quad t_N^n(\omega')\leq t<t_{N+1}^n(\omega')\,.$$

For almost all $\bar{\omega}$, $\bar{S}_n(\bar{\omega}\,;\,\chi_t\Phi)$ is continuous in t. Furthermore

$$c^2P(\sup_{0\leq t\leq a}|\bar{S}_n(\bar{\omega}\,;\,\chi_t\Phi)-\bar{S}_m(\bar{\omega}\,;\,\chi_t\Phi)|\geq c)$$
$$\leq\int_0^\infty E[\Phi^*(X_n(t,\,\omega'),\,\omega)-\Phi^*(X_m(t,\,\omega'),\,\omega)]^2\,dt$$

for almost all ω'.

So by the analogous calculation a subsequence $\bar{S}_{n_j}(\bar{\omega}\,;\,\chi_t\Phi)$ tends to a stochastic process whose sample functions are continuous with probability one.

On the other hand the following estimation

$$\bar{E}|S_n(\bar{\omega}\,;\,\chi_t\Phi)-\bar{S}_n(\bar{\omega}\,;\,\chi_t\Phi)|^2\leq\int_0^t E\Phi^2(s,\omega)(t-s)\,e^{-n(t-s)}n\,ds\to 0 \quad \text{for} \quad n\to\infty,$$

shows that $\bar{S}_n$ and S_n have the same limit for each t. Therefore our integral is continuous in t with probability one.

Now we shall show that our integral coincides with Itô's stochastic integral.

Theorem 2. For almost all ω, $\varphi^*(\omega)=\varphi(\omega)$.

Proof. Taking s_0 appropriately as in the proof of the existence theorem of $\varphi(\omega)$ [1], we have

$$\bar{E}\Big[S_n(\bar{\omega})-\sum_{s_0+\frac{k}{m}>0}\Phi(s_0+\frac{k}{m},m,\omega)\Big(Y\Big(s_0+\frac{k+1}{m},\omega\Big)-Y\Big(s_0+\frac{k}{m},\omega\Big)\Big)\Big]^2$$

$$=E'\int_0^\infty E[\Phi^*(X_n(t,\omega'),\omega)-\Phi(\phi_m(t-s_0)+s_0,\omega)]^2\,dt$$

where $\phi_m(t)=(k-1)/m$ for $(k-1)/m\leq t<k/m$, $k=0,\pm1,\pm2\cdots$

$$\leq 2E'\int_0^\infty E[\Phi^*(X_n(t,\omega'),\omega)-\Phi(t,\omega)]^2\,dt$$

$$+2\int_0^\infty E[\Phi(\phi_m(t-s_0)+s_0,\omega)-\Phi(t,\omega)]^2\,dt\to 0 \quad \text{for} \quad n,m\to\infty.$$

So $E(\varphi^*(\omega)-\varphi(\omega))^2=0$.

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