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p-primary components of homotopy groups

II. mod p Hopf invariant

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A mod p Hopf invariant (homomorphism) $H_p: \pi_{2pt}(S^{2t+1}) \longrightarrow Z_p$ is related to the double suspension E^2 by the exactness of the p-components of the sequence

$$\pi_{2pt-2}(S^{2t-1}) \xrightarrow{E^2} \pi_{2pt}(S^{2t+1}) \xrightarrow{H_p} Z_p.$$

The homomorphism H_p is onto if and only if there exists a cell complex $K = S^m \cup e^{m+2t(p-1)}$ in which $\mathscr{P}^t: H^m(K, Z_p) \longrightarrow H^{m+2t(p-1)}(K, Z_p)$ is an isomorphism.

One of the purposes of this section II is to prove

Theorem 2.11. *H_p is trivial except for $t=p^r$. If H_p is onto for $t=p^r$, then it is trivial for $t=p^{r+1}$.*

It is known that $H_p: \pi_{2p}(S^3) \longrightarrow Z_p$ is onto, then it follows that $H_p: \pi_{2p^2}(S^{2p+1}) \longrightarrow Z_p$ is trivial and $E^2: \pi_{2p^2-2}(S^{2p-1}) \longrightarrow \pi_{2p^2}(S^{2p+1})$ is an isomorphism of the p-components.

The above theorem is a consequence of an important theorem (Theorem 2.9) which will be applied in the next section to compute the homotopy groups, in particular, to determine the p-components (Z_p and Z_{p^2}) of the stable homotopy groups $\pi_{2p(p-1)-2}$ and $\pi_{2p(p-1)-1}$.

In the case $p=2$, $H_2: \pi_{4t}(S^{2t+1}) \longrightarrow Z_2$ is also defined and it is onto if and only if the usual Hopf homomorphism $H: \pi_{4t-1}(S^{2t}) \longrightarrow Z$ is onto. Then our theorem 2.11 is a modification of Adames' theorem (Theorem 2.15).

The notations and results in the previous section [9] are used and referred to such as (1.3), Lemma 1.3, etc.

§ mod p Hopf invariant.

The mod p Hopf invariant (homomorphism)

$$H_p: \pi_{m+n-1}(S^m) \longrightarrow Z_p, \quad n=2t(p-1), \tag{2.1}$$

may be defined in terms of the functional reduced power operations (*cf.* [7]). Here we introduce the homomorphism H_p in the following manner.

Denote by E^{r+1} the unit $(r+1)$-cube and by S^r the unit r-sphere of its boundary. Choose generators (orientations) $\iota \in H^m(S^m)$ and $\iota' \in H^{m+n}(E^{m+n}, S^{m+n-1})$. For any element α of $\pi_{m+n-1}(S^m)$, there is a cell complex

$$K_\alpha = S^m \cup e^{m+n} \tag{2.2}$$

such that the restriction $f|S^{m+n-1}$ of a characteristic map $f: (E^{m+n}, S^{m+n-1}) \longrightarrow (K_\alpha, S^m)$ of e^{m+n} represents α by the given orientations $\partial\iota'$ and ι. It is easy to see that such complexes K_α of (2. 2) have the same homotopy type.

Let $\iota_m \in H^m(K_\alpha, Z_p)$ and $\iota_{m+n} \in H^{m+n}(K_\alpha, Z_p)$ be the generators given by ι and $f^*\iota'$ respectively. Then the homomorphism H_p is defined by the following formula.

$$\mathscr{P}^t \iota_m = H_p(\alpha)\iota_{m+n}, \quad n=2t(p-1). \tag{2.3}$$

The proof of the formulas

$$H_p(\alpha+\beta) = H_p(\alpha) + H_p(\beta),$$
$$H_p \circ E = \pm H_p$$

is omitted.

Lemma 2. 1. i). *$H_p: \pi_{m+n-1}(S^m) \longrightarrow Z_p$, $n=2t(p-1)$ is onto if and only if there exists a cell complex $K_\alpha = S^m \cup e^{m+n}$ such that $\mathscr{P}^t: H^m(K_\alpha, Z_p) \longrightarrow H^{m+n}(K_\alpha, Z_p)$ is an isomorphism.*

ii). *$H_p: \pi_{m+n-1}(S^m) \longrightarrow Z_p$, $n=2t(p-1)$ is trivial for $m \leqq 2t$. For $m > 2t$, H_p is onto if and only if it is onto for $m=2t+1$: $(\pi_{2pt}(S^{2t+1}) \longrightarrow Z_p)$.*

iii). *If $H_p: \pi_{m+n-1}(S^m) \longrightarrow Z_p$ is onto for $n=2t(p-1)$, then $t=p^j$ for some integer $j \geqq 0$.*

Proof. i) is easy. For $m<2t$, $\mathscr{P}^t: H^m(K, Z_p) \longrightarrow H^{m+n}(K, Z_p)$ is trivial in general. For $m=2t$, $\mathscr{P}^t\alpha = \alpha^p = 0$, since $\alpha^2 \in H^{2m}(K, Z_p) = 0$. By i), it follows that H_p is trivial for $m \leqq 2t$. It is known that the double suspension $E^2: \pi_{m+n-1}(S^m) \longrightarrow \pi_{m+n+1}(S^{m+2})$ is a mod p isomorphism for odd m and for $m > 2t$ [4]. Then the second assertion of ii) follows from i) and from $H_p \circ E = \pm H_p$. Suppose that $p^j < t < p^{j+1}$. By Lemma 1.3, $\mathscr{S}^t \subset M_{j+1}$. Thus $\mathscr{S}^t \subset M^*_{j+1}$, in particular, $\mathscr{P}^t \in M^*_{j+1}$. Since $H^{m+i}(K, Z_p) = 0$ for $0 < i < n = 2t(p-1)$, it follows that $\Delta H^m(K, Z_p) = \mathscr{P}^{p^k} H^m(K, Z_p) = 0$ for $k = 0, 1, 2, \cdots, j$. Therefore M^*_{j+1} operates trivially on $H^m(K, Z_p)$ and thus $\mathscr{P}^t$ operates trivially on it. By i), it follows iii). q. e. d.

Next an alternative definition of the mod p Hopf invariant will be given.

Denote by $\Omega(X)$ the space of loops in X. Then there is an isomorphism $\Omega: \pi_i(X) \approx \pi_{i-1}(\Omega(X))$. Denote that $\Omega^2(X) = \Omega(\Omega(X))$ and $\Omega^2 = \Omega \circ \Omega: \pi_i(X) \approx \pi_{i-2}(\Omega^2(X))$. S^{2t-1} is imbedded canonically in $\Omega^2(S^{2t+1})$ and we have the following commutative diagram

(2. 4).
$$\begin{array}{ccccccc} & & & \pi_{i+1}(S^{2t+1}) & & & \\ & & {}^{E^2}\nearrow & & \searrow^{J} & & \\ \cdots \xrightarrow{\partial} \pi_{i-1}(S^{2t-1}) & & & \downarrow \Omega^2 & & & \pi_{i-1}(\Omega^2(S^{2t+1}), S^{2t-1}) \\ & & {}_{i_*}\searrow & & \nearrow_{j_*} & & \\ & & & \pi_{i-1}(\Omega^2(S^{2t+1})) & & & \end{array}$$
$$\xrightarrow{\partial} \pi_{i-2}(S^{2t-1}) \cdots,$$

where $J = j_* \circ \Omega^2$ and $E^2 = EE$ is the double suspension. It is known [4] that

$$H_i(\Omega^2(S^{2t+1}), Z_p) = \begin{cases} Z_p & \text{for } i = 2pt-2,\ 2pt-1, \\ 0 & \text{otherwise for } 2t-1 < i < 2(p+1)t-3. \end{cases}$$

Let

$$\tilde{H}_p: \pi_{2pt}(S^{2t+1}) \longrightarrow H_{2pt-2}(\Omega^2(S^{2t+2}), Z_p) = Z_p$$

be the composition of Ω^2 and the Hurewicz homomorphism $\tau: \pi_{2pt-2}(\Omega^2(S^{2t+1})) \longrightarrow H_{2pt-2}(\Omega^2(S^{2t+1}), Z_p)$.

Proposition 2. 2. *$\tilde{H}_p: \pi_{2pt}(S^{2t+1}) \longrightarrow Z_p$ is onto if and only if $H_p: \pi_{2pt}(S^{2t+1}) \longrightarrow Z_p$ is onto.*

Proof. We used the notation of [8]. Consider the case $t \geqq 2$. Let S^{2t}_{p-1} be $(p-1)$-reduced product of S^{2t} imbedded in $\Omega(S^{2t+1})$, then the injection homomorphism: $\pi_{2pt-1}(S^{2t}_{p-1}) \longrightarrow \pi_{2pt-1}(\Omega(S^{2t+1}))$ is onto. From the definition of $\tilde{H}_p$, it follows that $\tilde{H}_p$ is onto if and only if there is a mapping $g: S^{2pt-1} \longrightarrow S^{2t}_{p-1}$ such that $(\Omega g)_*$:

$H_{2pt-2}(S^{2p-2}) \longrightarrow H_{2pt-2}(\Omega(S^{2t}_{p-1}), Z_p)$ is onto, or equivalently, such that $(\Omega g)^* : H^{2pt-2}(\Omega(S^{2t}_{p-1}), Z_p) = H^{2pt-2}(\Omega^2(S^{2t+1}), Z_p) \longrightarrow H^{2pt-2}(S^{2pt-2}, Z_p)$ is an isomorphism. Let $S^{2t}_{p-1} \cup e^{2pt}$ be a complex such that the attaching map of e^{2pt} is g. By Lemma (4. 5) of [8], $(\Omega g)^*$ is an isomorphism if and only if $\mathscr{P}^t(e_1) = e_1^p \neq 0$ for a generator e_1 of $H^{2t}(S^{2t}_{p-1} \cup e^{2pt}, Z_p)$. From the canonical mapping $S^{2t}_{p-1} \times I \longrightarrow S^{2t+1}$, we can construct a mapping of a suspension $S(S^{2t}_{p-1} \cup e^{2pt})$ of $S^{2t}_{p-1} \cup e^{2pt}$ onto $K_\alpha = S^{2t+1} \cup e^{2pt+1}$ such that it carries the cells of the dimensions $2t+1$ and $2pt+1$ with degree ± 1, where g represents $\pm \Omega(\alpha) \in \pi_{2pt-1}(\Omega(S^{2t+1}))$. Conversely, since the injection homomorphism $\pi_{2pt-1}(S^{2t}_{p-1}) \longrightarrow \pi_{2pt-1}(\Omega(S^{2t+1}))$ is onto, for arbitrary $\alpha \in \pi_{2pt}(S^{2t+1})$ there is a mapping g having the above properties. Since $\mathscr{P}^t$ is compatible with the suspension, it follows that $\mathscr{P}^t(e_1) \neq 0$ if and only if $\mathscr{P}^t : H^{2t+1}(K_\alpha, Z_p) \longrightarrow H^{2pt+1}(K_\alpha, Z_p)$ is an isomorphism. Consequently we have from i) of Lemma 2. 1 that the proposition is true for $t \geqq 2$.

Consider the case $t=1$. We prove that H_p and $\tilde{H}_p$ are onto. Let M_k be the k-dimensional complex projective space. Extend the injection $S^2 \subset M_{p-1}$ over a cellular mapping $f: S^2_{p-1} \longrightarrow M_{p-1}$. By Theorem (4. 1) of [8], for the class $\alpha \in \pi_{2p-1}(M_{p-1})$ of the attaching map of $e^{2p} = M_p - M_{p-1}$, there is an element β of $\pi_{2p-1}(S^2_{p-1})$ such that $f_*(\beta) = r\alpha$ for some $r \not\equiv 0 \mod p$. Let g be a representative of β and let a complex $S^2_{p-1} \cup e_0^{2p}$ be given by attaching e_0^{2p} by g. Then f is extendable over $f: S^2_{p-1} \cup e_0^{2p} \longrightarrow M_p$ such that e_0^{2p} is mapped to e^{2p} with the degree r. Since $\mathscr{P}^1(e_1) = e_1^p \neq 0$ in M_p, it follows that $\mathscr{P}^1(e_1) = e_1^p \neq 0$ in $S^2_{p-1} \cup e_0^{2p}$. Similarly to the above, we have a complex K from $S(S^2_{p-1} \cup e_0^{2p})$ such that $\mathscr{P}^1$ is not trivial in K. Therefore H_p is onto. By (4. 3) of [8], there is a mapping $g_0 : S^{2p-2} \longrightarrow \Omega(S^2_{p-1})$ such that $g_0^* : H^{2p-2}(\Omega(S^2_{p-1}), Z_p) \longrightarrow H^{2p-2}(S^{2p-2}, Z_p)$ is an isomorphism. Let $g: S^{2p-1} \longrightarrow S^2_{p-1}$ be a mapping (a suspension of g_0) such that $g_0 = \Omega g$. That $(\Omega g)^*$ is an isomorphism implies that $\tilde{H}_p$ is onto, q. e. d.

Corollary 2. 3. *$H_p : \pi_{2p}(S^3) \longrightarrow Z_p$ and $\tilde{H}_p : \pi_{2p}(S^3) \longrightarrow Z_p$ are isomorphisms of the p-components.*

Proposition 2. 4. *$H_p : \pi_{2pt}(S^{2t+1}) \longrightarrow Z_p$ is onto if and only if $J: \pi_{2pt}(S^{2t+1}) \longrightarrow \pi_{2pt-2}(\Omega^2(S^{2t+1}), S^{2t-1})$ is onto of the p-components.*

Proof. For the case $t=1$, this follows from Corollary 2. 3 and from $\pi_{2p-2}(S^1) = 0$. For the case $t \geqq 2$, this follows from the commutative diagram

$$\begin{array}{ccc} \pi_{2pt-2}(\Omega^2(S^{2t+1})) & \xrightarrow{j_*} & \pi_{2pt-2}(\Omega^2(S^{2t+1}), S^{2t-1}) \\ \downarrow \tau & & \downarrow \tau \\ H_{2pt-2}(\Omega^2(S^{2t+1})) & \xrightarrow{j_*} & H_{2pt-2}(\Omega^2(S^{2t+1}), S^{2t-1}) \end{array}$$

in which τ of the right side is an isomorphism of the p-components by the relative Hurewicz theorem of [6]. q. e. d.

Proposition 2. 5. *Let* ι_{2t} *be a generator of* $\pi_{2t}(S^{2t})$. Then $H_3: \pi_{6t}(S^{2t+1}) \longrightarrow Z_3$ *is onto if and only if* $[\iota_{2t}, [\iota_{2t}, \iota_{2t}]]=0$.

Proof. Assume that $[\iota_{2t}, \iota_{2t}, \iota_{2t}]]=0$. Then we construct as in [5] a complex $M_1=S^{2t+1} \cup e^{6t+1}$ in which $\mathscr{P}^t$ is not trivial. By Lemma 2. 1, H_3 is onto.

Next assume that $[\iota_{2t}, [\iota_{2t}, \iota_{2t}]] \neq 0$. By (3. 1) of [5], $3[\iota_{2t}, [\iota_{2t}, \iota_{2t}]]=0$. By [10], $E[\iota_{2t}, [\iota_{2t}, \iota_{2t}]]=0$. By (5. 1), b) of [3], $[\iota_{2t}, [\iota_{2t}, \iota_{2t}]]=E\gamma$ for some $\gamma \in \pi_{6t-3}(S^{2t-1})$. These equations show that the 3-component of the kernel of $E^2: \pi_{6t-3}(S^{2t-1}) \longrightarrow \pi_{6t-1}(S^{2t+1})$ is not zero. From the exactness of the sequence (2. 4), it follows that $J: \pi_{6t}(S^{2t-1}) \longrightarrow \pi_{6t-2}(\Omega^2(S^{2t+1}), S^{2t-1})$ is not onto of the 3-components. Therefore, by Proposition 2. 4, H_3 is trivial if $[\iota_{2t}, [\iota_{2t}, \iota_{2t}]] \neq 0$. q. e. d.

As is seen in this proof, it follows from the exactness of the sequence (2. 4) and from Proposition 2. 4 that there exists the following exact sequence of p-components:

$$(0) \longrightarrow \pi_{2pt-2}(S^{2t-1}) \xrightarrow{E^2} \pi_{2pt}(S^{2t+1}) \xrightarrow{H_p} Z_p \longrightarrow \pi_{2pt-3}(S^{2t-1})$$
$$\xrightarrow{E^2} \pi_{2pt-1}(S^{2t+1}) \longrightarrow 0 .$$

§ Iterated reduced join.

Denote by I^q and $\dot{I}^q$ the unit q-cube and its boundary. I^1 is the unit interval $I=[0, 1]$ and each point of I^q will be represented by a sequence $(t_1, t_2, \cdots, t_q)$ of real numbers $t_1, t_2, \cdots, t_q \in I$.

Let

$$\psi_q: (I^q, \dot{I}^q) \longrightarrow (S^q, y_0)$$

be an identification which shrinks $\dot{I}^q$ to a (base) point y_0 of S^q. S^q is a q-sphere.

A *reduced join* $A*B$ of two spaces A and B, with respect to their base points $a_0 \in A$ and $b_0 \in B$, is the image of the product space $A\times B$ under an identification which shrinks the subset $A\times b_0 \cup a_0\times B$ to a point $x_0 \in A*B$. The image of a point (a, b) of $A\times B$ will be denoted by a symbol $a*b$. We take the point $x_0 = a_0*b_0$ as a base point of $A*B$. In the case $B=S^q$, the reduced join $A*S^q$ will be denoted by S^qA and it is a *q-fold suspension* of A. In fact S^qA and $S^1(S^{q-1}A)$ are homeomorphic by the correspondence: $a*\psi_q(t_1, \cdots, t_{q-1}, t_q) \leftrightarrow (a*\psi_{q-1}(t_1, \cdots, t_{q-1}))*\psi_1(t_q)$ and S^1A is a suspension SA (with some sigularities) of A.

The *q-fold iterated reduced join* $A*A*\cdots*A$ will be denoted by the symbol $A^{(q)}$, each point of which may be represented by $a_1*a_2*\cdots*a_q$ for some $a_1, a_2, \cdots, a_q \in A$. Let σ be a permutation of q letters $\{1, 2, \cdots, q\}$. Define a homeomorphism $h_\sigma: A^{(q)} \longrightarrow A^{(q)}$ by the formula $h_\sigma(a_1*a_2*\cdots*a_q) = a_{\sigma(1)}*a_{\sigma(2)}*\cdots*a_{\sigma(q)}$. Then it holds the equality $h_\sigma \circ h_\tau = h_{\tau\sigma}$.

Consider the case $A=SB=B*S^1$. Let D be a (closed) subset of $A^{(q)}=(SB)^{(q)}=(B*S^1)^{(q)}$ which consists of the points

$$z = (b_1*\psi_1(t_1))*\cdots*(b_{q-1}*\psi_1(t_{q-1}))*(b_q*\psi_1(t_q))$$

such that $0<t_1\leqq\cdots\leqq t_{q-1}\leqq t_q$ and $b_1, \cdots, b_{q-1}, b_q \in B$. Consider the formula

$$(2.5). \quad k(z) = (b_1*\psi_1(t_1/t_2))*\cdots*(b_{q-1}*\psi_1(t_{q-1}/t_q))*(b_q*\psi_1(t_q)).$$

Lemma 2.6. *There exists uniquely a continuous mapping* k: $(SB)^{(q)}\longrightarrow(SB)^{(q)}$ *such that the formula* (2.5) *holds on* D *and the equality*

$$k\cdot h_\sigma = k$$

holds for all permutations σ. *Let* x_0 *be the base point of* $(SB)^{(q)}$, *then the inverse image* $k^{-1}((SB)^{(q)}-x_0)$ *is the union of* $q!$ *disjoint open subsets* h_σ (Int. D) *each of which is mapped by* k *homeomorphically onto* $(SB)^{(q)}-x_0$. *Let* k_0 *be a mapping of* $(SB)^{(q)}$ *on itself given by setting* $k_0|D=k|D$ *and* $k_0((SB)^{(q)}-D)=x_0$, *then* k_0 *is homotopic to the identity.*

Proof. Consider a mapping $g: I^q\longrightarrow S^q$ given by the formula

$$g(t_1, t_2, \cdots, t_q) = \psi_q(t_1t_2\cdots t_q, t_2\cdots t_q, \cdots, t_q)$$

and denote that $g(I^q)=\Delta$ and $g(\dot{I}^q)=\dot{\Delta}$. g maps $I^q-\dot{I}^q$ homeomorphically onto $\Delta-\dot{\Delta}$. Then the formula $k'(x)=\psi_q(g^{-1}(x))$, $x\in\Delta$, defines a mapping

$$k': \quad (\Delta, \dot{\Delta}) \longrightarrow (S^q, y_0)$$

which maps $\Delta - \dot{\Delta}$ homeomorphically onto $S^q - y_0$. The continuity of k' follows from the compactness of I^q, h_σ operates on S^q by the formula

$$h_\sigma(\psi_q(t_1, \cdots, t_q)) = \psi_q(t_{\sigma(1)}, \cdots, t_{\sigma(q)}) .$$

Since Δ is the set of all the points $\psi_q(t_1, t_2, \cdots, t_q)$ such that $t_1 \leqq t_2 \leqq \cdots \leqq t_q$, it follows that

$$\bigcup_\sigma h_\sigma(\Delta) = S^q .$$

$\dot{\Delta}$ is the boundary of Δ. $\Delta - \dot{\Delta}$ is the set of all $\psi_q(t_1, t_2, \cdots, t_q)$ such that $0 < t_1 < t_2 < \cdots < t_q < 1$. Then

$$h_\sigma(\Delta) \cap \Delta \subset \dot{\Delta}$$

if σ is not the identity. Consider a mapping

$$\tilde{k}_\sigma: \quad (B^{(q)} * h_\sigma^{-1}\Delta, \ B^{(q)} * h_\sigma^{-1}\dot{\Delta}) \longrightarrow (B^{(q)} * S^q, \ x_0 * y_0)$$

given by the formula $\tilde{k}_\sigma(x*y) = h_\sigma(x) * k'(h_\sigma(y))$, $x \in B^{(q)}$, $y \in h_\sigma^{-1}\Delta$. $B^{(q)} \times h_\sigma^{-1}\Delta$ is closed in $B^{(q)} \times S^q$ and $B^{(q)} * S^q$ has the topology of the identification, then $B^{(q)} * h_\sigma^{-1}\Delta$ is closed in $B^{(q)} * S^q$. Two mappings $\tilde{k}_\sigma$ and $\tilde{k}_\tau$ coincide on the intersection $B^{(q)} * h_\sigma^{-1}\Delta \cap B^{(q)} * h_\tau^{-1}\Delta$. For, $y \in h_\sigma^{-1}\Delta \cap h_\tau^{-1}\Delta$ implies $h_\sigma(y) \in \Delta \cap h_{\tau^{-1}\sigma}\Delta \subset \dot{\Delta}$ and also $h_\tau(y) \in \dot{\Delta}$, then $\tilde{k}_\sigma(x*y) = \tilde{k}_\tau(x*y) = x_0 * y_0$. Therefore a continuous mapping

$$\tilde{k}: \quad B^{(q)} * S^q \longrightarrow B^{(q)} * S^q$$

is defined by setting $\tilde{k} \mid (B^{(q)} * h_\sigma^{-1}\Delta) = \tilde{k}_\sigma$.

Since h_σ and $k' \mid (\Delta - \dot{\Delta})$ are homeomorphisms, it follows that $\tilde{k}$ maps each subsets $B^{(q)} * h_\sigma\Delta - B^{(q)} * h_\sigma\dot{\Delta} = \mathrm{Int}\,(B^{(q)} * h_\sigma\Delta)$ homeomorphically onto $B^{(q)} * S^q - x_0 * y_0$. Consider a homeomorphism

$$\varphi: \quad B^{(q)} * S^q \longrightarrow (SB)^q .$$

given by the formula $\varphi((b_1 * \cdots * b_q) * \psi_q(t_1, \cdots, t_q)) = (b_1 * \psi_1(t_1)) * \cdots * (b_q * \psi_1(t_q))$. Then $D = \varphi(B^{(q)} * \Delta)$ and the composition $k = \varphi \circ \tilde{k} \circ \varphi^{-1}$ satisfies the conditions of the lemma. The uniqueness of k is obvious.

Next define a mapping $k_0': (S^q, y_0) \longrightarrow (S^q, y_0)$ by setting

$k'_0|\Delta=k'$ and $k'_0(S^q-\Delta)=y_0$. It is easy to see that k' is a mapping of degree 1. Then there is a homotopy $k'_t: (S^q, y_0)\longrightarrow(S^q, y_0)$, $0\leqq t\leqq 1$, such that k'_1 is the identity. Consider the formula $\tilde{k}_t(x*y)=x*k'_t(y)$, $x\in B^{(q)}$, $y\in S^q$. $k_t=\varphi\circ\tilde{k}_t\circ\varphi^{-1}: (SB)^{(q)}\longrightarrow(SB)^{(q)}$ is a homotopy satisfying the condition that $k_0|D=k|D$, $k_0((SB)^{(q)}-D)=x_0$ and k_1 is the identity. This completes the proof, q. e. d.

Let $K=S^m\cup e^{m+n}$ be a cell complex which consists of cells e_0, e_1 and e_2 of the dimensions 0, m and $m+n$ respectively, where $m>0$, $n>0$, $S^m=e_0\cup e_1$ and $e_2=e^{m+n}$. The q-fold product $(K)^q=K\times K\times\cdots\times K$ of K is a cell complex of the cells $e_{i_1}\times e_{i_2}\times\cdots\times e_{i_q}$ for $i_1, i_2, \cdots, i_q=0, 1, 2$. The iterated reduced join $K^{(q)}$ is the image of $(K)^q$ under the identification $i: (K)^q\longrightarrow K^{(q)}$ given by $i(x_1, x_2, \cdots, x_q)=x_1*x_2*\cdots*x_q$. Then $K^{(q)}$ is a cell complex of the cells $x_0=e_0*e_0*\cdots*e_0$ and

$$e_{i_1}*e_{i_2}*\cdots*e_{i_q}=i(e_{i_1}\times e_{i_2}\times\cdots\times e_{i_q}),$$

for $i_1, i_2, \cdots, i_q=1$ or 2. The homeomorphism h_σ maps $e_{i_1}*e_{i_2}*\cdots*e_{i_q}$ onto $e_{i_{\sigma(1)}}*e_{i_{\sigma(2)}}*\cdots*e_{i_{\sigma(q)}}$. Denote by e_1^{qm+rn}, $0\leqq r\leqq q$, the cell $e_1*\cdots*e_1*e_2*\cdots*e_2=e_1^{(q-r)}*e_2^{(r)}$. Then the cells of the dimension $qm+rn$ in $K^{(q)}$ are $h_\sigma(e_1^{qm+rn})$ and the number of the different $(qm+rn)$-cells is $\binom{q}{r}=q!/r!(q-r)!$. Denote by e_1^{qm+rn}, e_2^{qm+rn}, $\cdots$, $e_{\binom{q}{r}}^{qm+rn}$ the different cells of the dimension $qm+rn$. Then we have a cell-decomposition:

$$K^{(q)}=x_0+\sum_{r=0}^{q}\sum_{i=1}^{\binom{q}{r}}e_i^{qm+rn}.$$

Taken orientations on e_1 and e_2, the orientations in $(K)^q$ are given by the cross products. The cells e_i^{qm+rn} of $K^{(q)}$ are oriented such that the identification i preserve the orientations.

Now we suppose that m and n are even. Then the homeomorphism h_σ preserves the orientations.

Also we suppose that K is a suspension $SB=B*S^1$ of a cell complex $B=e'_0\cup e'_1\cup e'_2$ such that $e_0=e'_0*y_0$, $e_1=e'_1*(S^1-y_0)$ and $e_2=e'_2*(S^1-y_0)$. It is remarkable that the mapping $k: K^{(q)}\longrightarrow K^{(q)}$ and the homotopy $k_t: K^{(q)}\longrightarrow K^{(q)}$ in the proof of the Lemma 2.6 are cellular.

Let x be a point of e_i^{qm+rn} and consider the local degree of $k|e_j^{qm+rn}$, $j=1, 2, \cdots, \binom{q}{r}$ about the point x. By (2.5), $k^{-1}(x)\cap D$

is a point, say $x_1 \in e_i^{qm+rn}$. From the homotopy k_t, it follows that the local degree of $k|e_i^{qm+rn} \cap D$ about x is 1. There are $r!(q-r)! = q!\Big/\binom{q}{r}$ points of $k^{-1}(x)$ in e_i^{qm+rn} and each of which is mapped by some orientation preserving homeomorphism h_σ to x_1. Therefore the local degree of $k|e_i^{qm+rn}$ about x is $r!(q-r)!$ (m, n: *even*). Also considering suitable h_σ, it follows that the local degree of $k|e_j^{qm+rn}$ about x is $r!(q-r)!$ for every $j=1, 2, \cdots, \binom{q}{r}$. Then we have a formula

$$(2.6). \quad k^* e_i^{qm+rn} = r!(q-r)! \sum_{j=1}^{\binom{q}{r}} e_j^{qm+rn}, \quad 0 \leqq r \leqq q, \quad 1 \leqq i \leqq \binom{q}{r},$$

where k^* is the endomorphism of $H^{qm+rn}(K^{(q)})$ induced by k and where we use the following convention. A cohomology class of $H^s(K, G)$ represented by a cell (cocycle) $e^s \subset K$ will denoted by the same symbol $e^s \in H^s(K, G)$ without any confusions.

Shrinking the subset S^m of K to a single point y_0, we obtain a mapping $P' : (K, S^m) \longrightarrow (S^{m+n}, y_0)$ which maps e^{m+n} homeomorphically onto $S^{m+n} - y_0$. Define a mapping

$$P: \quad K^{(q)} \longrightarrow S^{m+n} K^{(q-1)} = K^{(q-1)} * S^{m+n}$$

by the formula $P(x_1 * \cdots * x_{q-1} * x_q) = (x_1 * \cdots * x_{q-1}) * P'(x_q)$.

Denote that

$$\tilde{e}_i^{qm+rn} = e_i^{(q-1)m+(r-1)n} * (S^{m+n} - y_0), \quad 1 \leqq r \leqq q, \quad 1 \leqq i \leqq \binom{q-1}{r-1},$$

then we have a cell decomposition

$$S^{m+n} K^{(q-1)} = x_0 * y_0 + \sum_{r=1}^{q} \sum_{i=1}^{\binom{p-1}{r-1}} \tilde{e}_i^{qm+rn}.$$

If $e_j^{qm+rn} = e_i^{(q-1)m+(r-1)n} * e_2$, then P maps e_j^{qm+rn} homeomorphically onto $\tilde{e}_i^{qm+rn}$. Then we orient $\tilde{e}_i^{qm+rn}$ such that $P|e_j^{qm+rn}$ preserves the orientations. If $e_j^{qm+rn} = e^{(q-1)m+rn} * e_1$, then P maps e_j^{qm+rn} into the $((q-1)m+n)$-skeleton of $S^{m+n} K^{(q-1)}$. It follows easily that the induced homomorphism

$$P^* : \quad H^{qm+rn}(S^{m+n} K^{(q-1)}) \longrightarrow H^{qm+rn}(K^{(q)})$$

is an isomorphism into such that $P^* \tilde{e}_i^{qm+rn} = e_j^{qm+rn}$ for $e_j^{qm+rn} = e_i^{(q-1)m+(r-1)n} * e_2$. Let

$$\kappa = P \circ k: \quad K^{(q)} \longrightarrow S^{m+n}K^{(q-1)}$$

be the composition of k and P. Then from (2.6),

$$(2.7). \quad \kappa^* \hat{e}_j^{qm+rn} = r!(q-r)! \sum_{i=1}^{\binom{q}{r}} e_i^{qm+rn}, \quad 1 \leqq r \leqq q, \quad 1 \leqq j \leqq \binom{q-1}{r-1}.$$

Suppose that ($e_1 \in H_p(K, Z_p)$ and $e_2 \in H^{m+n}(K, Z_p)$)

$$\mathscr{P}^t e_1 = e_2 \pmod{p}, \quad n = 2t(p-1),$$

in the complex $K = S^m \cup e^{m+n}$. Then in the product complex $(K)^q = K \times K \times \cdots \times K$, it follows from the Cartan's formula $\mathscr{P}^k(x \times y) = \sum_{i+j=k} (\mathscr{P}^i x \times \mathscr{P}^j y)$ that

$$\mathscr{P}^{rt}(e_1 \times \cdots \times e_1) = \sum e_{i_1} \times \cdots \times e_{i_q} \pmod{p}$$

where the summation runs over the indices $(i_1, \cdots, i_q)$ such that $i_1, \cdots, i_q = 1, 2$ and $i_1 + \cdots + i_q = q + r$. Since the identification homomorphism $i^*: H^*(K^{(q)}, Z_p) \longrightarrow H^*((K)^q, Z_p)$ is an isomorphism into, it follows

$$(2.8). \quad \mathscr{P}^{rt} e_1^{qm} = \sum_{i=1}^{\binom{q}{r}} e_i^{qm+rn}, \quad \pmod{p} \textit{ for } 0 \leqq r \leqq q.$$

Similarly

$$(2.9). \quad \mathscr{P}^{rt} \hat{e}_1^{qm+n} = \sum_{i=1}^{\binom{q-1}{r}} \hat{e}_i^{qm+(r+1)n}, \quad \pmod{p} \textit{ for } 0 \leqq r \leqq q-1.$$

Identifying $K^{(q)} \cup S^{m+n}K^{(q-1)} \cup K^{(q)} \times I$ by the relation $(x, 0) = x$ and $(x, 1) = \kappa(x)$, $x \in K^{(q)}$, a mapping cylinder L_q of κ is obtained. L_q is a cell complex by the natural cell-decomposition:

$$L_q = K^{(q)} + S^{m+n}K^{(q)} + x_0 \times (I - \dot{I}) + \sum\sum e_i^{qm+rn} \times (I - \dot{I}).$$

By setting $h_\sigma(x, t) = (h_\sigma(x), t)$, $x \in K^{(q)}$, $t \in I$, we have a transformation (homeomorphism)

$$\bar{h}_\sigma: \quad (L_q, K^{(q)}, S^{m+n}K^{(q-1)}) \longrightarrow (L_q, K^{(q)}, S^{m+n}K^{(q-1)})$$

such that $\bar{h}_\sigma | K^{(q)} = h_\sigma$. The restriction $\bar{h}_\sigma | S^{m+n}K^{(q-1)}$ is the identity since $\kappa \circ h_\sigma = P \circ k \circ h_\sigma = P \circ k = \kappa$. Obviously $\bar{h}_\sigma \circ \bar{h}_\tau = \bar{h}_{\tau\sigma}$. Consider the following commutative diagram:

(2.10).

$$\begin{array}{ccccccc}
\cdots \to H^k(L_q, Z_p) & \xrightarrow{i^*} & H^k(K^{(q)}, Z_p) & \xrightarrow{\delta^*} & H^{k+1}(L_q, K^{(q)}, Z_p) & \xrightarrow{j^*} & H^{k+1}(L_q, Z_p) \to \cdots \\
i_0^* \downarrow \uparrow r^* & \nearrow \kappa^* & & & & \searrow j_0^* & i_0^* \downarrow \uparrow r^* \\
H^k(S^{m+n}K^{(q-1)}, Z_p) & & & & & & H^{k+1}(S^{m+n}K^{(q-1)}, Z_p),
\end{array}$$

where i, i_0, j and j_0 are injections and r is a retraction given by $r(x, t)=\kappa(x)$. Since i_0 and r are homotopy equivalences, i_0^* and r^* are isomorphisms.

Lemma 2.7. *Suppose that m is an even positive integer, $p=q$ is an odd prime, $n=2t(p-1)$ and that $\mathscr{P}^t e_1=e_2 \bmod p$ in the complex $K=e_0\cup e_1\cup e_2=S^m\cup e^{m+n}=EB$. Then we have the following properties in the diagram* (2.10).

i). $i^*r^*\bar{e}_1^{pm+n}=\kappa^*\bar{e}_1^{pm+n}=-\mathscr{P}^t e_1^{pm}$ (mod p).

ii). *$j^*: H^{p(m+n)}(L_p, K^{(p)}, Z_p)\longrightarrow H^{p(m+n)}(L_p, Z_p)$ and $\delta^*: H^{p(m+n)}(K^{(p)}, Z_p)\longrightarrow H^{p(m+n)+1}(L_p, K^{(p)}, Z_p)$ are isomorphisms. The Bockstein homomorphism $\Delta: H^{p(m+n)}(L_p, K^{(p)}, Z_p)\longrightarrow H^{p(m+n)+1}(L_p, K^{(p)}, Z_p)$ is an isomorphism and it carries $j^{*-1}(\mathscr{P}^{(p-1)t}r^*\bar{e}_1^{pm+n})$ to $\delta^*(\mathscr{P}^{pt}e_1^{pm})$.*

iii). *Let $1\geqq s\geqq p-1$. If an element α of $\delta^*H^{pm+sn}(K^{(p)}, Z_p)$ satisfies the equality $\bar{h}_\sigma^*\alpha=\alpha$ for all the permutations σ, then $\alpha=0$.*

Proof. i) follows from (2.7) and (2.8).

$H^{p(m+n)}(K^{(p)}, Z_p)$ and $H^{p(m+n)}(L_p, Z_p)$ are generated by $e_1^{p(m+n)}\neq 0$ and $r^*\bar{e}_1^{p(m+n)}\neq 0$ respectively. By (2.8) and (2.9), $e_1^{p(m+n)}=\mathscr{P}^{pt}e_1^{pm}$ and $r^*\bar{e}_1^{p(m+n)}=r^{*(p-1)t}\bar{e}_1^{pm+n}=\mathscr{P}^{(p-1)t}r^*\bar{e}_1^{pm+n}$. By (2.7), $i^*(r^*\bar{e}_1^{p(m+n)})=\kappa^*\bar{e}_1^{p(m+n)}=p!e_1^{p(m+n)}=0$ (mod p). Then $i^*: H^{p(m+n)}(L_p, Z_p)\longrightarrow H^{p(m+n)}(K^{(p)}, Z_p)$ is trivial. From the exactness of the sequence (2.10) and from $H^{p(m+n)-1}(K^{(p)}, Z_p)=H^{p(m+n)+1}(L_p, Z_p)=0$, it follows the first assertion of ii). Denote that $\bar{e}=e_1^{p(m+n)}\times(I-\dot{I})$ and orient the cell $\bar{e}$ such that $\delta e_1^{p(m+n)}=\bar{e}$. In the integral coefficient, by (2.7), $r^\#\bar{e}_1^{p(m+n)}=\bar{e}_1^{p(m+n)}+p!e_1^{p(m+n)}$. Since $r^\#\bar{e}_1^{p(m+n)}$ is a cocycle, it follows $\delta\bar{e}_1^{p(m+n)}=-p!\delta e_1^{p(m+n)}=-p!\bar{e}$. Then $\dfrac{\delta}{p}\bar{e}_1^{p(m+n)}=-(p-1)!\bar{e}=\bar{e} \bmod p$. Thus $\Delta(j^{*-1}\mathscr{P}^{(p-1)t}r^*\bar{e}_1^{pm+n})=\Delta(j_0^{*-1}\bar{e}_1^{p(m+n)})=\delta^*e_1^{p(m+n)}=\delta^*\mathscr{P}^{pt}e_1^{pm}$ (mod p), and then the second assertion of ii) follows.

Let $\beta=\sum_i b_ie_1^{pm+sn}$ be an element of $H^{pm+sn}(K^{(p)}, Z_p)$ such that $\delta^*\beta=\alpha$. Since the homomorphisms induced by $\bar{h}_\sigma$ commute with the sequence of (2.10), it follows that $\delta^*(h_\sigma^*\beta-\beta)=\bar{h}_\sigma^*\alpha-\alpha=0$. Thus $h_\sigma^*\beta-\beta\in i^*H^{pm+sn}(L_p, Z^p)$. By (2.7) and by $\kappa^*=i^*\circ r^*$, it follows that $i^*H^{pm+sn}(L_p, Z_p)$ is generated by the element $\sum_i e_i^{pm+sn}$. Therefore

$$h_\sigma^*\beta=h_\sigma^*\sum_i b_ie_i^{pm+sn}=\sum_i (b_i+c_\sigma)e_i^{pm+sn}$$

for some integer c_σ which depends on σ. For two indices i and j, there exists a permutation σ such that $h_\sigma(e_i^{pm+sn})=e_j^{pm+sn}$ and

$h_\sigma(e_j^{pm+sn}) = e_i^{pm+sn}$. Thus $b_i \equiv b_j + c_\sigma$ and $b_j \equiv b_i + c_\sigma$ mod p, and $2b_i \equiv 2b_j$ mod p. Since p is an odd prime, $b_i \equiv b_j$ mod p and then $\beta = b_i \sum_i e_i^{pm+sn} \in i^* H^{pm+sn}(L_p, Z_p)$. Therefore $\alpha = \delta^* \beta \in \delta^* i^* H^{pm+sn}(L_p, Z_p) = 0$. q. e. d.

§ Theorems.

We shall construct a space W_r^N having the following properties:

$$\pi_i(W_r^N) \approx \begin{cases} Z & \text{for } i = N, \\ Z_p & \text{for } i = N + 2p^j(p-1) - 1, \quad 0 \leqq j < r, \\ 0 & \text{otherwise}, \end{cases}$$

$$\mathscr{P}^{p^j} H^N(W_r^N, Z_p) = 0 \quad \text{for} \quad 0 \leqq j < r.$$

W_0^N is an Eilenberg-MacLane space of a type (Z, N). If a space W_r^N is given, we may imbed W_r^N into an Eilenberg-MacLane space X of a type $(Z_p, N+2p^r(p-1))$ such that the injection homomorphism maps a fundamental class of $H^{N+2p^r(p-1)}(X, Z_p)$ onto $\mathscr{P}^{p^r} u$, where u is a fundamental class of $H^N(W_r^N, Z_p)$. Let W_{r+1}^N be a space of the paths in X starting at a point and ending in W_r^N. Then W_{r+1}^N satisfies the above properties. Asscciating to each path of W_{r+1}^N its end point, we have a fibering

$$f_r: \quad W_{r+1}^N \longrightarrow W_r^N$$

whose fibre $F_r = \Omega(X)$ is an Eilenberg-MacLane space of a type $(Z_p, N+2p^r(p-1)-1)$. Let $(k < 2N-1 < 2N+2p^r(p-1)-2)$

$$\text{(2.11)} \qquad \cdots \xrightarrow{\delta^*} H^k(W_r^N, Z_p) \xrightarrow{f_r^*} H^k(W_{r+1}^N, Z_p) \xrightarrow{i^*} H^k(F_r, Z_p) \xrightarrow{\delta^*} \cdots$$

be the cohomology exact sequence associated with the above fibering. We choose a fundamental class u_r of $H^{N+2p^r(p-1)-1}(F_r, Z_p)$ such as $\delta^* u_r = \mathscr{P}^{p^r} u$. $(u = (f_{r-1}^* \circ \cdots \circ f_0^*) u_0)$

In the following, we take N sufficiently large such as the exactness of the sequences (2.11) holds in our considerations. Then, from [2], there is an $\mathscr{S}^*$-isomorphism:

$$H^k(F_r, Z_p) = H^k(Z_p, N+2p^r(p-1)-1, Z_p) \approx \mathscr{S}^{k-N-2p^r(p-1)+1}$$

(for sufficiently large N).

The homomorphisms of (2.11) are $\mathscr{S}^*$-homomorphisms, may be different in sign. Then the image of δ^* is $\mathscr{S}^* \mathscr{P}^{p^r} u$. It follows that the image of $H^*(W_0^N, Z_p)$ in $H^*(W_r^N, Z_p)$ under $f_{r-1}^*\circ\cdots\circ f_0^*$ is $\mathcal{S}^*u/M_r^*u=\mathcal{S}^*u/(\mathcal{S}^*\Delta u+\mathcal{S}^*\mathcal{P}^1u+\cdots+\mathcal{S}^*\mathcal{P}^{p^{r-1}}u)$. Then the kernel of δ^* is clarified by Proposition 1.6 and Proposition 1.7, and the following proposition is verified by the exactness of (2.11).

Proposition 2.8. *There exists an element b_{r+1} of $H^{N+2p^{r+1}(p-1)-1}(W_{r+1}^N, Z_p)$ such that $i^*b_{r+1}=c\mathcal{P}^{p^r(p-1)}u_r$. $\sum_k H^k(W_{r+1}^N, Z_p)$, $k<2N-1$, is an $\mathcal{S}^*$-module generated by b_{r+1} and elements of dimensions less than $N+2(2p^r+p^{r-1})(p-1)$ (less then $N+4(p-1)+1$ for $r=0$).*

Now our main result is stated as follows.

Theorem 2.9. *Suppose that the* mod p *Hopf homomorphism $H_p: \pi_{2pt}(S^{2t+1})\longrightarrow Z_p$ is onto for $t=p^r$. Then, for sufficiently large N, the element $\Delta b_{r+1}-\mathcal{P}^{p^{r+1}}u$ belongs to an $\mathcal{S}^*$-submodule $\sum_k \mathcal{S}^*H^k(W_{r+1}^N, Z_p)$, $N<k\leqq N+2(p^r+1)(p-1)$.*

By Corollary 2.3, our theorem is valuable for $r=0$, and the result is stated as follows.

Theorem 2.10. *$H^k(W_1^N, Z_p)$, $k<2N-1$, is an $\mathcal{S}^*$-module generated by elements u, a and b_1 of dimensions N, $N+4(p-1)$ and $N+2p(p-1)-1$, respectively, such that $i^*a=R_1u_0=2\mathcal{P}^1\Delta u_0-\Delta\mathcal{P}^1u_0$ and $i^*b_1=\mathcal{P}^{p-1}u_0$. There are relations $\Delta u=\mathcal{P}^1u=0$, $R_2a=0$ and $\Delta b_1=\mathcal{P}^pu+\mathcal{P}^{p-2}a$.*

This follows from Proposition 1.6, the above Theorem 2.9, the exact sequence (2.11) and from the fact that $i^*\Delta b_1=\Delta\mathcal{P}^{p-1}u_0=\mathcal{P}^{p-2}R_1u_0=i^*\mathcal{P}^{p-2}a$. (See also the proof of Theorem 2.9.)

Suppose that $H_p: \pi_{2pt}(S^{2t+1})\longrightarrow Z_p$ is onto for $t=p^r$ and for $t=p^{r+1}$. By Lemma 2.1, there is a cell complex $K=S^m\cup e^{m+n}$, $n=2p^{r+1}(p-1)$, such that $\mathcal{P}^{pt+1}$ is not trivial. Let $f: S^m\longrightarrow W_{r+1}^m$ be a mapping representing a generator of $\pi_m(W_{r+1}^m)\approx Z$. Since $\pi_{m+n-1}(W_{r+1}^m)=0$, we can extend the mapping f over whole of K. Consider the induced homomorphism:

$$f^*:\quad H^*(W_{r+1}^m, Z_p)\longrightarrow H^*(K, Z_p)\,.$$

By Theorem 2.9, $\mathcal{P}^{p^{r+1}}u$ is a sum of elements of $\mathcal{S}^*H^k(W_{r+1}^m, Z_p)$, $N<k<N+2p^{r+1}(p-1)$. Since $f^*H^k(W_{r+1}^m, Z_p)=H^k(K, Z_p)=0$, $f^*\mathcal{P}^{p^{r+1}}u=\mathcal{P}^{p^{r+1}}f^*u=0$. Since $f^*u\neq 0$, this contradicts to the non-triviality of $\mathcal{P}^{p^{r+1}}$ in K. Therefore the following theorem is established.

Theorem 2.11. *If $H_p: \pi_{2pt}(S^{2t+1})\longrightarrow Z_p$ is onto for $t=p^r$, then H_p is trivial for $t=p^{r+1}$.*

By Corollary 2.3,

Corollary 2.12. *$H_p: \pi_{2p^2}(S^{2p+1}) \longrightarrow Z_p$ is trivial.*

By Proposition 2.5,

Corollary 2.13. *If $[\iota_{2t}, [\iota_{2t}, \iota_{2t}]]=0$, then $[\iota_{2pt}, [\iota_{2pt}, \iota_{2pt}]] \neq 0$. In particular $[\iota_6, [\iota_6, \iota_6]] \neq 0$.*

Proof of Theorem 2.9.

From the definition of H_p, it follows that there is a cell complex $K=S^m \cup e^{m+n}$, $n=2p^r(p-1)$, such that $\mathscr{P}^{p^r}e_1=e_2 \pmod p$. Here we may suppose that, taking suspensions if it is necessary, K is a suspension SB and m is even and sufficiently large. According to the previous §, we construct the iterated reduced join $K^{(p)}$, the mapping $\kappa: K^{(p)} \longrightarrow S^{m+n}K^{(p-1)}$ and its mapping cylinder L_p.

Put $N=pm$, and consider spaces $W_r^N \subset X$ such as in the biginning of this §. Since $\pi_i(W_r^N)=0$ for $i \geqq pm+n-1 > N+2p^{r-1}(p-1)-1$, there exists a mapping

$$g_0: \quad K^{(p)} \longrightarrow W_r^N \subset X$$

such that $g_0^* u = e_1^{pm}$ for $g_0^*: H^N(W_r^N, Z_p) \longrightarrow H^N(K^{(p)}, Z_p)$. Also there exists a mapping

$$g_1: \quad S^{m+n}K^{(p-1)} \longrightarrow X$$

such that $g_1^* u' = -\bar{e}_1^{pm+n}$ where u' is a fundamental clase of $H^{pm+n}(X, Z_p)$ such that $i^*u' = \mathscr{P}^{p^r}u$. Consider the composition $g_1 \circ \kappa$, then we have the equality $(g_1 \circ \kappa)^* u' = g_0^* u'$. Since X is an Eilenberg-MacLane space, the mapping $g_1 \circ \kappa$ and g_0 are homotopic to each other. Let $g_t': K^{(p)} \longrightarrow X$ be a homotopy such that $g_0'=g_0$ and $g_1'=g_1 \circ \kappa$. Then the formula $g(x, t)=g_t'(x)$, $x \in K^{(p)}$, defines a mapping

$$g: \quad (L_p, K^{(p)}) \longrightarrow (X, W_r^N)$$

such that $g | K^{(p)} = g_0$ and $g | S^{m+n}K^{(p-1)} = g_1$. Now it will be proved

$$\text{(2.12).} \qquad g \quad h_\sigma \simeq g: \quad (L_p, K^{(p)}) \longrightarrow (X, W_r^N).$$

We shall construct a homotopy $G: (L_p \times I, K^{(p)} \times I) \longrightarrow (X, W_r^N)$ as follows. Put $G(x, 0)=g(x)$, $G(x, 1)=g(\bar{h}_\sigma(x))$, $x \in L_p$. Since $\bar{h}_\sigma | S^{m+n}K^{(p-1)}$ is the identity, we can put $G(x, t)=g(x)$ for $x \in S^{m+n}K^{(p-1)}$. Since h_σ preserve the orientations, G is extended over $(x_0 \cup e_1^{pm}) \times I$ into W_r^N. Since $\pi_i(W_r^N)=0$ for $i \geqq pm+n/p$, G is extended over $K^{(p)} \times I$ such that $G(K^{(p)} \times I) \subset W_r^N$. By the natural cell decomposition of $L_p \times I$, there are no cells of the dimension $pm+n+1=N+2p^r(p-1)+1$ in $L_p \times I - (L_p \times \dot{I} \cup (K^{(p)} \cup S^{m+n}K^{(p-1)}) \times I)$. Therefore G may be extended the whole of $L_p \times I$ into X. This completes the proof of (2.12).

Consider the following commutative diagram.

$$\begin{array}{ccccccc}
 & & H^k(W_{r+1}^N, Z_p) & \xrightarrow{i^*} & H^k(F_r, Z_p) & & \\
 & \nearrow f_r^* & \downarrow S & & S \downarrow & \searrow \delta^* & \\
H^k(W_r^N, Z_p) & \xrightarrow{\delta^*} & H^{k+1}(X, W_r^N, Z_p) & \xrightarrow{j^*} & H^{k+1}(X, Z_p) & \xrightarrow{i^*} & H^{k+1}(W, Z_p) \\
\downarrow g^* & & \downarrow g^* & & \downarrow g^* & & \downarrow g^* \\
H^k(K^{(p)}, Z_p) & \xrightarrow{\delta^*} & H^{k+1}(L_p, K^{(p)}, Z_p) & \xrightarrow{j^*} & H^{k+1}(L_p, Z_p) & \xrightarrow{i^*} & H^{k+1}(K^{(p)}, Z_p),
\end{array}$$

where S are suspension isomorphisms of contractible fibre spaces, and we choose S and δ^* of (2.11) such that the above diagram is commutative. Then $Su_r=u'$. Put $S\tilde{b}_{r+1}=b \in H^{p(m+n)}(X, W_r^N, Z_p)$, then $j^*\tilde{b}=c\mathscr{P}^{(p-1)p^r}u'$.

Remark that the case $r=1$ does not occur, since the assumption of the theorem fails for $r=1$ by Corollary 2.12.

Consider $j^*(\Delta\tilde{b})=\Delta c\mathscr{P}^{(p-1)p^r}u'=-c(\mathscr{P}^{(p-1)p^r}\Delta)u'$. By (1.3)′, $\mathscr{P}((p-1)p^r, \Delta)=\mathscr{P}(1, \Delta, (p-1)p^r-1)+\mathscr{P}(\Delta, (p-1)p^r)$ for $r \geqq 2$ and $\mathscr{P}(p-1, \Delta)=R(1)\mathscr{P}(p-2)$. Then by (1.7) and (1.8),

$$j^*(\Delta\tilde{b}) = \begin{cases} \alpha_1\Delta u' + \alpha_2\Delta\mathscr{P}^1 u' & \textit{if} \quad r \geqq 2, \\ \alpha_0 R_1 u' & \textit{if} \quad r=0 \end{cases}$$

for some $\alpha_1, \alpha_2, \alpha_0 \in \mathscr{S}^*$. By Propositions 1.5 and 1.7, $i^*\Delta\mathscr{P}^1u'=i^*\Delta u'=i^*R_1u'=0$. Then there are elements $w_1 \in H^{pm+n+1}(X, W_r^N, Z_p)$, $w_2 \in H^{pm+n+2(p-1)+1}(X, W_r^N, Z_p)$, $r \geqq 2$, and $w_0 \in H^{pm+2n+1}(X, W_r^N, Z_p)$ such that $j^*w_1=\Delta u'$, $j^*w_2=\Delta\mathscr{P}^1u'$ and $j'w_0=R_1u'$. It follows from $j^*(\Delta\tilde{b}-\alpha_1w_1-\alpha_2w_2)=0$, $r \geqq 2$ and $j^*(\Delta\tilde{b}-\alpha_0w_0)=0$ that

$$\Delta\tilde{b} = \begin{cases} \alpha_1w_1+\alpha_2w_2+\delta^*x, & r \geqq 2, \\ \alpha_0w_0+\delta^*x, & r=0 \end{cases}$$

for some $x \in H^{p(m+n)}(W_r^N, Z_p)$. By Proposition 2.8, $x=\beta u+\sum\beta_iy_i$ for some $\beta, \beta_i \in \mathscr{S}^*$ and $y_i \in H^{pm+k_i}(W_r^N, Z_p)$, $0<k_i<n=2p^r(p-1)$.

Obviously $\beta_i=0$ if $r=0$. In $\delta^*H^*(W_r^N, Z_p)$, there is a relation $M_{r+1}\delta^*u=0$. It follows from Lemma 1.3 and (1.9), i) that $\delta^*\beta u=\beta\delta^*u=d\mathscr{P}^{p^{r+1}}\delta^*u$ for some integer d. Then we have

$$(2.13). \quad \Delta\tilde{b}-d\mathscr{P}^{p^{r+1}}\delta^*u=\begin{cases}\alpha_1w_1+\alpha_2w_2+\sum\beta_i\delta^*y_i, & r\geqq 2,\\ \alpha_0w_0, & r=0.\end{cases}$$

By operating S^{-1}, it follows that $\Delta b_{r+1}-d\mathscr{P}^{p^{r+1}}\delta^*u\in\sum_k\mathscr{S}^*H^k(W_{r+1}^N, Z_p)$ for $N=pm<k\leqq pm+n+2(p-1)=N+2(p^r+1)(p-1)$. Then it is sufficient to determine the coefficient d such as $d\equiv 1$ mod p.

Consider the image of each term of (2.13) under the homomorphism g^*. Since $0<k_i<n$, $g^*\delta^*y_i=\delta^*g^*y_i\in\delta^*H^{N+k_i}(K^{(p)}, Z_p)=0$. Since $1<2(p-1)+1<n$ for $r\geqq 2$, $g^*w_2\in H^{pm+n+2(p-1)+1}(L_p, K^{(p)}, Z_p)=0$. Since $j^*g^*w_1\in H^{pm+n+1}(L_p, Z_p)=0$ and $j^*g^*w_0\in H^{pm+2n+1}(L_p, Z_p)=0$, the elements g^*w_1 and g^*w_0 are the image of δ^*. By (2.12), $h_\sigma^*(g^*w_1)=g^*w_1$ and $h_\sigma^*(g^*w_0)=g^*w_0$ for all σ. Then it follows from iii) of Lemma 2.7 that $g^*w_1=g^*w_0=0$. Next $j^*g^*\tilde{b}=g^*j^*\tilde{b}=g^*c\mathscr{P}^{(p-1)p^r}u'=c\mathscr{P}^{(p-1)p^r}g^*u'=c\mathscr{P}^{(p-1)p^r}r^*\tilde{e}_1^{pm+n}$, and then $g^*\tilde{b}=j^{*-1}c\mathscr{P}^{(p-1)p^t}r^*\tilde{e}_1^{pm+n}$. Consequently the following relation is obtained from (2.13):

$$\Delta(j^{*-1}c\mathscr{P}^{(p-1)p^r}r^*\tilde{e}_1^{pm+n})=g^*\Delta\tilde{b}=g^*d\mathscr{P}^{p^{r+1}}\delta^*u=d\delta^*\mathscr{P}^{p^{r+1}}e_1^{pm}.$$

Compairing this to the relation $\Delta(j^{*-1}\mathscr{P}^{(p-1)p^r}r^*\tilde{e}_1^{pm+n})=\delta^{*p^{r+1}}e_1^{pm}$ of Lemma 2.7, ii), it follows from the following (2.14) the required equality

$$d\equiv 1 \quad \text{mod. } p,$$

and this proves the theorem.

(2.14). *Suppose that* $n=2p^t(p-1)$ *and that* $H^{N+k}(Y, Z_p)=0$ *for* $k\not\equiv 0$ mod n. *Then* $\mathscr{P}^{rp^t}\alpha=(-1)^rc\mathscr{P}^{rp^t}\alpha$ *for* $\alpha\in H^N(Y, Z_p)$ *and for* $0\leqq r\leqq p-1$.

This is obvious for $r=0$. By (1.3), for $0\leqq i<p$,

$$\mathscr{P}(ip^t)\mathscr{P}(jp^t)\alpha=\sum_{k=0}^{ip^{t-1}}*\mathscr{P}((i+j)p^t-k)\mathscr{P}(k)\alpha$$
$$=(-1)^i\binom{jp^t(p-1)-1}{ip^t}\mathscr{P}((i+j)p^t)\alpha=\binom{i+j}{i}\mathscr{P}((i+j)p^t)\alpha.$$

Suppose that (2.14) is true for $r<s\leqq p-1$. Then by (1.7),

$$0=\sum_{i=0}^{sp^t}\mathscr{P}(sp^t-i)c\mathscr{P}(i)\alpha=\sum_{j=0}^{s}\mathscr{P}((s-j)p^t)c\mathscr{P}(jp^t)\alpha$$
$$=\sum_{j=0}^{s}(-1)^j\mathscr{P}((s-j)p^t)\mathscr{P}(jp^t)\alpha-(-1)^s\mathscr{P}^{sp^t}\alpha+c\mathscr{P}^{sp^t}\alpha.$$

Thus $(-1)^s\ \mathscr{P}^{sp^t}\alpha-c\mathscr{P}^{sp^t}\alpha=\sum_{j=0}^{s}(-1)^j\binom{s}{j}\mathscr{P}^{sp^t}\alpha=0$. By the induction, (2.14) is proved, and then the proof of the theorem is accomplished. q. e. d.

§ **The case $p=2$.**

The mod 2 Hopf homomorphism

$$H_2:\quad \pi_{m+n-1}(S^m)\longrightarrow Z_2,\qquad n=2t,$$

is also defined similarly by using Sq^{2t} in place of $\mathscr{P}^t$.

Meny properties of H_p are established for H_2 replacing $\mathscr{P}^t$ by Sq^{2t}. The exceptions are the followings. ii) of Lemma 2.1 has to be rewritten such as

ii). *H_2: $\pi_{m+n-1}(S^m)\longrightarrow Z_2$ is trivial for $m<n$. For $m\geqq n$ H_2 is onto if and only if it is onto for $m=n$(: $\pi_{2n-1}(S^n)\longrightarrow Z_2$).*

Instead of Proposition 2.5, we have

(2.16). *H_2: $\pi_{2n-1}(S^n)\longrightarrow Z_2$ is onto if and only if $[\iota_{n-1},\iota_{n-1}]=0$.*

W_r^N is defined also for $p=2$. Then

Proposition 2.8′. *There exists an element b_{r+1} of $H^{N+2^{r+2}-1}(W_r^N, Z_2)$ such that $i^*b_{r+1}=Sq^{2^{r+1}}u_r$. $\sum_k H^k(W_{r+1}^N, Z_2)$, $k<2N-1$, is an A^*-module generated by b_{r+1} and elements of dimensions less than $N+2^{r+1}+2^{r-1}$.*

Regarding the proof of Theorem 2.9, for the case $p=2$, it is seen that the only difficulty is to use Proposition 1.9 in place of Proposition 1.7. Then, in the proof, we take the relation $Sq^{2^{r+1}}Sq^1=Sq^2Sq^{2^{r+1}-1}+Sq^1Sq^{2^{r+1}}$ in place of $\mathscr{P}^{(p-1)p^r}\Delta=\cdots$. To be contained Sq^2 and Sq^1 in the kernel of $(Sq^{2^{r+1}})^*: A^*\longrightarrow A^*/M_{r+1}^*$, it is necessary to hold $r\geqq 2$. Then the modification of Theorem 2.9 is stated as follows.

Thorem 2.14. *Suppose that $H_2:\pi_{4t-1}(S^{2t})\longrightarrow Z_2$ is onto for $t=2^r$ and $r\geqq 2$. Then for sufficiently large N, the element $Sq^1\tilde{b}_{r+1}-Sq^{2^{r+2}}u$*

belongs to an A^**-submodule* $\sum_k A^* H^k(W^N_{r+1}, Z_2)$, $N < k \leqq N+2^{r+1}+4$.

It follows from this the following

Theorem 2.15. (*Adames* [1]). *If* $H_2: \pi_{4t-1}(S^{2t}) \longrightarrow Z_2$ *is onto for* $t \geqq 4$, *then* $H_2: \pi_{8t-1}(S^{4t}) \longrightarrow Z_2$ *is trivial.*

Finally, as is seen in [7], $H_2: \pi_{4t-1}(S^{2t}) \longrightarrow Z_2$ is onto if and only if the usual Hopf homomorphism $H: \pi_{4t-1}(S^{2t}) \longrightarrow Z$ is onto.

REFERENCES

[1] J. F. Adames, *Une relation entre groupes d'homotopie et groupes de cohomologie.* Comptes Rendus, 245 (1957), 24–25.

[2] H. Cartan, *Algébres d'Eilenberg-MacLane et homotopie.* Seminaire de E. N. S. Paris (1955).

[3] I. M. James, *On the suspension sequence.* Ann. of Math., 65 (1957), 74–107.

[4] J. C. Moore, *Some applications of homology theory to homotopy problems.* Ann. of Math., 58 (1953) 325–350.

[5] M. Nakaoka and H. Toda, *On Jacobi identity for Whitehead products.* J. of Osaka City Univ., 5 (1954) 1–13.

[6] J. P. Serre, *Groupes d'homotopie et classe groupes abélien.* Ann. of Math., 58 (1953) 258–294.

[7] N. E. Steenrod, *Cohomology invariants of mappings.* Ann. of Math., 50 (1949) 954–988.

[8] H. Toda, *On the double suspension* E^2. J. of Osaka City Univ. 7 (1956) 103–145.

[9] H. Toda, *p-components of Homotopy groups, I Exact sequences in Steenrod algebra.* This memoir, 31 (1958), 131–144.

[10] G. W. Whitehead, *On products in homotopy groups.* Ann. of Math., 47 (1946) 460–475.