MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES A
Vol. XXXI, Mathematics No. 2, 1958.

Relative Riemannian geometry

II. On the metric connections

By

Makoto MATSUMOTO

(Received May 3, 1958)

In a previous paper [1] we introduced the notion of the relative affine connection of the pair of manifolds (M, N), and developed the theory of the torsions and curvatures. We shall treat, in this Part 2, the g-related metrics of the (M, N), and establish the foundation of the relative Riemannian geometry.

1. The metric tensors.

Continuing the first part [1], we consider two differentiable manifolds M and N of dimension n. Let (x^i) and $(y^{i'})$ be the local coordinates of any points $P\in M$ and $Q\in N$ respectively. We assume first that a positive-definite quadratic differential form

$$ds^2 = g_{ij}(x, y)\, dx^i dx^j \tag{1.1}$$

is associated with every point $P(x)$ of the M, where the coefficients $g_{ij}(x, y)$ are the functions of (x) and furthermore *depend upon* (y) *of the observing point* $Q(y)$. This form is called the *metric form* of the M, and the tensor (g_{ij}) of the (x)-order $(0, 2)$ the *metric tensor* of the M. By means of this form we can define the scalar product of two vectors and hence the length of a vector by the usual manners.

According to the definition [1], § 1 of the g-related tensors, the tensor, which is g-related to the metric tensor of the M, is given by

$$g_{i'j'} = g_{ij} g^i_{i'} g^j_{j'}\,. \tag{1.2}$$

It is clear that the quadratic form

$$ds'^2 = g_{i'j'}dy^{i'}dy^{j'}$$

is positive-definite. We define the metric of the N by the above form. Thus we may say that *the metrics of the M and N are g-related.*

We consider the vectors $u^{i'}$ and $v^{i'}$ of the (y)-order, which are g-related to the vectors u^i and v^i of the (x)-order respectively, namely

$$u^{i'} = g_i^{i'}u^i, \quad v^{i'} = g_i^{i'}v^i.$$

The scalar product of the vectors u' and v' is given by

$$g_{i'j'}u^{i'}v^{j'} = (g_{ij}g_{i'}^{i}g_{j'}^{j})(g_k^{i'}u^k)(g_l^{j'}u^l) = g_{kl}u^k v^l.$$

If follows that the scalar product of the vectors u' and v' is equal to the one of the vectors u and v, which are g-related to the formers. Hence the length of the vector u' is the same as the one of the u.

Let g^{ij} and $g^{i'j'}$ be the inverses of the metric tensors g_{ij} and $g_{i'j'}$ respectively. Making use of the g-tensors [1], § 1 and the metric tensors, we can introduce two tensors $g_{ij'}$ and $g^{ij'}$, which will be useful in the following. Namely we define

$$g_{ij'} = g_{j'i} = g_{ik}g_{j'}^{k}, \quad g^{ij'} = g^{j'i} = g^{ik}g_k^{j'}, \tag{1.3}$$

and then the following identities are the direct results of the above definitions.

$$\begin{aligned} &g_{ij}g^{ik'} = g_j^{k'}, \quad g_{ij'}g^{ik} = g_{j'}^{k}, \quad g_{ij'}g^{ik'} = \delta_{j'}^{k'}, \\ &g_{i'j}g^{i'k'} = g_j^{k'}, \quad g_{i'j'}g^{i'k} = g_{j'}^{k}, \quad g_{i'j}g^{i'k} = \delta_j^k. \end{aligned} \tag{1.4}$$

We take the natural frames (e_i) and $(e_{i'})$, such that the equations

$$e_i e_j = g_{ij}, \quad e_{i'}e_{j'} = g_{i'j'}$$

are satisfied. In the Riemannian geometry, the condition, that the connection preserves the metric, is imposed and leads us to the Ricci's formula

$$\frac{\partial g_{ij}}{\partial x^k} = g_{lj}\Gamma_{ik}^{l} + g_{il}\Gamma_{jk}^{l},$$

where the Γ's are the coefficients of the connection-form. In our case also, we assume that this conditions, that is,

$$d(e_i e_j) = dg_{ij}\,, \quad d(e_{i'} e_{j'}) = dg_{i'j'}$$

are satisfied. Since we defined the connection in the [1], § 1 as

$$de_i = \omega_i^j e_j\,, \quad \omega_i^j = \Gamma_{ik}^j dx^k + C_{ik'}^j dy^{k'}\,,$$

the Ricci's formula is given by the system of the following equations.

$$\frac{\partial g_{ij}}{\partial x^k} = g_{lj}\,\Gamma_{ik}^l + g_{il}\,\Gamma_{jk}^l\,, \tag{1.5}$$

$$\frac{\partial g_{ij}}{\partial x^{k'}} = g_{lj} C_{ik'}^l + g_{il} C_{jk'}^l\,, \tag{1.6}$$

and furthermore we have the similar equations for the metric tensor of the N. These facts mean that *the metric tensors of the M and N are covariantly constant* with respect to the covariant differentiation (,). It is immediately seen that the inverses of the metric tensors have the same property.

2. Existence of the metric connections.

Putting together the above results, we assumed that the g-tensors and the metric tensors were both covariantly constant. In this section, we shall show that there exists the connection $(\omega_i^j, \omega_{i'}^{j'})$, such that the above assumptions hold good. Such a connection is called to be *metric.*

[A]. For this purpose we pay attention first to the Δ-differentiation, which was defined in the [1], § 4. From (1.5) and (1.6) we obtain

$$\frac{\Delta g_{ij}}{\Delta x^k} = g_{lj}\,\Lambda_{ik}^l + g_{il}\,\Lambda_{jk}^l\,. \tag{2.1}$$

Moreover we get from the [1], (1.6) and (1.7)

$$\frac{\Delta g_{j'}^i}{\Delta y^{k'}} = -g_{j'}^l\,\Lambda_{lk'}^i + g_{l'}^i\,\Lambda_{j'k'}^{l'}\,, \tag{2.2}$$

where we put

$$\begin{aligned} \Lambda_{jk}^i &= \Gamma_{jk}^i + C_{jl'}^i g_k^{l'}\,, \quad \Lambda_{j'k'}^{i'} = \Gamma_{j'k'}^{i'} + C_{j'l}^{i'} g_{k'}^l\,, \\ \Lambda_{jk'}^i &= \Lambda_{jl}^i g_{k'}^l\,. \end{aligned} \tag{2.3}$$

The skew-symmetric parts S_{ijk} of the $\Lambda_{ijk} = g_{jl}\,\Lambda_{ik}^l$ were called the components of the g-torsion of the M, and their symmetric parts will be denoted by λ_{ijk}. Then it follows that

(2.4) $$\Lambda^i_{jk}=\lambda^i_{jk}+S^i_{jk}, \quad \Lambda^{i'}_{j'k'}=\lambda^{i'}_{j'k'}+S^{i'}_{j'k'},$$

where putting $\lambda^i_{jk}=g^{il}\lambda_{jlk}$ and $S^i_{jk}=g^{il}S_{jlk}$. Substituting from (2.4) in (2.1), we get

(2.5) $$\frac{\Delta g_{ij}}{\Delta x^k}=\lambda_{(ij)k}+S_{(ij)k}.$$

Hence, if we take

(2.6) $$\Delta_{ijk}=\frac{1}{2}\left(\frac{\Delta g_{ij}}{\Delta x^k}+\frac{\Delta g_{jk}}{\Delta x^i}-\frac{\Delta g_{ik}}{\Delta x^j}\right),$$

then these quantities are determined by the g-tensor and the metric tensor alone, and the equations

(2.7) $$\lambda_{ijk}=\Delta_{ijk}-S_{j(ik)}$$

are obtained from (2.5), and we have conversely (2.5) from (2.7). By the same process from the equations

(2.1′) $$\frac{\Delta g_{i'j'}}{\Delta y^{k'}}=g_{l'j'}\Lambda^{l'}_{i'k'}+g_{i'l'}\Lambda^{l'}_{j'k'},$$

we obtain

(2.7′) $$\lambda_{i'j'k'}=\Delta_{i'j'k'}-S_{j'(i'k')},$$

where the $\Delta_{i'j'k'}$ are defined by the similar equations to (2.6).

Next, the equation [1], (4.22) is written in terms of the g-torsions as follows:

(2.8) $$S^{i'}_{j'k'}=\frac{1}{2}g^{i'}_{i}\frac{\Delta g^{i}_{[j'}}{\Delta y^{k']}}+S^{i}_{jk}g^{i'}_{i}g^{j}_{j'}g^{k}_{k'}.$$

Now we shall prove the

Lemma 1. *If we take a skew-symmetric tensor* S^i_{jk} $(=-S^i_{kj})$ *arbitrarily, and define the quantities* $S^{i'}_{j'k'}$, λ_{ijk} *and* $\lambda_{i'j'k'}$ *by the equations* (2.8), (2.7) *and* (2.7′) *respectively, then the equations* (2.1) *and* (2.2) *are satisfied, where the* Λ*'s are given by the equation* (2.4).

Since the equation (2.1) is immediate results from (2.6) and (2.7), we shall show that the equation (2.2) is satisfied, making use of the relation (2.8) between the g-torsions S^i_{jk} and $S^{i'}_{j'k'}$. We have first from the definition of the Δ_{ijk}

$$\Delta_{ijk}g^i_{i'}g^j_{j'}g^k_{k'}=\frac{1}{2}\left(\frac{\Delta g_{ij}}{\Delta x^k}+\frac{\Delta g_{jk}}{\Delta x^i}-\frac{\Delta g_{ik}}{\Delta x^j}\right)g^i_{i'}g^j_{j'}g^k_{k'},$$

from which we obtain in virtue of the [1], (4.13)

$$\begin{aligned}(2.9)\qquad &\Delta_{ijk}g^i_{i'}g^j_{j'}g^k_{k'}\\ &=\frac{1}{2}\left(\frac{\Delta g_{ij}}{\Delta y^{k'}}g^i_{i'}g^j_{j'}+\frac{\Delta g_{ij}}{\Delta y^{i'}}g^i_{j'}g^j_{k'}-\frac{\Delta g_{ij}}{\Delta y^{j'}}g^i_{i'}g^j_{k'}\right).\end{aligned}$$

Making use of (1.2), (1.3) and (1.4), the first term of the right-hand members of the above equation is rewritten as follows:

$$\begin{aligned}(2.10)\qquad \frac{1}{2}\frac{\Delta g_{ij}}{\Delta y^{k'}}g^i_{i'}g^j_{j'}&=\frac{1}{2}\left(\frac{\Delta g_{ji'}}{\Delta y^{k'}}-g_{ij}\frac{\Delta g^i_{i'}}{\Delta y^{k'}}\right)g^j_{j'}\\ &=\frac{1}{2}\frac{\Delta g_{ji'}}{\Delta y^{k'}}g^j_{j'}-\frac{1}{2}g_{ij'}\frac{\Delta g^i_{i'}}{\Delta y^{k'}}\\ &=\frac{1}{2}\frac{\Delta g_{i'j'}}{\Delta y^{k'}}-\frac{1}{2}g_{ii'}\frac{\Delta g^i_{j'}}{\Delta y^{k'}}-\frac{1}{2}g_{ij'}\frac{\Delta g^i_{i'}}{\Delta y^{k'}}.\end{aligned}$$

Substitution of the above and the similar two equations in (2.9) gives

$$\begin{aligned}(2.11)\qquad \Delta_{i'j'k'}&=\Delta_{ijk}g^i_{i'}g^j_{j'}g^k_{k'}\\ &+\frac{1}{2}\left(g_{i(i'}\frac{\Delta g^i_{j')}}{\Delta y^{k'}}+g_{i(k'}\frac{\Delta g^i_{j')}}{\Delta y^{i'}}-g_{i(i'}\frac{\Delta g^i_{k')}}{\Delta y^{j'}}\right).\end{aligned}$$

Since we have from (2.8)

$$(2.8')\qquad S_{i'j'k'}=S_{ijk}g^i_{i'}g^j_{j'}g^k_{k'}+\frac{1}{2}g_{ij'}\frac{\Delta g^i_{[i'}}{\Delta y^{k']}},$$

summation of this and (2.11) gives on account of (2.7)

$$(2.12)\qquad \lambda_{i'j'k'}=\lambda_{ijk}g^i_{i'}g^j_{j'}g^k_{k'}+\frac{1}{2}g_{ij'}\frac{\Delta g^i_{(i'}}{\Delta y^{k')}}.$$

Hence, making use of (2.4), (2.8) and (2.12), the right-hand side of (2.2) is written in the form

$$\begin{aligned}&g^{il'}(\lambda_{j'l'k'}-\lambda_{jlk}g^j_{j'}g^l_{l'}g^k_{k'})\\ &\qquad+g^i_{l'}(S^{l'}_{j'k'}-S^l_{jk}g^j_{j'}g^k_{k'}g^{l'}_l)=\frac{\Delta g^i_{j'}}{\Delta y^{k'}}.\end{aligned}$$

Thus we have (2.2) and so prove the lemma.

We note here that the expressions of the Δ-derivatives of the g-tensors and the metric tensors are all satisfied as the consequences of (2.1) and (2.2). In fact, according to the [1], (4.13), we have immediately from (2.1) and (2.2)

$$\frac{\Delta g_{ij}}{\Delta y^{k'}}=g_{lj}\Lambda^l_{ik'}+g_{il}\Lambda^l_{jk'}\,. \tag{2.13}$$

$$\frac{\Delta g^i_{j'}}{\Delta x^k}=-g^l_{j'}\Lambda^i_{lk}+g^i_{l'}\Lambda^{l'}_{j'k}\,. \tag{2.14}$$

Since the g^{ij} and $g^{i'}_j$ are the inverses of the g_{ij} and $g^i_{j'}$ respectively, we obtain easily

$$\frac{\Delta g^{ij}}{\Delta x^k}=-g^{lj}\Lambda^i_{lk}-g^{il}\Lambda^j_{lk}\,,\qquad \frac{\Delta g^{ij}}{\Delta y^{k'}}=-g^{lj}\Lambda^i_{lk'}-g^{il}\Lambda^j_{lk'}\,, \tag{2.15}$$

and

$$\frac{\Delta g^{i'}_j}{\Delta x^k}=-g^{l'}_j\Lambda^{i'}_{l'k}+g^{i'}_l\Lambda^l_{jk}\,,\qquad \frac{\Delta g^{i'}_j}{\Delta y^{k'}}=-g^{l'}_j\Delta^{i'}_{l'k'}+g^{i'}_l\Lambda^l_{jk'}\,. \tag{2.16}$$

Finally, operating the Δ-differentiation to (1.2), we have

$$\frac{\Delta g_{i'j'}}{\Delta x^k}=g_{l'j'}\Lambda^{l'}_{i'k}+g_{i'l'}\Lambda^{l'}_{j'k}\,,\qquad \frac{\Delta g_{i'j'}}{\Delta y^{k'}}=g_{l'j'}\Lambda^{l'}_{i'k'}+g_{i'l'}\Lambda^{l'}_{j'k'}\,. \tag{2.17}$$

[**B**]. Now we shall prove the existence of a metric connection. We put

$$C_{ijk'}=E_{ijk'}+F_{ijk'}\,,\quad C_{i'j'k}=E_{i'j'k}+F_{i'j'k}\,. \tag{2.18}$$

where $C_{ijk'}=g_{jl}C^l_{ik'}$, $C_{i'j'k}=g_{j'l'}C^{l'}_{i'k}$, and the E's and F's are respectively the symmetric and skew-symmetric parts of the C's with respect to the first two indices. Substituting (2.18) in (1.6) we get

$$E_{ijk'}=\frac{1}{2}\frac{\partial g_{ij}}{\partial y^{k'}},\quad E_{i'j'k}=\frac{1}{2}\frac{\partial g_{i'j'}}{\partial x^{k}}. \tag{2.19}$$

Thus the symmetric parts E's are determined by the metric tensors alone. On the other hand, the equation [1], (4.4) is written in the from

$$C^{*}_{i'j'k}=-g_{l'j}g^{i}_{i'}\left(\frac{\partial g^{l'}_{i}}{\partial x^{k}}-g^{l'}_{j}\Lambda^{j}_{ik}\right),$$

where $C^{*}_{i'j'k}=g_{j'l'}C^{*l'}_{i'k}$. It follows easily from the above that

$$C^{*}_{i'j'k}=g_{ij'}\frac{\partial g^{i}_{i'}}{\partial x^{k}}+g^{i}_{i'}g^{j}_{j'}\Lambda_{ijk}. \tag{2.20}$$

The symmetric and skew-symmetric parts of the $C^{*}_{i'j'k}$ are clearly given by

$$E^{*}_{i'j'k}=E_{i'j'k}+E_{ijk'}g^{i}_{i'}g^{j}_{j'}g^{k'}_{k},$$
$$F^{*}_{i'j'k}=F_{i'j'k}+F_{ijk'}g^{i}_{i'}g^{j}_{j'}g^{k'}_{k},$$

respectively. Therefore we obtain from (2.20)

$$F^{*}_{i'j'k}=-\frac{1}{2}g_{i[i'}\frac{\partial g^{i}_{j']}}{\partial x^{k}}+\frac{1}{2}g^{i}_{[i'}g^{j}_{j']}\Lambda_{ijk}, \tag{2.21}$$

from which it follows that

$$F_{i'j'k}=-F_{ijk'}g^{i}_{i'}g^{j}_{j'}g^{k'}_{k}-\frac{1}{2}g_{i[i'}\frac{\partial g^{i}_{j']}}{\partial x^{k}}+\frac{1}{2}g^{i}_{[i'}g^{j}_{j']}\Lambda_{ijk}. \tag{2.21'}$$

Now we shall prove the

Lemma 2. *If the E's are defined by* (2.19) *and the $F_{i'j'k}$ by* (2.21′) *for an arbitrary choice of the $F_{ijk'}$ $(=-F_{jik'})$, then the equation* [1], (1.6), *that is,*

$$\frac{\partial g^{i}_{j'}}{\partial x^{k}}=-g^{l}_{j'}\Gamma^{i}_{lk}+g^{i}_{l'}C^{l'}_{j'k} \tag{2.22}$$

and (1.6) *are satisfied, where the Γ's are given by the equation* (2.3) *and the C's by* (2.18).

The equation (1.6) is clearly satisfied, and so we shall verify (2.22). It follows from (2.19) that

$$E^{*}_{i'j'k}=\frac{1}{2}\frac{\partial g_{i'j'}}{\partial x^{k}}+\frac{1}{2}\frac{\partial g_{ij}}{\partial y^{k'}}g^{i}_{i'}g^{j}_{j'}g^{k'}_{k}\,.$$

Applying the calculation similar to (2. 10), we get

$$=\frac{1}{2}\frac{\partial g_{i'j'}}{\partial x^{k}}+\frac{1}{2}\left(\frac{\partial g_{i'j'}}{\partial y^{k'}}-g_{i(i'}\frac{\partial g^{i}_{j')}}{\partial y^{k'}}\right)g^{k'}_{k}$$

$$=\frac{1}{2}\frac{\Delta g_{i'j'}}{\Delta x^{k}}-\frac{1}{2}g_{i(i'}\left(\frac{\Delta g^{i}_{j')}}{\Delta x^{k}}-\frac{\partial g^{i}_{j')}}{\partial x^{k}}\right).$$

Substitution of (2. 14) and (2. 17) gives

$$E^{*}_{i'j'k}=\frac{1}{2}g_{i(i'}\frac{\partial g^{i}_{j')}}{\partial x^{k}}+\frac{1}{2}g^{i}_{(i'}g^{j}_{j')}\Lambda_{ijk}\,. \tag{2. 23}$$

Summing (2. 21) and (2. 23), we have (2. 20), which is written in the form

$$C_{i'j'k}+C_{ijk'}g^{i}_{i'}g^{j}_{j'}g^{k'}_{k}=g_{ij'}\frac{\partial g^{i}_{i'}}{\partial x^{k}}+g^{i}_{i'}g^{j}_{j'}\Lambda_{ijk}\,,$$

that is,

$$\frac{\partial g^{i}_{j'}}{\partial x^{k}}=-g^{j}_{j'}(\Lambda^{i}_{jk}-C^{i}_{jk'}g^{k'}_{k})+g^{i}_{i'}C^{i'}_{j'k}\,.$$

Hence, remembering (2. 3), we obtain (2. 22) and then prove the lemma.

Next, making use of the equations

$$\frac{\partial g_{ij}}{\partial x^{k}}=\frac{\Delta g_{ij}}{\Delta x^{k}}-\frac{\partial g_{ij}}{\partial y^{k'}}g^{k'}_{k}\,,\qquad \frac{\partial g^{i}_{j'}}{\partial y^{k'}}=\frac{\Delta g^{i}_{j'}}{\Delta y^{k'}}-\frac{\partial g^{i}_{j'}}{\partial x^{k}}g^{k}_{k'}\,,$$

and furthermore (2. 1) and (2. 2), we obtain easily (1. 5) and [1], (1. 7), and hence all of the expressions of the partial derivatives of the g-tensors and the metric tensors are established. Consequently we obtain the

Theorem. *There exists a metric connection* $(\omega^{j}_{i},\ \omega^{j'}_{i'})$, *and it is uniquely determined, when the g-torsion* S^{i}_{jk} *of the M and the skew-symmetric parts* $F_{ijk'}$ *of the rotation of the connection of the M are given arbitrarily.*

Finally the coefficients of the connection form are expressed explicitly in terms of the g-tensors, the metric tensors, quantities S_{ijk} and $F_{ijk'}$ as follows.

$$\Gamma_{ijk} = \Delta_{ijk} - \frac{1}{2}\frac{\partial g_{ij}}{\partial y^{k'}} g^{k'}_{k} + S_{ijk} - S_{jki} + S_{kij} - F_{ijk'} g^{k'}_{k},$$

$$C_{ijk'} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^{k'}} + F_{ijk'},$$

$$\Gamma_{i'j'k'} = \Delta_{i'j'k'} + \frac{1}{2}\Delta_{[i'j']k} - \frac{1}{2} g_{i[i'} \frac{\partial g^{i}_{j']}}{\partial y^{k'}}$$
$$- \frac{1}{2}\frac{\partial g_{i'j'}}{\partial x^{k}} g^{k}_{k'} + g_{i[i'} \frac{\Delta g^{i}_{k'}}{\Delta y^{j']}} + g_{ik'} \frac{\Delta g^{i}_{[i'}}{\Delta y^{j']}}$$
$$+ 2 g^{i}_{i'} g^{j}_{j'} g^{k}_{k'} (S_{ijk} - S_{jki} + S_{kij}) + F_{ijk'} g^{i}_{i'} g^{j}_{j'},$$

$$C_{i'j'k} = \frac{1}{2}\frac{\partial g_{i'j'}}{\partial x^{k}} + \frac{1}{2} g^{i}_{[i'} g^{j}_{j']} \Delta_{ijk} - \frac{1}{2} g_{i[i'} \frac{\partial g^{i}_{j']}}{\partial x^{k}}$$
$$+ \frac{1}{2} g^{i}_{[i'} g^{j}_{j']} (S_{ijk} - S_{jki} + S_{kij}) - F_{ijk'} g^{i}_{i'} g^{j}_{j'} g^{k'}_{k}.$$

Thus we can have the metric connections, but the above arbitrarinesses are left. In order to determine some useful metric connections, we shall remember the cases of the Riemannian and the Finslerian spaces. In the following sections we shall give the effective ideas to determine an useful metric connection.

3. The metric connection without torsions.

The torsion tensor of the M is given from (2. 3) by

$$T^{i}_{jk} = S^{i}_{jk} - \frac{1}{2} C^{i}_{[jk'} g^{k'}_{k]}.$$

Following the case of the Riemannian geometry, we assume that *the torsion tensors of the M and N vanish.*

Then this assumption is expressed analytically by

$$2 S_{ijk} - F_{[ijk'} g^{k'}_{k]} = E_{[ijk'} g^{k'}_{k]}, \tag{3. 1}$$

and

$$2 S_{i'j'k'} - F_{[i'j'k} g^{k}_{k']} = E_{[i'j'k} g^{k}_{k']}, \tag{3. 2}$$

where the E's and F's are the symmetric and skew-symmetric parts of the C' respectively as used in the last section, and the E's have been already determined by the metric tensors as the equations (2. 19). We shall show that the equations (3. 1) and (3. 2) determine uniquely the quantities S's and F's, so that the *metric connection without torsions* is uniquely determined, according to the above theorem.

In the first place, contracting (3.1) by $g^i_{i'} g^j_{j'} g^k_{k'}$ and making use of (2.8′), (2.21) and (2.23), we get

$$2S_{i'j'k'} - g_{ij'} \frac{\Delta g^i_{[i'}}{\Delta y^{k']}} + F_{[i'j'k} g^k_{k']} \tag{3.3}$$
$$= -E_{[i'j'k} g^k_{k']} + (P_{[i'j'k} + Q_{[i'j'k}) g^k_{k']},$$

where the P's and Q's are the right-hand sides of the equations (2.21) and (2.23) respectively. Hence we obtain from (3.3) by means of (3.2)

$$4S_{i'j'k'} = g_{ij'} \frac{\Delta g^i_{[i'}}{\Delta y^{k']}} + (P_{[i'j'k} + Q_{[i'j'k}) g^k_{k']}. \tag{3.4}$$

It follows from the definitions of the P's and Q's that

$$(P_{i'j'k} + Q_{i'j'k}) g^k_{k'} = g_{ij'} g^j_{k'} \frac{\partial g^i_{i'}}{\partial x^j} + g^i_{i'} g^j_{j'} g^k_{k'} \Lambda_{ijk}.$$

Substituting from (2.4) and (2.7) and making use of (2.8′), then the above equation is written in the form

$$= g_{ij'} g^j_{k'} \frac{\partial g^i_{i'}}{\partial x^j} + g^i_{i'} g^j_{j'} g^k_{k'} \Delta_{ijk} - S_{j'(k'i')} + S_{i'j'k'}$$
$$- \frac{1}{2} g_{ij'} \frac{\Delta g^i_{[i'}}{\Delta y^{k']}} + \frac{1}{2} g_{i(i'} \frac{\Delta g^i_{j'}}{\Delta y^{k')}} - \frac{1}{2} g_{i(i'} \frac{\Delta g^i_{k')}}{\Delta y^{j'}}.$$

Accordingly we obtain

$$(P_{[i'j'k} + Q_{[i'j'k}) g^k_{k']}$$
$$= g_{ij'} \frac{\partial g^i_{[i'}}{\partial x^j} g^j_{k']} + 2S_{i'j'k'} - g_{ij'} \frac{\Delta g^i_{[i'}}{\Delta y^{k']}}.$$

Substitution of the above into (3.4) gives finally the expression of the g-torsin $S_{i'j'k'}$ as follows:

$$S_{i'j'k'} = \frac{1}{2} g_{ij'} \frac{\partial g^i_{[i'}}{\partial x^j} g^j_{k']}. \tag{3.5′}$$

Similarly we have the expression of the g-torsion S_{ijk} as

$$S_{ijk} = \frac{1}{2} g_{i'j} \frac{\partial g^{i'}_{[i}}{\partial y^{j'}} g^{j'}_{k]}. \tag{3.5}$$

Thus the S_{ijk} are determined. On the other hand, the quantities $F_{ijk'}$ are directly given from (3. 1) by

$$F_{ijk'} = (E_{k[ji'}g^{i'}_{i]} + S_{ijk} - S_{jki} + S_{kij})\, g^{k}_{k'} \,. \tag{3. 6}$$

Consequently we obtain the metric without torsions. The coefficients Γ's of the connection are given by

$$\Gamma'^{i}_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{il}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right), \tag{3. 7}$$

because of the equation (1. 5) and symmetry of the Γ's. The coefficients C's are expressible in the forms

$$\begin{aligned} C_{ijk'} = \frac{1}{2} \Big(& \frac{\partial g_{ij}}{\partial y^{k'}} + g^{i'}_{[i} \frac{\partial g_{j]k}}{\partial y^{i'}} g^{k}_{k'} - g_{i'[i} \frac{\partial g^{i'}_{j]}}{\partial y^{k'}} \\ & - g^{i'}_{[i} \frac{\partial g^{j'}_{j]}}{\partial y^{i'}} g_{j'k'} + g^{i'}_{[i} g_{j]k} \frac{\partial g^{k}_{k'}}{\partial y^{i'}} \Big) . \end{aligned} \tag{3. 8}$$

We have also the similar expressions for the coefficients $\Gamma'^{i'}_{j'k'}$ and $C_{i'j'k}$ of the N.

4. The normal metric connection.

In this section we shall define more important class of metric connections. In the Finsler space, *E. Cartan* gave the five geometrical axioms [2], § 7, and then determined uniquely the connection, satisfying the axioms. The last axiom saies that the components Γ'^{*i}_{jk} are symmetric, where the Γ'^{*i}_{jk} are used to express the absolute differential of a vector when its supporting element enjoys a parallel displacement to itself. On the other hand, in our case, the quantities Λ^{i}_{jk} are thought to correspond to the Γ'^{*i}_{jk} in Finsler spaces, because these quantities are used to express the absolute differential of a vector when the displacement of the observing point is g-related to the one of the origin of the vector. On account of these considerations we assume first that the Λ^{i}_{jk} are symmetric, that is,

I. *The g-torsion S^{i}_{jk} of the manifold M vanishes.*

Since the g-torsion of the N is determined by (2. 8) in the general case, we have under the assumption 1

$$S^{i'}_{j'k'} = \frac{1}{2} g^{i'}_{i} \frac{\Delta g^{i}_{[j'}}{\Delta y^{k']}} \,. \tag{4. 1}$$

It follows from (2.4) and (2.7) that the components Λ^i_{jk} are simply given by

$$(4.2)\qquad \Lambda^i_{jk} = \frac{1}{2} g^{il}\left(\frac{\Delta g_{jl}}{\Delta x^k} + \frac{\Delta g_{kl}}{\Delta x^j} - \frac{\Delta g_{jk}}{\Delta x^l}\right).$$

Next, in order to determine the quantities $F_{ijk'}$, we consider again the infinitesimal parallelograms $\pi(P)$ in the M and $\pi'(Q)$ in the N, which were used in the [1], §2. If the sides $QQ_1 = dy$ and $QQ_1' = \delta y$ of the $\pi'(Q)$ are g-related to the sides $PP_1 = dx$ and $PP_1' = \delta x$ of the $\pi(P)$ respectively, then the $\pi(P)$ and $\pi'(Q)$ are called to be g-related. As was shown in the [1], (2.9), the torsion ΔP associating with the $\pi(P)$ with respect to the fixed observing point Q is equal to $\Delta P = 2T^i_{jk} dx^j \delta x^k e_i$. The expression of the torsion ΔQ associating with the $\pi'(Q)$ with respect to the fixed observing point P is similarly given by $\Delta Q = 2T^{i'}_{j'k'} dy^{j'} \delta y^{k'} e_{i'}$. Hence, if the $\pi(P)$ and $\pi'(Q)$ are g-related, then we have

$$\Delta Q = 2T^{i'}_{j'k'} g^{j'}_j g^{k'}_k dx^j \delta x^k e_{i'}.$$

Now we give the second assumption.

II. *If the parallelograms $\pi(P)$ and $\pi'(Q)$ are g-related, the torsion ΔP associating with the $\pi(P)$ is carried into the torsion ΔQ associating with the $\pi'(Q)$ by the mapping g: $T_P \to T_Q$.*

Consequently the analytic expression of the assumption is given by

$$(4.3)\qquad T^i_{jk} = T^{i'}_{j'k'} g^i_{i'} g^{j'}_j g^{k'}_k,$$

which means that the torsion tensors of the M and N are g-related. Making use of (2.3) and taking care of symmetry of the Λ^i_{jk}, we get

$$(4.4)\qquad T^i_{jk} = \frac{1}{2} g^{i'}_{[j} C^i_{k]i'}, \quad T^{i'}_{j'k'} = \frac{1}{2} g^i_{[j'} C^{i'}_{k']i} + S^{i'}_{j'k'}.$$

Then the above (4.3) are expressed in the form

$$(4.3')\qquad C_{[j|i|k]} = (C_{[j'|i'|k']} - 2S_{j'i'k'}) g^{i'}_i g^{j'}_j g^{k'}_k,$$

where putting $C_{jik} = C_{jik'} g^{k'}_k$. Substitution (2.18) in (4.3′) gives

$$(4.5)\qquad \begin{aligned} &F_{i[jk]} - F_{i'[j'k']} g^{i'}_i g^{j'}_j g^{k'}_k \\ &\quad = E_{i[jk]} - (E_{i'[j'k']} - 2S_{j'i'k'}) g^{i'}_i g^{j'}_j g^{k'}_k, \end{aligned}$$

where we take $E_{ijk}=E_{ijk'}g_k^{k'}$ and $F_{ijk}=F_{ijk'}g_k^{k'}$. As well as in the last section, we set the right-hand side of (2.21) by $P_{i'j'k}$ and substitute into (4.5) from (2.21'). Then the equations

$$(4.5') \qquad 2F_{i[jk]}=P_{i[jk]}+E_{i[jk]}-(E_{i'[j'k']}-2S_{j'i'k'})\, g_i^{i'}g_j^{j'}g_k^{k'}$$

are obtained, where we put $P_{ijk}=P_{i'j'k}g_i^{i'}g_j^{j'}$. If we construct from the above equations the forms $2(F_{i[jk]}+F_{j[ki]}-F_{k[ij]})$, and make use of skew-symmetry of the F_{ijk} and P_{ijk}, and further symmetry of the E_{ijk}, then the equations

$$2F_{ijk}=P_{ijk}-E_{k[ij]} + \left[E_{k'[i'j']}-\frac{1}{2}\,(S_{[i'j']k'}+S_{i'k'j'})\right] g_i^{i'}g_j^{j'}g_k^{k'}$$

are obtained. Therefore we get

$$2F_{ijk'}=P_{i'j'k}g_i^{i'}g_j^{j'}g_{k'}^{k}+g_{[i}^{i'}E_{j]ki'}g_{k'}^{k}-E_{k'i'[i}g_{j]}^{i'} -\frac{1}{2}\,(S_{[i'j']k'}+S_{i'k'j'})\, g_i^{i'}g_j^{j'}\,.$$

We see easily $P_{i'j'k}g_i^{i'}g_j^{j'}g_{k'}^{k}=P_{ijk'}$, where the quantities $P_{ijk'}$ are defined similar to the $P_{i'j'k}$, namely

$$(4.6) \qquad P_{ijk'}=-\frac{1}{2}\,g_{i'[i}\frac{\partial g_{j]}^{i'}}{\partial y^{k'}}+\frac{1}{2}\,g_{[i}^{i'}g_{j]}^{j'}\Delta_{i'j'k'}\,.$$

Consequently we obtain the final expressions of the quantities $F_{ijk'}$ as follows:

$$(4.7) \qquad F_{ijk'}=\frac{1}{2}\,(P_{ijk'}+g_{[i}^{i'}E_{j]ki'}g_{k'}^{k}-E_{k'i'[i}g_{j]}^{i'}) -\frac{1}{4}\,(S_{[i'j']k'}+S_{i'k'j'})\, g_i^{i'}g_j^{j'}\,.$$

Hereon, a metric connection, which satisfies the assumptions *I* and *II*, has been determined by the metric tensors, the g-tensors, and their derivatives of the first degree. It follows from (4.1) that the g-torsion $S_{j'k'}^{i'}$ of the N is generally not equal to zero. However, it is conjectured that this circumstance will be undesirable for the following discussions. Therefore we assume, instead of the *I*

I'. *Both of the g-torsions of the M and N vanish.*
It is clear that the condition

$$\frac{\Delta g^{i}_{j'}}{\Delta y^{k'}}=\frac{\Delta g^{i}_{k'}}{\Delta y^{j'}} \tag{4.8}$$

is necessary and sufficient for the assumption I'. We shall call the metric connection, satisfying the assumptions I' and II, to be *normal*. The coefficients Γ's and C's of the normal metric connection are given by the equations

$$\begin{aligned} \Gamma^{i}_{jk} = {} & \frac{1}{2} g^{il}\left(\frac{\partial g_{il}}{\partial x^{k}}+\frac{\partial g_{kl}}{\partial x^{j}}-\frac{\partial g_{jk}}{\partial x^{l}}\right)-\frac{1}{2} g^{ii'}\frac{\partial g_{jk}}{\partial y^{i'}} \\ & +\frac{1}{4} g^{j'}_{(j}\left(\frac{\partial g_{k)l}}{\partial y^{j'}} g^{il}+\frac{\partial g_{k)j'}}{\partial y^{i'}} g^{ii'}-\frac{\partial g_{k)i'}}{\partial y^{j'}} g^{ii'}\right) \\ & +\frac{1}{4} g^{j'}_{[j} g^{k'}_{k]}\frac{\partial g^{i}_{j'}}{\partial y^{k'}}, \end{aligned} \tag{4.9}$$

$$C_{ijk'}=\frac{1}{2}\frac{\partial g_{ij}}{\partial y^{k'}}-\frac{1}{4} g^{i'}_{[i}\left(\frac{\partial g_{j]i'}}{\partial y^{k'}}-\frac{\partial g_{j]k}}{\partial y^{i'}} g^{k}_{k'}+g^{j'}_{j]}\frac{\partial g_{i'k'}}{\partial y^{j'}}\right), \tag{4.10}$$

and the similar equations for the $\Gamma^{i'}_{j'k'}$ and $C_{i'j'k'}$.

5. Geodesics.

In the first place we shall introduce the notion of the g-related curves. Let $P(x_0)$ and $Q(y_0)$ be any points of the M and N respectively, and the $C:[0, 1]\rightarrow M$ a curve in the M, issuing from the point $P(x_0)=C(0)$. The equations of the C are given by $x^i=x^i(t)$. We consider a system of the ordinal differential equations

$$\frac{dy^{i'}}{dt}=g^{i'}_{i}(x, y)\frac{dx^i}{dt} \tag{5.1}$$

for unknown functions $y^{i'}(t)$ of one variable t. Then there exists an unique set of solutions $y^{i'}(t)$, satisfying the initial conditions $y^{i'}(0)=y^{i'}_0$. The curve $y^{i'}=y^{i'}(t)$ as thus defined is called to be g-related to the curve C in the M. In the last section of the [1], we used already this notion, by means of which we defined the path [1], (5.3). Since the metric of the N is g-related to the one of the M, we see easily from (1.2) that *the g-related curves have the same line-elements.*

Now we consider the extremal C of the integral of the line-element

$$\int\sqrt{g_{ij}(x, y)\frac{dx^i}{dt}\frac{dx^j}{dt}}\,dt\,,$$

where we suppose that the curve (y) in the N is g-related to the one (x) in the M. If we put $F=\sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}$, then it is well known that the extremal C is given by the solution of a system of the differential equations

$$\frac{\partial F}{\partial x^i}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^i}\right)=0\,, \qquad \left(\dot{x}^i=\frac{dx^i}{dt}\right).$$

We have

$$\frac{\partial F}{\partial x^i}=\frac{1}{2F}\left(\frac{\partial g_{kl}}{\partial x^i}+\frac{\partial g_{kl}}{\partial y^{i'}}\frac{\partial y^{i'}}{\partial x^i}\right)\dot{x}^k\dot{x}^l\,.$$

From (5.1) we have

$$\frac{\partial F}{\partial x^i}=\frac{1}{2F}\left(\frac{\partial g_{kl}}{\partial x^i}+\frac{\partial g_{kl}}{\partial y^{i'}}g_i^{i'}\right)\dot{x}^k\dot{x}^l\,.$$

Therefore the equations

$$\frac{\partial F}{\partial x^i}=\frac{1}{2F}\frac{\Delta g_{kl}}{\Delta x^i}\dot{x}^k\dot{x}^l$$

are obtained. Similarly we get

$$\frac{\partial F}{\partial \dot{x}^i}=\frac{1}{F}g_{ik}\dot{x}^k\,,$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^i}\right)=\frac{1}{F}\left(\frac{\Delta g_{ik}}{\Delta x^l}\dot{x}^k\dot{x}^l+g_{ik}\ddot{x}^k\right)-\frac{1}{F}\frac{dF}{dt}g_{ik}\dot{x}^k\,.$$

Consequently, if we take the arc-length s as the parameter, then we have the equations of the extremal C as

$$\frac{d^2x^i}{ds^2}+\Delta^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0\,. \tag{5.2}$$

We call such an extremal the *geodesic* in the M.

If the connection under consideration is normal, then the coefficients Δ^i_{jk} are equal to the Λ^i_{jk} from (4.2), so that the above equations are written in the forms

$$\frac{d^2x^i}{ds^2}+\Lambda^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0\,. \tag{5.2'}$$

By means of the [1], (5. 3), we have

Theorem I. *If the metric connection is normal, then the notion of a geodesic coincides with the one of a path.*

Let C and C' be the g-related curves in the M and N respectively, and the C be the geodesic. Since the line-elements of these curves are equal, we have

$$\frac{dy^{i'}}{ds}=g_i^{i'}(x, y)\frac{dx^i}{ds},$$

$$\frac{d^2y^{i'}}{ds^2}=\left(\frac{\partial g_i^{i'}}{\partial x^k}+\frac{\partial g_i^{i'}}{\partial y^{k'}}\frac{\partial y^{k'}}{\partial x^k}\right)\frac{dx^i}{ds}\frac{dx^k}{ds}+g_i^{i'}\frac{d^2x^i}{ds^2}.$$

Substituting from (5. 2′) and making use of (5. 1), we obtain

$$\frac{d^2y^{i'}}{ds^2}=\frac{\Delta g_i^{i'}}{\Delta x^k}\frac{dx^i}{ds}\frac{dx^k}{ds}+g_i^{i'}\frac{d^2x^i}{ds^2}$$

$$=(-g_i^{l'}\Lambda_{l'k}^{i'}+g_l^{i'}\Lambda_{ik}^{l})\frac{dx^i}{ds}\frac{dx^k}{ds}-g_l^{i'}\Lambda_{ik}^{l}\frac{dx^i}{ds}\frac{dx^k}{ds}$$

$$=-\Lambda_{l'k}^{i'}g_i^{l'}g_{j'}^{i}g_{k'}^{k}\frac{dy^{j'}}{ds}\frac{dy^{k'}}{ds}.$$

By means of the definition $\Lambda_{l'k'}^{i'}=\Lambda_{l'k}^{i'}g_{k'}^{k}$, we get finally

$$\frac{d^2y^{i'}}{ds^2}+\Lambda_{j'k'}^{i'}\frac{dy^{j'}}{ds}\frac{dy^{k'}}{ds}=0.$$

Thus the curve C' is given as the solution of the above equations, and hence, the C' is geodesic in the N. Therefore

Theorem 2. *If the curves C' in the N is g-related to the geodesic in the M, then C' is also the geodesic in the N.*

Mathematical Institute, University of Kyoto

REFERENCES

[1] M. Matsumoto: *Relative Riemannian geometry I. On the affine connection,* This Memoirs, vol. 31 (1958), pp. 65-82.

[2] E. Cartan: Les espaces de Finsler, Actualités Sci., 79 (1934).