MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES A
Vol. XXXI, Mathematics No. 2, 1958.

Some results in the theory of the differential forms of the first kind on algebraic varieties II.

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(Received April 15, 1958)

This is a suit of our previous paper [3][1] with the same title, and we shall prove the following:

THEOREM. *Let V^n be a hypersurface in a projective space L^{n+1}, and let s be the minimum number among the codimensions of the singular subvarieties of V (we shall put $s=n+1$ when V^n has no singular point). Now assume that $s>2$, then V has no differential form of degree $\leqq s-2$ which is relatively of the first kind on V*[2]. *In particular V has no differential form of the first kind of degree $\leqq s-2$.*

This is the generalization of the well known classical results that *a non singular surface in a projective 3-space has no differential form of the first kind of degree* 1.[3] Moreover it will be shown by an example that the hypersurface V^n in L^{n+1} may have a differential form of the first kind which is of degree $>s-2$, where s has the same meaning as above. Thus the estimation $s-2$ is the best one in the above sence.

The method of the proof is based on the representation of the differential forms of the first kind given in [3] and some auxiliary lemmas. Let V^n be a hypersurface in a projective space L^{n+1} and assume that V has no singular subvariety of codimension 1.

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1) The number in the bracket refers to the bibliography at the end of the paper.

2) A differential form ω is called relatively of the first kind on V when ω is finite at every simple point of V. A differential form ω is of the first kind, if ω is relatively of the first kind on every birationally equivalent model of V. This is equivalent to say that ω is relatively of the first kind on a non-singular birational model of V (cf. [2]).

3) An algebraic proof can be found, e. g., in [4], pp. 119-120. But the proof appeared there is not satisfactory.

Let k be a field of definition for V and $X_0, X_1, \cdots, X_{n+1}$ be indeterminates in L. Let u_{ij} $(i, j=0, 1, \cdots, n+1)$ be $(n+2)^2$ independent variables over k and put

$$Y_i=\sum_{j=0}^{n+1} u_{ij}X_j \quad (i=0, 1, \cdots, n+1)$$

Let $f(X_0, X_1, \cdots, X_{n+1})=0$ be a defining equation for V with coefficients in k and we shall put

$$F(Y_0, Y_1, \cdots, Y_{n+1})=f(\sum_{j=0}^{n+1} v_{0j}Y_j, \sum_{j=0}^{n+1} v_{1j}Y_j, \cdots, \sum_{j=0}^{n+1} v_{n+1j}Y_j)$$

where (v_{ij}) is the inverse matrix of the matrix (u_{ij}). Let $P=(\xi_0, \xi_1, \cdots, \xi_{n+1})$ be a generic point of V over the field $K=k(u_{ij})$ in the coordinate system (X). Then in the new coordinate system (Y), P can be represented by a homogeneous coordinates $(\eta_0, \eta_1, \cdots, \eta_{n+1})$, where $\eta_i=\sum_{j=0}^{n+1} u_{ij}\xi_j$ $(i=0, 1, \cdots, n+1)$, and $F(Y)=0$ is the defining equation for V, and P is also a generic point of V over K. We shall put $y_i=\eta_i/\eta_0$. As is easily seen the function field $K(y)=K(P)$ of V over K is separably generated over $K(y_{i_1}, \cdots, y_{i_n})$ for any choice $i_1, \cdots, i_n$ taken from $1, \cdots, n+1$. Hence n differentials $dy_{i_1}, \cdots, dy_{i_n}$ form a base of the Grassmann algebra of the differential forms on V. Let H_i be the hyperplane defined by the equation $Y_i=\sum_{j=0}^{n+1} u_{ij}X_j=0$ and $C_i=V\cdot H_i$. Then C_i is an irreducible variety without singular subvariety of codimension 1 for any index $i\,(0\leq i\leq n+1)$. For any subvariety A^{n-1} of V different from C_0 we can find n indices $i_1, \cdots, i_n$ such that $y_{i_1}, \cdots, y_{i_n}$ $(1\leq i_\alpha\leq n+1)$ is a set of uniformizing parameters along A. Let F_i $(i=0, 1, \cdots, n+1)$ be the partial derivatives of the form $F(Y)$ with respect to Y_i, then F_i is either identically zero or a form of degree $m-1$, where $m=\deg V$. We shall also denote by $F_i(y)$ the polynomial in $K[y]$ defined by $F_i(y)=F_i(1, y_1, \cdots, y_{n+1})$. Let $G(Y)$ be a form in (Y), then we shall denote by (G) the hypersurface defined by the equation $G(Y)=0$. Now we shall restate a part of the Theorem 1 of [3] in the

PROPOSITION. *Under the same notations and assumptions as above, let ω be a differential form on V which is relatively of the first kind on V and of degree $q(<n)$. Moreover we shall assume that ω is defined over k. Then ω must be written in the form*

$$\omega = \sum_{i_1<\cdots<i_q} \frac{A_{i_1\cdots i_q}(y)}{F_{n+1}(y)} dy_{i_1} \wedge \cdots \wedge dy_{i_q}$$

where the sum is extended over all indices $i_1 < \cdots < i_q$ *taken from* $1, \cdots, n$, *and* $A_{i_1\cdots i_q}(y)$*'s are polynomials in* $K[y]$ *satisfying the following conditions*:

(1) *degrees of* $A_{i_1\cdots i_q}(y)$*'s as a polynomials in* y*'s are at most equal to* $m-q-1$, *where* m *is the projective degree of* V.

(2) $\sum_{\alpha \neq i_1, \cdots, i_{q-1}} y_\alpha A_{\alpha i_1 \cdots i_{q-1}} = A^*_{i_1\cdots i_{q-1}}(y)$

is a polynomial of degree $\leqq m-q-1$.

(3) *There exist the polynomials* $A^{**}_{i_0 i_1 \cdots i_q}(y)$ *of degree* $\leqq m-q-1$ *such that*

$$\sum_{\alpha=0}^{q} (-1)^\alpha F_{i_\alpha}(y) A_{i_0 \cdots \hat{i}_\alpha \cdots i_q}(y) = A^{**}_{i_0 i_1 \cdots i_q}(y) F_{n+1}(y) .$$

Conversely if ω *satisfies these three conditions,* ω *is relatively of the first kind on* V.

We shall remark here the following. In the proposition we take the hyperplane H_0 as a plane at infinity, but the similar formulation are valid when we take any one of the hyperplanes H_i $(i=1, \cdots, n+1)$ as a plane at infinity.

LEMMA 1. *Let* V^n *be a variety in a projective space* L *and let* $G_1, \cdots, G_t$ *be hypersurfaces in* L. *Assume that the components of the set-theoretic intersection* $\bigcap_{i=1}^{t} G_i \cap V$ *has the dimension* $\leqq n-r$, *and there exists at least one component in that intersection which is exactly of dimension* $n-r$. *Then there exist indices* $i_1, \cdots, i_r$ *among* $1, \cdots, t$ *such that any component of* $V \cap G_{i_1} \cap \cdots \cap G_{i_r}$ *is of dimension exactly* $n-r$.

PROOF. We shall use the induction on the number t of the hypersurfaces. When $t=1$ the assertion is trivial. We shall denote by $\dim(G_1 \cap \cdots \cap G_t \cap V)$ the highest dimension of the components in that intersection. Then by our assumption $\dim(G_1 \cap \cdots \cap G_t \cap V) = n-r$. We shall pay attension to the intersection $G_1 \cap \cdots \cap G_{t-1} \cap V$. Then either $\dim(G_1 \cap \cdots \cap G_{t-1} \cap V) = n-r+1$ or $\dim(G_1 \cap \cdots \cap G_{t-1} \cap V) = n-r$. Now assume that the second case take place, then the asertion is valid by the induction assumption. When the first case occurs, then G_t does not contain any component of $G_1 \cap \cdots \cap G_{t-1} \cap V$ which is of dimension $n-r+1$. By the induction assumption we can find $r-1$ indices $i_1, \cdots, i_{r-1}$ from $1, \cdots, t-1$

such that any component of $G_{i_1} \cap \cdots G_{i_{r-1}} \cap V$ is of dimension exactly $n-r+1$. Moreover these components are not contained in G_t, hence $G_{i_1}, \cdots, G_{i_{r-1}}, G_t$ satisfy the condition of our Lemma.

LEMMA 2. *Let V, $f(X)$ and $F(Y)$ be as before and assume that V has no singular subvariety of codimension less than s. Then for any integer $a \leqq s$, we have* $\dim((F_{i_1}) \cap \cdots \cap (F_{i_a}) \cap V) = n-a$, *where $i_1, \cdots, i_a$ are arbitrary a indices taken from* $0, 1, \cdots, n+1$.

PROOF. Since any singular subvariety has the dimension $\leqq n-s$, we have $\dim((f_0) \cap \cdots \cap (f_{n+1}) \cap V) = n-r \leqq n-s$. Then we can find r indices $i_1, \cdots, i_r$ such that $\dim((f_{i_1}) \cap \cdots \cap (f_{i_r}) \cap V) = n-r$. We shall assume for the sake of simplicity that $i_\alpha = \alpha$ $(\alpha = 1, \cdots, r)$. Let $j_1 < \cdots < j_a$ be the indices taken form $1, \cdots, r$, then $\dim((f_{j_1}) \cap \cdots \cap (f_{j_a}) \cap V) = n-a$. Let $i_1, \cdots, i_a$ be arbitrary indices taken from $0, 1, \cdots, n+1$. Then since $F_\alpha = \sum_{\beta=0}^{n+1} f_\beta v_{\beta\alpha}$ and $(v_{\beta\alpha})$ are independent variables over k which contains all the coefficients of f_β's, the hypersurfaces $(F_{i_1}), \cdots, (F_{i_a})$ can be specialized simultaneously to the hypersurfaces $(f_{j_1}), \cdots, (f_{j_a})$ over k. Hence we must have $\dim((F_{i_1}) \cap \cdots \cap (F_{i_a}) \cap V) \leqq n-a$. Combining the inverse inequality which holds true in general we get the Lemma.

LEMMA 3. *Let V^n be a hypersurface of degree m in a projective space L^{n+1} and $F(Y)=0$ be the defining equation for V. Let F_i be the partial derivative of $F(Y)$ with respect to the indeterminate Y_i and assume that* $\dim((F_{i_1}) \cap \cdots \cap (F_{i_s}) \cap V) = n-s$. *Then there cannot exist the relation of the form*

$$\sum_{\alpha=1}^{s} A_\alpha F_{i_\alpha} \equiv 0 \pmod{F(Y)}$$

with the forms A's of degrees $< m-1$, unless all the forms are identically zero.

PROOF. Without losing any generality we can assume that $i_\alpha = \alpha - 1$ $(\alpha = 1, \cdots, s)$. Let K be a field containing all the coefficients of $F(Y)$. Let w_{ij} $(i = 1, \cdots, s-1;\ j = 0, 1, \cdots, s-1)$ be $s(s-1)$ independent variables over K. Let

$$X_0 = Y_0,\ X_{s-1+j} = Y_{s-1+j} \quad (j = 1, \cdots, n-s+2)$$
$$X_i = \sum_{j=0}^{s-1} w_{ij} Y_j \quad (i = 1, \cdots, s-1)$$

For the sake of simplicity we shall put $w_{00}=1$ and $w_{0i}=0$ for $i>0$. Let $(\overline{w}_{ij})$ be the inverse matrix of the matrix (w_{ij}), then we have

$$Y_i=\sum_{j=0}^{s-1}\overline{w}_{ij}X_j \quad (0\leqq i\leqq s-1)$$

Substituting these relations in the equation $F(Y)=0$, we get the equation $G(X)=0$ for V in the new coordinate system (X). We shall denote as before by F_i and G_α, the partial derivatives of $F(Y)$ with respect to Y_i and the partial derivatives of $G(X)$ with respect to X_α respectively. Then we have the relations

$$F_i=\sum_{\alpha=0}^{s-1}G_\alpha w_{\alpha i} \quad (0\leqq i\leqq s-1)$$

$$G_\alpha=\sum_{i=0}^{s-1}F_i\overline{w}_{i\alpha} \quad (0\leqq \alpha\leqq s-1)$$

Hence

$$\sum_{\alpha=0}^{s-1}A_\alpha F_\alpha=\sum_{\alpha,\beta}A_\alpha G_\beta w_{\beta\alpha}=\sum_{\beta=0}^{s-1}(\sum_{\alpha=0}^{s-1}w_{\beta\alpha}A_\alpha)G_\beta=\sum_{\beta=0}^{s-1}B_\beta G_\beta$$

where $B_\beta=\sum_{\alpha=0}^{s-1}w_{\beta\alpha}A_\alpha$, in particular $B_0=A_0$.

Let C be a component of $V\cap(G_1)\cap\cdots\cap(G_{s-2})$. Then $\dim C=n-s+2$. In fact if $\dim C>n-s+2$, we have $\dim(V\cap(G_0)\cap\cdots\cap(G_{s-1}))>n-s$. But this is a contradiction to the assumption, since $(V\cap(G_0)\cap\cdots\cap(G_{s-1}))=(V\cap(F_0)\cap\cdots\cap(F_{s-1}))$. We shall show that if the form A_0 is not identically zero, C cannot be contained in the hypersurface (A_0). Let x be a generic point of C over the field $\overline{K}_1$, where $K_1=K(w_{ij},\ 1\leqq i\leqq s-1,\ 1\leqq j\leqq s-2)$. We shall show that $\dim_K(x)=n$ which will prove our assertion since the hypersurface (A_0) is defined over K and does not contain V. Since w_{ij} $(1\leqq i\leqq s-1,\ 1\leqq j\leqq s-1)$ are independent variables over K, $\overline{w}_{ij}$ $(1\leqq i\leqq s-1,\ 1\leqq j\leqq s-1)$ are also independent variables over K, and hence $\dim_K K_1=(s-1)(s-2)$. From this we get the equality $\dim_K K_1(x)=\dim_K K_1+\dim_{K_1}(x)=(s-1)(s-2)+(n-s+2)=n+(s-2)^2$. Since $\dim((F_0)\cap\cdots\cap(F_{s-1})\cap V)=n-s$, C is not contained in at least one of the hypersurfaces F_i $(i=1,\cdots,s-1)$. Now assume that C is not contained in the hypersurface (F_1). Then we can solve the linear equations $0=G_\beta(x)=\overline{w}_{1\beta}F_1(x)+\cdots+\overline{w}_{s-1\beta}F_{s-1}(x)$ with respect to $\overline{w}_{1\beta}$ $(\beta=1,\cdots,s-2)$ and we see that $\dim_{K(x)}K_1\leqq(s-2)^2$. Combining the inequalities $\dim_K K_1(x)=n+(s-2)^2$ and $\dim_{K(x)}K_1\leqq(s-2)^2$ we get the required result $\dim_K(x)\geqq n$.

Let l be a linear form such that the hyperplane (l) does not contain C. Then $(\sum_{\alpha=0}^{s-1} B_\alpha G_\alpha)/l^a$ is a function of the ambiant space for a suitable integer a. Restricting this function on C we get $(B_0G_0+B_{s-1}G_{s-1})/l^a=0$ on C. Now assume that $A_0=B_0$ is not identically zero. Then by the preceding considerations the hypersurface (B_0) does not contain C. On the other hand (G_0) does not contain C. Then we get the relation

$$(B_0)\cdot C+(G_0)\cdot C=(B_{s-1})\cdot C+(G_{s-1})\cdot C$$

Since $\dim((G_0)\cap\cdots\cap(G_{s-1})\cap V)=n-s$, $(G_0)\cdot C$ and $(G_{s-1})\cdot C$ have no common component. Then we must have $(B_0)\cdot C>(G_{s-1})\cdot C$. But this is impossible because the degree of B_0 is less that that of G_{s-1}. Thus we have shown that under our assumption A_0 must be identically zero. Changing the rolles of the forms A_i's, we see that all the forms A_i $(i=1,\cdots,s-1)$ must be identically zero. This proves the Lemma.

THE PROOF OF THE THEOREM. Let V, $f(X)$ and $F(Y)$ have the same meaning as before and let ω be a differential form of degree $q(\leqq s-2)$ defined over k which is relatively of the first kind on V. Let us represent ω in the form (*) of the Proposition. Then the coefficients $A_{i_1\cdots i_q}$ of ω must satisfy the equation of the type (3) in the Proposition, i.e.

$$\sum_{\alpha=1}^{q+1}(-1)^\alpha F_{i_\alpha}(y)A_{i_1\cdots\hat{i}_\alpha\cdots i_{q+1}}(y)=F_{n+1}(y)A^{**}_{i_1\cdots i_{q+1}}(y)\,.$$

The above equality implies the existence of the forms of degree $<m-1$ such that $\sum_{\alpha=1}^{q+1}F_{i_\alpha}A_{i_\alpha}+F_{n+1}A^*=0$ on V. Since $q+2\leqq s$, the Lemma 2 implies that $\dim((F_1)\cap\cdots\cap(F_{q+1})\cap(F_{n+1})\cap V)=n-q+2$. Thus the proof of the theorem is reduced to the Lemma 3. Moreover if there exists a differential form which is relatively of the first kind on V, then we can find such one among the differential forms defined over a given field of definition for V ([2]). Hence the proof is complete.

COROLLARY 1. *Let V^n be a non-singular hypersurface in a projective $(n+1)$-space, then there cannot exist the differential form of the first kind on V which is of degree$<n$.*

As is known the irregularity of a normal variety V (=the dimension of the Picard variety attached to V) is not greater than the number of the linearly independent differential forms of the first kind of degree 1 ([1]). Hence we have the

COROLLARY 2. *Let V^n be a hypersurface in L^{n+1} such that any singular subvariety has the codimension >2, then V is a regular variety.*

At the end of the paper we shall give an example which shows that a hypersurface V^n in a projective space L^{n+1} whose singular subvarieties have the codimensions s at least, may have a differential form of the first kind of degree$>s-2$. Let U^{s-1} be a non-singular variety contained in a linear subspace L^s and assume that U has a differential form of the first kind of degree $s-1$. Let T^{n-s} be a linear subspace in L^{n+1} such that $T\frown L^s=\phi$. Let $\tilde{U}$ be the variety which is composed of the straight lines connecting the points on U and T. Then $\tilde{U}$ is an irreducible variety of dimension n, i.e. a hypersurface in L^{n+1}, and whose singular loci are all contained in T, i.e. T is the largest singular subvariety of $\tilde{U}$ which is of codimension s on $\tilde{U}$. Moreover $\tilde{U}$ is birationally equivalent to the product of U and a projective space of dimension $n-s+1$. Since U has a differential form of the first kind of degree $s-1$, $\tilde{U}$ has also a differential form of the first kind of degree $s-1$. This is a required example.

BIBLIOGRAPHY

[1] Igusa, J. A fundamental inequality in the theory of Picard varieties, Proc. N. A. Sc., U. S. A. 41 (1955), 317-320.

[2] Koizumi, S. On the differential forms of the first kind on algebraic varieties, Jour. M. S. of Japan, 1 (1949), 273-280.

[3] Nakai, Y. Some results in the theory of the differential forms of the first kind on algebraic varieties, Proc. Inter. Symp. on algebraic number theory, Tokyo-Nikko (1955), 155-178.

[4] Picard, E. and Simard, G. Théorie des fonctions algébriques de deux variables indepéndantes, Tome 1, Paris (1897).