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p-primary components of homotopy groups

III. Stable groups of the sphere

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Denote by $\pi_{N+k}(S^N; p)$ the p-primary component of the $(N+k)$-th homotopy group $\pi_{N+k}(S^N)$ of N-sphere S^N. In this paper, N denotes always a sufficiently large integer, in particular $N>k+1$ for the group $\pi_{N+k}(S^N)$ which does not depend on $N(>k+1)$ and is the k-th stable homotopy group $\pi_k(\mathfrak{S})$ of the sphere.

For $k<2p^2(p-1)-3$, the stable groups $\pi_{N+k}(S^N; p)$ are determined and stated as follows (p: *an odd prime*):

(A) $\pi_{N+2rp(p-1)-1}(S^N; p)=Z_{p^2}$ *for* $1\leqq r\leqq p-2$ *and* $=Z_{p^2}+Z_p$ *for* $r=p-1$;

(B) $\pi_{N+k}(S^N; p)=Z_p$ *for the following values of* k:

$k=2t(p-1)-1$ *where* $1\leqq t<p^2$ *and* $t\not\equiv 0 \pmod p$,

$=2(rp+s)(p-1)-2(r-s)$ *where* $0\leqq s<r\leqq p-1$,

$=2p^2(p-1)-2p$,

$=2(rp+s+1)(p-1)-2(r-s)-1$ *where* $0\leqq s<r\leqq p-1$ *and* $r-s\neq p-1$;

(C) $\pi_{N+k}(S^N; p)=0$ *for the other values of* $k<2p^2(p-1)-3$.

For example, $\pi_{N+2p(p-1)-2}(S^N; p)=Z_p$ and $\pi_{N+2p(p-1)-1}(S^N; p)=Z_{p^2}$.

Methods emploied here are the same as in [4] by determining the $\mathscr{S}^*$-module structure of stable cohomology groups of Postnikov complexes K_k over the sphere. Several difficulties may occur, the first one is related closely with the values of the above example, and it is removed by the aid of the results in the preceeding paper [6]. The second difficulty occurs in the dimensions about

$k=2p^2(p-1)-2$. We may show many possibilities, in these dimensions, which depend on mod p Hopf invariant $H_p: \pi_{N+k}(S^N)\to Z_p$, $k=2p^2(p-1)-1$, and some others, however, we do not know how to determine the groups $\pi_{N+k}(S^N)$ for $k=2p^2(p-1)-3$, $2p^2(p-1)-2, \cdots$.

The results and the notations in the preceeding sections [5] and [6] are refered such as Proposition 1.5, Theorem 2.9, etc.

§ Preliminaries and lemmas.

Let S^N be an N-sphere. According to §4 of [4], take a sequence of CW-complexes

$$K_1 \supset K_2 \supset \cdots \supset K_{k-1} \supset K_k \supset \cdots \supset S^N$$

such that $\pi_i(K_k)=0$ for $i\geqq N+k$ and the injection homomorphisms $i_*: \pi_i(S^N)\to\pi_i(K_k)$ are isomorphisms for $i<N+k$. The sequence may be regarded as a realization of Postnikov system over S^N.

Let $K_k{}^*$ be a complex obtained from K_k by shrinking K_{k+1} to a point. Obviously, (co)homology groups of the pair (K_k, K_{k+1}) are isomorphic to those of $K_k{}^*$ under the homomorphisms induced by the shrinking map. As is easily seen, $\pi_i(K_k, K_{k+1})$ vanishes except for $\pi_{N+k+1}(K_k, K_{k+1})\approx\pi_{N+k}(S_N)$. Since K_{k+1} is $(N-1)$-connected and since (K_k, K_{k+1}) is $(N+k)$-connected, the shrinking map induces isomorphisms $\pi_i(K_k, K_{k+1})\approx\pi_i(K_k{}^*)$ for $i\leqq 2N+k-1$ and onto-homomorphism for $i=2N+k$ [1]. Thus $\pi_i(K_k{}^*)=0$ for $i\neq N+k+1$ and $i\leqq 2N+k$.

Imbedding $K_k{}^*$ in an Eilenberg-MacLane complex of the type $(\pi_{N+k}(S^N), N+k+1)$, we see that $H^{N+i}(K_k, K_{k+1}, Z_p)$ and $H^{N+i}(K_k^*, Z_p)$ are naturally isomorphic to $H^{N+i}(\pi_{N+k}(S^N), N+k+1, Z_p)$ for $i<N+k$, and isomorphic to the stable group $A^{i-k-1}(\pi_{N+k}(S^N), Z_p)$ for sufficiently large $N(>i-k>i-2(k+1))$. It follows from the cohomology sequence of (K_k, K_{k+1}) that the following sequence is exact:

$$\cdots \xrightarrow{j^*} H^{N+i}(K_k, Z_p) \xrightarrow{i^*} H^{N+i}(K_{k+1}, Z_p) \xrightarrow{\delta^*} A^{i-k}(\pi_{N+k}(S^N), Z_p)$$
$$\xrightarrow{j^*} H^{N+i+1}(K_k, Z_p) \xrightarrow{i^*} \cdots,$$

where we identify $H^{N+i+1}(K_k, K_{k+1}, Z_p)$ with $A^{i-k}(\pi_{N+k}(S^N), Z_p)$ by the above isomorphism.

Since the Steenrod operations $\mathscr{P}^i$ and the Bockstein operator Δ commute with the homomorphisms of the above sequence, it follows that the homomorphisms are $\mathscr{S}^*$-homomorphisms.

By suspension methods as in [4], we take stable groups $A^i(K_k, Z_p)$, $\pi_k(\mathfrak{S})$ and $\pi_k(\mathfrak{S}; p)$ given by

$$A^i(K_k, Z_p) = H^{N+i}(K_k, Z_p), \qquad 0 \leqq i < N+k,$$
$$\pi_k(\mathfrak{S}) = \pi_{N+k}(S^N), \quad \pi_k(\mathfrak{S}; p) = \pi_{N+k}(S^N; p), \qquad N > k+1.$$

Then it follows the following exact sequence of *$\mathscr{S}^*$-homomorphisms.*

(3.1). $$\cdots \xrightarrow{j^*} A^i(K_k, Z_p) \xrightarrow{i^*} A^i(K_{k+1}, Z_p) \xrightarrow{\delta^*} A^{i-k}(\pi_k(\mathfrak{S}; p), Z_p) \xrightarrow{j^*} A^{i+1}(K_k, Z_p) \xrightarrow{i^*} \cdots.$$

Let δ/p^r: δ/p^{r-1}-*kernel* $(\subset H^i(X, Z_p)) \to H^{i+1}(X, Z_p)/(\delta/p^{r-1}$-*image*) be the Bockstein homomorphism. Put

$$\Delta_r = (-1)^i \delta/p^r \quad (\Delta_1 = \Delta)$$

then Δ_r commutes with the suspension homomorphisms. Thus Δ_r may be defined in the stable group $A^*(K_k, Z_p)$ and $A^*(\pi, Z_p)$. Δ_r commutes with the homomorphisms of (3.1).

From the exactness of the homotopy sequence of the pair (K_k, S^N), it follows that $\pi_i(K_k, S^N) = 0$ for $i \leqq N+k$ and $\partial: \pi_{i+1}(K_k, S^N) \approx \pi_i(S^N)$ for $i \geqq N+k$. Since (K_k, S^N) is $(N+k)$-connected, there are Hurewicz isomorphisms $\pi_i(K_k, S^N) \approx H_i(K_k, S^N)$ for $i \leqq N+k+1$. Obviously $H_i(K_k, S^N) \approx H_i(K_k)$ for $i \neq 0, N$. Thus

(3.2). $$H_i(K_k) \approx \begin{cases} \pi_{N+k}(S^k), & i = N+k+1, \\ 0, & N < i < N+k+1, \\ Z, & i = N. \end{cases}$$

The first isomorphism is also given by the composition $H_{N+k+1}(K_k) \approx H_{N+k+1}(K_k, K_{k+1}) \approx H_{N+k+1}(\pi_{N+k}(S^N), N+k+1) \approx \pi_{N+k}(S^N)$. Then by the duality, we have

(3.3). *$A^i(K_k, Z_p) = 0$ for $0 < i < k+1$ and $j^*: A^0(\pi_k(\mathfrak{S}; p), Z_p) \to A^{k+1}(K_k, Z_p)$ is an isomorphism. $j^*: A^1(\pi_k(\mathfrak{S}; p), Z_p) \to A^{k+2}(K_k, Z_p)$ is an isomorphism into.*

The last assertion follows from the fact that $A^1(\pi, Z_p)$ is spanned by Δ_r-*images* and from the following lemma established easily from (3.2).

Lemma 3.1. *The number of the direct factors of* $\pi_k(\mathfrak{S};p)$ *isomorphic to* Z_{p^r} *is the rank of (the image of)* $\Delta_r:\Delta_{r-1}$*-kernel* $(\subset A^{k+1}(K_k, Z_p))\rightarrow A^{k+2}(K_k, Z_p)/(\Delta_{r-1}$*-image).*

In particular, if $A^{k+1}(K_k, Z_p)=0$ then $\pi_k(\mathfrak{S};p)=0$, and it follows that the homomorphisms i^* of (3.1) are isomorphisms. Thus,

Lemma 3.2. *If* $A^i(K_k, Z_p)=0$ *for* $0<i\leqq k+r(r>0)$, *then* $\pi_j(\mathfrak{S};p)=0$ *for* $k\leqq j<k+r$ *and* $i^*:A^*(K_k, Z_p)\rightarrow A^*(K_j, Z_p)$ *are isomorphisms for* $k<j\leqq k+r$.

Lemma 3.3. *Assume that* $A^{k+1}(K_k, Z_p)=\{a\}$ *and* $\Delta a\neq 0$. *Let* $\{\alpha_s a=0,\ s=1, 2, \cdots\}$ *be a system of relations in the submodule* $\mathcal{S}^*a$ *of* $A^*(K_k, Z_p)$ *generated by* a. *Let* $\{\sum_s\beta_{st}\alpha_s=0,\ t=1, 2, \cdots\}$ *be a system of relations in the submodule* $\sum_s\mathcal{S}^*\alpha_s$ *of* $\mathcal{S}^*$ *generated by* $\{\alpha_s\}$. *Then there exist elements* b_s *of* $A^*(K_{k+1}, Z_p)$ *and* w_t *of* $A^*(K_k, Z_p)$ *such that*

$$\delta^*b_s=\alpha_s(j^{*-1}a) \quad \text{and} \quad \sum_s\beta_{st}b_s=i^*w_t\,.$$

Let $\{a_m\}$ *and* $\{r_n=0\}$ *be systems of generators and relations of* $A^*(K_k, Z_p)$, *then* $A^*(K_{k+1}, Z_p)$ *has a system* $\{i^*a_m, b_s\}$ *of generators and a system* $\{i^*r_n=0,\ i^*a=0,\ \sum_s\beta_{st}b_s-i^*w_t=0\}$ *of relations.*

Proof. By the exactness of the sequence (3.1), $j^*(\alpha_s(j^{*-1}a))=\alpha_s a=0$ implies the existence of b_s such that $\delta^*b_s=\alpha_s(j^{*-1}a)$. Similarly there exist w_t such that $\sum_s\beta_{st}b_s=i^*w_t$, where it is to be remarked that $\pi_k(\mathfrak{S};p)\approx Z_p$ and $A^*(\pi_k(\mathfrak{S};p), Z_p)=\mathcal{S}^*(j^{*-1}a)\approx\mathcal{S}^*$. The second part of the lemma is also proved by the exactness of the sequence (3.1) of $\mathcal{S}^*$-homomorphisms. q.e.d..

Similarly the following lemma is established.

Lemma 3.4. *The previous lemma* 3.3 *is also true under the following replacement of the assumption, relations and notations:*

$A^{k+1}(K_k, Z_p)=\{a\}$ *and* $\Delta a\neq 0$ *by* $A^{k+1}(K_k, Z_p)=\{a'\}$, $A^{k+2}(K_k, Z_p)=\{a''\}$ *and* $\Delta a'=\Delta a''=0$ *(resp. by* $A^{k+1}(K_k, Z_p)=\{a, a'\}$, $A^{k+2}(K_k, Z_p)=\{\Delta a, a''\}$ *and* $\Delta a'=\Delta a''=0$),

$\alpha_s a=0$ *by* $\alpha_s' a_s' + \alpha_s'' a_s'' = 0$ (*resp.* $\alpha_s a_s + \alpha_s' a_s' + \alpha_s'' a_s'' = 0$),

$\mathscr{S}^* a$ *by* $\mathscr{S}^* a' + \mathscr{S}^* a''$ (*resp.* $\mathscr{S}^* a + \mathscr{S}^* a' + \mathscr{S}^* a''$),

$\delta^* b_s = \alpha_s(j^{*-1}a)$ *by* $\delta^* b_s = \alpha_s'(j^{*-1}a') + \alpha_s''(j^{*-1}a'')$

(*resp.* $\delta^* b_s = \alpha_s(j^{*-1}a) + \alpha_s'(j^{*-1}a') + \alpha_s''(j^{*-1}a'')$),

$\sum_s \beta_{st}\alpha_s = 0$ *in* $\mathscr{S}^*$ *by* $\sum_s \beta_{st}(\alpha_s', \alpha_s'') = 0$ *in* $\mathscr{S}^*/\mathscr{S}^*\Delta \oplus \mathscr{S}^*/\mathscr{S}^*\Delta$

(*resp.* $\sum_s \beta_{st}(\alpha_s, \alpha_s', \alpha_s'') = 0$ *in* $\mathscr{S}^* \oplus \mathscr{S}^*/\mathscr{S}^*\Delta \oplus \mathscr{S}^*/\mathscr{S}^*\Delta$),

and by adding the relation $i^*a' = i^*a'' = 0$ *in the last assertion.*

A remark in the proof is that $A^*(\pi_k(\mathfrak{S}\,;\,p), Z_p) \approx \mathscr{S}^*/\mathscr{S}^*\Delta \oplus \mathscr{S}^*/\mathscr{S}^*\Delta$ (resp. $\approx \mathscr{S}^* \oplus \mathscr{S}^*/\mathscr{S}^*\Delta \oplus \mathscr{S}^*/\mathscr{S}^*\Delta$), which follows from (3. 3).

The following lemma is established from § 3 of [7] (*cf.* Lemma 3. 3 of [4]), by taking stable groups.

Lemma 3. 5. *Let* i^*, j^* *and* δ^* *be the homomorphisms of* (3. 1).

i). *For* $a \in A^{i-k}(\pi_k(\mathfrak{S}\,;\,p), Z_p)$ *and* $b \in A^i(K_k, Z_p)$, *assume that* $\Delta_r b = \{j^*a\}$. *Then there is an element* $\tilde{a} \in A^{i+1}(K_{k+1}, Z_p)$ *such that* $\delta^*\tilde{a} = \Delta a$ *and* $\Delta_{r+1} j^* b = \{\tilde{a}\}$ ($r \geqq 1$).

ii). *For* $a \in A^{i-k}(\pi_k(\mathfrak{S}\,;\,p), Z_p)$, *assume that* $j^*a \in \Delta_{r-1}$*-kernel. Then there are elements* $\tilde{a} \in A^{i+1}(K_{k+1}, Z_p)$ *and* $c \in A^{i+2}(K_k, Z_p)$ *such that* $\delta^*\tilde{a} = \Delta a$, $\Delta_r j^* b = \{c\}$ *and* $\Delta_{r-1}\tilde{a} = \{i^*c\}$ ($r \geqq 2$).

iii). *For* $a \in A^{i-k}(\pi_k(\mathfrak{S}\,;\,p), Z_p)$ *and* $d \in A^{i-k+1}(\pi_k(\mathfrak{S}\,;\,p), Z_p)$, *assume that* $\Delta_r(j^*a) = \{j^*d\}$. *Then there are elements* $\tilde{a} \in A^{i+1}(K_{k+1}, Z_p)$ *and* $\tilde{d} \in A^{i+2}(K_{k+1}, Z_p)$ *such that* $\delta^*\tilde{a} = \Delta a - p^{r-1}d$, $\delta^*\tilde{d} = \Delta d$ *and* $\Delta_r\tilde{a} = \{\tilde{d}\}$ ($r \geqq 1$).

§ Cohomology of $\mathbf{K}_{2p-2}$ and $\mathbf{K}_{4p-4}$.

The complex K_1 is an Eilenberg-MacLane space of the type (Z, N). Thus

$$A^*(K_1, Z_p) \approx A^*(Z, Z_p) \approx \mathscr{S}^*/\mathscr{S}^*\Delta\,.$$

Denote by

$$a_0 \in A^0(K_1, Z_p)$$

a fundamental element. Then the $\mathscr{S}^*$-module $A^*(K_1, Z_p)$ is generated by a_0 and has the relations generated by $\Delta a_0 = 0$. $A^*(K_1, Z_p) = \{a_0, \mathscr{P}^1 a_0, \Delta\mathscr{P}^1 a_0, \mathscr{P}^2 a_0, \cdots\}$, and thus $A^i(K_1, Z_p)$

vanishes for $1 \leqq i \leqq 2p-3$. It follows from Lemma 3.2 and Lemma 3.1 that

$$\begin{aligned} &\pi_i(\mathfrak{S}\,;\,p)=0 && for\ \ 1\leqq i<2p-3\,, \\ (3.4).\quad &i^*: A^*(K_1,\,Z_p)\approx A^*(K_i,\,Z_p) && for\ \ 1<i\leqq 2p-3\,, \\ &\pi_{2p-3}(\mathfrak{S}\,;\,p)=Z_p\,. \end{aligned}$$

For the convenience, we write the image $i^*\alpha$ of an element $\alpha \in A^*(K_k,\,Z_p)$ by the same symbol $\alpha \in A^*(K_{k+1},\,Z_p)$. Then $A^*(K_k,\,Z_p)$, for $1 \leqq k \leqq 2p-3$, is generated by a_0 and has a system $\{\Delta a_0=0\}$ of relations.

Now we apply Lemma 3.3 for the case $k=2p-3$. $A^{2p-2}(K_{2p-3},\,Z_p)=\{\mathscr{P}^1 a_0\}$ and $\Delta\mathscr{P}^1 a_0 \neq 0$. By the isomorphism $A^*(K_{2p-3},\,Z_p)\approx \mathscr{S}^*/\mathscr{S}^*\Delta$, $\mathscr{S}^*(\mathscr{P}^1 a_0)$ corresponds to the image of $(\mathscr{P}^1)^*: \mathscr{S}^* \to \mathscr{S}^*/\mathscr{S}^*\Delta$, the kernel of which is $\mathscr{S}^*R_1+\mathscr{S}^*\mathscr{P}^{p-1}$ by Proposition 1.6 [5]. Consider a relation $\alpha_1 R_1+\alpha_2\mathscr{P}^{p-1}=0$ in $\mathscr{S}^*R_1+\mathscr{S}^*\mathscr{P}^{p-1}$. By Proposition 1.6, $\alpha_2\mathscr{P}^{p-1}\equiv 0 \bmod \mathscr{S}^*R_1$ implies $\alpha_2=\beta_1\Delta+\beta_2\mathscr{P}^1$ for some $\beta_1,\,\beta_2\in\mathscr{S}^*$. Since $\mathscr{P}^1\mathscr{P}^{p-1}=0$ and $\Delta\mathscr{P}^{p-1}=-\mathscr{P}^{p-2}R_1$, it follows $(\alpha_1-\beta_1\mathscr{P}^{p-2})\,R_1=0$. Then, by Proposition 1.5, $\alpha_1-\beta_1\mathscr{P}^{p-1}=\beta_3 R_2+\beta_4\Delta\mathscr{P}^1\Delta$ for some $\beta_3,\,\beta_4\in\mathscr{S}^*$ such that $\beta_4=0$ if $p>3$. Consequently it is proved that the relations in $\mathscr{S}^*R_1+\mathscr{S}^*\mathscr{P}^{p-1}$ are generated by the following relations:

$$R_2R_1=0\,,\quad (\Delta\mathscr{P}^1\Delta R_1=0\ \ if\ \ p=3)\,,\quad \Delta\mathscr{P}^{p-1}+\mathscr{P}^{p-2}R_1=0$$
$$and\quad \mathscr{P}^1\mathscr{P}^{p-1}=0\,.$$

By Lemma 3.3 there exist elements

$$a_2\in A^{4p-4}(K_{2p-2},\,Z_p)\quad and \quad b_1\in A^{2p(p-1)-1}(K_{2p-2},\,Z_p)$$

such that

$$\delta^*a_2=R_1(j^{*-1}a_0)\quad and \quad \delta^*b_1=\mathscr{P}^{p-1}(j^{*-1}a_0)\,,$$

and there are relations $R_2a_2=i^*w_3$, $(\Delta\mathscr{P}^1\Delta a_2=i^*w_4$ if $p=3)$, $\Delta b_1+\mathscr{P}^{p-2}a_2=i^*w_1$ and $\mathscr{P}^1 b_1=i^*w_2$ for some $w_1,\,w_2,\,w_3,\,w_4 \in A^*(K_{2p-3},\,Z_p)$.

Theorem 3.6. *$A^*(K_{2p-2},\,Z_p)$ is an $\mathscr{S}^*$-module generated by a_0, a_2 and b_1 having a system $\{\Delta a_0=\mathscr{P}^1 a_0=R_2a_2=\mathscr{P}^1 b_1=0$, $\Delta b_1=\mathscr{P}^p a_0-\mathscr{P}^{p-2}a_2$ (adding $\Delta\mathscr{P}^1\Delta a_2=0$ if $p=3$)$\}$ of relations.*

Proof. By Lemma 3.3, it is sufficient to prove that

$$i^*w_2=i^*w_3=i^*w_4=0 \quad \text{and} \quad i^*w_1=\mathscr{P}^p a_0\,.$$

$w_2\in A^{2(p+1)(p-1)-1}(K_{2p-3},\ Z_p)=0$, and thus $i^*w_2=0$. $w_3\in A^{6(p-1)+1}(K_{2p-3},\ Z_p)=\{\Delta\mathscr{P}^3 a_0\}$. Since $\Delta\mathscr{P}^3 a_0=\mathscr{P}^2\Delta(\mathscr{P}^1 a_0)$, $i^*w_3\in i^*A^{6(p-1)+1}(K_{2p-3},\ Z_p)=0$ and thus $i^*w_3=0$. Also it follows from $A^{6(p-1)+2}(K_{2p-3},\ Z_p)=0$ $(p=3)$ that $i^*w_4=0$. Since $w_1\in A^{2p(p-1)}(K_{2p-3},\ Z_p)=\{\mathscr{P}^p a_0\}$ and since $\mathscr{P}^p a_0$ does not vanish in K_{2p-2}, we have to determine the coefficient x in the equation

$$i^*w_1=x\mathscr{P}^p a_0\,.$$

Consider theorem 2.9 of the case $r=0$, where W_1^N is a fibre space over an Eilenberg-MacLane space W_0^N of the type (Z, N) such that $\pi_i(W_1^N)$ vanishes for $i\neq 0$, $N+2p-3$ and $\pi_{N+2p-3}(W_1^N)=Z_p$, $\pi_N(W_1^N)=Z$. Let X be a mapping-cylinder of the fibering: $W_1^N\to W_0^N$. As in seen in § 4 of [4], we may take K_{2p-2} and K_{2p-3} such that $K_{2p-2}^{N+2p-2}=K_{2p-3}^{N+2p-3}=S^N$. Let $f: S^N\to W_1^N$ be a representative of a generator of $\pi_N(W_1^N)$. Since $\pi_i(W_1^N)$ vanishes for $i\geqq N+2p-2$, the mapping f is extendable over the whole of K_{2p-2} and the result is denoted by $f_1: K_{2p-2}\to W_1^N$. Also f_1: $K_{2p-2}\to W_1^N\subset X$ is extended over f_2: $K_{2p-3}\to X$ such that $f_1=f_2|K_{2p-2}$.

It is easy to see that f_1 and f_2 induces mod p isomorphisms of the homotopy groups and thus isomorphisms of the cohomology groups in the following diagram.

$$\begin{array}{ccccccc}\cdots\to H^i(X,Z_p) & \to & H^i(W_1^N,\ Z_p) & \to & H^{i+1}(X,\ W_1^N,\ Z_p) & \to & \cdots\\ \downarrow f_2^* & & \downarrow f_1^* & & \downarrow f_2^* & & \\ \cdots\to H^i(K_{2p-3},\ Z_p) & \to & H^i(K_{2p-2},\ Z_p) & \to & H^{i+1}(K_{2p-3},\ K_{2p-2},\ Z_p) & \to & \cdots.\end{array}$$

Choose the element u of Theorem 2.9 such that $f_1^*u=a_0$, then it is verified easily that the element b_1 of Theorem 2.9 is mapped by f_1^* to our element b_1. It follows from Theorem 2.9 that $\Delta b_1-\mathscr{P}^p a_0$ is in $\mathscr{S}^*a_2$. Therefore we have $\Delta b_1-\mathscr{P}^p a_0=-\mathscr{P}^{p-2}a_2$, $x=1$ and thus $i^*w_1=\mathscr{P}^p a_0$. q.e.d.

Added in proof. *In Theorem* 2.10 [6], *read* $\Delta b_1=\mathscr{P}^p u-\mathscr{P}^{p-2}a$ *in place of* $\Delta b_1=\mathscr{P}^p u+\mathscr{P}^{p-2}a$. $H^*(W_1^N,\ Z_p)$ *is naturally isomorphic to* $A^*(K_{2p-2},\ Z_p)$.

By Lemma 3.1 and Lemma 3.2, it follows from $A^*(K_{2p-2},\ Z_p)=\{a_0,\ a_2,\ \Delta a_2,\cdots\}$ that

$$\pi_i(\mathfrak{S}\,;\,p)=0 \qquad for\quad 2p-2\leqq i<4p-5\,,$$
$$(3.5).\quad i^*: A^*(K_{2p-2},\,Z_p)\approx A^*(K_i,\,Z_p) \quad for\quad 2p-2<i\leqq 4p-5\,,$$
$$\pi_{4p-5}(\mathfrak{S}\,;\,p)=Z_p\,.$$

Then Theorem 3.6 is also true for K_{4p-5}.

Theorem 3.7. i). *Let $p>3$. There exists an element $a_3\in A^{6(p-1)}(K_{4p-4},\,Z_p)$ such that $\delta^*a_3=R_2(j^{*-1}a_2)$. The $\mathcal{S}^*$-module $A^*(K_{4p-4},\,Z_p)$ has a system $\{a_0,\,b_1,\,a_3\}$ of generators and a system $\{\Delta a_0=\mathcal{P}^1a_0=\mathcal{P}^1b_1=R_3a_3=0,\ \Delta b_1=\mathcal{P}^pa_0\}$ of relations.*

ii). *(Let $p=3$.) There exist elements $a_3\in A^{12}(K_8,\,Z_3)$ and $a_3'\in A^{13}(K_8,\,Z_3)$ such that $\delta^*a_3=R_2(j^{*-1}a_2)=\Delta\mathcal{P}^1(j^{*-1}a_2)$, $\delta^*a_3'=\Delta\mathcal{P}^1\Delta(j^{*-1}a_2)$ and $\Delta\mathcal{P}^1a_3=\mathcal{P}^1a_3'$. The $\mathcal{S}^*$-module $A^*(K_8,\,Z_3)$ has a system $\{a_0,\,b_1,\,a_3,\,a_3'\}$ of generators and a system $\{\Delta a_0=\mathcal{P}^1a_0=\mathcal{P}^1b_1=\Delta a_3=\Delta a_3'=0,\ \Delta b_1=\mathcal{P}^pa_0,\ \Delta\mathcal{P}^1a_3=\mathcal{P}^1a_3'\}$ of relations.*

Proof. i). Consider Lemma 3.3. $A^{4p-4}(K_{4p-5},\,Z_p)=\{a_2\}$ and $\Delta a_2\neq 0$. By Theorem 3.6, the relations in $\mathcal{S}^*a_2$ is generated by $R_2a_2=0$. By Proposition 1.5, the relations in $\mathcal{S}^*R_2$ is generated by $R_3R_2=0$. Then it follows from Lemma 3.3 that the theorem is true for $p>3$, by concerning the relation $R_3a_3=i^*w\in i^*A^{8(p-1)+1}(K_{4p-5},\,Z_p)=\{i^*\Delta\mathcal{P}^4a_2\}=0$.

ii). $A^8(K_7,\,Z_3)=\{a_2\}$ and $\Delta a_2\neq 0$. By Theorem 3.6, the relations in $\mathcal{S}^*a_2$ is generated by $\Delta\mathcal{P}^1a_2=0$ and $\Delta\mathcal{P}^1\Delta a_2=0$. By Proposition 1.5, the relations in $\Delta\mathcal{P}^1\mathcal{S}^*+\Delta\mathcal{P}^1\Delta\mathcal{S}^*$ is generated by $\Delta(\Delta\mathcal{P}^1)=0$, $\Delta(\Delta\mathcal{P}^1\Delta)=0$ and $\Delta\mathcal{P}^1(\Delta\mathcal{P}^1)-\mathcal{P}^1(\Delta\mathcal{P}^1\Delta)=0$. By Lemma 3.3, there exist elements $\tilde{a}_3\in A^{12}(K_8,\,Z_3)$ and $a_3'\in A^{13}(K_8,\,Z_3)$ such that $\delta^*\tilde{a}_3=\Delta\mathcal{P}^1(j^{*-1}a_2)$ and $\delta^*a_3'=\Delta\mathcal{P}^1\Delta(j^{*-1}a_2)$ and there are relations $\Delta\tilde{a}_3=i^*w_1$, $\Delta a_3'=i^*w_2$ and $\Delta\mathcal{P}^1\tilde{a}_3-\mathcal{P}^1a_3'=i^*w_3$. Since $w_3\in A^{17}(K_7,\,Z_3)=\{\Delta\mathcal{P}^4a_0,\,\Delta\mathcal{P}^2a_2\}$ and $i^*a_2=0$, $i^*w_3=x\Delta\mathcal{P}^4a_0$ for some coefficient x. Since $i^*A^{13}(K_7,\,Z_3)=i^*A^{14}(K_7,\,Z_3)=0$, $i^*w_1=i^*w_2=0$. Put $a_3=\tilde{a}_3-x\mathcal{P}^3a_0$, then it is verified easily that $\delta^*a_3=\Delta\mathcal{P}^1(j^{*-1}a_2)$, $\Delta a_3=0$ and $\Delta\mathcal{P}^1a_3=\mathcal{P}^1a_3'$. Now the theorem, for $p=3$, is established by Lemma 3.3. q.e.d.

§ Some adding relations from Steenrod algebra.

By Theorem 1.7, the kernel of the homomorphism $(\mathcal{P}^p)^*:\mathcal{S}^*\to\mathcal{S}^*/(\mathcal{S}^*\Delta+\mathcal{S}^*\mathcal{P}^1)$ is $\mathcal{S}^*\Delta+\mathcal{S}^*\mathcal{P}^2+\mathcal{S}^*c(2\mathcal{P}^{p+1}-\mathcal{P}^p\mathcal{P}^1)$

$+\mathcal{S}^*c(\mathcal{P}^{p(p-1)})$. By (1.7) and (1.8), $c(2\mathcal{P}^{p+1}-\mathcal{P}^p\mathcal{P}^1)=2\mathcal{P}^p\mathcal{P}^1-\mathcal{P}^{p+1}$.

By Lemma 1.3 and by (1.9), $\mathcal{P}(p^2)\notin\mathcal{S}^*\Delta+\mathcal{S}^*\mathcal{P}^1+\mathcal{S}^*\mathcal{P}^p$ and $\mathcal{P}(p^2-i,i)\in\mathcal{S}^*\Delta+\mathcal{S}^*\mathcal{P}^1$ for $1\leqq i\leqq p-1$. Thus $(\mathcal{P}^p)^*\mathcal{P}(p(p-1)-i,i)=0$ for $0\leqq i\leqq p-2$. By (1.3)$'$, $\mathcal{P}(p(p-1)-i,i)\in\mathcal{S}^*\mathcal{P}^2$ for $2\leqq i\leqq p-1$. By (1.3), $\mathcal{P}(p(p-1)-(p-1),p-1)=\sum_{i=0}^{p-2}(-1)^{i+1}i\mathcal{P}(p(p-1)-i,i)$ and thus $\mathcal{P}(p(p-1)-1,1)\in\mathcal{S}^*\mathcal{P}^2$. It follows from $\mathcal{P}(p(p-1))\notin\mathcal{P}^2\mathcal{S}^*$ that $c(\mathcal{P}(p(p-1)))\notin\mathcal{S}^*\mathcal{P}^2$. Therefore $c(\mathcal{P}^{p(p-1)})-x\mathcal{P}^{p(p-1)}\in\mathcal{S}^*\mathcal{P}^2$ for some $x\not\equiv 0 \pmod p$. This shows $\mathcal{S}^*\mathcal{P}^2+\mathcal{S}^*c(\mathcal{P}^{p(p-1)})=\mathcal{S}^*\mathcal{P}^2+\mathcal{S}^*\mathcal{P}^{p(p-1)}$. We have proved

(3.6). *The kernel of* $(\mathcal{P}^p)^*:\mathcal{S}^*\to\mathcal{S}^*(/\mathcal{S}^*\Delta+\mathcal{S}^*\mathcal{P}^1)$ *is* $\mathcal{S}^*\Delta+\mathcal{S}^*\mathcal{P}^2+\mathcal{S}^*(2\mathcal{P}^p\mathcal{P}^1-\mathcal{P}^{p+1})+\mathcal{S}^*\mathcal{P}^{p(p-1)}$.

Now consider the submodule $\mathcal{S}^*b_1$ of $A^*(K_{4p-4},Z_p)$ generated by b_1. Let $\alpha b_1=0$ be a relation, $\alpha\in\mathcal{S}^*$. Then, by Theorem 3.7, $\alpha=\beta\mathcal{P}^1+\gamma\Delta$ for some $\beta,\gamma\in\mathcal{S}^*$ such that $\gamma\mathcal{P}^p a_0=0$, i.e., $(\mathcal{P}^p)^*\gamma=0$. By (3.6) and by the relations $\Delta\Delta=0$ and $\mathcal{P}^2\Delta=(\mathcal{P}^1\Delta-\frac{1}{2}\Delta\mathcal{P}^1)\mathcal{P}^1$, it follows that $\alpha=\beta\mathcal{P}^1+\gamma\Delta\in\mathcal{S}^*\mathcal{P}^1+\mathcal{S}^*(2\mathcal{P}^p\mathcal{P}^1-\mathcal{P}^{p+1})\Delta+\mathcal{S}^*\mathcal{P}^{p(p+1)}\Delta$. Conversely $\mathcal{P}^1b_1=(2\mathcal{P}^p\mathcal{P}^1-\mathcal{P}^{p+1})\Delta b_1=\mathcal{P}^{p(p-1)}\Delta b_1=0$. Therefore the following lemma is established.

Lemma 3.8. $\mathcal{S}^*b_1$ *has a system* $\{\mathcal{P}^1b_1=W_1b_1=\mathcal{P}^{p(p-1)}\Delta b_1=0\}$ *of relations, where* $W_1=2\mathcal{P}^p\mathcal{P}^1\Delta-\mathcal{P}^{p+1}\Delta$.

Denote that

$$W_k=(k+1)\mathcal{P}^p\mathcal{P}^1\Delta-k\mathcal{P}^{p+1}\Delta+(k-1)\Delta\mathcal{P}^{p+1},$$
$$W(k)=c(W_k)=k\Delta\mathcal{P}^p\mathcal{P}^1-(k+1)\Delta\mathcal{P}^{p+1}-(k-1)\mathcal{P}^p\mathcal{P}^1\Delta.$$

By use of (1.3), the following relations mod $\mathcal{P}^1\mathcal{S}^*$ are verified.

$W(k)\mathcal{P}^1\equiv 0$, *in fact*, $W(k)\mathcal{P}^1=\mathcal{P}^1W_{p-k}$,

$$W(k)\mathcal{P}^{pt}\mathcal{P}^s\equiv(k+s)\Delta\mathcal{P}^{p(t+1)}\mathcal{P}^{s+1}-(k-1)\mathcal{P}^{p(t+1)}\mathcal{P}^{s+1}\Delta-(k+t+1)\Delta\mathcal{P}^{p(t+1)+1}\mathcal{P}^s,$$

$$W(k)\mathcal{P}^{pt}\mathcal{P}^s\Delta\equiv(k+s)\Delta\mathcal{P}^{p(t+1)}\mathcal{P}^{s+1}\Delta-(k+t+1)\Delta\mathcal{P}^{p(t+1)+1}\mathcal{P}^s\Delta,$$

$$W(k)\Delta\mathcal{P}^{pt}\mathcal{P}^s\equiv k\Delta\mathcal{P}^{p(t+1)}\mathcal{P}^{s+1}\Delta-(k+t+1)\Delta\mathcal{P}^{p(t+1)+1}\Delta\mathcal{P}^s,$$

$$W(k)\Delta\mathcal{P}^{pt+1}\mathcal{P}^{s-1}\equiv k\Delta\mathcal{P}^{p(t+1)+1}\mathcal{P}^{s}\Delta-(k+s)\,\Delta\mathcal{P}^{p(t+1)+1}\Delta\mathcal{P}^{s},$$
$$W(k)\Delta\mathcal{P}^{pt}\mathcal{P}^{s}\Delta\equiv-(k+t+1)\Delta\mathcal{P}^{p(t+1)+1}\Delta\mathcal{P}^{s}\Delta,$$
$$W(k)\Delta\mathcal{P}^{pt+1}\mathcal{P}^{s-1}\Delta\equiv-(k+s)\Delta\mathcal{P}^{p(t+1)+1}\Delta\mathcal{P}^{s}\Delta,$$
$$W(k)\Delta\mathcal{P}^{pt+1}\Delta\mathcal{P}^{s-1}=-(k+1)\Delta\mathcal{P}^{p(t+1)+1}\Delta\mathcal{P}^{s}\Delta,$$
$$W(k)\Delta\mathcal{P}^{pt+1}\Delta\mathcal{P}^{s-1}\Delta\equiv 0,$$
$$\mathcal{P}^{p(pt-2)}\mathcal{P}^{pr-1}\equiv\mathcal{P}^{p(2p-2)}\mathcal{P}^{p-1}(\mathcal{P}^{p^2(t-2)}\mathcal{P}^{p(r-1)}).$$
$$\Delta\mathcal{P}^{p(pt+1)+1}\Delta\mathcal{P}^{ps+1}\Delta=\Delta\mathcal{P}^{p+1}\Delta\mathcal{P}^{1}\Delta(\mathcal{P}^{p^2t}\mathcal{P}^{ps}).$$

It follows from these relations that the kernel of the homomorphism

$$W(k)_*:\mathcal{S}^*\longrightarrow\mathcal{S}^*/\mathcal{P}^1\mathcal{S}^*$$

is
$$\begin{aligned}
&\mathcal{P}^1\mathcal{S}^*+W(2)\mathcal{S}^*+\mathcal{P}^{2p(p-1)}\mathcal{P}^{p-1}\mathcal{S}^* && \text{if } k=1,\\
&\mathcal{P}^1\mathcal{S}^*+W(k+1)\mathcal{S}^* && \text{if } 1<k<p-2,\\
&\mathcal{P}^1\mathcal{S}^*+W(p-1)\mathcal{S}^*+\Delta\mathcal{P}^{p+1}\Delta\mathcal{P}^1\Delta\mathcal{S}^* && \text{if } k=p-2,\\
&(\mathcal{P}^1\mathcal{S}^*+W(2)\mathcal{S}^*+\mathcal{P}^{2p(p-1)}\mathcal{P}^{p-1}\mathcal{S}^*+\Delta\mathcal{P}^{p+1}\Delta\mathcal{P}^1\Delta\mathcal{S}^* && \\
& && \text{if } p=3 \text{ and } k=1),\\
&\mathcal{P}^1\mathcal{S}^*+W(0)\mathcal{S}^*+\{\Delta\mathcal{P}^{pt+1}\Delta\mathcal{P}^{s}\cdots,\ \Delta\mathcal{P}^{p^2t}\mathcal{P}^{ps+1}\Delta\cdots, && \\
&\Delta\mathcal{P}^{p^2t+1}\mathcal{P}^{ps}\Delta\cdots\} && \text{if } k=p-1,\\
&\mathcal{P}^1\mathcal{S}^*+W(1)\mathcal{S}^*+\{\mathcal{P}^{p(pt-1)}\mathcal{P}^{pr}\Delta\cdots,\ \Delta\mathcal{P}^{p(pt-1)}\mathcal{P}^{pr}\cdots, && \\
&\Delta\mathcal{P}^{p(pt-1)+1}\mathcal{P}^{pr-1}\cdots && \text{if } k=0,
\end{aligned}$$

where we remark that $\mathcal{P}^{p(pt-1)}\mathcal{P}^{pr}\Delta\cdots\equiv\Delta\mathcal{P}^{p(pt-1)}\mathcal{P}^{pr}\cdots$ mod $\mathcal{P}^1\mathcal{S}^*$ if $r=0$.

Let Im $W(k)_*$ be the image of $W(k)_*$. Then the above results show the exactness of the sequence

$$0\to\operatorname{Im} W(k+1)_*\to\mathcal{S}^*/\mathcal{P}^1\mathcal{S}^*\to\operatorname{Im} W(k)_*\to 0$$

for some lower dimensions. Consider the exact sequence of H^{Δ} associated with this sequence. Since $H^{\Delta}(\mathcal{S}^*/\mathcal{P}^1\mathcal{S}^*)=0$ by Proposition 1.2, we have the following isomorphisms.

$$H^{i+2(p+1)(p-1)}(\operatorname{Im} W(1)_*)\approx H^i(\operatorname{Im} W(2)_*)$$
$$\text{for } i<2(2p^2-p-1)(p-1)-1,\ p>3,$$
$$H^{i+2(p+1)(p-1)}(\operatorname{Im} W(k)_*)\approx H^i(\operatorname{Im} W(k+1)_*)$$
$$\text{for all } i \text{ and } 1<k<p-2,$$

$$H^{i+2(p+1)(p-1)}(\mathrm{Im}\ W(p-2)_*) \approx H^i W(\mathrm{Im}\ (p-1)_*) \quad \text{for} \quad i<2(p+2)(p-1)+2\,,$$

$$H^{i+2(p+1)(p-1)}(\mathrm{Im}\ W(p-1)_*) \approx H^i(\mathrm{Im}\ W(0)_*) \quad \text{for} \quad i<2(p+1)(p-1)+1\,,$$

$$H^{i+2(p+1)(p-1)}(\mathrm{Im}\ W(0)_*) \approx H^i(\mathrm{Im}\ W(1)_*) \quad \text{for} \quad i<2(p^2-p)(p-1)\,.$$

Since $H^i(\mathrm{Im}\ W(k)_*)=0$ for $i<0$, it follows that

$$H^i(\mathrm{Im}\ W(k)_*)=0 \quad i<2(p^2-(k-1)(p+1))(p-1)+2\,, \quad 0\leqq k<p\,.$$

By operating $c=c^{-1}$, the following lemma is obtained,

Lemma 3.9. *$\mathscr{P}^1 W_k = W(p-k)\mathscr{P}^1$. The kernel of the homomorphism $W_k^*: \mathscr{S}^* \to \mathscr{S}^*/\mathscr{S}^*\mathscr{P}^1$, $1\leqq k\leqq p-2$, is generated by $\mathscr{P}^1$ and W_{k+1} adding $\mathscr{P}^{p-1}c\mathscr{P}^{2p(p-1)}$ if $k=1$ and adding $\Delta\mathscr{P}^{p+1}\Delta\mathscr{P}^1\Delta$ if $k=p-2$. The kernel of W_0^* corresponds with the relations of $\mathscr{S}^*b_1$ for dimensions less then $4p^2(p-1)+1$. $H_i(\mathrm{Im}\ W_k^*)$ vanishes for $i<2(p^2-(k-1)(p+1))(p-1)+2$, $0\leqq k<p$.*

(3.7). *The kernel of $(\mathscr{P}^{p(p-1)}\Delta)^*: \mathscr{S}^* \to \mathscr{S}^*/(\mathscr{S}^*\mathscr{P}^1+\mathscr{S}^*W_1)$ contains $\mathscr{S}^*\Delta+\mathscr{S}^*\mathscr{P}^1$ and thus $\mathscr{S}^i$ for $0<i<2p(p-1)$ and it does not contain $\mathscr{P}^p$.*

This follows from the following relations mod $\mathscr{P}^1\mathscr{S}^*$ by operating c.

$$\Delta\mathscr{P}^{p(p-1)}\Delta\equiv 0\,, \quad \Delta\mathscr{P}^{p(p-1)}\mathscr{P}^1\equiv W(1)\mathscr{P}^{p(p-2)}\,,$$

$$\Delta\mathscr{P}^{p(p-1)}\mathscr{P}^p\equiv\Delta\mathscr{P}^{p(p-1)+1}\mathscr{P}^{p-1}\notin\mathscr{P}^1\mathscr{S}^*+W(1)\mathscr{S}^*\,.$$

§ $\mathscr{S}^*$-structure of $\mathbf{A}^*(K_k, Z_p)$.

If an element $\alpha\in A^*(K_k, Z_p)$ is defined, then we denote by

$$\alpha \text{ in } K_l \text{ or simply by } \alpha \ (l\geqq k)$$

the image of α under the injection homomorphism $i^*: A^*(K_k, Z_p) \to A^*(K_l, Z_p)$.

Consider the following notations and relations.

$$(3.8).\quad \begin{aligned} &a_t\in A^{d(a_t)}(K_k, Z_p), && k\geqq h(a_t)\,, \quad 2\leqq t<p^2\,,\\ &a'_{rp}\in A^{d(a'_{rp})}(K_k, Z_p)\,, && k\geqq h(a'_{rp})\,, \quad 1\leqq r<p\,,\\ &b_r^{(s)}\in A^{d(b_r^{(s)})}(K_k, Z_p)\,, && k\geqq h(b_r^{(s)})\,, \quad 0\leqq s<r<p\,,\\ &c_r^{(s)}\in A^{d(c_r^{(s)})}(K_k, Z_p)\,, && k\geqq h(c_r^{(s)})\,, \quad 0\leqq s<r<p\,,\end{aligned}$$

where

$$d(a_t)=2t(p-1)\,,\qquad d(a'_{rp})=2rp(p-1)+1\,,$$
$$d(b_r^{(s)})=2(rp+s)(p-1)-2(r-s)+1,\quad d(c_r^{(s)})=2(rp+s+1)(p-1)-2(r-s),$$
$$h(a_t)=2(t-1)(p-1)\,,\qquad h(a'_{rp})=2(rp-1)(t-1)\,,$$
$$h(b_r^{(s)})=\begin{cases}2(p-1)\,, & \textit{if}\ \ r=s+1=1\,,\\ 2(sp+s-1)(p-1)-1\,, & \textit{if}\ \ r=s+1>1\,,\\ 2((r-1)p+s+1)(p-1)-2(r-1-s)\,, & \textit{if}\ \ r>s+1\,,\end{cases}$$
$$h(c_r^{(s)})=2(rp+s)(p-1)-2(r-s)+1\,.$$

In the following relations (3.9), a)-c), δ^* and j^* are the homomorphisms of (3.1) for the case $k=h(\alpha)-1$, where α is the element in $\delta^*(\ )$.

(3.9), *a*): $\delta^*(a_2)=R_1 j^{*-1}(\mathscr{P}^1 a_0)\,,$

$\delta^*(a_t)=R_{t-1}(j^{*-1}a_{t-1})\,,\quad t\not\equiv 1 \pmod p$ *and* $2<t<p^2\,,$

$\delta^*(a'_{rp})=\Delta\mathscr{P}^1\Delta(j^{*-1}a_{rp-1})\,,\quad 1\leqq r<p\,,$

$\delta^*(a_{rp+1})=\Delta\mathscr{P}^1(j^{*-1}a_{rp})-\mathscr{P}^1(j^{*-1}a'_{rp})\,,\quad 1\leqq r<p\,.$

(3.9), *b*): $\delta^*(b_1^{(0)})=\mathscr{P}^{p-1}j^{*-1}(\mathscr{P}^1 a_0)\,,$

$\delta^*(b_r^{(s)})=\mathscr{P}^{p-1}(j^{*-1}c_{r-1}^{(s)})\,,\quad 1\leqq s+1<r<p\,,$

$\delta^*(b_{s+1}^{(s)})=W_s(j^{*-1}b_s^{(s-1)})\,,\quad 1<s+1<p\,.$

(3.9), *c*): $\delta^*(c_r^{(s)})=\mathscr{P}^1(j^{*-1}b_r^{(s)})\,,\quad 0\leqq s<r<p\,.$

(3.10), *a*): $R_t a_t=0\,,\quad 2\leqq t<p^2\,,$

$\Delta a_{rp}=\Delta a'_{rp}=\Delta\mathscr{P}^1 a_{rp}-\mathscr{P}^1 a'_{rp}=0\,,\quad 1\leqq r<p\,,$

$\Delta\mathscr{P}^1\Delta a_{rp-1}=0\,,\quad 1\leqq r\leqq p\,.$

(3.10), *b*): $\mathscr{P}^1 b_1^{(0)}=\mathscr{P}^1 b_r^{(s)}=0\,,\quad 1\leqq s+1<r<p\,,$

$\mathscr{P}^1 b_{s+1}^{(s)}=W(p-s)c_s^{(s-1)}\,,\quad 1<s+1<p\,,$

$W_1 b_1^{(0)}=0\,,$

$W_{s+1}b_{s+1}^{(s)}=U_{s+1}c_s^{(s-1)}+V_{s+1}a_{sp+s-1}\,,\quad 1<s+1<p\,,$

$\mathscr{P}^{p(p-1)}\Delta b_1^{(0)}=0\,,$

$\mathscr{P}^{p-1}c\mathscr{P}^{2p(p-1)}b_2^{(1)}=Uc_1^{(0)}+Va_p+V'a_p'\,,$

$\Delta\mathscr{P}^{p+1}\Delta\mathscr{P}^1\Delta b_{p-1}^{(p-2)}=U_0 c_{p-2}^{(p-3)}+V_0 a_{p(p-1)-3}+V_0' a'_{p(p-1)-3}\,,$

where $W_{s+1}W_s=U_{s+1}\mathscr{P}^1$ *and* $\mathscr{P}^{p-1}c\mathscr{P}^{2p(p-1)}W_1=U\mathscr{P}^1$, $\Delta\mathscr{P}^{p+1}\Delta\mathscr{P}^1\Delta W_{p-2}=U_0\mathscr{P}^1$, *and if* $p>3$ *then* $V=V'=V_0=V_0'=0$.

(3.10), *c*): $\mathscr{P}^{p-1}c_r^{(s)}=0\,,\quad 0\leqq s<r<p\,.$

(3. 11) $\alpha=0$ *in* K_k *for each element* α *of* (3. 8) *sucn that* $d(\alpha)\leqq k$.

Denote by

$$B^*(k)$$

the $\mathscr{S}^*$-submodule of $A^*(K_k, Z_p)$ generated by the elements α of (3. 8) such that $h(\alpha)\leqq k<d(\alpha)$ and that the relations (3. 9) and (3. 10) are satisfied.

The purpose of this § is to prove the following theorem.

Theorem 3. 10. *Let* $4p-4\leqq k$. *There exists the elements* α (*in* K_k) *of* (3. 8) *satisfying* (3. 9) *and* (3. 10). *The relations in* $B^*(k)$ *are generated by the corresponding relations of* (3. 10) *and* (3. 11). *The module* $A^*(K_k, Z_p)/B^*(k)$ *has the following form.*

$$\begin{aligned} A^*(K_k,\ & Z_p)/B^*(k) \\ &= \{a_0,\ \mathscr{P}^{p^2}a_0,\ \mathscr{P}^{p^2+p}a_0, \cdots\} \approx \mathscr{S}^*/(\mathscr{S}^*\Delta+\mathscr{S}^*\mathscr{P}^1+\mathscr{S}^*\mathscr{P}^p) \\ &\qquad\qquad\qquad \textit{for}\quad 4p-4\leqq k\leqq 2p(p-1)-2, \\ &= \{a_0,\ d,\ \mathscr{P}^{p^2}a_0,\ \mathscr{P}^p d,\ \mathscr{P}^{p^2+p}a_0, \cdots\} \\ &\qquad\qquad\qquad \textit{for}\quad 2p(p-1)-1\leqq k\leqq 2(p^2-p)(p-1)-1, \\ &= \{a_0,\ b_p,\ \Delta b_p,\ d,\ \mathscr{P}^{p^2}a_0,\ \mathscr{P}^1\Delta b_p,\ \Delta\mathscr{P}^1\Delta b_p, \mathscr{P}^p b_p,\ \mathscr{P}^p d, \\ &\qquad \mathscr{P}^{p^2+p}a_0, \cdots\} \qquad \textit{for}\quad 2(p^2-p)(p-1)\leqq k\leqq 2(p^2-1)(p-1)-2, \end{aligned}$$

where $d\in A^{2p^2(p-1)-1}(K_{2p(p-1)-1}, Z_p)$ *and* $b_p\in A^{2(p^2-1)(p-1)-1}(K_{2(p^2-p)(p-1)-1}, Z_p)$ *are given by* $\delta^*(d)=\mathscr{P}^{p(p-1)}\Delta(j^{*-1}b_1)$ *and* $\delta^*(b_p)=\mathscr{P}^{p-1}(j^{-1}c_{p-1}^{(0)})$.

Proof. i) *The case* $4p-4\leqq k\leqq 2p(p-1)-2$.

The $\mathscr{S}^*$-module $A^*(K_{4p-4}, Z_p)$ is already determined in Theorem 3. 7, and this shows that the following proposition (3. 12) is true for $k=4p-4$.

(3. 12). $A^*(K_k, Z_p)$ *has the following system of generators and relations respectively* ($b_1=b_1^{(0)}$).

$\{a_0,\ a_t,\ b_1\}$ *and* $\{\Delta a_0=\mathscr{P}^1a_0=\mathscr{P}^1b_1=R_ta_t=0$, (*adding* $\Delta\mathscr{P}^1\Delta a_{p-1}=0$ *when* $t=p-1$), $\Delta b_1=\mathscr{P}^pa_0\}$ *for* $2(t-1)(p-1)\leqq k<2t(p-1)$ *and* $2<t<p$;

$\{a_0,\ b_1,\ a_p,\ a_p'\}$ *and* $\{\Delta a_0=\mathscr{P}^1a_0=\mathscr{P}^1b_1=\Delta a_p=\Delta a_p'=0,\ \Delta b_1=\mathscr{P}^pa_0,\ \Delta\mathscr{P}^1a_p=\mathscr{P}^1a_p'\}$ *for* $2(p-1)^2\leqq k\leqq 2p(p-1)-2$.

The proof is done by the induction on k, using Lemma 3. 2, Lemma 3.3 and Proposition 1.5, and it is quite similar to one of Theorem 3.7 and omitted.

It follows from (3.12) and from Lemma 3.8 that Theorem 3.10 is true for $4p-4\leqq k\leqq 2p(p-1)-2$.

ii) *The case* $k=2p(p-1)-1$.

From the result for $k=2p(p-1)-2$, $A^{2p(p-1)-1}(K_{2p(p-1)-2}, Z_p)=\{b_1\}$, $\Delta b_1\neq 0$ and $\{\mathscr{P}^1 b_1=W_1 b_1=\mathscr{P}^{p(p-1)}\Delta b_1=0\}$ is a system of relations in $\mathscr{S}^* b_1$. By Proposition 1.6, Lemma 3.9 and by (3.7), it is seen that $\mathscr{S}^*\mathscr{P}^1+\mathscr{S}^* W_1+\mathscr{S}^*\mathscr{P}^{p(p-1)}\Delta$ has a system of relations $\{\mathscr{P}^{p-1}\mathscr{P}^1=0,\quad \mathscr{P}^1 W_1=W(p-1)\mathscr{P}^1,\quad W_2 W_1=U_2\mathscr{P}^1,$ $\mathscr{P}^{p-1}c\mathscr{P}^{2p(p-1)}W_1=U\mathscr{P}^1,\ \Delta\mathscr{P}^{p(p-1)}\Delta=V\mathscr{P}^1,\ \mathscr{P}^1\mathscr{P}^{p(p-1)}\Delta=V'\mathscr{P}^1+V''W_1,\ \alpha_s\mathscr{P}^{p(p-1)}\Delta=V_s'\mathscr{P}^1+V_s''W_1\}$ for some $U_2, U, V, \cdots, \alpha_s$ such that the dimensions of α_s are not less than $2(p+1)(p-1)$.

Then by Lemma 3.3, there are elements $c_1^{(0)}$, $b_2^{(1)}$ and d such that $\delta^*(c_1^{(0)})=\mathscr{P}^1(j^{*-1}b_1)$, $\delta^*(b_2^{(1)})=W_1(j^{*-1}b_1)$ and $\delta^*(d)=\mathscr{P}^{p(p-1)}\Delta(j^{*-1}b_1)$, and $A^*(K_{2p(p-1)-1}, Z_p)$ is obtained from $A^*(K_{2p(p-1)-2}, Z_p)$ by adding the generators $c_1^{(0)}$, $b_2^{(1)}$, d and the relations $b_1=0$, $\mathscr{P}^{p-1}c_1^{(0)}=i^*w_1,\quad \mathscr{P}^1 b_2^{(1)}=W(p-1)c_1^{(0)}+i^*w_2,\quad W_2 b_2^{(1)}=U_2 c_1^{(0)}+i^*w_3,$ $\mathscr{P}^{p-1}c\mathscr{P}^{2p(p-1)}b_2^{(1)}=Uc_1^{(0)}+i^*w_4,\quad \Delta d=Vc_1^{(0)}+i^*w_5,\quad \mathscr{P}^1 d=V'c_1^{(0)}+V'b_2^{(1)}+i^*w_6$, and $\alpha_s d=V_s'c_1^{(0)}+V_s''b_2^{(1)}+i^*w_s'$.

From the fact that the i^*-images vainish for the corresponding dimensions to i^*w_1 and i^*w_2, it follows $i^*w_1=i^*w_2=0$. From $i^*A^{(3p+2)(p-1)}(K_{2p(p-1)-2}, Z_p)=\{\mathscr{P}^{2p+2}a_p,\ \mathscr{P}^{2p+1}\mathscr{P}^1 a_p\}$, it follows that $i^*w_3=V_2 a_p$ for some V_2, Similarly we have that $i^*w_4=Va_p+V'a_p'$ for some V, V' such that $V=V'=0$ if $p>3$.[1]

Consequently Theorem 3.10 is proved for $k=2p(p-1)-1$.

iii) *The case* $2p(p-1)\leqq k\leqq 2(p^2-p)(p-1)-1$.

The proof is done by the induction on k. The following four cases are considered.

$$
\begin{aligned}
A^{k+1}(K_k, Z_p) &= \{a_t\}, && \textit{for}\ \ k=d(a_t)-1,\\
&= \{b_r^{(s)}\}, && \textit{for}\ \ k=d(b_r^{(s)})-1,\\
&= \{c_r^{(s)}\}, && \textit{for}\ \ k=d(c_r^{(s)})-1,\\
&= 0 && \textit{otherwise}.
\end{aligned}
$$

For the first case Lemma 3.3 ($t\neq 0$) and Lemma 3.4 ($t=0$) are applied, for the next two cases Lemma 3.3, and for the last one Lemma 3.2 is applied. Then it is sufficient to prove that

1) If $p=3$, the last relation of (3.10), b) has to be added.

(3.13) *for each steps from* K_k *to* K_{k+1}, *we may take new generators* α *of* $h(\alpha)=k+1$ *and new relations given by* (3.8), (3.9), (3.10) *and* (3.11).

The case $A^{k+1}(K_k, Z_p)=0$ is trivial.

Consider the case $A^{k+1}(K_k, Z_p)=\{a_t\}$. By Proposition 1.5, Lemma 3.3, Lemma 3.4 and by (3.10), *a*), new generators are a_{t+1} and a'_{rp} (*if* $rp=t+1$) and new relations are the followings:

$$R_{t+1}a_{t+1}=i^*w_1\ (t+1\not\equiv 0),\quad \Delta\mathscr{P}^1\Delta a_{rp-1}=i^*w_2\ (t+1=rp-1),$$
$$\Delta a_{rp}=i^*w_3,\quad \Delta a'_{rp}=i^*w_4,\quad \Delta\mathscr{P}^1a_{rp}-\mathscr{P}^1a'_{rp}=i^*w_5\ (t+1=rp)$$

i^*w_1 and i^*w_5 belong to $i^*A^{2(t+2)(p-1)+1}(K_{2t(p-1)-1}, Z_p)$ which is generated by some $b_r^{(s)}$ and $c_r^{(s)}$ such that $2t(p-1)\leqq d(b_r^{(s)})\leqq 2(t+2)(p-1)+1$, $2t(p-1)\leqq d(c_r^{(s)})\leqq 2(t+2)(p-1)+1$ and $h(c_r^{(s)})=d(c_r^{(s)})-(2p-3)\geqq 2t(p-1)$. Then the possibility of $i^*w_1\neq 0$ or $i^*w_5\neq 0$ is the followings.

$$R_sa_{(s+1)p+s}=i^*w_1=x\Delta\mathscr{P}^1\Delta b_{s+1}^{(s)},\quad 1<s<p-2,$$
$$\Delta\mathscr{P}^1a_p-\mathscr{P}^1a_p'=i^*w_5=x\Delta\mathscr{P}^1\Delta b_1,$$
$$R_{p-2}a_{(p-1)p-2}=i^*w_1=xb_{p-1}.$$

In the first case we replace $a_{(s+1)p+s}$ by $a_{(s+1)p+s}-(x/s)\Delta b_{s+1}^{(s)}$, and in the second case we replace a_p by $a_p+x\Delta b_1$. Then we see that $i^*w_1=i^*w_5=0$. In the last case, it follows from $\Delta\mathscr{P}^1\Delta R_{p-2}=0$ and $\Delta\mathscr{P}^1\Delta b_{p-1}\neq 0$ that $x=0$ and thus $i^*w_1=0$.

Similarly it is verified that the possibility of $i^*w_2\neq 0$, $i^*w_3\neq 0$ or $i^*w_4\neq 0$ is the followings.

$$\Delta\mathscr{P}^1\Delta a_{(p-1)p-1}=i^*w_2=x\Delta b_{p-1},$$
$$\Delta\mathscr{P}^1\Delta a_{(p-2)p-1}=i^*w_2=xc_{p-2},$$
$$\Delta a'_{(p-2)p}=i^*w_4=xc_{p-2},$$
$$\Delta a'_{(p-1)p}=i^*w_4=x\mathscr{P}^1\Delta b_{p-1}.$$

In the first case, it follows from $\Delta\mathscr{P}^1(\Delta\mathscr{P}^1\Delta)=0$ and $\Delta\mathscr{P}^1\Delta b_{p-1}\neq 0$ that $x=0$ and $i^*w_2=0$. Also in the other three cases, it follows from $\Delta(\Delta\mathscr{P}^1\Delta)=\Delta\Delta=0$, $\Delta c_{p-2}\neq 0$ and $\Delta\mathscr{P}^1\Delta b_{p-1}\neq 0$ that $x=0$ and $i^*w_2=i^*w_4=0$.

Consequently, by a suitable chice of a_t, the relations (3.10), *a*) are satisfied and thus (3.13) is established for the case $A^{k+1}(K_k, Z_p)=\{a_t\}$.

Next consider the case $A^{k+1}(K_k, Z_p)=\{c_r^{(s)}\}$. By (3.10), the module $\mathscr{S}^*c_r^{(s)}$ has a system $\{\mathscr{P}^{p-1}c_r^{(s)}\}$ of relations for $r>s+1$. This is also true for $r=s+1$, because δ^* maps $\mathscr{S}^*c_{s+1}^{(s)}$ isomorphically onto $\mathscr{S}^*\mathscr{P}^1(j^{*-1}b_{s+1}^{(s)})\approx\mathscr{S}^*/\mathscr{S}^*\mathscr{P}^{p-1}$. Then, by Proposition 1.6 and Lemma 3.3, new generator and relation are $b_{r+1}^{(s)}$ and $\mathscr{P}^1b_{r+1}^{(s)}=i^*w$. The only possibility of $i^*w\neq 0$ is $\mathscr{P}^1b_{p-1}=x\mathscr{P}^{p+1}\Delta a_{(p-2)p+1}$. In the case, we raplace b_{p-1} by $b_{p-1}+x\mathscr{P}^p\Delta a_{(p-2)p+1}$, then $i^*w=0$. Therefore (3.13) is established for the case $A^{k+1}(K_k, Z_p)=\{c_r^{(s)}\}$.

Similarly, we have new generator $c_r^{(s)}$ and new relation $\mathscr{P}^{p-1}c_r^{(s)}=i^*w$ for the case $A^{k+1}(K_k, Z_p)=\{b_r^{(s)}\}$ and $r>s+1$. There is no possibility of $i^*w\neq 0$, in this case, and (3.13) is established.

Finally consider the case $A^{k+1}(K_k, Z_p)=\{b_{s+1}^{(s)}\}$ where $1\leqq s<p-2$. Let $1<s<p-2$, then $\mathscr{S}^*b_{s+1}^{(s)}$ has a system $\{\mathscr{P}^1b_{s+1}^{(s)}=W_{s+1}b_{s+1}^{(s)}=0\}$ of relations. By Lemma 3.9 and Lemma 3.3, new generators are $c_{s+1}^{(s)}$ and $b_{s+2}^{(s+1)}$ and new relations are

$$\mathscr{P}^{p-1}c_{s+1}^{(s)}=i^*w_1,\quad \mathscr{P}^1b_{s+2}^{(s+1)}=i^*w_2,\quad W_{s+2}b_{s+2}^{(s+1)}=U_{s+2}c_{s+1}^{(s)}+i^*w_3,$$
$$\Delta\mathscr{P}^{p+1}\Delta\mathscr{P}^1\Delta b_{p-1}^{(p-2)}=U_0c_{p-2}^{(p-3)}+i^*w_4.$$

There is no possibility of $i^*w_1\neq 0$ or $i^*w_2\neq 0$. There are possibilities of $i^*w_3=V_{s+2}a_{sp+s-1}$ and $i^*w_4=V_0a_{p(p-1)-3}+V_0'a'_{p(p-1)-3}$.

In the case $s=1$, there is an adding relation $\mathscr{P}^{p-1}c\mathscr{P}^{2p(p-1)}b_2^{(1)}=0$, and there is a corresponding new generator, however, the dimension of which is so hight that it may be neglected in Theorem 3.10. (3.13) is established for the case $A^{k+1}(K_k, Z_p)=\{b_{s+1}^{(s)}\}$.

iv) *The case* $2(p^2-p)(p-1)\leqq k\leqq 2(p^2-1)(p-1)-2$.

In the case $k=2(p^2-p)(p-1)$, $A^{k+1}(K_k, Z_p)=\{a_{(p-1)p}, c_{p-1}^{(0)}\}$ and Lemma 3.4 may be applied. Then new generators are $a_{(p-1)p+1}$ and b_p and new relations are $R_1a_{(p-1)p+1}=i^*w_1$ and $\mathscr{P}^1b_p=i^*w_2$. It is easy to see that $i^*w_1=0$ and $i^*w_2=xd$ for some integer x. Then Theorem 3.10 is established for $k=2(p^2-p)(p-1)$. The proof of the other caces is similar to the above iii) and rather easy. q.e.d.

§ Stable groups.

Proposition 3.11. *The group $A^{k+1}(K_k, Z_p)$ has the following basis:*

$\{a_t\}$, $\Delta a_t \neq 0$, *for* $k=2t(p-1)-1$, $t\not\equiv 0 \bmod p$ *and* $2\leqq t<p^2$,
$\{\mathscr{P}^1 a_0\}$, $\Delta\mathscr{P}^1 a_0 \neq 0$, *for* $k=2p-3$,
$\{b_r^{(s)}\}$, $\Delta b_r^{(s)} \neq 0$, *for* $k=2(rp+s)(p-1)-2(r-s)$
and $0\leqq s<r\leqq p-1$ *or* $r=p=s+p$,
$\{c_r^{(s)}\}$, $\Delta c_r^{(s)} \neq 0$, *for* $k=2(rp+s+1)(p-1)-2(r-s)-1$,
$r-s\neq p-1$ *and* $0\leqq s<r\leqq p-1$,
$\{a_{rp}\}$, $\Delta a_{rp}=0$, *for* $k=2rp(p-1)-1$ *and* $1\leqq r<p-1$,
$\{a_{(p-1)p}, c_{p-1}\}$, $\Delta a_{(p-1)p}=0$, $\Delta c_{p-1}\neq 0$, *for* $k=2(p^2-p)(p-1)-1$,
$\{\ \}$ (*empty*), *otherwise for* $0<k<2p^2(p-1)-3$.

For $k\leqq 2(p^2-1)(p-1)-2$, this proposition follows directly from Theorem 3.6 and Theorem 3.10. For $2(p^2-1)(p-1)-2<k<2p^2(p-1)-3$, it is proved easily.

By Lemma 3.1, for the first four cases of the above proposition, $\pi_k(\mathfrak{S}\,;\,p)=Z_p$ and, for the last case, $\pi_k(\mathfrak{S}\,;\,p)=0$. In order to determine the groups $\pi_k(\mathfrak{S}\,;\,p)$ of the other two cases, we shall verify the Bockstein operator Δ_2 in $A^*(K_k, Z_p)$.

Let $H_\Delta(A^*)$ be the cohomology group of an $\mathscr{S}^*$-module A^* with respect to the homomorphism $\Delta_*: A^*\to A^*$. Then we may regard that Δ_2 is essentially a homomorphism of $H_\Delta(A^*)$ in itself.

Let $C^*(k)$ be a submodule of $B^*(k)$ generated by a_t, a_{rp}, a'_{rp}, $c_r^{(s)}$ and $b_{r'}^{(s')}$ such that $r'>s'+1$. Then

(3.14). $H_\Delta(C^*(k))=\{c\mathscr{P}^{pi-t}a_t,\ c\mathscr{P}^{pi-t}\Delta a_t\}$ *or* $\{c\mathscr{P}^{pi}a_{rp}, c\mathscr{P}^{pi}a'_{rp}\}$.

Proof. By (3.10), $C^*(k)$ is a direct sum of some $\mathscr{S}^*a_t$, $\mathscr{S}^*a_{rp}+\mathscr{S}^*a'_{rp}$, $\mathscr{S}^*c_r^{(s)}$ and $\mathscr{S}^*b_{r'}^{(s')}$. By Proposition 1.6 and by (3.10), $H_\Delta(\mathscr{S}^*c_r^{(s)})\approx H_\Delta(\mathscr{S}^*/\mathscr{S}^*\mathscr{P}^{p-1})\approx H_\Delta(\mathscr{S}^*\mathscr{P}^1)=0$ and $H_\Delta(\mathscr{S}^*b_{r'}^{(s')})\approx H_\Delta(\mathscr{S}^*/\mathscr{S}^*\mathscr{P}^1)\approx H_\Delta(\mathscr{S}^*\mathscr{P}^{p-1})=0$. Thus $H_\Delta(C^*(k))$ is isomorphic to $H_\Delta(\mathscr{P}^*a_t)$, $t\not\equiv 0 \bmod p$, $H_\Delta(\mathscr{S}^*a_{rp}+\mathscr{S}^*a'_{rp})$ or 0.

Let $t\not\equiv 0, 1$. Then, by Proposition 1.5 and (3.10), *a*), $H_\Delta(\mathscr{S}^*a_t)\approx H_\Delta(\mathscr{S}^*/\mathscr{S}^*R_t)\approx H_\Delta(\mathscr{S}^*R_{t-1})=\{\Delta c\mathscr{P}^{pi-t+1},\ \Delta c\mathscr{P}^{pi-t+1}\Delta\}$. As is seen in the proof of Proposition 1.1, $R^*_{t-1}(c\mathscr{P}^{pi-t})=c\mathscr{P}^{pi-t}R_{t-1}=c(R(t-1)\mathscr{P}^{pi-t})=c((1-t)\mathscr{P}^{pi-t+1}\Delta)=(1-t)\Delta c\mathscr{P}^{pi-t+1}$ and also $R^*_{t-1}(c\mathscr{P}^{pi-t}\Delta)=t\Delta c\mathscr{P}^{pi-t+1}\Delta$. Then (3.14) is proved for this case $t\not\equiv 0, 1$. The other cases are proved similarly. q.e.d.

By (3.10), b), $B^*(k)/C^*(k)$ is generated by the class of $b_{s+1}^{(s)}$ and it is isomorphic to 0, $\mathscr{S}^*/(\mathscr{S}^*\mathscr{P}^1+\mathscr{S}^*W_{s+1})$, $\mathscr{S}^*/(\mathscr{S}^*\mathscr{P}_1$ $\mathscr{S}^*W_1+\mathscr{S}^*\mathscr{P}^{p(p-1)}\Delta)$, $\mathscr{S}^*/(\mathscr{S}^*\mathscr{P}^1+\mathscr{S}^*W_2+\mathscr{S}^*\mathscr{P}^{p-1}c\mathscr{P}^{2p(p-1)})+$

or $\mathcal{S}^*/(\mathcal{S}^*\mathcal{P}^1+\mathcal{S}^*W_{p-1}+\mathcal{S}^*\Delta\mathcal{P}^{p+1}\Delta\mathcal{P}^1\Delta)$. As is seen in Lemma 3.9, the last four modules are isomorphic to the image of W_s^* for dimensions less than $2(2p^2+p)(p-1)$. By the last conclusion of Lemma 3.9,

$$H_\Delta^i(B^*(k)/C^*(k))=0 \qquad for \quad i<2(p^2+2p-1)(p-1)+1,$$

and thus

$$H_\Delta^i(C^*(k))\approx H_\Delta^i(B^*(k)) \quad for \quad i<2(p^2+2p-1)(p-1).$$

Since $A^*(K_k, Z_p)/B^*(k)=0$ *for* $i<2p^2(p-1)-3$,

$$(3.15). \quad H_\Delta^i(C^*(k))\approx H_\Delta^i(A^*(K_k, Z_p)) \qquad for \quad i<2p^2(p-1)-4.$$

Now we shall prove the following important lemma.

Lemma 3.12. $\Delta_2: H_\Delta(A^{2jp(p-1)}(K_k, Z_p))\approx H_\Delta(A^{2jp(p-1)+1}(K_k, Z_p))$ *for* $2p-2\leqq k<2jp(p-1)$ *and* $j<p$.

Proof. First consider the case $k=2p-2$. Apply Lemma 3.5, i) to the sequence (3.1) of $k=2p-3$. (See the proof of Theorem 3.6). Since $\Delta c\mathcal{P}^{pi}a_0=c\mathcal{P}^{pi-1}\Delta\mathcal{P}^1a_0=j^*(c\mathcal{P}^{pi-1}\Delta(j^{*-1}\mathcal{P}^1a_0))$, it follows that, in K_{2p-2}, $\Delta_2c\mathcal{P}^{pi}a_0$ is the class of an element $\tilde{a}_i$ such that $\delta^*\tilde{a}_i=\Delta c\mathcal{P}^{pi-1}\Delta(j^{*-1}\mathcal{P}^1a_0)=\frac{1}{2}c\mathcal{P}^{pi-2}\Delta R_1(j^{*-1}\mathcal{P}^1a_0)$ $=\delta^*\left(\frac{1}{2}c\mathcal{P}^{pi-2}\Delta a_2\right)$. Here we remark that (3.14) and (3.15) are true for K_{2p-2}. From $\Delta b_1=\mathcal{P}^pa_0-\mathcal{P}^{p-2}a_2$, we have $\Delta c\mathcal{P}^{pi-p}b_1$ $=c\mathcal{P}^{pi-p}(\mathcal{P}^pa_0-\mathcal{P}^{p-2}a_2)=-ic\mathcal{P}^{pi}a_0+c\mathcal{P}^{pi-2}a_2$. Since $\Delta_2\Delta=0$, we have $\Delta_2(ic\mathcal{P}^{pi}\Delta a_0)=\Delta_2c\mathcal{P}^{pi-2}a_2$. Therefore

$$\Delta_2(c\mathcal{P}^{pi-2}a_2)\neq 0 \qquad for \quad 1\leqq i<p,$$

and this is a class of $xc\mathcal{P}^{pi-2}\Delta a_2$ for some $x\not\equiv 0 \bmod p$, since $c\mathcal{P}^{pi-2}\Delta a_2$ is a generator of $H_\Delta(A^{2ip(p-1)+1}(K_{2p-2}, Z_p))$, $1\leqq i<p$.

Now the lemma will be proved by the induction on k. If the lemma is true for some $k\not\equiv -1 \bmod 2(p-1)$, then it is also true for $k+1$ by the naturality of Δ_2. If the lemma is true for $k=2t(p-1)-1$, we apply iii) of Lemma 3.5 to (3.1) of $k=2t(p-1)-1$. Since a_t (and a'_{rp} if $t=rp$) are j^*-images, the non-triviality of Δ_2 in K_k implies the non-triviality of Δ_2 in K_{k+1}. Then the lemma is proved. q.e.d.

From the above lemma, $\Delta_2a_{rp}\neq 0$ for $1\leqq r<p$. Thus, by Lemma 3.1, $\pi_{2rp(p-1)-1}(S;p)$ has a direct factor isomorphic to Z_{p^2}.

Consequently the following theorem is established.

Theorem 3.13.

(A) $\pi_{2rp(p-1)-1}(\mathfrak{S}\,;\,p)=Z_{p^2}$ for $1\leqq r<p-1$,
$=Z_{p^2}+Z_p$ for $r=p-1$,

(B) $\pi_{2t(p-1)-1}(\mathfrak{S}\,;\,p)=Z_p$ for $1\leqq t<p^2$ and $t\not\equiv 0 \bmod p$,
$\pi_{2(rp+s)(p-1)-2(r-s)}(\mathfrak{S}\,;\,p)=Z_p$ for $0\leqq s<r\leqq p-1$,
$\pi_{2(rp+s+1)(p-1)-2(r-s)-1}(\mathfrak{S}\,;\,p)=Z_p$ for $0\leqq s<r\leqq p-1$ and $r-s\neq p-1$,
$\pi_{2p^2(p-1)-2p}(\mathfrak{S}\,;\,p)=Z_p$,

(C) $\pi_k(\mathfrak{S}\,;\,p)=0$ *otherwise for* $k<2p^2(p-1)-3$.

To compute the group $\pi_{2p^2(p-1)-3}(\mathfrak{S}\,;\,p)$, one has to be determine the coefficient x of

$$\mathscr{P}^1 b_p = xd \quad in \quad K_{h(b_p)}.$$

It is verified easily that

(3.16). $\pi_{2p^2(p-1)-3}(\mathfrak{S}\,;\,p)=Z_p$ *if* $\mathscr{P}^1 b_p=0$ *and* $\pi_{2p^2(p-1)-3}(\mathfrak{S}\,;\,p)=0$ *if* $\mathscr{P}^1 b_p\neq 0$.

The second undetermined factor is Δd. It is a reasonable conjecture that

$$\Delta d = \mathscr{P}^{p^2} a_0 \quad \bmod B^*(k).$$

This is the case that the conclusion of Theorem 2.9 is true for $r=1$, and this implies the tiriviality of mod p Hopf invariant $H_p:\pi_{N+2p^2(p-1)-1}(S^N)\to Z_p$. Under this conjecture it will be computed that

(3.17). $\pi_{2p^2(p-1)-2}(\mathfrak{S}\,;\,p)=Z_p$ *if* $\mathscr{P}^1 b_p=0$ *and* $\pi_{2p^2(p-1)-2}(\mathfrak{S}\,;\,p)=0$ *if* $\mathscr{P}^1 b_p\neq 0$.

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