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An example of deformations of complex analytic bundles

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In a recent paper Y. Matsushima studied the family of complex analytic bundles over a complex torus T which have $GL(r, C)$ as the structural group and which have holomorphic connections (see [5]). Particularly definitive are the results that if $r=2$, then those bundles which have holomorphic connections are those which are associated to the universal covering space of T, with respect to various representations of its fundamental group into $GL(2, C)$, and that the set of the indecomposable $GL(2, C)$-bundles over T with holomorphic connections are in one-to-one correspondence, in a natural way, with the product of the Picard variety $\mathfrak{P}$ of T and a complex projective space P of dimension one less than that of T. (See theorems 4 and 5 in [5].)

On the other hand it is clear that the decomposable $GL(2, C)$-bundles with holomorphic connections over a compact Kähler manifold M are in one-to-one correspondence with the points of the symmetric product V of the Picard variety $\mathfrak{P}$ of M with itself. We shall construct a non-singular variety W by a monoidal transformation applied to V. It will be shown that W contains a submanifold X homeomorphic to $\mathfrak{P}\times P$, that there exists an analytic $GL(2, C)$-bundle E over $M\times W$ which gives rise to a family of $GL(2, C)$-bundles over M parametrized by W, each of its member having holomorphic connections, and that $w\in X$ gives an indecomposable bundle.

Thus we see the true nature of Matsushima's theorem 5 and, at the same time, we see that the indecomposable bundles can be considered as limits of decomposable ones. Furthermore, if we compare this family with that of decomposable bundles parametrized by V, we have another example of the phenomenon of non-Hausdorff parameter variety in the theory of deformations of complex structures, as was first revealed by Kodaira and Spencer [4].

§1. Symmetric products of domains in C^n.

We shall first consider the symmetric product of the complex n-space C^n with itself ($n \geqq 2$). It is the quotient space of $C^n \times C^n$ by the group $\{1, \tau\}$, where 1 denotes the identity automorphism of $C^n \times C^n$ and τ denotes the automorphism defined by $\tau(x, y) = (y, x)$. Let $(\xi_1, \cdots, \xi_n)$ be linear coordinates of C^n and let $u_i = \xi_i \circ p_1$ and $v_i = \xi_i \circ p_2$, where p_1 and p_2 denote projections from $C^n \times C^n$ to the first and the second factors respectively. Put

$$(1) \qquad \begin{cases} x_i = \dfrac{1}{2}(u_i + v_i) & i = 1, 2, \cdots, n, \\ y'_{ij} = \dfrac{1}{2}(u_i v_j + u_j v_i) & 1 \leqq i, j \leqq n. \end{cases}$$

M. Nagata has shown ([6]) that the functions x_i and y'_{ij} on the symmetric product define a normal model of this variety. His argument refers to the algebraic structure of the symmetric product, but it is not hard to see that the same conclusion holds when we replace C^n by an open set U of it and consider the analytic structure of the symmetric product V of U with itself. Thus V has a normal model given by (1). If we put

$$(2) \qquad \begin{cases} x_i = \dfrac{1}{2}(u_i + v_i) & i = 1, 2, \cdots, n, \\ y_{ij} = \dfrac{1}{2}(u_i - v_i)(u_j - v_j) & 1 \leqq i, j \leqq n, \end{cases}$$

then we have $y'_{ij} = x_i x_j - \dfrac{1}{2} y_{ij}$, and hence (2) also define a normal model of V. This model is defined by the equations

$$(3) \qquad \begin{cases} y_{ij} = y_{ji} \\ y_{ij} y_{kl} = y_{il} y_{kj} & 1 \leqq i, j, k, l \leqq n. \end{cases}$$

This shows that V is the product of a domain in the space of x's and a neighbourhood of the vertex in a quadric cone in the space of y's. The latter has a singularity at the vertex.

Because of this simple structure, it is easy to "de-singularize" V by a monoidal transformation. We take a projective $(n-1)$-space P^{n-1} and denote by $(t_1, \cdots, t_n)$ a set of homogeneous coordinates of this space. We then define

$$(4) \qquad W=\{(x, y, t)\in V\times P^{n-1} \mid y_{ij}t_k=y_{ik}t_j\}\ .$$

W is a non-singular manifold. In fact, around a point (x, y) of V where $(y)\neq(0)$, (t) may be solved as holomorphic functions in (y) and W is locally homeomorphic to V in analytic sense. If $(y)=(0)$, then any point $(x, 0, t)$ belongs to W and some of t's, say t_n, differs from zero at this point. Then we may put $t_n=1$ and the set $(x_1, \cdots, x_n, y_{nn}, t_1, \cdots, t_{n-1})$ serves as a system of local parameters at the point in question. We see that the subvariety S of V composed of its singular points is "blown up" into a divisor X defined by $(y)=(0)$.

§2. Symmetric products of complex tori.

Now let us consider a complex torus T of dimension n. We can represent T as C^n/D, where D is a discrete subgroup of C^n of rank $2n$. Let V be the symmetric product of T with itself. The singular subvariety S of V is the image of the diagonal under the natural projection $T\times T\to V$. To examine V in the neighbourhood of S, we take a sufficiently fine simple open covering $\{U_\alpha\}$ of T. Let π denote the canonical projection $C^n\to T$. We choose, for each α, a sheet $\tilde{U}_\alpha$ in $\pi^{-1}(U_\alpha)$, which maps homeomorphically onto U_α. If $(u_1, \cdots, u_n)$ is a set of linear coordinates in C^n, then $u_i^\alpha=u_i\circ\pi_\alpha^{-1}$ (for $i=1, 2, \cdots, n$) form a set of local parameters on U_α, where π_α^{-1} denotes the inverse of $\pi: \tilde{U}_\alpha\to U_\alpha$. We have

$$(5) \qquad u^\alpha-u^\beta=a^{\alpha\beta}\in D \quad \text{for points in } U_\alpha\cap U_\beta\,,$$

where u^α denotes the point $(u_1^\alpha, \cdots, u_n^\alpha)$ in C^n.

We take a replica of T and define v^α in the same way as u^α. Consider u_i^α and v_j^α as functions on $U_\alpha\times$(its replica) in an obvious manner. Form the symmetric product V_α of U_α with itself as in the preceeding paragraph. $\bigcup_\alpha V_\alpha$ is a neighbourhood of S in V. We shall distinguish the quantities refering to V_α by superscript α. Then by (5) and the corresponding equation $v^\alpha-v^\beta=a^{\alpha\beta}$, we see

$$x^{\alpha}-x^{\beta}=a^{\alpha\beta},\quad y_{ij}^{\alpha}=y_{ij}^{\beta}\qquad \text{for points in } V_{\alpha}\cap V_{\beta}.$$

The way how V_{α} and V_{β} are patched together within V is now obvious, and we see that there exists a neighbourhood of S which is the product of T and a neighbourhood of the vertex in the cone (3). Monoidal transforms W_{α} of V_{α} with centres $S\cap V_{\alpha}$ are also patched together to form, together with $V-\bigcup V_{\alpha}$, the monoidal transform W of V with the centre S, and S *is blown up into* $X=T\times P^{n-1}$ *in this transform.*

§ 3. Construction of a class of bundles.[1]

Let M be a manifold and $\hat{M}$ be a principal bundle over M with the structural group Γ. ($\hat{M}$ and M have complex analytic, differentiable or C^0 structures according to the kind of structures we are interested in. The same should be understood for all objects and maps in this paragraph.) Let a group G be given. We consider a map $f:\hat{M}\times\Gamma\rightarrow G$ with the property

$$(6)\qquad f(\hat{x},\sigma\tau)=f(\hat{x}\sigma,\tau)\cdot f(\hat{x},\sigma)\qquad \text{for } \hat{x}\in\hat{M},\quad \sigma,\tau\in\Gamma.$$

In this equation, dot in the right hand side denotes the group multiplication in G. Two such functions f and g are said to be equivalent if and only if there exists a map $\varphi:\hat{M}\rightarrow G$ such that

$$(7)\qquad f(\hat{x},\sigma)=\varphi(\hat{x}\sigma)^{-1}\cdot g(\hat{x},\sigma)\cdot\varphi(\hat{x}).$$

It is easy to see that this is an equivalence relation.

We shall point out the following fact:

The G-bundles over M which induce the product bundle over $\hat{M}$ under the canonical projection $\pi:\hat{M}\rightarrow M$ are in one-to-one correspondence with the equivalence classes of the maps f with property (6). *This correspondence is given in the following way: Given a map f we consider the quotient space F obtained from $\hat{M}\times G$ by the equivalence relation*

$$(8)\qquad (\hat{x},\zeta)\sim(\hat{x}\sigma, f(\hat{x},\sigma)\cdot\zeta),$$

then F is the G-bundle we associate to f.

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1) The contents of this and the next paragraphs are essentially contained in Kodaira [3], § 2. But it is convenient to have a presetation as given here.

In fact it is easy to see that (8) is actually an equivalence relation. Next, (8) is compatible with the projection $\hat{M}\times G\ni(\hat{x}, \zeta) \to x=\pi(\hat{x})\in M$. This map defines the projection $p:F\to M$. Right translations on G are also compatible with (8). Thus G operates on F on the right. These structures define F as a principal G-bundle over M. If we denote by Π the canonical map from $\hat{M}\times G$ to F, then the diagram

$$\begin{array}{ccc} \hat{M}\times G & \xrightarrow{\Pi} & F \\ p\downarrow & & \downarrow p \\ \hat{M} & \xrightarrow{\pi} & M \end{array}$$

is commutative, and right translations by elements of G commute with Π. This shows that Π is a bundle map.

Conversely let a principal G-bundle F over M be given, and suppose F induces the product bundle on $\hat{M}$. The induced bundle is the subset $\hat{F}$ of $\hat{M}\times F$ defined by

$$\hat{F}=\{(\hat{x}, a)\in\hat{M}\times F\mid\pi(\hat{x})=p(a)\}\ ,$$

where p denotes the projection of the bundle F. Since $\hat{F}$ has global cross sections, we take one given by

$$\hat{M}\ni\hat{x}\to a(\hat{x})\in F \qquad \text{where } p(a)=\pi(\hat{x})\,.$$

For $\sigma\in\Gamma$, $a(\hat{x}\sigma)$ is a point of $p^{-1}(\pi(\hat{x}))\subset F$. Hence there exists an element $g=g(\hat{x}, \sigma)\in G$ such that

$$(9) \qquad a(\hat{x}\sigma)=a(\hat{x})\cdot g(\hat{x}, \sigma)\,.$$

From $a(\hat{x}(\sigma\tau))=a((\hat{x}\sigma)\cdot\tau)$, we deduce $g(\hat{x}, \sigma\tau)=g(\hat{x}, \sigma)\cdot g(\hat{x}\sigma, \tau)$. Hence if we put

$$(10) \qquad f(\hat{x}, \sigma)=g(\hat{x}, \sigma)^{-1},$$

f satisfies the equation (6). Now the cross section $a(\hat{x})$ define a bundle map $\hat{M}\times G\ni(\hat{x}, \zeta)\leftrightarrow(\hat{x}, a(\hat{x})\cdot\zeta)\in\hat{F}$. Since F comes out from $\hat{F}$ by identifying $(\hat{x}, b)$ with $(\hat{x}\sigma, b)$ (for $b\in F$), F also comes out from $\hat{M}\times G$ by identifying $(\hat{x}, \zeta)$ with $(\hat{x}\sigma, f(\hat{x}, \sigma)\cdot\zeta)$. This shows that F is associated to f.

Finally suppose f and g give rize to equivalent bundles F and

F'. F is the quotient space of $\hat{M}\times G$ with respect to the relation $(\hat{x}, \zeta)\sim(\hat{x}\sigma, f(\hat{x}, \sigma)\cdot\zeta)$ and F' is similarly defined. Equivalence between F and F' is brought into expression by a correspondence

$$F \ni \text{the class of } (\hat{x}, \xi) \leftrightarrow \text{the class of } (\hat{x}, \eta) \in F',$$

which is compatible with right translation of G. Hence we must have

$$(7') \qquad \eta = \varphi(\hat{x})\cdot\xi \qquad \text{where } \varphi \text{ is a map } \hat{M}\to G\,.$$

If $(\hat{x}\sigma, \xi')$ and $(\hat{x}\sigma, \eta')$ belong to the classes of $(\hat{x}, \xi)$ and $(\hat{x}, \eta)$ respectively, then we have $\xi'=f(\hat{x}, \sigma)\cdot\xi$, $\eta'=g(\hat{x}, \sigma)\cdot\eta$ and $\eta'=\varphi(\hat{x}\sigma)\cdot\xi'$. This gives the relation

$$f(\hat{x}, \sigma) = \varphi(\hat{x}\sigma)^{-1}\cdot g(\hat{x}, \sigma)\cdot\varphi(\hat{x})\,.$$

The converse is easily seen.

Let us remark that if f is independent of $\hat{x}$, then f is an anti-representation of Γ into G and the corresponding bundle F is the one associated to the bundle $\hat{M}\to M$.

§4. Family of decomposable bundles.

Let M be a compact Kähler manifold and let $\mathfrak{P}$ be the Picard variety of M. $\mathfrak{P}$ has the structure of a complex torus and is the set of all complex line bundles over M which are associated to representations of the fundamental group of M into the multiplicative group of non-zero complex numbers.[2] Kodaira has given a line bundle over $M\times\mathfrak{P}$, which explicitly gives an analytic family of line bundles over M parametrized by $\mathfrak{P}$. (See [3], § 2.) We want to construct a family of GL(2, C)-bundles over M, parametrized by the symmetric product V of $\mathfrak{P}$ with itself.

Let n be half the first Betti number of M and let $\omega_1, \cdots, \omega_n$ be a basis of the module A of linear differential forms of the first kind on M. The vector space $\bar{A}$ of the conjugates of the forms in A can be considered as the universal covering space of $\mathfrak{P}$. Namely

$$(11) \qquad \mathfrak{P} = \bar{A}/D\,,$$

2) A line bundle (or a C^*-bundle) is associated to a representation of fundmental group if and only if it has a holomorphic connection.

where $\bar{\beta} \in D$ if and only if $\frac{1}{2\pi\sqrt{-1}}\int_Z(\bar{\beta}-\beta)\in \mathbf{Z}$ for all $Z\in H_1(M, \mathbf{Z})$. We take linear coordinates $(u_1, \cdots, u_n)$ in $\bar{A}$ with respect to the basis $\bar{\omega}_1, \cdots, \bar{\omega}_n$ and apply notations in § 2 for $\mathfrak{P}=T$ and (u).

Let U_1 and U_2 be any two, say U_α and U_β, among $\{U_\alpha\}$ and let (u) and (v) stand for u^α and v^β. We set

$$f((u, v), \gamma)=\begin{pmatrix} \exp\left(\int_\gamma \sum_j v_j\bar{\omega}_j\right) & 0 \\ 0 & \exp\left(\int_\gamma \sum_j u_j\bar{\omega}_j\right)\end{pmatrix} \tag{12}$$

for $(u, v)\in U_1\times U_2$, $\gamma\in\Gamma=$ the fundamental group of M, where $\int_\gamma\bar{\omega}$ denotes the integral of $\bar{\omega}$ on the homology class determined by γ.

We look on (12) as a map from $\hat{M}\times U_1\times U_2\times\Gamma$ into GL(2, C) where $\hat{M}$ is the universal covering space of M and where $\hat{M}\times U_1\times U_2$ is considered as a principal Γ-bundle over $M\times U_1\times U_2$. Since f is independent of the variables describing $\hat{M}$, the analytic bundle over $M\times U_1\times U_2$ defined by f represents an analytic family of bundles over M, parametrized by $U_1\times U_2$, and each member of the family having holomorphic connections.

If we take another open set $U_1'\times U_2'$ instead of $U_1\times U_2$, we see the bundles constructed over $M\times U_1\times U_2$ and $M\times U_1'\times U_2'$ induce equivalent bundles over the intersection, by virtue of (11) and (5). In fact we denote by (u') and (v') local coordinates on U_1' and U_2' respectively, and set

$$\varphi(\hat{x}, (u, v))=\begin{pmatrix} \exp\left(\int_{\hat{o}}^{\hat{x}} \sum_j v_j\omega_j\right) & 0 \\ 0 & \exp\left(\int_{\hat{o}}^{\hat{x}} \sum_j u_j\omega_j\right)\end{pmatrix},$$

where $\hat{x}$ is a variable point on $\hat{M}$ and $\hat{o}$ is a fixed point on $\hat{M}$, then we have

$$f((u', v'), \gamma)=\varphi(\hat{x}\gamma, (u-u', v-v'))^{-1}\cdot f((u, v), \gamma)\cdot\varphi(\hat{x}, (u-u', v-v'))\,.$$

Thus we have principal GL(2, C)-bundles over $M\times U_\alpha\times U_\beta$ for all α, β and any of them induces equivalent bundle on an open set of $M\times\mathfrak{P}\times\mathfrak{P}$ where it is defined. *Under this circumstance we have a principal GL(2, C)-bundle F over whole $M\times\mathfrak{P}\times\mathfrak{P}$. It is clear* *that F induces on* $M\times u\times v$ *the bundle* $E_u\oplus E_v$, *where* E_u *denotes the* C^**-bundle corresponding to* $u\in\mathfrak{P}$.

Now let us consider the automorphism τ of $\mathfrak{P}\times\mathfrak{P}$ which is defined by $\tau(u, v)=(v, u)$. τ induces an automorphism $\tilde{\tau}$ of the bundle F defined in the following way: τ maps $M\times U_1\times U_2$ onto $M\times U_2\times U_1$, and in (12) it holds that

$$f((v, u), \gamma)=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} f((u, v), \gamma)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

We have only to take $\varphi=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ in (7′), to establish required bundle map $\tilde{\tau}$ of the part of F over $M\times U_1\times U_2$ onto that over $M\times U_2\times U_1$.

If we disregard the part of F which lies over $M\times\Delta_{\mathfrak{P}}$, where $\Delta_{\mathfrak{P}}$ denotes the diagonal in the product $\mathfrak{P}\times\mathfrak{P}$, then *formation of the quotient with respect to the group* $\{1, \tau\}$ *gives a bundle* F' *over* $M\times(V-S)$, where S denotes the singular subvariety of the symmetric product V. This family F' has an advantage that we have inequivalent bundles over M for different points of V. (A consequence of a theorem of M. F. Atiyah, see [1], theorem 3, or [5], prop. 4.1.)

Unfortunately, this method breaks down when we take S into account. It seems we have to content ourselves with the following remark: *the bundle F defines a V-bundle on the V-mainfold* $M\times V$ *in the sense of W. L. Bailey* [2], *and we may say we have an analytic family of bundles parametrized by V, in some weaker sense.*

§5. Family parametrized by W.

We have constructed the bundle F' over $M\times(V-S)$, or, if we consider the monoidal transform W (see §2), over $M\times(W-X)$. We want to see what will happen in a neighbourhood of a point w_0 of X. Suppose the homogeneous coordinate t_n which appeared in (4) is not zero at w_0, then in a neighbourhood N of w_0 in W, we may assume that $t_n=1$ and that $x_1, \cdots, x_n, y_{nn}$ and $t_1, \cdots, t_{n-1}$ form a system of local parameters on N. (For the meaning of symbols, see (2).) On the other hand, u_i and v_j for $1\leqq i, j\leqq n$ form a system of local coordinates in $N-X$, and (12) gives the bundle

F' restricted on $N-X$. We first transform f by $\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}$ $\in GL(2, C)$, and obtain

$$g((u, v), \gamma)=\frac{1}{2}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\cdot f((u, v), \gamma)\cdot\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}$$

$$=\exp\left(\int_\gamma \sum x_i\bar{\omega}_i\right)\otimes\begin{pmatrix}\frac{1}{2}(e^{\alpha}+e^{-\alpha}) & \frac{1}{2}(e^{\alpha}-e^{-\alpha})\\ \frac{1}{2}(e^{\alpha}-e^{-\alpha}) & \frac{1}{2}(e^{\alpha}+e^{-\alpha})\end{pmatrix},$$

where $\alpha=\frac{1}{2}\int_\gamma \sum (v_i-u_i)\bar{\omega}_i$. Next transform g by $\begin{pmatrix}1 & 0\\ 0 & v_n-u_n\end{pmatrix}$, then we have

$$h((u, v), \gamma)=\begin{pmatrix}1 & 0\\ 0 & v_n-u_n\end{pmatrix}\cdot g\cdot\begin{pmatrix}1 & 0\\ 0 & 1/(v_n-u_n)\end{pmatrix} \tag{13}$$

$$=\exp\left(\int_\gamma \sum x_i\bar{\omega}_i\right)\otimes\begin{pmatrix}\frac{1}{2}(e^{\alpha}+e^{-\alpha}) & \frac{1}{2(v_n-u_n)}(e^{\alpha}-e^{-\alpha})\\ \frac{(v_n-u_n)}{2}(e^{\alpha}-e^{-\alpha}) & \frac{1}{2}(e^{\alpha}+e^{-\alpha})\end{pmatrix}.$$

All these are maps from $(N-X)\times\Gamma$ into $GL(2, C)$ which define F' over $M\times(N-X)$. But from the invariance of h by $\tau: (u, v)\rightarrow(v, u)$, and from the fact that h remains bounded in N and that $\det(h)$ remains away from zero in N, it follows that (13) can be extended to a holomorphic map from the whole $N\times\Gamma$ into $GL(2, C)$ satisfying (6), and thus F' can be extended to an analytic bundle over $M\times N$. For different N's these bundles are in coherence, and *we have an analytic principal $GL(2, C)$-bundle E over $M\times W$.* For a point $w_0\in X$ where $t_n=1$, we have

$$h(w_0, \gamma)=\exp\left(\int_\gamma \sum_i x_i\bar{\omega}_i\right)\otimes\begin{pmatrix}1 & \frac{1}{2}\int_\gamma\sum_i t_i\bar{\omega}_i\\ 0 & 1\end{pmatrix}. \tag{14}$$

The first factor in (14) gives the line bundle which corresponds to the projection of w_0 to the factor $\mathfrak{P}$ in the product $X=\mathfrak{P}\times P^{n-1}$, and the second factor represents the projection of w_0 to P. As one sees in Matsushima's paper (or in Weil's lectures [7]), different points of X correspond to different indecomposable bundles.

Matsushima's theorem 5 tells that if M is a complex torus T, we obtain the set of all GL(2, C)-bundles with holomorphic connections as the union $V \cup W$ (with obvious identification of $V-S$ and $W-X$). The complex structure he gave to the set of indecomposable bundles is the same as that of our X. This can readily be seen from (14). Namely $\int_{\gamma} \sum t_j \bar{\omega}_j$, considered as a function of $\gamma \in \pi_1(T) = H_1(T, \mathbf{Z})$, represents an element of Matsushima's A_0^{*C}/A^* (see [5], last section), since we integrate $\bar{\omega}$ and not ω.

The final remark is the following: If we consider decomposable and indecomposable bundles together, we have to consider the space $V \cup W$, where points of $V-S$ are identified with the corresponding points of $W-X$, while points of S and X are left separately. *Thus this space is not a Hausdorff space but a pre-analytic variety* as Serre calls in his Faisceaux algébriques cohérents. In view of what has been shown by Kodaira and Spencer ([4] Chapter IV, 14, γ), it is no wonder that we have such a situation. But it seems interesting to the author to have another simple and concrete example of the phenomenon like this one.

REFERENCES

[1] M. F. Atiyah, On the Krull-Schmidt theorem with application to sheaves, Bull. Soc. Math. France, vol. 84 (1956) pp. 307-317.
[2] W. L. Bailey, The decomposition theorem for V-manifolds, Amer. Jour. Math., vol. 78 (1956) pp. 862-888.
[3] K. Kodaira, Characteristic linear systems of complete continuous systems, Ibid., vol. 78 (1956) pp. 716-744.
[4] K. Kodaira ans D. C. Spencer, On deformations of complex analytic structures, Ann. of Math., vol. 67 (1958) pp. 328-466.
[5] Y. Matsushima, Fibrés holomorphes sur un tore complex, to appear in Nagoya Math. Jour.
[6] M. Nagata, On the normality of the Chow variety of positive 0-cycles of degree m in an algebraic variety, These memoirs, vol. 29 (1955) pp. 165-176.
[7] A. Weil, Fibre spaces in algebraic geometry, Lecture notes, Chicago (1952).