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Contributions to Riemann-Roch's theorem

By

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Introduction.

In the present paper we shall give at first a functiontheoretic new proof of the well-known Riemann-Roch's theorem for closed Riemann surfaces. Main ideas lie in the making use of Riemann's periods relation and the theory of linear spaces, that is, by considering some vector spaces consisting of Abelian differentials, linear functionals over them and dual spaces (vector spaces of linear functionals) we get two converse inequalities, therefore the equality, which implies our conclusion. From this point of view the relations between these spaces will be clarified.

In the second paragraph we treat, under the following restrictions, the case of non compact Riemann surfaces by the same method and we obtain an extension of R. Nevanlinna's theorem (Th. 2.3) which is valid for square integrable differentials on Riemann surfaces $\in O_G$ of finite genus, where O_G denotes the class of Riemann surfaces which do not possess a Green's function. Here our restrictions are as follows;

(i) The basic Riemann surface (of finite or infinite genus) R belongs to O_{HD}, i.e. the class of Riemann surfaces admitting no non-vanishing total harmonic differential which is square integrable on R.

(ii) The Abelian differentials should be square integrable on R except the neighborhoods of a finite number of possible singularities. It is known that the inclusion relation $O_G \subset O_{HD}$ is proper only if the genus of R is infinite[1].

Finally, as an application of our theorem, a representation of

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1) Ahlfors and Royden [3] or Tôki [15].

open Riemann surfaces $\in O_{HD}-O_G$ (which are necessarily of infinite genus) will be given.

§ I. Classical Riemann-Roch's theorem.

1. Let R be a closed Riemann surface of genus p and δ be an arbitrary divisor on R given by

$$(1) \qquad \delta=\frac{\delta_{(P)}}{\delta_{(Q)}}=\frac{P_1^{m_1}P_2^{m_2}\cdots P_r^{m_r}}{Q_1^{n_1}Q_2^{n_2}\cdots Q_s^{n_s}}, \qquad m=\sum_{i=1}^{r}m_i, \quad n=\sum_{j=1}^{s}n_j,$$

where $P_1, P_2, \cdots, P_r, Q_1, Q_2, \cdots, Q_s$ denote mutually distinct points on R and $m_1, \cdots, m_r, n_1, \cdots, n_s$ are non negative integers. We denote the total order of δ by

$$(1)' \qquad G=m-n$$

We define now four vector spaces in the *complex field* as follows;

E: The vector space consisting of Abelian differentials on R which are multiples of $1/\delta_{(Q)}$.

D: The vector space consisting of Abelian differentials on R which are multiples of $\delta=\delta_{(P)}/\delta_{(Q)}$.

M: The vector space consisting of Abelian integrals (of the second kind) which are multiples of $1/\delta_{(P)}$ and have no periods along the cycles A_i $(i=1, \cdots, p)$, where (A_i, B_i) $(i=1, \cdots, p)$ are the canonical homology basis of R. We normalize[2)] by additive constants such that these integrals vanish at Q_1.

S: The vector space consising of the single-valued meromorphic functions on R which are multiples of $1/\delta=\delta_{(Q)}/\delta_{(P)}$.

Obviously $D\subset E$, $S\subset M$. Let φ_i $(i=1, \cdots, p)$ be normalized elementary differentials of the first kind and let $\psi_P^{(\nu)}$ and ϕ_{PQ} be normalized elementary differentials of the second resp. third kind, such that

$$(2) \qquad \int_{A_i}\varphi_j=\delta_j^i \text{ (Kronecker)}, \quad \int_{A_i}\psi_P^{(\nu)}=\int_{A_i}\phi_{PQ}=0 \quad (i, j=1, \cdots, p)$$

2) We have no need of normalization (at Q_1) if δ is an integral divisor, i.e. $\delta=\delta_{(P)}$, $\delta_{(Q)}=1$.

and the integral $\int \psi_P^{(\nu)}$ is regular except the point P where it has the expansion of the form $z^{-\nu}+\text{reg. term}$ (z is a local parameter at P), and vanishes at Q_1[2], while $\int \phi_{PQ}$ has two logarithmic singularities P and Q with residues -1 (at P) and $+1$ (at Q)[3]. If it is noticed that these differentials φ_i ($i=1, \cdots, p$), $\psi_{Q_\nu}^{(\mu-1)}$ ($\mu=2, \cdots, n_\nu$; $\nu=1, \cdots, s$), $\psi_{P_\nu}^{(\mu)}$ ($\mu=1, \cdots, m_\nu$; $\nu=1, \cdots, r$) and $\phi_{Q_1Q_2}, \phi_{Q_1Q_3}, \cdots, \phi_{Q_1Q_s}$ are linearly independent each other, and the differentials of the first kind reduce to identically zero, provided that they have no period along A_i ($i=1, \cdots, p$), we find that

$$(3) \qquad \dim E = \begin{cases} p+n-1 \\ p, & \text{if } \delta \text{ is an integral divisor.} \end{cases}$$

$$\dim M = \begin{cases} m \\ m+1, & \text{if } \delta \text{ is an integral divisor.} \end{cases}$$

In fact, for instance, any $\varphi \in E$ can be expressed as

$$(4) \qquad \varphi = \sum_{i=1}^{p} a_i\varphi_i + \sum_{\nu=1}^{s}\sum_{\mu=2}^{n_\nu} -\frac{b_{\mu\nu}}{(\mu-1)}\psi_{Q_\nu}^{(\mu-1)} + \sum_{j=2}^{s} c_j\phi_{Q_1Qj},$$

where $\int_{A_i}\varphi=a_i$, $\varphi=(\sum_{\mu=2}^{n_\nu} b_{\mu\nu}/z_\nu^\mu+c_\nu/z_\nu+\text{reg. term.})dz_\nu$ at Q_ν (z_ν is a uniformizer at Q_ν) ($\nu=1, \cdots, s$), but φ should satisfy the residue relation $\sum_{\nu=1}^{s} c_\nu=0$. We shall write $\dim D=B$, $\dim S=A$.

2. With any pair of two elements $\varphi \in E$ and $\Omega \in M$ we associate the bilinear form

$$(5) \qquad \langle\varphi, \Omega\rangle = 2\pi i \sum_{i=1}^{r} \operatorname*{Res.}_{P_i} \varphi\Omega$$

where $\operatorname*{Res.}_{P_i}$ means the residue at P_i. It is seen immediately that (5) is a bilinear form over the spaces E and M, and it does not be affected by additive constants of the integral Ω, because $\varphi \in E$ is regular at every point P_i. For any $\varphi \in E$ the bilinear form (5) induces a linear functional

$$\varphi[\Omega] = \langle\varphi, \Omega\rangle$$

over the space M. Moreover it permits us to define a linear

3) For the existence of such differentials see Weyl [16], Schiffer-Spencer [14].

functional for each element of the quotient space $F=E/D$, indeed, if $\psi\equiv\varphi_1-\varphi_2\in D$ for $\varphi_1, \varphi_2\in E$, then, since ψ and Ω are multiples of the divisors $\delta_{(P)}/\delta_{(Q)}$ resp. $1/\delta_{(P)}$, the differential $\psi\Omega$ becomes regular at P_i $(i=1, \cdots, r)$, we have therefore

$$\psi[\Omega]=2\pi i\sum_{i=1}^{r}\operatorname*{Res.}_{P_i}\psi\Omega=0\,, \quad \text{i.e.} \quad \varphi_1[\Omega]=\varphi_2[\Omega]\,.$$

Thus each element $\mathcal{G}$ of F is considered as a linear functional $\mathcal{G}[\Omega]$ over the space M, which is defined by

$$\mathcal{G}[\Omega]=\varphi[\Omega]$$

where φ is any element of E which belongs to the class $\mathcal{G}$. The linear functionals $\{\mathcal{G}[\Omega]\}$ corresponding to all elements of F constitute a vector space F^* by a usual rule $(\alpha\mathcal{G}_1^*+\beta\mathcal{G}_2^*)[\Omega]=\alpha\mathcal{G}_1^*[\Omega]+\beta\mathcal{G}_2^*[\Omega]$, $\mathcal{G}_1^*$, $\mathcal{G}_2^*\in\{\mathcal{G}[\Omega]\}$ (α, β are complex numbers). Obviously $(\alpha\mathcal{G}_1+\beta\mathcal{G}_2)[\Omega]=\alpha\mathcal{G}_1[\Omega]+\beta\mathcal{G}_2[\Omega]$, $\mathcal{G}_1, \mathcal{G}_2\in F$. Now we shall prove that the mapping $F\to F^*$ is one-to-one. To see this it suffices to prove that if

$$\mathcal{G}[\Omega]=0 \quad \text{for all} \quad \Omega\in M\,,$$

then we have $\mathcal{G}=0$, i.e. $\varphi\in D$ for any element $\varphi\in\mathcal{G}$. Assume that φ is not a multiple of $\delta=\delta_{(P)}/\delta_{(Q)}$, hence there exists at least one point P_k where it is not a multiple of $P_k^{m_k}$, e.g. $\varphi=z^{m_k'}(c_0+c_1z+\cdots)dz$, $c_0\neq 0$, $0\leq m_k'<m_k$ (z is a local parameter at P_k). Then if we take $\Omega=\int\psi_{P_k}^{(m_k'+1)}\in M$, it follows that

$$0=2\pi i\sum_{i=1}^{r}\operatorname*{Res.}_{P_i}\varphi\int\psi_{P_k}^{(m_k'+1)}=2\pi i c_0\neq 0\,,$$

which is absurd. Hence $\varphi\in D$. We have therefore

$$(6) \qquad \dim F^*=\dim F=\dim E-\dim D\,.$$

3. We consider the (algebraic) dual space M^* of M, i.e. the space of linear functionals over M. Then for every pair of elements $x\in M$ and $y\in M^*$ the bilinear form $\langle y, x\rangle$ is defined by $y[x]$. It is said that two spaces $N(\subset M)$ and $N^*(\subset M^*)$ are mutually orthogonal provided that $\langle y, x\rangle=0$ for any pair of $y\in N^*$ and $x\in N$. We call the set of $y\in N^*$ (resp. $x\in N$) which are orthogonal to N(resp. N^*) the orthogonal space of N(resp. N^*). Now we shall prove that the spaces $F^*=F$ and S are mutually

orthogonal with respect to the bilinear form (5).

For any pair of two elements $\varphi \in E$ and $\Omega \in M$ we have by the well known Riemann's period relation

$$(7) \quad <\varphi, \Omega> = 2\pi i \sum_{i=1}^{r} \underset{P_i}{\mathrm{Res.}}\, \varphi\Omega$$
$$= \sum_{i=1}^{p}\left[\int_{B_i}\varphi\int_{A_i}d\Omega - \int_{A_i}\varphi\int_{B_i}d\Omega\right] - 2\pi i \sum_{j=1}^{s} \underset{Q_j}{\mathrm{Res.}}\, \varphi\Omega\,.$$

This becomes zero if $\Omega \in S$, for the first summation vanishes on account of the single-valuedness of Ω and the latter also does, because φ and Ω are multiples of $1/\delta_{(Q)}$ resp. $\delta_{(Q)}/\delta_{(P)}$, hence $\varphi\Omega$ is regular at every Q_j. Therefore the space F^*, the subspace of the dual space M^* of M, is contained in the orthogonal space $\hat{S}$ of S. Therefore

$$(8) \qquad \dim \hat{S} \geqq \dim F^* = \dim E - \dim D\,.$$

While,

$$\dim \hat{S} = \dim M - \dim S\,.^{4)}$$

Thus in any case we have by (1)′ and (3)

$$(8)' \qquad G+1-A \geqq p-B\,.$$

4. To obtain the inverse inequality of (8)′ we proceed as before, but as all circumstances are not symmetrical, we shall state simply again.

Every element $\Omega \in M$ can be considered as a linear functional

$$\Omega[\varphi] = <\varphi, \Omega>, \quad \varphi \in E$$

over the space E. If $\Omega \equiv \Omega_1 - \Omega_2 \in S$ for $\Omega_1,\ \Omega_2 \in M$ we see $\Omega[\varphi]=0$, i.e. $\Omega_1[\varphi]=\Omega_2[\varphi]$. Therefore to each element $\mathfrak{Q}$ of the quotient space $T=M/S$ corresponds a linear functional $\mathfrak{Q}[\varphi]$. To see that the mapping $T \to T^* = \{\mathfrak{Q}[\varphi]\}$ is the isomorphism it is sufficient to prove that if

$$(9) \qquad \mathfrak{Q}[\varphi] = 0 \quad \text{for all } \varphi \in E\,,$$

we have $\Omega \in S$ for any $\Omega \in \mathfrak{Q}$. First if we take as φ the normalized

4) Cf. Bourbaki [6] p. 48.

differentials φ_i ($\in E$) of the first kind, then we have by (2), (7) and (9)

$$\int_{B_i} d\Omega = 0 \quad (i = 1, \cdots, p)$$

because $\Omega \in M$ has no A-periods and $\varphi_i \Omega$ are regular at every point Q_j. This implies that the integral Ω is single-valued on R. Therefore our relation reduces to

$$0 = \Omega[\varphi] = -2\pi i \sum_{j=1}^{s} \underset{Q_j}{\mathrm{Res.}}\, \varphi\Omega \quad \text{for all } \varphi \in E.$$

Now if $\Omega(Q_t) \neq 0$ ($t \neq 1$), we choose $\varphi = \phi_{Q_1 Q_t} \in E$ then we have $\Omega(Q_t) = \Omega(Q_1) = 0$ which is absurd. Consequently we can conclude as before that Ω is a multiple of $1/\delta$ under suitable choices of normalized differentials of the second kind. These imply $\Omega \in S$. Finally we find at once that the orthogonal space $\hat{D}$ of D contains T^*. Hence we get

$$(10) \quad \dim M - \dim S = \dim T = \dim T^* \leqq \dim \hat{D} = \dim E - \dim D,$$

i.e.

$$(10)' \qquad G + 1 - A \leqq p - B.$$

THEOREM 1 (Riemann-Roch)—*Let δ be a divisor of total order G given on a closed Riemann surface of genus p. Let B (resp. A) denote the numbers of linearly independent differentials (resp. single-valued functions) which are multiples of δ (resp. $1/\delta$), then*

$$(11) \qquad A = B + (G + 1 - p).$$

The equalities in (8) and (10) imply

THEOREM 1′—$\hat{D} = T$, $\hat{S} = F$, *that is, the orthogonal space of D (resp. S) in the dual space E^* of E (resp. M^*) is identical with the quotient space M/S (resp. E/D). Equivalently we can say that two spaces M/S and E/D are mutually dual.*

5. In the above procedure we have used the differentials normalized with respect to A-periods, but of course we can use as usual

(i) the differentials $\varphi_{A_i}, \varphi_{B_i}$ ($i = 1, \cdots, p$) of the first kind normalized in the real sense such that

$$\text{(12)} \quad \operatorname{Re}\int_{B_j}\varphi_{A_i} = -\operatorname{Re}\int_{A_j}\varphi_{B_i} = \delta^i_j, \quad \operatorname{Re}\int_{A_j}\varphi_{A_i} = \operatorname{Re}\int_{B_j}\varphi_{B_i} = 0 \qquad (i, j = 1, \cdots, p)$$

(ii) the normalized differentials $\psi_P^{(\nu)}$, $\tilde{\psi}_P^{(\nu)}$ $(\nu \geq 1)$ of the second kind whose integrals have single-valued real parts and have singularities at $P(z)$ such that

$$\text{(13)} \qquad \int \psi_P^{(\nu)} = z^{-\nu} + \text{reg. term}, \qquad \int \tilde{\psi}_P^{(\nu)} = i z^{-\nu} + \text{reg. term},$$

(iii) the normalized differentials ϕ_{PQ}, $\tilde{\phi}_{PQ}$ of the third kind such that

$$\text{(14)} \qquad \phi_{PQ} = \begin{cases} (-1/z + \text{reg. term})dz & \text{at} \quad P(z), \\ (1/\zeta + \text{reg. term})d\zeta & \text{at} \quad Q(\zeta) \end{cases}$$

$$\tilde{\phi}_{PQ} = \begin{cases} (-i/z + \text{reg. term})dz & \text{at} \quad P(z) \\ (i/\zeta + \text{reg. term})d\zeta & \text{at} \quad Q(\zeta), \end{cases}$$

and $\operatorname{Re}\int \phi_{PQ}$, $\operatorname{Re}\int \tilde{\phi}_{PQ}$ are single-valued on R except a curve running from P to Q.

We consider analogously four vector spaces E', D', M' and S' in the real field, but only the space M' must be taken somewhat in the different way, i.e. we take as M' the vector space consisting of Abelian integrals (of the second kind) which are multiples of $1/\delta_{(P)}$ and have single-valued real parts, moreover they vanish at Q_1 (if δ is not an integral divisor). In this case if we consider the bilinear form

$$\text{(15)} \qquad \langle \varphi, \Omega \rangle = \operatorname{Im}\left[2\pi i \sum_{i=1}^{r} \operatorname*{Res.}_{P_i} \varphi\Omega\right]$$

we can proceed as before and get the same conclusion[5]. In the following paragraph this method will be rather available, for it will connect easily with the uniqueness theorem on non-compact surfaces.

§ II. Non compact cases.

Now if we step from closed Riemann surfaces to open surfaces, how becomes of the Riemann-Roch's theorem? An arbitrary open Riemann surface R may have the genus of infinity. Moreover a

5) Cf. the next paragraph, especially the proof of Theorem 2.

single-valued meromorphic function on R may have an infinite number of poles and zeros clustering nowhere in R, hence we are able to give a divisor consisting of an infinite number of points. While, on any R there exists always single-valued meromorphic functions having exactly prescribed singularities and zeros (Behnke-Stein [5]), therefore the dimension becomes infinity for the space consisting of single-valued functions which are multiples of the divisor with an infinite number of poles. In order to obtain those which are analogous to the classical formula therefore we shall have to put some restrictions to basic surface, divisor or differentials.

6. Let R be an arbitrary open Riemann surface and G be any subregion of R. We shall write Dirichlet integral of a function u (or differential du) taken over G as

$$D_G[u]=\iint_G |\operatorname{grad} u|^2\, dxdy=\iint_G du \wedge {}^*du$$

where $z=x+iy$ is a local uniformizer. In the following we restrict our differentials to those of the class $\mathfrak{D}$, which is a vector space consisting of Abelian differentials (or integrals) whose Dirichlet integrals taken over R are finite except the neighborhoods of possible singularities at a finite number of points. We shall denote by $\mathfrak{D}_i$ $(i=1,2,3)$ the subsets of $\mathfrak{D}$ which consist of differentials of the i-th kind respectively. Now let $\{A_i, B_i\}$ $(i=1,2,\cdots)$ be a canonical homology basis of R[6], i.e. $\{A_i, B_i\}$ $(i=1,2,\cdots)$ is a homology basis on R such that the intersection numbers N satisfy the conditions $N(A_n, A_m)=N(B_n, B_m)=0$, $N(A_n, B_m)=\delta_m^n$ $(m, n=1,2,\cdots)$.

The existence of the following fundamental differentials[7] which play important roles in the theory of open Riemann surfaces is known, which will be used in the sequel.

(i) φ_{A_i}, φ_{B_i} $(i=1,2,\cdots)\in\mathfrak{D}_1$ such that $\mathrm{Re}\int\varphi_{A_i}$, $\mathrm{Re}\int\varphi_{B_i}$ have no periods along the cycles except B_i resp. A_i along which they satisfy the same period-conditions as (12), but here $i, j=1,2,\cdots$.

(ii) $\psi_P^{(\nu)}$, $\tilde{\psi}_P^{(\nu)}$ $(\nu=1,2,\cdots)\in\mathfrak{D}_2$ whose integrals have single-valued real parts on R and singularities at P where they have the same expansions as (13).

6) Ahlfors [1].
7) Sario [13] or R. Nevanlinna [12].

(iii) ϕ_{PQ}, $\tilde{\phi}_{PQ}$ $(P \neq Q) \in \mathfrak{D}_3$ such that they have singularities with the expansions (14) at P and Q, and $\mathrm{Re}\int\phi_{PQ}$, $\mathrm{Re}\int\tilde{\phi}_{PQ}$ are single-valued on R except a curve running from P to Q.

7. Hereafter we confine our open Riemann surfaces R to those belonging to the class O_{HD}. Let W be a subregion of $R \in O_{HD}$ such that each component of the boundary ∂W is a Jordan closed curve and divides R into two disjoint parts. Suppose $\delta = \delta_{(P)}/\delta_{(Q)}$ ((1), (1)') is the divisor given on W. Now four vector spaces $E(W)$, $D(W)$, $M(W)$ and $S(W)$ in the *real field* are defined as follows:

$E(W)$: The vector space consisting of Abelian differentials $\varphi \in \mathfrak{D}$ with the properties (i) φ are multiples of $1/\delta_{(Q)}$, (ii) $\mathrm{Re}\int\varphi$ are single-valued on $R-W$, and (iii) residue relations $\sum \mathrm{Res.}\,\varphi = 0$ are satisfied.

$D(W)$: The vector space consisting of Abelian differentials $\in E(W)$ which are multiples of $\delta = \delta_{(P)}/\delta_{(Q)}$.

$M(W)$: The vector space consisting of Abelian integrals $\Omega \in \mathfrak{D}$ which are multiples of $1/\delta_{(P)}$ and $\mathrm{Re}\,\Omega$ are single-valued on R. In case of non-integral divisor we normalize such that $\Omega(Q_1) = 0$.

$S(W)$: The vector space of Abelian integrals $\in M(W)$ which are multiples of $1/\delta = \delta_{(Q)}/\delta_{(P)}$ and single-valued on W.

Let $\{A_i, B_i\}$ $(i = 1, 2, \cdots, p(W)$; $p(W)$ denotes the genus of $W)$ be a canonical homology basis of W mod ∂W. Since $R \in O_{HD}$ it is easily seen (cf. § I. sec. 1) that the space $E(W)$ is composed of normalized elementary differentials $\varphi_{A_i}, \varphi_{B_i} \in \mathfrak{D}_1$ $(i = 1, \cdots, p(W))$, $\psi^{(\mu-1)}_{(Q_\nu)}$, $\tilde{\psi}^{(\mu-1)}_{(Q_\nu)} \in \mathfrak{D}_2$ $(\mu = 2, \cdots, n_\nu,\ \nu = 1, \cdots, s)$ and $\phi_{Q_1Q_2}$, $\tilde{\phi}_{Q_1Q_2}, \cdots,$ $\phi_{Q_1Q_s}$, $\tilde{\phi}_{Q_1Q_s} \in \mathfrak{D}_3$ and the space $M(W)$ is composed of constants and integrals $\int\psi^{(\mu)}_{P_\nu}$, $\int\tilde{\psi}^{(\mu)}_{P_\nu}$ $(\mu = 1, \cdots, m_\nu,\ \nu = 1, \cdots, r)$[8]. We shall state at first the following theorem, which will be proved in sec. 9.

THEOREM 2.—*Let R be an open Riemann surface $\in O_{HD}$ and W be the subregion of R whose boundary ∂W consists of a finite number of Jordan closed curves, each of which divides R into two disjoint*

8) If δ is not an integral divisor, we normalize such that these vanish at Q_1, hence in this case M consists of these integrals only.

parts. Then, for the divisor $\delta=\delta_{(P)}/\delta_{(Q)}$ ((1), (1)') *given on* W *the orthogonal space of* $D(W)$ (*resp.* $S(W)$) *in the dual space* $E(W)^*$ *of* $E(W)$ (*resp.* $M(W)^*$) *is identical with the quotient space* $M(W)/S(W)$ (*resp.* $E(W)/D(W)$), *in other words, two spaces* $M(W)/S(W)$ *and* $E(W)/D(W)$ *are mutually dual. Thus, we have the formula*

$$p(W)-B(W)=G-A(W)+1\,, \tag{16}$$

where $p(W)$ *is the genus of* W *and* $2B(W)$ (*resp.* $2A(W)$) *denotes the dimension of the space* $D(W)$, (*resp.* $S(W)$).

For our later purposes we note that the real parts of integrals of the elementary differentials ((i) (ii) (iii) p. 168) on $R\in O_{HD}$ respectively identical, except constants, with the normalized potentials u[9] which possess the same behaviors as (i), (ii), (iii) respectively and have the following property; let $\{R_n\}_{n=1,2,\cdots}$ be any exhaustion of R and u_n be harmonic functions defined on $R_n-\bar{R}_1$ which vanish on ∂R_n and are identical with u on ∂R_1, then $\lim\limits_{n\to\infty}u_n=u$ uniformly on every compact subset of $R-R_1$, where we suppose that R_1 contains the singularities of u.

Now to approach the Riemann-Roch's relation for R we consider an arbitrary divisor

$$\delta=\frac{\delta_{(P)}}{\delta_{(Q)}}=\frac{P_1^{h_1}P_2^{h_2}\cdots}{Q_1^{k_1}Q_2^{k_2}\cdots}$$

where h_i, k_i $(i=1,2,\cdots)$ are non-negative integers. Let $R_1\subset R_2\subset\cdots\subset R_n\subset\cdots\to R$ be an exhaustion of R such that each component of the boundary $\Gamma_n\equiv\partial R_n$ is a closed analytic curve and divides R (such an exhaustion always exists). Without loss of generality we may assume that every Γ_n does not contain any P_i, Q_i. Let

$$\delta^n=\frac{\delta^n_{(P)}}{\delta^n_{(Q)}}=\frac{P_1^{h_1}P_2^{h_2}\cdots P_{r_n}^{h_{r_n}}}{Q_1^{k_1}Q_2^{k_2}\cdots Q_{s_n}^{k_{s_n}}}\qquad \sum_{i=1}^{r_n}h_i=m_n\,,\quad \sum_{j=1}^{s_n}k_j=\kappa_n \tag{17}$$

be the restriction of δ to R_n i.e. $\delta^n=\delta\cap R_n$, $\delta^n_{(P)}=\delta_{(P)}\cap R_n$ and $\delta^n_{(Q)}=\delta_{(Q)}\cap R_n$. Then for every subregion R_n and δ^n (16) shows

$$\kappa_n+p_n-\mathrm{B}_n=m_n-\mathrm{A}_n+1\,.\qquad (n=1,2,\cdots) \tag{16'}$$

where $p_n=p(R_n)$, $\mathrm{B}_n=B(R_n)$ and $\mathrm{A}_n=A(R_n)$. Here we note that it is possible to choose the canonical homology basis $\{A_i, B_i\}$ of

9) R. Nevanlinna [12] p. 320-333.

R such that every section $\{A_i, B_i\}$ $(i=1, \cdots, p_n)$ is a (relative) canonical homology basis on R_n mod Γ_n.[6] Next we consider the the semi-infinite divisor

$$(18) \qquad \delta = \frac{\delta_{(P)}}{\delta_{(Q)}} = \frac{P_1^{h_1} P_2^{h_2} \cdots P_r^{h_r}}{Q_1^{k_1} Q_2^{k_2} \cdots} \qquad \sum_{i=1}^{r} h_i = m < \infty$$

and the space S which is defined as follows

S: The vector space consisting of mermorphic functions $\in \mathfrak{D}$[10] which are single-valued on R and multiples of $1/\delta = \delta_{(Q)}/\delta_{(P)}$.

We may suppose that R_1 already contains the points $P_1, \cdots, P_r, Q_1$. Then it is easily seen that

$$M \equiv M_1 = M_2 = \cdots \supset S_1 \supset S_2 \supset \cdots \supset S_n \supset \cdots \supset S,$$

where $M_n = M(R_n)$, $S_n = S(R_n)$, and that

$$S = \bigcap_{n=1}^{\infty} S_n.$$

Let σ_n be the dimension of S_n, then $2m+2 \geq \dim M \geq \sigma_1 \geq \cdots \geq \sigma_n \geq \cdots \geq 0$, hence $\lim_{n\to\infty} \sigma_n = \sigma$ exists. Since all σ_n are integers, $\sigma_n = \sigma$ for all $n \geq N$ which imply the equalities

$$S_N = S_{N+1} = \cdots = \bigcap_{n=1}^{\infty} S_n = S.$$[11]

Therefore by Theorem 2 we have

THEOREM 2. 1[12]—*Let* δ *be a semi-infinite divisor* (18) *given on* $R \in O_{HD}$. *Then*

$$\kappa_n + p_n - \mathrm{B}_n = m - A + 1 \qquad \text{for } n \geq N.$$

We find that $A = \mathrm{A}_n$ $(n \geq N)$ *indicates the number of single-valued functions* $\in \mathfrak{D}$ *which are multiples of* $1/\delta$ *and linearly independent in the complex sense, and that* B_n *are integers and* $\lim_{n\to\infty} (\kappa_n + p_n - \mathrm{B}_n)$ *is independent of the exhaustion of* R.

If $R \in O_G$, for any $\varphi \in \mathfrak{D}$ the residue relation is automatically satisfied.[13] Hence

10) These always belong to $\mathfrak{D}_2$, because they are single-valued on $R \in O_{HD}$.

11) Cf. Bourbaki [6] p. 37.

12) Cf. the remark in sec. 10.

13) R. Nevanlinna [12], but if $R \in O_{HD} - O_G$, this does not hold in general, for example there exists a Green differential on R.

THEOREM 2. 2—*Let* δ *be a finite divisor* (1), (1)′ *given on* $R \in O_G$. *Then*

$$p_n - \mathrm{B}_n = G - A + 1 \quad \textit{for} \quad n \geq N,$$

where $2\mathrm{B}_n$ *denotes the number of differentials* $\varphi \in \mathfrak{D}$ *linearly independent in the real sense which are multiples of* δ *and* $\mathrm{Re}\int\varphi$ *are single-valued on* $R - R_n$.

Further let p (genus of $R \in O_G$) be finite, then for large n the complementary domains $R - R_n$ become of planar character and $p_n = p$ $(n \geq N)$, therefore for any $\varphi \in \mathfrak{D}$ $\mathrm{Re}\int\varphi$ are single-valued on $R - R_n$[14] and $E_n \equiv E(R_n) = E$ $(n \geq N)$. Thus, we have

THEOREM 2. 3 (R. Nevanlinna)[15]—*Let* R *be an open Riemann surface* $\in O_G$ *and* p, *the genus of* R, *be finite. Then for the divisor* δ (1), (1)′, *we have Riemann-Roch's formula*

$$p - B = G - A + 1\,,$$

where B (*resp.* A) *is the number of differentials* $\in \mathfrak{D}$ (*resp. single-valued functions* $\in \mathfrak{D}$) *linearly independent in the complex sense which are multiples of* δ (*resp.* $1/\delta$).

REMARK.—An open Riemann surface $R \in O_G$ of finite genus can be imbedded in a closed Riemann surface R^* of the same genus, where the ideal boundary of R appears as a set $\Delta(\subset R^*)$ of capacity zero. Hence every element of spaces E or S becomes regular even on Δ by the generalized continuation principle, because integrals become single-valued, bounded in the neighborhood of Δ. Conversely if the divisor given on R^* has no intersection with Δ, every element of E and S for R^* belong obviously to E resp. S for R. Hence if we transform a closed surface to an open one by rejecting one point which does not appear in δ, then we can obtain again the classical Riemann-Roch's theorem from the above Theorem 2. 3.

8. For the proof of Theorem 2 we prepare two lemmas.

LEMMA 1.—*Let* R *be a Riemann surface* $\in O_{HD}$ *and* $dw = du + idv$ *be a differential* $\in \mathfrak{D}$. *If the total sum of its residues is zero and*

14) R. Nevanlinna [11] or next Lemma 1.

15) R. Nevanlinna [11] p. 32.

the function u is single-valued on $R-K$, then for every dividing curve $C \subset R-K$, we have

$$\int_C dw = 0 .$$

where K denotes a compact subregion containing all the singularities of dw.

Proof. We may assume that C is an analytic Jordan closed curve not bounding a compact region. The other cases are trivial. Now suppose that our assertion is not valid, hence for such a curve $C_1 \int_{C_1} dw = \int_{C_1} idv \neq 0$, e.g.

$$\int_{C_1} dv > 0 . \tag{19}$$

Consider a finite number of dividing curves $C_2, \cdots, C_m$, which bound together with C_1 a compact subregion $B \supset K$, then, by the residue relation, we have

$$\sum_{i=1}^{m} \int_{C_i} dw = i \sum_{i=1}^{m} \int_{C_i} dv = 0 .$$

Hence there exists at least a curve, C_2 say, such that

$$\int_{C_2} dv < 0 , \tag{20}$$

where the integrations are taken in the positive direction with resp. to B. Let G_1 and G_2 be non-compact regions on R whose relative boundaries on R consist of C_1 resp. C_2 only. $G_1 \cap G_2 = \phi$. Let $\{G_j^n\}_{n=1,2,\cdots}$ $(j=1, 2)$ be the exhaustions of G_j and G_j^n be bounded by analytic curves C_j and Γ_j^n. Let us choose a constant $k \geq 0$ such that

$$\min_{p \in C_1} u(p) + k > 0 , \tag{21}$$

and construct the harmonic functions U_n $(n=1, 2, \cdots)$ on G_1^n such that

$$U_n(p) = \begin{cases} u(p) + k , & p \in C_1 \\ 0 , & p \in \Gamma_1^n , \end{cases}$$

then the sequence $\{U_n\}$ is monotone increasing and uniformly bounded; $0 \leq U_n \leq \max_{p \in C_1} u(p) + k$. Therefore by Harnack's theorem the limit function U_0 becomes harmonic on G_1 and the convergence is uniform on every compact set of G_1. Moreover this holds on $G_1 \cup C_1$. For the functions $V_n \equiv U_n - (u+k)$ vanish on the analytic curve C_1, hence V_n can be harmonically continued across C_1 by the reflection principle. The bounded sequence $\{V_n\}$ of increasing functions therefore tends to $V_0 = U_0 - (u+k)$ uniformly even on $G_1 \cup C_1$. Hence the derivatives $\frac{\partial U_n}{\partial \nu} = \frac{\partial V_n}{\partial \nu} + \frac{\partial u}{\partial \nu}$ on C_1 converge uniformly to $\frac{\partial U_0}{\partial \nu}$ where $\frac{\partial}{\partial \nu}$ denotes the differentiation in the direction of the inner normal of G_1^n. We find that Dirichlet integral $D_{G_1}[U_0]$ is finite. In fact, fix an integer n, then we have for $m > n$

$$0 \leq D_{G_1^n}[U_m] \leq D_{G_1^m}[U_m] = -\int_{C_1} U_m \frac{\partial U_m}{\partial \nu} ds = -\int_{C_1} (u+k) \frac{\partial U_m}{\partial \nu} ds,$$

Letting $m \to \infty$, successively $n \to \infty$, it follows under above remarks that

$$0 \leq D_{G_1}[U_0] \leq -\int_{C_1} (u+k) \frac{\partial U_0}{\partial \nu} ds < \infty .$$

Now we distinguish two cases;

$$(\alpha) \quad V_0 (= U_0 - (u+k)) \equiv 0 \qquad (\beta) \quad V_0 \not\equiv 0 .$$

Here we shall show that (α) is not the case. Suppose $U_0 \equiv u+k$. Let ω_n be harmonic measures of G_1^n which vanish on C_1 and $=1$ on Γ_1^n. By Green's formula we have

$$\int_{C_1} (u+k) \frac{\partial \omega_n}{\partial \nu} ds = \int_{\Gamma_1^n} \frac{\partial U_n}{\partial \nu} ds = -\int_{C_1} \frac{\partial U_n}{\partial \nu} ds$$

hence for $n \to \infty$

$$\int_{C_1} (u+k) \frac{\partial \omega}{\partial \nu} ds = -\int_{C_1} \frac{\partial U_0}{\partial \nu} ds = -\int_{C_1} \frac{\partial u}{\partial \nu} ds = -\int_{C_1} dv .$$

which is a contradiction. For the right hand side is negative by (19), while the left hand side is non-negative by (21) and the fact $\frac{\partial \omega}{\partial \nu} \geq 0$ on C_1. Therefore the case (β) happens, which implies the existence of a non-constant harmonic function V_0 on G_1 which $=0$ on C_1 and has a finite Dirichlet integral over G_1. On the other hand by (20) we have $\int_{C_2} d(-v) > 0$, hence we also see the existence of a non-constant harmonic function V_0' on G_2 which $=0$ on C_2 and $D_{G_2}[V_0'] < \infty$. While, since $G_1 \cap G_2 = \phi$, the existence of such two functions V_0 and V_0' implies $R \notin O_{HD}$ (Bader-Parreau [4] or Mori [10]), which contradicts with our hypothesis, q.e.d.

LEMMA 2.—*Let* $df_j = du_j + idv_j$ $(j=1, 2)$ *be any two differentials* $\in \mathfrak{D}$, *defined on a Riemann surface* $R \in O_{HD}$. *If each total sum of residues of* df_j *is zero and the functions* u_1, u_2 *are single-valued on* $R-K$, *where* K *denotes a compact subregion containing the singularities of* df_1 *and* df_2, *then for any dividing curve* $C \subset R-K$ *we have*

$$\mathrm{Im}\,[\int_C f_1 df_2] = 0\,.$$

Proof. Let $\{R_n\}$ be an exhaustion of the domain[17] ($\not\supset K$) bounded by a relative boundary C, such that each component Γ_n^i $(i=1, \cdots, t_n)$ of $\Gamma_n = \partial R_n - C$ is an analytic curve dividing R. Since u_1, u_2 are single-valued on $R-K$, we have easily by Riemann's first period relation[18]

$$\mathrm{Im}\int_C f_1 df_2 = \mathrm{Im}\int_{\Gamma_n} f_1 df_2\,.$$

Here we shall prove that

$$\lim_{n\to\infty} \mathrm{Im}\int_{\Gamma_n} f_1 df_2 = 0\,.$$

By the remark in sec. 7 (p. 170) we have

$$u_j = U_j + \alpha_j\,,\ v_j = V_j + \beta_j\ (\alpha_j, \beta_j \text{ are real constants}) \quad (j=1, 2),$$

where U_j are normalized potentials and V_j the conjugate harmonic functions of U_j. By the Lemma 1, we have

$$\begin{aligned}\mathrm{Im}\int_{\Gamma_n} f_1 df_2 &= \sum_{i=1}^{t_n}\int_{\Gamma_n^i} V_1 dU_2 + U_1 dV_2 + \alpha_1 dV_2 + \beta_1 dU_2\\ &= \sum_{i=1}^{t_n}\int_{\Gamma_n^i} V_1 dU_2 + U_1 dV_2\,.\end{aligned}$$

At first fix the integer n. Since U_j $(j=1, 2)$ are normalized potentials, they are respectively the uniform limits of harmonic functions U_j^m $(m=n+1, n+2, \cdots)$ which $=0$ on Γ_m and $=U_j$ on C. Since U_1^m are single-valued on $G_{m,n}^i = G_n^i \cap R_m$, we have

$$\int_{\Gamma_n^i} U_1^m dV_2 = D_{G_{m,n}^i}[U_1^m, U_2]$$

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17) We suppose this is non compact, for the other case is trivial.

18) Cf. Kusunoki [9].

where each G_n^i ($\not\supset C$) denotes the non compact domain whose relative boundary consists of Γ_n^i only and the second term is the mixed Dirichlet integral of U_1^m and U_2. Hence for $m\to\infty$

$$L_n^i\equiv\lim_{m\to\infty} D_{G_{m,n}^i}[U_1^m, U_2]=\int_{\Gamma_n^i} U_1 dV_2$$

On the other hand, by Schwarz's inequality, we have

$$D_{G_{m,n}^i}[U_1^m, U_2]^2\leq D_{G_{m,n}^i}[U_1^m]\,D_{G_{m,n}^i}[U_2]\leq\int_{\Gamma_n^i} U_1^m\frac{\partial U_1^m}{\partial\nu}ds\,D_{G_n^i}[U_2].$$

Since $\dfrac{\partial U_1^m}{\partial\nu}$ converge to $\dfrac{\partial U_1}{\partial\nu}$ uniformly on Γ_n^i, we have $m\to\infty$

$$(L_n^i)^2\leq\int_{\Gamma_n^i} U_1 dV_1\ D_{G_n^i}[U_2].$$

While

$$\left|\int_{\Gamma_n^i} U_1^m dV_1\right|^2=D_{G_{m,n}^i}[U_1^m, U_1]^2\leq D_{G_{m,n}^i}[U_1^m]\,D_{G_{m,n}^i}[U_1]$$
$$\leq\int_{\Gamma_n^i} U_1^m{}^* dU_1^m D_{G_n^i}[U_1],$$

hence for $m\to\infty$ we have

$$\left|\int_{\Gamma_n^i} U_1 dV_1\right|\leq D_{G_n^i}[U_1].$$

Therefore

$$\sum_{i=1}^{t_n}|L_n^i|\leq\sum_{i=1}^{t_n}\sqrt{D_{G_n^i}[U_1]D_{G_n^i}[U_2]}\leq(D_{G_n}[U_1]+D_{G_n}[U_2])/2$$

where $G_n=\sum_{i=1}^{t_n}G_n^i$. Since $U_1, U_2\in\mathfrak{D}$, the right hand side tends to 0 for $n\to\infty$, i.e. we have

$$\int_{\Gamma_n} U_1 dV_2\to 0.$$

As for $\int V_1 dU_2$, by Riemann's second period relation[19], we have

$$\int_{\Gamma_n^i} V_1 dU_2^m=-D_{G_{m,n}^i}[U_1, U_2^m]$$

which tends to $\int_{\Gamma_n^i} V_1 dU_2$ for $m\to\infty$. Thus it is analogously concluded that

19) Ahlfors [1] or Kusunoki [9].

$$\int_{\Gamma_n} V_1 dU_2 \to 0 \quad \text{for } n\to\infty, \qquad \text{q.e.d.}$$

9. Proof of Theorem 2. Now we take vector spaces $E=E(W)$, $D=D(W)$, $M=M(W)$ and $S=S(W)$ defined before, and proceed as in § I, where the scalar product is defined by (15). We shall show here only the key points of the proof. By the remark in sec. 7 (p. 169) we have

$$\dim M=\begin{cases} 2m \\ 2(m+1), & \text{if } \delta \text{ is integral.}\end{cases}$$

$$\dim E=\begin{cases} 2(n+p-1), & p=p(W) \\ 2p, & \text{if } \delta \text{ is integral.}\end{cases}$$

Each element of the space $F=E/D$ is considered as a linear functional over the space M, and the spaces F and S are orthogonal with resp. to (15), because for $\varphi\in E$, $\Omega\in M$ we have

$$\varphi[\Omega]=\Omega[\varphi]=\operatorname{Im}\left[\sum_{i=1}^{p}\left(\int_{B_i}\varphi\int_{A_i}d\Omega-\int_{A_i}\varphi\int_{B_i}d\Omega\right)\right]$$
$$+\operatorname{Im}\sum_{i=1}^{t}\int_{\Gamma^i}\Omega\varphi-\operatorname{Im}(2\pi i\sum_{j=1}^{s}\operatorname{Res}_{Q_j}\Omega\varphi),$$

where $\sum_{i=1}^{t}\Gamma^i=\partial W$. The second term $=0$ by Lemma 2, hence

$$\Omega[\varphi]=\sum_{i=1}^{p}\left(\operatorname{Re}\int_{B_i}\varphi\ \operatorname{Im}\int_{A_i}d\Omega-\operatorname{Re}\int_{A_i}\varphi\ \operatorname{Im}\int_{B_i}d\Omega\right) \tag{22}$$
$$-2\pi\operatorname{Re}\sum_{j=1}^{s}\operatorname{Res}_{Q_j}\Omega\varphi.$$

If $\Omega\in S$, it follows easily that $\varphi[\Omega]=0$, hence

$$\dim E-\dim D\leq\dim M-\dim S.$$

To obtain the converse inequality we prove $T=M/S\simeq T^*$, i.e. we should conclude $\Omega\in S$ from the assumption that $\Omega[\varphi]=0$ for $\varphi\in E$. If $\varphi=\varphi_{A_i}, \varphi_{B_i}$ are chosen we have by (22)

$$\operatorname{Im}\int_{A_i}d\Omega=\operatorname{Im}\int_{B_i}d\Omega=0.$$

Therefore $d\Omega$ has no period along the cycles (A_i, B_i) $(i=1,\cdots,p)$, moreover along every boundary cycle Γ^i by Lemma 1, hence Ω becomes single-valued on W and our condition reduces to

$$0=\Omega[\varphi]=-2\pi\,\mathrm{Re}\sum_{j=1}^{s}\underset{Q_j}{\mathrm{Res.}}\,\Omega\varphi \quad \text{for all } \varphi\in E.$$

If we choose successively $\varphi=\psi_{Q_\nu}^{(\mu)}, \tilde{\psi}_{Q_\nu}^{(\mu)}$ and $\phi_{Q_1Q_2}, \tilde{\phi}_{Q_1Q_2}, \cdots, \phi_{Q_1Q_s}, \tilde{\phi}_{Q_1Q_s}$, then we see that Ω is a multiple of $1/\delta$. Therefore $\Omega\in S$. Finally the spaces M and D are obviously orthogonal, thus we have

$$\dim E-\dim D\geq\dim M-\dim S, \qquad \text{q.e.d.}$$

10. REMARKS AND APPLICATIONS. *In the Theorem* 2.1 *we have indeed* $A=1$ (*i.e. every function* $\in S$ *reduces to a constant*) *for the following cases*:

(i) *The case that* δ *is the simplest integral divisor* $\delta=P$, *where* P *is an arbitrary point on* R ($\in O_{HD}$) *of genus* ≥ 1. *Then*

$$p_n-\mathrm{B}_n=1 \quad \textit{for} \quad n\geq N. \tag{23}$$

(ii) *The case that* δ *is any integral divisor* (*of order* $m<\infty$) *given on* $R\in O_{HD}-O_G$. *Then we have*

$$p_n-\mathrm{B}_n=m \quad \textit{for} \quad n\geq N. \tag{24}$$

To prove this, let $\delta=P$ be a divisor given on $R\in O_G$. If $df\in\mathfrak{D}_1$, $f\in S$, it is trivial, hence we suppose now that $f\in S$ has a simple pole at P. Then there exists a sufficiently small neighborhood U of P which is mapped univalently by $w=f$. Since $D_{R-U}[f]<\infty$, f is bounded on $R-U$, hence there is a subset U' of U such that the image $f(R)$ covers $f(U')$ exactly once. This is possible only if $f(R)$ covers the w-plane at most once, which is absurd if the genus of $R\geq 1$. Next, suppose δ is any integral divisor $P_1^{m_1}\cdots P_r^{m_r}$ given on $R\in O_{HD}-O_G$. According to Kuramochi's theorem (cf. [8] or Cornea [7]), for every compact domain K on $R\in O_{HD}-O_G$, we have $R-K\in O_{AD}$ (i.e. every single-valued analytic function with finite Dirichlet integral over $R-K$ reduces to a constant). Therefore if we choose a compact domain K such that it contains the points $P_1,\cdots,P_r$ and $R-K$ is connected, then every element of S has a finite Dirichlet integral over $R-K$, hence it reduces to a constant, q.e.d.

Let the cycles A_n, B_n ($n=1,2,\cdots$) be denoted K_{2n-1}, K_{2n} respectively. Here, by the same notation φ_{K_i} we understand the covariants corresponding to previous differentials φ_{K_i}. From above remarks it follows:

(i) The relation (23) implies that the matrix

$$\left\|\begin{matrix} \operatorname{Re}\varphi_{K_1}(z) & \operatorname{Re}\varphi_{K_2}(z) & \cdots & \operatorname{Re}\varphi_{K_{2pn}}(z) \\ \operatorname{Im}\varphi_{K_1}(z) & \cdots\cdots & \cdots\cdots & \operatorname{Im}\varphi_{K_{2pn}}(z) \end{matrix}\right\| \quad (z \text{ is a local parameter at } P)$$

has rank 2. Therefore φ_{K_i} $(i=1, 2, \cdots)$ *on* $R \in O_{HD}$ *never have common zero points.*

(ii) Let $\delta = PQ$ be a divisor given on $R \in O_{HD} - O_G$, where $P(z)$ and $Q(\zeta)$ are arbitrary two distinct points on R. Then from (24) the matrix

$$\left\|\begin{matrix} \operatorname{Re}\varphi_{K_1}(z) & \cdots & \operatorname{Re}\varphi_{K_{2pn}}(z) \\ \operatorname{Im}\varphi_{K_1}(z) & \cdots & \\ \operatorname{Re}\varphi_{K_1}(\zeta) & \cdots & \\ \operatorname{Im}\varphi_{K_1}(\zeta) & \cdots & \operatorname{Im}\varphi_{K_{2pn}}(\zeta) \end{matrix}\right\|$$

has rank 4. Hence at least one of the determinants

$$\begin{vmatrix} \varphi_{K_i}(z) & \varphi_{K_j}(z) \\ \varphi_{K_i}(\zeta) & \varphi_{K_j}(\zeta) \end{vmatrix} \quad i \neq j, \quad i, j = 1, 2, \cdots$$

is different from zero. From this we can conclude that if we denote by $x_n(p)$ $(n=1, 2, \cdots)$ a countable number of non-constant meromorphic functions (quotients of square integrable covariants)

$$\frac{\varphi_{K_i}(z)}{\varphi_{K_j}(z)} = \frac{\displaystyle\int_{K_i} \frac{\partial^2 g(z, \zeta)}{\partial z \partial \nu_\zeta} ds_\zeta}{\displaystyle\int_{K_j} \frac{\partial^2 g(z, \zeta)}{\partial z \partial \nu_\zeta} ds_\zeta}, \quad i \neq j, \quad p = p(z).$$

where $\dfrac{\partial}{\partial z}$ denotes a usual complex derivative and g a Green function of R, there exists a function x_m such that $x_m(P) \neq x_m(Q)$ for each pair of two points P and Q on R. Thus, *any open Riemann surface* $R \in O_{HD} - O_G$ *(necessarily, of infinite genus) can be mapped univalently into the product space* $C \times C \times \cdots$ *of a countable number of complex planes* C *by the vector-valued function*

$$f = (x_1(p), x_2(p), \cdots),$$

where $x_n(p)$ *are the above functions on* R.

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