Memoirs of the College of Science, University of Kyoto, Series A
Vol. XXXII, Mathematics No. 1, 1959.

On the equiasymptotic stability in the large

By

Taro Yoshizawa

(Received Feb. 27, 1959)

In the foregoing paper [3]* we have discussed a necessary and sufficient condition for the equi-ultimate boundedness of solutions of the differential system. In this paper we will discuss the asymptotic stability in the large by the idea in [3].

Now we consider a system of differential equations,

$$\frac{dx}{dt}=F(t, x)\,, \tag{1}$$

where x denotes an n-dimensional vector and $F(t, x)$ is a given vector field which is defined and continuous in the domain

$$\Delta:\quad 0\leqq t<\infty\,,\qquad \|x\|<\infty\,.$$

We adopt the notations in [3]. Moreover for the purpose of discussing the stability, we assume that

$$F(t, 0)\equiv 0\,. \tag{2}$$

Definition. *The solution* $x(t)\equiv 0$ *of* (1) *is said to be equi-asymptotically stable in the large if there exists a positive constant* $T(t_0, \alpha, \varepsilon)$, *defined for any* $\varepsilon>0$ *and any non-negative value of* α *and* $t_0\geqq 0$, *such that* $\|x_0\|\leqq\alpha$, $t_0\geqq 0$ *and* $t>t_0+T(t_0, \alpha, \varepsilon)$ *imply* $\|x(t\,;x_0, t_0)\|<\varepsilon$.

In the case where the solutions of (1) are uniformly bounded and the solution $x(t)\equiv 0$ is uniformly stable, if $T(t_0, \alpha, \varepsilon)$ is determined depending only on α and ε and independent of t_0, the solution $x(t)\equiv 0$ is uniformly asymptotically stable in the large (cf. [1]).

---

* Numbers in [ ] refer to the bibliography at the end of the paper.

When every solution is unique for the Cauchy-problem, if the solution $x(t)\equiv 0$ of (1) is equiasymptotically stable in the large, it is stable in the sense of Liapounoff and hence it is equiasymptotically stable in the sense of Liapounoff. Moreover by Lemma 1 in [3] we can see that the equiasymptotic stability in the large implies the equi-boundedness. Therefore we need not add the stability and the equi-boundedness to the definition.

As a simple case, if $F\in\bar{C}_0$ with respect to x (cf. [2]) and the solutions of (1) are uniformly bounded and T is independent of t_0, we can easily see that the equiasymptotic stability in the large implies the uniform asymptotic stability in the large.

Now we will obtain a condition for the equiasymptotic stability in the large. At first we have the following theorem which gives a sufficient condition.

Theorem 1. *We suppose that there exists a continuous function $\varphi(t, x)$ satisfying the following conditions in the domain Δ; namely*

1° $\varphi(t, x)>0$, *if* $\|x\|\neq 0$,

2° $\lambda(\|x\|)\leqq\varphi(t, x)$, *where* $\lambda(u)$ *is a continuous increasing function such that* $\lambda(u)>0$ *for* $u>0$ *and* $\lambda(u)\to\infty$ *as* $u\to\infty$,

3° $\varphi(t, x)$ *belongs to the class* C_0 *with respect to* (t, x) *and we have*

$$D_F\varphi(t, x)=\overline{\lim_{h\to 0}}\frac{1}{h}\{\varphi(t+h, x+hF)-\varphi(t, x)\}\leqq-\varphi(t, x).$$

Then the solution $x(t)\equiv 0$ *of* (1) *is equiasymptotically stable in the large.*

Proof. For a given positive constant α, we put

$$\max_{\|x_0\|\leqq\alpha}\varphi(t_0, x_0)=M(t_0, \alpha).$$

By the condition 2°, there exists a positive number $\beta(>\alpha)$ such that

$$\min_{\|x\|=\beta}\varphi(t, x)>M(t_0, \alpha).$$

If we suppose that for some t, say t_1, we have

$$\|x(t_1; x_0, t_0)\|=\beta,$$

we have

$$\varphi(t_1, x(t_1; x_0, t_0))>M(t_0, \alpha)\geqq\varphi(t_0, x_0). \tag{3}$$

On the other hand, by the condition 3°, we have

$$(4) \qquad \varphi(t_1, x(t_1; x_0, t_0)) \leqq \varphi(t_0, x_0)$$

which contradicts (3). Therefore we have $\|x(t; x_0, t_0)\| < \beta$.

Now in the domain

$$\Delta_\beta: \quad t_0 \leqq t < \infty, \quad \|x\| \leqq \beta,$$

we consider a function $\psi(t, x)$ such as

$$\psi(t, x) = e^t \varphi(t, x).$$

It is clear that this function satisfies

$$(5) \qquad \lambda(\|x\|)e^t \leqq \psi(t, x),$$

$$(6) \qquad D_F \psi(t, x) \leqq 0$$

and belongs to the class C_0 with respect to (t, x). If for some $\varepsilon > 0$ (ε: small), there exists a divergent sequence $\{t_m\}$ such that for some solution, we have $\|x(t_m; x_0, t_0)\| \geqq \varepsilon$, we have

$$(7) \qquad \psi(t_m, x(t_m; x_0, t_0)) \geqq e^{t_m} \lambda(\varepsilon).$$

On the other hand, we have

$$(8) \qquad \psi(t_0, x(t_0; x_0, t_0)) \leqq e^{t_0} M(t_0, \alpha)$$

and

$$\psi(t_m, x(t_m; x_0, t_0)) \leqq \psi(t_0, x(t_0; x_0, t_0)) \qquad \text{(by (6))}$$

and hence by (7) and (8) we find

$$e^{t_m} \lambda(\varepsilon) \leqq e^{t_0} M(t_0, \alpha).$$

Since $\lambda(\varepsilon) > 0$ and $t_m \to \infty$ $(m \to \infty)$, there arises a contradiction. Therefore we have $\|x(t; x_0, t_0)\| < \varepsilon$ for $t > t_0 + \log \dfrac{M(t_0, \alpha)}{\lambda(\varepsilon)}$. This completes the proof of Theorem.

Next we will obtain a necessary condition. Namely we have the following theorem.

Theorem 2. *We assume that* $F(t, x)$ *in* (1) *belongs to the class* C_0 *with respect to* x. *Then if the solution* $x(t) \equiv 0$ *of* (1) *is equi-asymptotically stable in the large, there exists a continuous function* $\varphi(t, x)$ *satisfying the following conditions in the domain* Δ; *namely*

1° *$\varphi(t, x) > 0$ if $\|x\| \neq 0$,*
2° *$\varphi(t, 0) \equiv 0$,*
3° *$\lambda(\|x\|) \leqq \varphi(t, x)$, where $\lambda(u)$ is the same as one in Theorem 1,*
4° *$\varphi(t, x)$ belongs to the class C_0 with respect to (t, x),*
5° *$D_F\varphi(t, x) \leqq -\varphi(t, x)$.*

Proof. Let I_η and X_η be the interval $0 \leqq t \leqq \eta$ and the space $\|x\| \leqq \eta$ respectively. And we represent the product space $I_\eta \times X_\eta$ by Ω_η. Since the solution $x(t) \equiv 0$ is equiasymptotically stable in the large, the solutions of (1) are equi-ultimately bounded for any positive number ε. Hence, by Lemma 2 in [3], they are locally uniformly ultimately bounded[(1)]. Namely for the solution issuing from $(t_0, x_0) \in \Omega_\eta$ to the right, there exists $T(\varepsilon, \eta)$ such that we have $\|x(t; x_0, t_0)\| < \varepsilon$ for $t > t_0 + T(\varepsilon, \eta)$. We assume that $T(\varepsilon, \eta) = T(1, \eta)$ for $\varepsilon > 1$. Then $T(\varepsilon, \eta)$ is defined in the quadrant $0 < \varepsilon$, $0 \leqq \eta$ and it is non-negative. Moreover we can assume that it is monotone increasing in η for each fixed ε and monotone decreasing in ε for each fixed η and that it is a continuous function of (ε, η)[(2)].

By Lemma 1 in [3], when $x(t) \equiv 0$ is equiasymptotically stable in the large, the solutions of (1) are equi-bounded. Therefore there exists $\beta(\eta)$ such as $\|x(t; x_0, t_0)\| \leqq \beta(\eta)$ for $(t_0, x_0) \in \Omega_\eta$[(3)], and we can assume that $\beta(\eta)$ is a positive continuous increasing function.

If we put

$$\max_{\substack{0 \leqq t \leqq \eta \\ 0 \leqq \|x\| \leqq \beta(\eta)}} \|F(t, x)\| = F^*(\eta),$$

(1) Even if we do not assume the uniqueness for the Cauchy-problem, this lemma is true.

(2) $T(\varepsilon, \eta)$ is defined in the quadrant $0 < \varepsilon$, $0 \leqq \eta$ and it is increasing in η and decreasing in ε. Therefore it is integrable in $0 < a \leqq \varepsilon \leqq b$, $0 \leqq c \leqq \eta \leqq d$. Thus if we put

$$\frac{2}{\varepsilon} \int_{\frac{\varepsilon}{2}}^{\varepsilon} \int_{\eta}^{\eta+1} T(\sigma, \xi) d\sigma\, d\xi = \tilde{T}(\varepsilon, \eta),$$

$\tilde{T}(\varepsilon, \eta)$ is a continuous function of (ε, η) and it is increasing in η and decreasing in ε. Moreover we have $T(\varepsilon, \eta) \leqq \tilde{T}(\varepsilon, \eta)$. Hence we represent this $\tilde{T}(\varepsilon, \eta)$ by $T(\varepsilon, \eta)$ again.

(3) We suppose that the solutions of (1) are equi-bounded (every solution need not be unique for the Cauchy-problem). Then every solution issuing from $(t_0, x_0) \in \Omega_\eta$ to the right intersects the hyperplane $t = \eta$ and hence $\|x(t; x_0, t_0)\|$ $(0 \leqq t \leqq \eta)$ is bounded by a suitable positive number $\alpha(\eta)$. Since we can consider this solution as the solution issuing from $t = \eta$, $\|x\| \leqq \alpha(\eta)$ to the right, by the equi-boundedness we have a suitable positive number $\gamma(\eta)$ which is the bound of $\|x(t; x_0, t_0)\|$ on $\eta \leqq t < \infty$. Consequently we can find a suitable positive number $\beta(\eta)$.

$F^*(\eta)$ is continuous increasing and we may assume that $F^*(\eta)\geqq 1$. Moreover let $D_{\varepsilon,\eta}$ be the set of points (t,x) such as

$$0\leqq t\leqq \eta+T(\varepsilon,\eta),\quad 0\leqq \|x\|\leqq \beta(\eta).$$

Since $F(t,x)$ belongs to the class C_0 with respect to x, there is a positive constant $L(\varepsilon,\eta)$ such that

$$\|F(t,x)-F(t,x')\|\leqq L(\varepsilon,\eta)\|x-x'\|,$$

where $(t,x)\in D_{\varepsilon,\eta}$, $(t,x')\in D_{\varepsilon,\eta}$. We can also assume that $L(\varepsilon,\eta)$ is a positive continuous function of (ε,η) and that it is increasing in η for each fixed ε and decreasing in ε for each fixed η[4].

Now we consider a function as follows; namely

$$G(\varepsilon,\zeta)=\begin{cases}\zeta-\varepsilon & (\zeta\geqq\varepsilon)\\ 0 & (0\leqq\zeta<\varepsilon).\end{cases}$$

Then this is a non-negative continuous function defined for $0\leqq\zeta$, $0<\varepsilon$ and it tends to infinity as $\zeta\to\infty$ when ε is fixed, and we have

$$|G(\varepsilon,\zeta)-G(\varepsilon,\zeta')|\leqq|\zeta-\zeta'|. \tag{9}$$

If we put

$$F^*(\eta)\exp\{(L(\varepsilon,\eta)+1)T(\varepsilon,\eta)\}+G(\varepsilon,\beta(\eta))\exp\{T(\varepsilon,\eta)+\eta\} = M(\varepsilon,\eta), \tag{10}$$

$M(\varepsilon,\eta)$ is a positive continuous function defined for $0<\varepsilon$, $0\leqq\eta$. Since $\dfrac{1}{M(\varepsilon,\eta)}$ is also a positive continuous function, by Massera's lemma in [1] there are two functions $g(\varepsilon)$, $k(\eta)\in C_\infty$ in $[0,+\infty)$ such that $k(\eta)>0$, $g(\varepsilon)>0$ when $\varepsilon>0$, $g(0)=0$ and that

$$g(\varepsilon)\,k(\eta)\leqq\frac{1}{M(\varepsilon,\eta)}.$$

If we put $\dfrac{1}{k(\eta)}=h(\eta)$, we have

$$g(\varepsilon)M(\varepsilon,\eta)\leqq h(\eta). \tag{11}$$

To simplify the descriptions, we represent $T(\varepsilon,\eta)$, $L(\varepsilon,\eta)$, $G(\varepsilon,\zeta)$

(4) cf. (2).

and $g(\varepsilon)$ for $\varepsilon=\frac{1}{n}$ $(n=1, 2, \cdots)$ by $T_n(\eta)$, $L_n(\eta)$, $G_n(\zeta)$ and g_n respectively.

Now we put

$$\varphi_n(t, x) = g_n \sup_{\tau} [G_n(\|x(t+\tau\,;\, x,\, t)\|)e^{\tau}\,;\; 0 \leqq \tau]. \tag{12}$$

It is clear that we have

$$g_n G_n(\|x\|) \leqq \varphi_n(t, x) \tag{13}$$

and

$$\varphi_n(t, 0) \equiv 0. \tag{14}$$

For $(t, x) \in \Omega_\eta$, that is, for the solutions issuing from Ω_η to the right we have

$$\varphi_n(t, x) \leqq g_n G_n(\beta(\eta)) e^{T_n(\eta)}$$

and hence by (10) and (11) we have

$$\varphi_n(t, x) \leqq h(\eta). \tag{15}$$

Next we will show that $\varphi_n(t, x)$ belongs to the class C_0 with respect to (t, x). We suppose that $(t, x) \in \Omega_\eta$, $(t', x') \in \Omega_\eta$. If we put $\varphi_n(t, x) = g_n G_n(\|x(t+\tau\,;\, x,\, t)\|)e^{\tau}$,

$$\begin{aligned}\varphi_n(t, x) - \varphi_n(t, x') &\leqq g_n\{G_n(\|x(t+\tau\,;\, x,\, t)\|)e^{\tau} - G_n(\|x(t+\tau\,;\, x',\, t)\|)e^{\tau}\} \\ &\leqq g_n e^{\tau} \|x(t+\tau\,;\, x,\, t) - x(t+\tau\,;\, x',\, t)\| \qquad \text{(by (9))} \\ &\leqq g_n e^{\tau} \|x - x'\| e^{L_n(\eta)\tau} \\ &\leqq g_n \|x - x'\| e^{(L_n(\eta)+1)T_n(\eta)}.\end{aligned}$$

In the similar way we have

$$\varphi_n(t, x) - \varphi_n(t, x') \geqq -g_n \|x - x'\| e^{(L_n(\eta)+1)T_n(\eta)}.$$

Therefore by (10) and (11) we have

$$|\varphi_n(t, x) - \varphi_n(t, x')| \leqq h(\eta)\, \|x - x'\|. \tag{16}$$

Now we shall consider $\varphi_n(t, x) - \varphi_n(t', x)$, where $t < t'$. If we put $\varphi_n(t, x) = g_n G_n(\|x(t+\tau\,;\, x,\, t)\|)e^{\tau}$ and we assume that $t+\tau \geqq t'$ and $t+\tau = t'+\tau'$, we have

$$\begin{aligned}&\varphi_n(t, x) - \varphi_n(t', x) \\ &\leqq g_n\{G_n(\|x(t+\tau\,;\, x,\, t)\|)e^{\tau} - G_n(\|x(t'+\tau'\,;\, x,\, t')\|)e^{\tau'}\}\end{aligned}$$

$$
\begin{aligned}
&\leqq g_n e^{\tau}\{G_n(\|x(t+\tau\,;\,x,t)\|)-G_n(\|x(t'+\tau'\,;\,x,t')\|)\} \\
&\quad +g_n G_n(\|x(t'+\tau'\,;\,x,t')\|)(e^{\tau}-e^{\tau'}) \\
&\leqq g_n e^{\tau}\|x(t+\tau\,;\,X,t')-x(t'+\tau'\,;\,x,t')\| \\
&\quad +g_n G_n(\|x(t'+\tau'\,;\,x,t')\|)e^{\tau'}(e^{\tau-\tau'}-1) \\
&\leqq g_n e^{T_n(\eta)}\|X-x\|e^{L_n(\eta)T_n(\eta)}+g_n G_n(\beta(\eta))e^{T_n(\eta)}e^{\eta}(t'-t)\,,
\end{aligned}
$$

where $X=x(t'\,;\,x,t)$. Since $\|X-x\|\leqq F^*(\eta)(t'-t)$, we find

$$
\varphi_n(t,x)-\varphi_n(t',x)\leqq g_n\{e^{(L_n(\eta)+1)T_n(\eta)}F^*(\eta)+G_n(\beta(\eta))e^{T_n(\eta)+\eta}\}(t'-t)\,.
$$

In the case where $t+\tau<t'$, we have

$$
\begin{aligned}
&\varphi_n(t,x)-\varphi_n(t',x) \\
&\leqq g_n\{G_n(\|x(t+\tau\,;\,x,t)\|)e^{\tau}-G_n(\|x\|)\} \\
&\leqq g_n G_n(\|x(t+\tau\,;\,x,t)\|)(e^{\tau}-1)+g_n G_n(\|x(t+\tau\,;\,x,t)\|)-g_n G_n(\|x\|) \\
&\leqq g_n G_n(\|x(t+\tau\,;\,x,t)\|)(e^{t'-t}-1)+g_n\|x(t+\tau\,;\,x,t)-x\| \quad (\tau<t'-t) \\
&\leqq g_n G_n(\beta(\eta))e^{\eta}(t'-t)+g_n F^*(\eta)(t'-t) \\
&\leqq g_n\{G_n(\beta(\eta))e^{\eta}+F^*(\eta)\}(t'-t)\,.
\end{aligned}
$$

If we put $\varphi_n(t',x)=g_n G_n(\|x(t'+\tau'\,;\,x,t')\|)e^{\tau'}$ and $t'+\tau'=t+\tau$, we have

$$
\begin{aligned}
&\varphi_n(t,x)-\varphi_n(t',x) \\
&\geqq g_n\{G_n(\|x(t+\tau\,;\,x,t)\|)e^{\tau}-G_n(\|x(t'+\tau'\,;\,x,t')\|)e^{\tau'}\} \\
&\geqq g_n e^{\tau}\{G_n(\|x(t+\tau\,;\,X,t')\|)-G_n(\|x(t'+\tau'\,;\,x,t')\|)\} \qquad (\tau\geqq\tau') \\
&\geqq -g_n e^{T_n(\eta)}\|X-x\|e^{L_n(\eta)T_n(\eta)} \\
&\geqq -g_n e^{(L_n(\eta)+1)T_n(\eta)}F_n^*(\eta)(t'-t)\,.
\end{aligned}
$$

In the end, by (10) and (11), we have

$$
|\varphi_n(t,x)-\varphi_n(t',x)|\leqq h(\eta)|t-t'|\,.
$$

From this and (16) we obtain

$$
\text{(17)} \qquad |\varphi_n(t,x)-\varphi_n(t',x')|\leqq h(\eta)[\|x-x'\|+|t-t'|]\,.
$$

Finally, if $h>0$, $x'=x(t+h\,;\,x,t)$ and τ' is such that $\varphi_n(t+h,x')=g_n G_n(\|x(t+h+\tau'\,;\,x',t+h)\|)e^{\tau'}$ and if $\tau=\tau'+h$

$$
\begin{aligned}
\varphi_n(t+h,x') &= g_n G_n(\|x(t+h+\tau'\,;\,x',t+h)\|)e^{\tau'} \\
&= g_n G_n(\|x(t+\tau\,;\,x,t)\|)e^{\tau}\cdot e^{\tau'-\tau} \\
&\leqq \varphi_n(t,x)e^{\tau'-\tau}\,,
\end{aligned}
$$

whence

$$\frac{\varphi_n(t+h,\ x')-\varphi_n(t,\ x)}{h} \leqq \varphi_n(t,\ x)\frac{e^{-h}-1}{h}$$

and by letting $h \to 0$, we find

$$D_F\varphi_n(t,\ x) \leqq -\varphi_n(t,\ x)\,. \tag{18}$$

We are now going to obtain the desired function. If we put

$$\varphi(t,\ x) = \sum_{n=1}^{\infty} \frac{1}{2^n}\varphi_n(t,\ x)\,,$$

$\varphi(t, x)$ is a continuous function defined in Δ, because $\varphi_n(t, x) \leqq h(\eta)$ when $(t, x) \in \Omega_\eta$ and hence the series is uniformly convergent in Ω_η and moreover η is arbitrary. We can see easily that we have

$$\varphi(t,\ x) \geqq 0 \quad \text{and} \quad \varphi(t,\ 0) \equiv 0\,.$$

For x such as $\frac{1}{n} \leqq \|x\| < \frac{1}{n-1}$,

$$\varphi(t,\ x) \geqq \frac{1}{2^{n+1}}\varphi_{n+1}(t,\ x) \geqq \frac{1}{2^{n+1}}\left(\|x\| - \frac{1}{n+1}\right)g_{n+1} \geqq \frac{1}{2^{n+1}}\,\frac{1}{n(n+1)}g_{n+1}$$

and for x such as $\|x\| > 1$,

$$\varphi(t,\ x) \geqq \frac{1}{2}g_1(\|x\|-1)$$

and hence there exists a continuous increasing function $\lambda(u)$ such that $\lambda(u) > 0$ when $u > 0$ and $\lambda(u) \to \infty$ as $u \to \infty$ and $\lambda(\|x\|) \leqq \varphi(t, x)$.

This function $\varphi(t, x)$ belongs to the class C_0 with respect to (t, x). For $(t, x) \in \Omega_\eta$ and $(t', x') \in \Omega_\eta$,

$$\begin{aligned}
&|\varphi(t,\ x) - \varphi(t',\ x')| \\
&= \Big|\sum_{n=1}^{\infty} \frac{1}{2^n}\varphi_n(t,\ x) - \sum_{n=1}^{\infty} \frac{1}{2^n}\varphi_n(t',\ x')\Big| \\
&= \Big|\sum_{n=1}^{\infty} \frac{1}{2^n}\{\varphi_n(t,\ x) - \varphi_n(t',\ x')\}\Big| \\
&\leqq \sum_{n=1}^{\infty} \frac{1}{2^n}|\varphi_n(t,\ x) - \varphi_n(t',\ x')| \\
&\leqq \sum_{n=1}^{\infty} \frac{1}{2^n}h(\eta)[\|x-x'\| + |t-t'|] \\
&\leqq h(\eta)[\|x-x'\| + |t-t'|]\sum_{n=1}^{\infty} \frac{1}{2^n}\,.
\end{aligned}$$

Therefore we find

$$|\varphi(t, x)-\varphi(t', x')| \leqq h(\eta)[\|x-x'\|+|t-t'|],$$

that is, $\varphi(t, x) \in C_0$.

Finally, we have

$$\begin{aligned}
D_F\varphi(t, x) &= \overline{\lim_{h\to 0}} \frac{1}{h}\{\varphi(t+h, x+hF)-\varphi(t, x)\} \\
&= \overline{\lim_{h\to 0}} \frac{1}{h}\left\{\sum_{n=1}^{\infty}\frac{1}{2^n}\varphi_n(t+h, x+hF)-\sum_{n=1}^{\infty}\frac{1}{2^n}\varphi_n(t, x)\right\} \\
&= \overline{\lim_{h\to 0}} \sum_{n=1}^{\infty}\frac{1}{2^n}\frac{1}{h}\{\varphi_n(t+h, x+hF)-\varphi_n(t, x)\} \\
&\leqq \sum_{n=1}^{\infty}\overline{\lim_{h\to 0}}\frac{1}{2^n}\frac{1}{h}\{\varphi_n(t+h, x+hF)-\varphi_n(t, x)\}^{(5)} \\
&\leqq \sum_{n=1}^{\infty}\frac{1}{2^n}(-\varphi_n(t, x)) = -\sum_{n=1}^{\infty}\frac{1}{2^n}\varphi_n(t, x) = -\varphi(t, x),
\end{aligned}$$

whence we obtain

$$D_F\varphi(t, x) \leqq -\varphi(t, x).$$

Therefore we can see that this function $\varphi(t, x)$ is the desired.

Then we have the following theorem which gives a necessary and sufficient condition.

Theorem 3. *We assume that $F(t, x)$ of* (1) *belongs to the class C_0 with respect to x. In order that the solution $x(t)\equiv 0$ of* (1) *is equiasymptotically stable in the large, it is necessary and sufficient that there exists a continuous function $\varphi(t, x)$ satisfying the conditions in Theorem 2 in the domain Δ.*

---

(5) For instance, if $f_n(h)$ is continuous in $0<|h-a|\leqq b$ and the series $\sum_{n=1}^{\infty} f_n(h)$ is uniformly convergent (this condition is not essential) and $\sum_{n=1}^{\infty}\overline{\lim_{h\to a}} f_n(h)$ is convergent, we have

$$\overline{\lim_{h\to a}}\sum_{n=1}^{\infty} f_n(h) \leqq \sum_{n=1}^{\infty}\overline{\lim_{h\to a}} f_n(h).$$

In our case, since for $(t, x)\in\Omega_\eta$ and $(t', x')\in\Omega_\eta$, we have $|\varphi_n(t, x)-\varphi_n(t', x')|\leqq h(\eta)$ $[\|x-x'\|+|t-t'|]$ and for a sufficiently small $|h|$, we may assume that both (t, x) and $(t+h, x+hF)$ belong to Ω_η for a suitable η, we have

$$\left|\frac{1}{h}\{\varphi_n(t+h, x+hF)-\varphi_n(t, x)\}\right| \leqq \frac{1}{|h|}h(\eta)[|h|\,\|F\|+|h|] \leqq \tilde{h}(\eta)$$

and hence we can regard $\frac{1}{2^n}\frac{1}{h}\{\varphi_n(t+h, x+hF)-\varphi_n(t, x)\}$ as $f_n(h)$.

Remark 1. As a sufficient condition, we need not $\varphi(t, 0) \equiv 0$. As a necessary condition, we obtain $\varphi(t, 0) \equiv 0$, but if we take the function $\psi(t, x)$ such as

$$\psi(t, x) = \varphi(t, x) + e^{-t},$$

this function $\psi(t, x)$ is positive for all x and $\psi(t, x) \in C_0$ with respect to (t, x) and we have

$$\lambda(\|x\|) \leqq \psi(t, x),$$
$$D_F \psi(t, x) \leqq -\psi(t, x)$$

and hence $\psi(t, x)$ has the same properties as those of $\varphi(t, x)$ in Theorem 1. Therefore we may consider that the function $\varphi(t, x)$ in Theorem 3 is positive.

Remark 2. When, without assuming (2), we consider the case where $x(t) \to 0$ $(t \to \infty)$ under the condition in *Definition*, we can follow the proof of Theorem 3. But we have not necessarily $\varphi_n(t, 0) \equiv 0$ and hence we have not necessarily $\varphi(t, 0) \equiv 0$.

BIBLIOGRAPHY

[1] J. L. Massera, "*Contributions to stability theory*", Ann. of Math., Vol. 64 (1956), pp. 182–206.
[2] T. Yoshizawa, "*On the necessary and sufficient condition for the uniform boundedness of solutions of $x' = F(t, x)$*", These Memoirs, Vol. 30 (1957), pp. 217–226.
[3] T. Yoshizawa, "*Note on the equi-ultimate boundedness of solutions of $x' = F(t, x)$*", These Memoirs, Vol. 31 (1958), pp. 211–217.