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A topological proof of theorems of Bott and Borel-Hirzeburch for homotopy groups of unitary groups

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§ 1. Introduction.

It is the purpose of this paper to give a simple and topological proof of the following two theorems [4], [3].

Theorem of Bott.

$$\pi_{2n+1}(SU(m)) \approx Z \qquad \textit{for} \quad m > n \geqq 1\,,$$
$$\pi_{2n}(SU(m)) = 0 \qquad \textit{for} \quad m > n\,.$$

Theorem of Borel-Hirzeburch.

$$\pi_{2n}(SU(n)) \approx Z_{n!} \qquad \textit{for} \quad n \geqq 2\,.$$

The CW-complex $SU(\infty) = \bigcup_m SU(m)$ has the following properties:

(U_1) *it is simply connected,*

(U_2) *it is an H-space with a homotopy-associative multiplication,*

(U_3) *its integral cohomology ring is an exterior algebra* $\Lambda(e_1, e_2, \cdots)$ *generated by elements* $e_i \in H^{2i+1}$, $i = 1, 2, \cdots$,

(U_4) *and there exists a mapping f of EM into it such that the induced homomorphisms f^* of the cohomology groups are onto,*

where $M = \bigcup_n M_n$ denotes the infinite dimensional complex projective space and EM denotes a suspension of M.

The last property is provided from Yokota's cellular-decomposition of $SU(\infty)$ [9] in which EM is a subcomplex of $SU(\infty)$ and generates multiplicatively the cells of $SU(\infty)$.

Denote by $\Omega(X)$ the space of the loops in X with a fixed origin. Denote by (X, n) an $(n-1)$-connective fibre space over X.

Consider a space $X'=\Omega((\Omega(X), 3))=(\Omega^2(X), 2)$, then $\pi_i(X')\approx\pi_{i+2}(X)$ for $i>1$.

Now our main theorem is stated as follows.

Main theorem. *If a space X has the properties (U_1)-(U_4), then $X'=\Omega((\Omega(X), 3))$ has also the same properties (U_1)-(U_4).*

As a corollary we have that (Theorem 4.1), for a space X satisfying (U_1)-(U_4), $\pi_{2n+1}(X)\approx Z$ and $\pi_{2n}(X)=0$, $n\geqq 1$. Then it follows the theorem of Bott.

The proof of the main theorem will be done as follows. First it will be shown that the homology ring $H_*(X)$ is an exterior algebra $\Lambda(\bar{e}_1, \bar{e}_2, \cdots)$, and the homology ring $H_*(\Omega(X))$ is a polynomial algebra $P[b_1, b_2, \cdots]$ over some $b_i\in H^{2i}$.

Next, by the aid of the structure of $H^*(M)$, it will be shown that the cohomology rings $H^*(\Omega(X))$ and $H^*((\Omega(X), 3))$ are polynomial algebras $P[a_1, a_2, a_3, \cdots]$ and $P[p^*a_1, p^*a_2, \cdots]$, $a_i\in H^{2i}(\Omega(X))$. Finally, the property (U_3) will be proved for X' by the aid of a new mapping $f': EM\rightarrow X'$ satisfying (U_4). The properties (U_1) and (U_2) are obvious.

The above new mapping f' is constructed from a cellular mapping

$$\zeta: E^3M \longrightarrow EM$$

of degree k at each $(2k+1)$-cells such that the double suspension of f' is homotopic to the composition $f\circ\zeta$. The homotopy class of the composition

$$\zeta_n=\zeta\circ E^2\zeta\circ\cdots\circ E^{2(n-2)}\zeta: S^{2n+1}\subset E^{2n-1}M\rightarrow\cdots\rightarrow E^3M\rightarrow EM\subset SU(\infty)$$

is a generator of $\pi_{2n+1}(SU(\infty))$ (Proposition 4.2), and the degree of $\zeta_{n*}: H_{2n+1}(S^{2n+1})\rightarrow H_{2n+1}(EM)$ is $n!$. It follows the theorem of Borel-Hirzeburch. In Theorem 4.3, we shall give a method to calculate the groups $\pi_i(SU(k))$ for $2k<i\leqq 4k+1$, and Theorem 4.4 shows the results for the case $i=2k+1$ and $i=2k+2$. For further calculations of $\pi_i(SU(k))$, in particular on their p-primary components, we may expect to forthcoming papers.

§2. Topological preliminaries.

i) *Suspension, reduced join and join.* By *an n-fold suspension* E^nX of a space X with a base point x_*, we mean a space obtained from $X\times I^n$ by shrinking the susbet $X\times\dot{I}^n\cup x_*\times I^n$ to a point, where $I^n=\{(t_1, \cdots, t_n)\,|\,0\leqq t_i\leqq 1\}$ is the unit n-cube and $\dot{I}^n$ is the boundary of I^n. We represent by $\{x, t\}$, $x\in X$, $t\in I^n$ a point of E^nX corresponding to the point (x, t) of $X\times I^n$. When X is a cell-complex of the cells x_* and e_α^r, then E^nX is also a cell-complex of the cells $x_*=\{x_*, 0\}$ and $E^n e_\alpha^r=\{\{x, t\}\,|\,x\in e_\alpha^r,\ t\in I^n-\dot{I}^n\}$. EX denotes a suspension E^1X of X and we identify E^nX with $EE^{n-1}X$ by $\{x, (t_1, \cdots, t_n)\}=\{\{x, (t_1, \cdots, t_{n-1})\}, t_n\}$. For a mapping $f:(X, x_*)\to(Y, y_*)$, we denote by

$$E^nf:E^nX\longrightarrow E^nY$$

an n-fold suspsnsion of f, given by the formule $E^nf\{x, t\}=\{f(x), t\}$. Denote that $E^1f=Ef$, then $E^nf=EE^{n-1}f$.

By *a reduced join* $A\divideontimes B$ of two spaces A and B, with base points a_* and b_*, we mean a space obtained from $A\times B$ by shrinking $A\vee B=A\times b_*\cup a_*\times B$ to a single point. We represent by $\{a, b\}$ a point corresponding to $(a, b)\in A\times B$. When $A=a_*+\bigcup e_\alpha^r$ and $B=b_*+\bigcup e_\beta^s$ are cell-complexes then $A\divideontimes B$ is a cell-complex of the cells $\{a_*, b_*\}$ and $e_\alpha^r\divideontimes e_\beta^s=\{\{a, b\}\,|\,a\in e_\alpha^r,\ b\in e_\beta^s\}$, In the case B is an n-sphere $S^n=y_*+e^n$, we chose a mapping $\psi:(I^n, \dot{I}^n)\to(S^n, y_*)$ which is homeomorphism of $I^n-\dot{I}^n$ onto $e^n=S^n-y_*$. Then E^nX is homeomorphic to $X\divideontimes S^n$ by the correspondence $\{x, t\}\leftrightarrow\{x, \psi(t)\}$.

A join $A*B$ of two spaces A and B is obtained from $A\times B\times I$ by identifying $A\times b\times 0$ and $a\times B\times 1$ with $b\in B$ and $a\in A$ respectively. We represent by $\{a, b, t\}$ a point corresponding to $(a, b, t)\in A\times B\times I$. When $A=\bigcup e_\alpha^r$ and $B=\bigcup e_\beta^s$ are cell-complexes, then $A*B$ is a cell-complex of the cells e_α^r, e_β^s and $e_\alpha^r*e_\beta^s=\{\{a, b, t\}\,|\,a\in e_\alpha^r,\ b\in e_\beta^s,\ t\in I-\dot{I}\}$.

By setting $\phi\{a, b, t\}=\{\{a, b\}, t\}$, we have a mapping

$$\phi:A*B\longrightarrow E(A\divideontimes B) \tag{2.1}$$

which shrinks $A*b_*\cup a_**B$ to a point. If $A=a_*+\bigcup e_i^r$ and $B=b_*+\bigcup e_j^s$ are locally finite CW-complexes, then ϕ is cellular and a homotopy equivalence, because $A*b_*\cup a_**B$ is a subcomplex contractible to a point in itself. Thus there exists a cellular homotopy equivalence

$$\bar{\phi}:E(A\divideontimes B)\longrightarrow A*B. \tag{2.1$'$}$$

More precisely, we may take $\bar{\phi}$ such as $\bar{\phi}=\phi^{-1}$ on the outside of a neibourhood of the point $\phi(A*b_*\bigcup a_* * B)$.

ii) *Suspensions of complex projective spsces.* Denote by M_k the k-dimensional complex projective space. M_{k-1} is naturally imbedded in M_k and M_k-M_{k-1} is an open $2k$-cell e^{2k}. Then $M=e^0\cup e^2\cup \cdots \cup e^{2k}$ is a cell-complex, and, as a limit, a CW-complex

$$M=\bigcup_k M_k$$

is defined. It is well known that the cohomology ring $H^*(M)$ is a polynomial ring generated by an element u of $H^2(M)$. The cell e^{2k} is oriented such that e^{2k} represents u^k. M is an H-space, i.e., there exists a cellular mapping

$$\zeta_0 : M\times M \longrightarrow M$$

satisfying $\zeta_0(x, e^0)=\zeta_0(e^0, x)=x$ for each $x\in M$. This mapping is given from $\zeta_0 | M\vee M$ by using the fact $\pi_i(M)=0$ for $i\neq 2$. In the induced homomorphism

$$\zeta_0^* : H^*(M) \longrightarrow H^*(M\times M) \cong H^*(M)\otimes H^*(M),$$

we have $\zeta_0^*(u)=u\otimes 1+1\otimes u$ and thus $\zeta_0^*(u^k)=\sum_{i=0}^k\binom{k}{i} u^i\otimes u^{k-i}$.

By setting $\zeta_1\{x, y, t\}=\{\zeta_0(x, y), t\}$, we have a cellular mapping

$$\zeta_1 : M*M \longrightarrow EM.$$

Denote by

$$\zeta : E(M\times\!\!\!\!\times M) \longrightarrow EM \tag{2.2}$$

the composition $\zeta_0\circ\bar{\phi}$, then it is easily verified that

$$\zeta^*(Ee^{2k})=\sum_{i=1}^{k-1}\binom{k}{i} E(e^{2j}\times\!\!\!\!\times e^{2(k-i)}), \tag{2.3}$$

where Ee^{2k} and $E(e^{2i}\times\!\!\!\!\times e^{2(k-i)})$ indicate the cohomology classes represented by themselves with the natural orientations given from those of $e^{2k}\times I$ and $e^{2i}\times e^{2(k-i)}\times I$. $M_1=e^0\bigcup e^2$ is a 2-sphere S^2, thus $M\times\!\!\!\!\times M_1=M\times\!\!\!\!\times S^2$ may be indentify with E^2M. Then the restriction of ζ on $E(M\times\!\!\!\!\times M_1)=E^3M$ is denoted by the same symbol

$$\zeta : E^3 M \longrightarrow EM, \tag{2.4}$$

This mapping is cellular and satisfies the following relation

$$\zeta^*(Ee^{2k})=k\cdot E^3 e^{2(k-1)} \qquad \textit{for} \quad k\geqq 2. \tag{2.5}$$

iii) *Special unitary groups.* Denote by $SU(k)$ k-th special unitary group. I. Yokota has given a cellular-decomposition of $SU(k)$ [9], in which EM_{k-1} is a subcomplex and the cells of $SU(k)$ is generated by the cells of EM_{k-1} by means of product of cells. In his decomposition, $SU(k-1)$ is a subcomplex of $SU(k)$ and $SU(k-1)\bigwedge EM_{k-1}=EM_{k-2}$. Then a CW-complex

$$SU(\infty)=\bigvee SU(k)$$

is defined naturally, and we have an injection

(2.6) $\quad i: EM\subset SU(\infty)$,

such that i induces isomorphisms into of homology groups, or in duality, i induces homomorphisms onto of cohomology groups. Obviously $SU(\infty)$ is simply connected and has an associative multiplication. As is well-known (*cf.* [2]), the cohomology ring $H^*(SU(\infty))$ is an exterior algebra $\Lambda(e_1, e_2, \cdots)$ over elements e_i of $H^{2i+1}(SU(\infty))$, $i=1, 2, \cdots$.

It is known also that $H^*(SU(n+1))=\Lambda(e_1, \cdots, e_n)$, $H^*(SU(n+1)/SU(k))=\Lambda(e_k, \cdots, e_n)$ and the projection homomorphism $p^*: H^*(SU(n+1)/SU(k))\rightarrow H^*(SU(n+1))$ carries e_i onto e_i for $k\leqq i\leqq n$. We remark that the projection $p: SU(n+1)\rightarrow SU(n+1)/SU(k)$ shrinks the subset EM_{k-1} of EM_n to a point and p is homeomorphic at EM_n-EM_{k-1}.

iv) *H-space and Pontrjagin product.* Let X be an H-space, i.e., X has a multiplication (continuous on compacts subsets) $\mu: X\times X\rightarrow X$ such that $\mu(x_*, x)\simeq\mu(x, x_*)\simeq x$ for each $x\in X$ and a fixed point x_* (identity). By the composition $H_*(X)\otimes H_*(X)\subset H_*(X\times X)\xrightarrow{\mu_*} H_*(X)$, Pontrjagin product $\alpha*\beta=\mu_*(\alpha\otimes\beta)$ is defined. Obviously the product x is bilinear and has the identity represented by the point x_*. If the multiplication μ is homotopy-associative, then the product $*$ is associative and $H_*(X)$ becomes a ring, Pontrjagin ring. If the multiplication μ is homotopy-commutative, then the product $*$ is anti-commutative.

Let E_X be a space of the paths in X ending at x_*. Then E_X is a fibre space over X with a projection p associating the starting points to each paths, and the fibre $p^{-1}(x_*)$ is the loop-space $\Omega(X)$. Let $(E_r^{p,q})$ be a homological spectral sequence associated with this fibering. The multiplication μ in X defines naturally a multiplication $\bar{\mu}$ in E_X compatible with the projection p. Then a multiplication (μ_r) is defined in the spectral sequence $(E_r^{p,q})$. μ_r maps $E_r^{p,q}\otimes E_r^{p',q'}$ into $E_r^{p+p',q+q'}$ and this induces μ_{r+1}. Under some conditions, $E_2^{p,q}=H_p(X)\otimes H_q(\Omega(X))$ and μ_2 is equivalent to the tensor product of Pontrjagin products of $H_*(X)$ and $H_*(\Omega(X))$. It is known that the multiplication $\bar{\mu}$ in $\Omega(X)$ is homotopic to the loop-multiplication and they are homotopy-commutative. Thus $H_*(\Omega(X))$ is an anti-commutative ring if X is an H-space. For the details, see [6], § 1.

§ 3. Proof of Main theorem.

In the followings, all the homology and cohomology groups are free abelian and finitely generated for each dimensions. So, there are canonical isomorphisms between homology groups H_i and cohomology groups $H^i=\mathrm{Hom}(H_i, Z)$. For an element a of H^i, we shall denote by $\bar{a}\in H_i$ the corresponding element, the dual of a.

i) *Homology ring of X.* Let X be a space satisfying the conditions (U_1)-(U_4), in particular $H^*(X)=\Lambda(e_1', e_2', \cdots)$ for some $e_i'\in H^{2i+1}(X)$. The multiplication in X defines a homomorphism $\mu^*: H^*(X)\to H^*(X\times X)=H^*(X)\otimes H^*(X)$, and $H^*(X)$ becomes an associative Hopf algebra with respect to μ^*. The associativity means that the relation $(\mu^*\otimes 1)\circ\mu^*=(1\otimes\mu^*)\circ\mu^*$ holds.

Lemma 3.1. *There exist primitive elements* $e_i\in H^{2i+1}(X)$, $i=1, 2, \cdots$, *such that* $H^*(X)=\Lambda(e_1, e_2, \cdots)$ *and* $\mu^*(e_i)=e_i\otimes 1+1\otimes e_i$.

Proof. Set $e_1=e_1'$, then obviously $\Lambda(e_1)=\Lambda(e_1')$ and $\mu^*(e_1)=e_1\otimes 1+1\otimes e_1$. Assume that it is already proved the existence of e_i for $i=1,\cdots,k-1$ such that $\mu^*(e_i)=e_i\otimes 1+1\otimes e_i$ and $\Lambda(e_1,\cdots,e_{k-1})=\Lambda(e_1',\cdots,e_{k-1}')$. For a subset I of $\{1, 2,\cdots,k-1\}$, we denote by e_I the element $e_{i_1}e_{i_2}\cdots e_{i_a}$ for $i_1<i_2<\cdots<i_a$ and $I=\{i_1, i_2,\cdots,i_a\}$. Then we have $\mu^*(e_I)=\sum_{J,K}\mathrm{Sgn}(J,K)\,e_J\otimes e_K$, where $I=J\cup K$, $\mathrm{Sgn}(J,K)=0$ if $J\cap K\neq\phi$, and if $J\cap K=\phi$ then $\mathrm{Sgn}(J,K)$ indicates the sign of the permutation which rearrange $J+K=\{j_1,\cdots,j_b, k_1,\cdots,k_c\}$ into the natural order of I. Now the element $\mu^*(e_k')$ has a form $e_k'\otimes 1+1\otimes e_k'+\sum_{I,J}\lambda_{I,J}e_I\otimes e_J$ for some coefficients $\lambda_{I,J}$, where I and J run over the non-empty subsets of $\{1,\cdots,k-1\}$. It is calculated directly that $0=(\mu^*\otimes 1)\,\mu^*(e_k')-(1\otimes\mu^*)\,\mu^*(e_k')=\sum_{I,J,K}(\lambda_{I+J,K}\,\mathrm{Sgn}(I,J)-\lambda_{I,J+K}\,\mathrm{Sgn}(J,K))\,e_I\otimes e_J\otimes e_K$. Thus $\lambda_{I+J,K}\,\mathrm{Sgn}(I,J)=\lambda_{I,J+K}\,\mathrm{Sgn}(J,K)$ and $\lambda_{I+J,K}\,\mathrm{Sgn}(I+J,K)$

$=\lambda_{I,J+K}\,\mathrm{Sgn}\,(I, J+K)$ for non-emply subsets I, J, K of $\{1, \cdots, k-1\}$, since $\mathrm{Sgn}\,(I+J, K)\,\mathrm{Sgn}\,(I, J)=\mathrm{Sgn}\,(I, J+K)\,\mathrm{Sgn}\,(J, K)$. It follows easily that $\lambda_{I,J}=0$ if $I\cap J\neq\phi$. Also $\lambda_{I,J}$ vanishes if $2(i_1+\cdots+i_a+j_1+\cdots+j_b)+a+b\neq 2k+1$. Denote that $\lambda(I, J)=\lambda_{I,J}\,\mathrm{Sgn}\,(I, J)$, $I\neq\phi$, $J\neq\phi$, then $\lambda(I+J, K)=\lambda(I, J+K)$ and $\lambda(I, J)\neq 0$ only if $I\cap J=\phi$ and $I+J$ has at least three indices. It may be proved from these properties of $\lambda(I, J)$ that, for fixed I, $\lambda(J, K)$ are independent of decompositions $J+K=I$, $J\cap K=\phi$ of I, and therefore it may be denoted by λ_I. For example, if $J'+J''$ is a non-trivial decomposition of J, then $\lambda(J, K)=\lambda(J', J''+K)=\lambda(J'+K, J'')=\lambda(K, J)$. By setting $e_k=e_k'-\sum_I\lambda_I e_I$, we have easily that $\mu^*(e_k)=e_k\otimes 1+1\otimes e_k$. Obviously $\Lambda(e_1, \cdots, e_k)=\Lambda(e_1', \cdots, e_{k-1}', e_k)=\Lambda(e_1', \cdots, e_k')$. Consequently the lemma is proved by the induction on k.

q. e. d.

Using the notations in the above proof, we have $\mu^*(e_I)=\sum_{J\subset I}\mathrm{Sgn}\,(J, I-J)\,e_J\otimes e_{I-J}$. Since the dual of μ^* defines the Pontrjagin product, it follows $\bar{e}_I*\bar{e}_J=\mathrm{Sgn}\,(I, J)\,\bar{e}_{I+J}$. Therefore,

Proposition 3.2. *$H^*(X)$ is an exterior algebra $\Lambda(e_1, e_2, \cdots, e_k, \cdots)$ and $\bar{e}_{i_1}*\bar{e}_{i_2}*\cdots*\bar{e}_{i_a}=\overline{e_{i_1}e_{i_2}\cdots e_{i_a}}$.*

Consider the mapping $f: EM\to X$ of (U_4). As is well known, the cup products are trivial in the suspensions. Thus the image of f^* is spanned by $f^*(e_i)$, and the kernel of f^* is spanned by the decomposable elements. Since f^* is onto, $f^*(e_i)=Ee^{2i}$ by changing the sign of e_i if it is necessary. By duality,

(3.1) $\qquad f_*(\overline{Ee^{2i}})=\bar{e}_i\,.$

ii) *Homology of* $\Omega(X)$. The mapping f defines a mapping Ωf: $\Omega(EM)\to\Omega(X)$ of loop-spaces. Then the diagram

$$\begin{tikzcd} H_i(M) \arrow[r, "i_*"] \arrow[dr, "E"', "\approx"] & H_i(\Omega(EM)) \arrow[r, "\Omega f_*"] \arrow[d, "\Sigma"] & H_i(\Omega(X)) \arrow[d, "\Sigma"] \\ & H_{i+1}(EM) \arrow[r, "f_*"] & H_{i+1}(X) \end{tikzcd}$$

is commutative, where $\sum$ denote the suspension homomorphisms of contractible fibre spaces. Let $(E_r^{p,q})$ be the homological spectral sequence associated with a contractible fibre space over X with the fibre $\Omega(X)$. Then $\sum$ is equivalent to the composition:

$E_2^{0,i} \longrightarrow E_{i+1}^{0,i} \underset{\approx}{\overset{d}{\longleftarrow}} E_{i+1}^{i+1,0} \longrightarrow E_2^{i+1,0}$. Denote that

$$\bar{b}_k = \Omega f^* i^* \bar{e}^{2k},$$

then by the commutativity of the above diagram, $\sum \bar{b}_k = f_* E\bar{e}^{2k} = \bar{e}_k$, or in words of the spectral sequence,

(3.2) $\quad d_{2k+1}(\bar{e}_k \otimes 1) = 1 \otimes \bar{b}_k$.

Let $P[\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_k, \cdots]$ be the polynomial ring on the indeterminants $\{\bar{b}_k\}$ and we construct a formal spectral sequence $('E_r^{p,q})$, $r \geqq 2$, having a product, by setting $'E_2 = \Lambda(\bar{e}_1, \cdots, \bar{e}_k, \cdots) \otimes P[\bar{b}_1, \cdots, \bar{b}_k, \cdots]$, $'d_i(\bar{e}_k \otimes 1) = 0$ for $i = 2, 3, \cdots, 2k$ and $'d_{2k+1}(\bar{e}_k \otimes 1) = 1 \otimes \bar{b}_k$. Then we see that $'E_{2k} = 'E_{2k+1} = \Lambda(\bar{e}_k, \bar{e}_{k+1}, \cdots) \otimes P[\bar{b}_k, \bar{b}_{k+1}, \cdots]$ and $'E_\infty = 0$. By (3.2) and by iv) of §2, the natural correspondence gives a homomorphism $(h_r^{p,q}) : ('E_r^{p,q}) \to (E_r^{p,q})$ such that $d_r^{p,q} \circ h_r^{p,q} = h_r^{p-r,q+r-1} \circ d_r^{p,q}$ and $h_r^{p,q}$ induces $h_{r+1}^{p,q} : 'E_{r+1}^{p,q} = H('E_r^{p,q}) \to E_{r+1}^{p,q} = H(E_r^{p,q})$, where the anticommutativity of $H^*(\Omega(X))$ is need for the construction of h_2.

Lemma 3.3. *Let $H : 'E \to E$ be a homomorphism of homological spectral sequences as above. Assume that $h_2^{p,q}$ is an isomorphism if $h_2^{p,0}$ and $h_2^{0,q}$ are isomorphisms. If $h_2^{p,0}$ and $h_\infty^{p,q}$ are all isomorphisms, then h is also an isomorphism ($h_r^{p,q}$ are all isomorphisms). This is ture for the cohomological case.*

Proof. Obviously $h_2^{0,0}$ is an isomorphism. Assume that $h_2^{0,q}$ are isomorphisms for $q \leqq n$, and then we shall prove that $h_2^{0,n+1}$ is an isomorphism. First we have that $h_r^{p,q}$ are isomorphisms for $q \leqq n-r+2$ and homomorphisms onto for $q \leqq n$. This is obvious for $r=2$. and in the general case it is proved easily by the induction on r. The following diagram is commutative and the horizontal lines are exact.

$$\begin{array}{ccccccc}
'E_r^{2r,n-2r+3} & \longrightarrow & Ker.\, 'd_r^{r,n-r+2} & \longrightarrow & 'E_{r+1}^{r,n-r+2} & \longrightarrow & 0 \\
\downarrow h & & \downarrow h & & \downarrow h & & \downarrow h \\
E_r^{2r,n-2r+3} & \longrightarrow & Ker.\, d_r^{r,n-r+2} & \longrightarrow & E_{r+1}^{r,n-r+2} & \longrightarrow & 0\,.
\end{array}$$

The first and third h are onto and the last h is an isomorphism. Then $h : Ker.\, 'd_r^{r,n-r+1} \to Ker.\, d_r^{r,n-r+1}$ is onto by Lemma 4.5 of [5]. Next, in the diagram

$$\begin{array}{ccccccccc}
Ker.\,'d_r^{r,n-r+2} & \longrightarrow & 'E_r^{r,n-r+2} & \xrightarrow{'d_r} & 'E_r^{0,n+1} & \longrightarrow & 'E_{r+1}^{0,n+1} & \longrightarrow & 0 \\
\downarrow h & & \downarrow h & & \downarrow h_r^{0,n+1} & & \downarrow h_{r+1}^{0,n+1} & & \downarrow h \\
Ker.\ d_r^{r,n-r+2} & \longrightarrow & E_r^{r,n-r+2} & \xrightarrow{d_r} & E_r^{0,n+1} & \longrightarrow & E_{r+1}^{0,n+1} & \longrightarrow & 0
\end{array}$$

the first h is onto and second h is an isomorphism. Then by the five lemma (Lemma 4. 5 and 4. 6 of [5]), it follows that if $h_{r+1}^{0,n+1}$ is an isomorphism then $h_r^{0,n+1}$ is an isomorphism. Since $h_{n+2}^{0,n+1}=h_\infty^{0,n+1}$ is an isomorphism, we conclude that $h_2^{0,n+1}$ is an isomorphism. By the induction on q, we have proved that $h_2^{0,q}$ are isomorphisms. By the assumptions, $h_2^{p,q}$ are all isomorphisms and therefore $h_r^{p,q}$ are all isomorphisms.

For the cohomological case, the lemma is proved similarly, by interchanging the words "homomorphism onto" and "isomorphism into" to each other, by reversing the horizontal arrows of the above two diagrams and by replacing *Ker.* by *Coker.*

q. e. d.

Applying this lemma to our case, we have isomorphisms $E\approx 'E$ and $E_{2k}=E_{2k+1}=\Lambda(\bar{e}_k,\ \bar{e}_{k+1},\cdots)\otimes P[\bar{b}_k,\ \bar{b}_{k+1},\cdots]$. For the ideal I_k generated by $\bar{b}_1,\cdots,\bar{b}_{k-1}$, $1\otimes I_k$ vanishes by $\kappa_2^{2k+1}:E_2\to E_{2k+1}$ and thus I_k vanishes by the suspension homomorphism $\sum: H_{2k}(\Omega(X))\to H_{2k+1}(X)$. Consequently the following proposition is established.

Proposition 3. 4. *The Pontrjagin ring* $H_*(\Omega(X))$ *is the polynomial ring* $P[\bar{b}_1,\ \bar{b}_2,\cdots,\bar{b}_k,\cdots]$ *over* $\bar{b}_k=\Omega f_* i_* \bar{e}^{2k}$, *where* $\Omega f_*\circ i_*: H_{2k}(M)\to H_{2k}(\Omega(EM))\to H_{2k}(\Omega(X))$. *The suspension homomorphism* $\sum$ *maps* $\bar{b}_k$ *onto* $\bar{e}_k$ *and it vanishes on the ideal generated by the decomposable elements, i.e., the ideal is the kernel of* $\sum$.

iii) *Cohomology ring of* $\Omega(X)$. Let $(M)^k=M\times\cdots\times M$ be the iterated k-fold product of M. The loop-multiplication in $\Omega(X)$ defines a mapping

$$(\Omega f)^k:(M)^k\longrightarrow\Omega(X)$$

and this induces a homomorphism

$$(\Omega f)_*^k: H_*((M)^k)\longrightarrow H_*(\Omega(X))$$

such that $(\Omega f)_*^k(\bar{e}^{2i_1}\times\cdots\times\bar{e}^{2i_k})=\Omega f_*\bar{e}^{2i_1}*\cdots*\Omega f_*\bar{e}^{2i_k}=\bar{b}_{i_1}*\cdots*\bar{b}_{i_k}$ $(b_0=1)$. By the duality, it follows

$$(3.3)\qquad (\Omega f)^{k*}(\overline{\bar{b}_{i_1}*\cdots*\bar{b}_{i_k}})=\sum(e^{2i_{\sigma(1)}}\times\cdots\times e^{2i_{\sigma(k)}})$$

for $(\Omega f)^{k*}: H^*(\Omega(X)) \to H^*((M)^k)$, where the summation $\sum$ runs over all the permutation σ of $\{1, 2, \cdots, k\}$.

Proposition 3.5. *Denote by $a_k \in H^{2k}(\Omega(X))$ the dual of the iterated k-fold Pontrjagin product $\bar{b}_1 * \cdots * \bar{b}_1$ of $\bar{b}_1$. Then $H^*(\Omega(X))$ is the polynomial ring $P[a_1, a_2, \cdots, a_k, \cdots]$ over $\{a_k\}$. Let I_D be the ideal generated by the decomposable elements of $H^*(\Omega(X))$ then $b_k \equiv (-1)^{k-1} k \cdot a_k$* mod. I_D.

Proof. For the simplicity, denote that $e^{2i_1} \times \cdots \times e^{2i_k} = x_1^{i_1} \cdots x_k^{i_k}$, then $H^*((M)^k)$ is a polynomial ring over k indeterminants $x_1, \cdots, x_k \in H^2((M)^k)$. (3.3) shows that $(\Omega f)^{k*}$ is an isomorphism into for dimensions less than $2(k+1)$ and the image of $(\Omega f)^{k*}$ is the set of the symmetric functions. Then, as is well-known, for dimensions less than $2(k+1)$, each image is represented uniquely by a polynomial over the elementary symmetric functions $\sigma_i = x_1 x_2 \cdots x_i + \cdots$, $i = 1, \cdots, k$. By (3.3), $\sigma_i = (\Omega f)^{k*} a_i$, $i \leqq k$. By taking k large, it follows that $H^*(\Omega(X))$ is the polynomial ring over $a_1, a_2, \cdots, a_k, \cdots$.

Next, $(\Omega f)^{k*} b_k = x_1^k + \cdots + x_k^k$ by (3.3) and this equals to $F(\sigma_1, \cdots, \sigma_{k-1}) + x\sigma_k$ for a polynomial F and a coefficient x. To determine the coefficient x, we take that $x_1, \cdots, x_k$ are the roots of the equation $x^k - 1 = 0$. Then $x_1^k = \cdots = x_k^k = 1$, $\sigma_i = \cdots = \sigma_{k-1} = 0$ and $\sigma_k = (-1)^{k-1}$. Thus $x_1^k + \cdots + x_k^k = F(\sigma_1, \cdots, \sigma_{k-1}) + x\sigma_k$ implies $k = x(-1)^{k-1}$. Since $(\Omega f)^*$ is an isomorphism into, it follows that $b_k = F(a_1, \cdots, a_{k-1}) + (-1)^{k-1} k \cdot a_k \equiv (-1)^{k-1} k \cdot a_k$ mod I_D.

q. e. d.

iv) *Cohomology of* $(\Omega(X), 3)$. Let $(\Omega(X), 3)$ be a 2-connective fibre space over $\Omega(X)$. The fibre is an Eilenberg-MacLane space of the type $(\pi_2(\Omega(X)), 1)$. Since $\pi_2(\Omega(X)) \approx \pi_3(X) \approx H_3(X) \approx Z$, the fibre has the same homology as 1-sphere S^1. Thus there is Gysin's exact sequence [7]

$$\cdots \longrightarrow H^i(\Omega(X)) \xrightarrow{h} H^{i+2}(\Omega(X)) \xrightarrow{p^*} H^{i+2}((\Omega(X), 3)) \longrightarrow H^{i+1}(\Omega(X)) \longrightarrow \cdots,$$

where p is the projection of the fibering and h satisfies the equality $h(\alpha) = h(1) \cdot \alpha$. Since $H^2((\Omega(X), 3)) = 0$, h is onto for $i = 2$ and $h(1) = \pm a_1$, and thus $h(\alpha) = \pm a_1 \cdot \alpha$. It follows from Proposition 3.5 that h is an isomorphism into and the image is an ideal generated by a_1. Therefore we have

Proposition 3.6. *$H^*((\Omega(X), 3))$ is the polynomial ring $P[p^*a_2, \cdots, p^*a_k, \cdots]$.*

Next, we shall prove

Lemma 3.7. *There exists a mapping $\xi: E^2M \to (\Omega(X), 3)$ such that $p\circ\xi$ is homotopic to the composition $\Omega f\circ\Omega\zeta\circ i: E^2M \subset \Omega(E^3M) \to \Omega(EM) \to \Omega(X)$. These mappings ξ are homotopic to each other. For the induced homomorphism $\xi^*: H^*((\Omega(X), 3)) \to H^*(E^2M)$, we have*

$$\xi^*(p^*a_k) = (-1)^{k-1}E^2e^{2(k-1)}, \qquad k = 2, 3, \cdots.$$

Proof. Since E^2M has no 2-cells and since $\pi_i(p^{-1}(x_*)) = 0$ for $i \neq 1$, there are no obstructions to lift the mapping $\Omega f\circ\Omega\zeta\circ i$ up to ξ. Thus ξ exists. Similarly these ξ are homotopic to each other.

For the simplicity, we set $\xi' = \Omega f\circ\Omega\zeta\circ i$, then $p\circ\xi \simeq \xi'$ and the following diagram is commutative.

$$\begin{array}{ccc} H_{2k}(E^2M) & \xrightarrow{\xi'_*} & H_{2k}(\Omega(X)) \\ \downarrow E & & \downarrow \Sigma \\ H_{2k+1}(E^3M) & \xrightarrow{(f\circ\zeta)_*} & H_{2k+1}(X). \end{array}$$

By Proposition 3.4, $\xi'_*(\overline{E^2e^{2(k-1)}}) = x\cdot\bar{b}_k + F(\bar{b}_1, \cdots, \bar{b}_{k-1})$ for a coefficient x and a polynomial F. By (2.5) and by Proposition 3.4,

$$\begin{aligned} x\cdot\bar{e}_k &= \sum(x\cdot\bar{b}_k + F(\bar{b}_1, \cdots, \bar{b}_{k-1})) = \sum \xi'_*(\overline{E^2e^{2(k-1)}}) \\ &= f_*(\zeta_*(\overline{E^3e^{2(k-1)}})) = f_*(k\cdot\overline{Ee^{2k}}) = k\cdot\bar{e}_k. \end{aligned}$$

Thus $x = k$. By the duality, $\xi'^*b_k = k\cdot E^2e^{2(k-1)}$. Since the cup product is trivial in $H^*(E^2M)$, $\xi'^*I_D = 0$. By Proposition 3.5, $\xi'^*(-1)^{k-1}k\cdot a_k = \xi'^*b_k = k\cdot E^2e^{2(k-1)}$. Since $H^{2k}(E^2M)$ is free, it follows that $\xi^*(p^*a_k) = \xi'^*a_k = (-1)^{k-1}E^2e^{2(k-1)}$.

q. e. d.

v) *Cohomology of* $X' = \Omega((\Omega(X), 3))$. Similarly to ii), we consider a cohomological spectral sequence $(E_r^{p,q})$ associated with a contractible fibre space over $(\Omega(X), 3)$ such that $E_r^{p,q} = H^2((\Omega(X), 3)) \otimes H^q(X')$, $E_\infty^{p,q} = 0$ for $(p, q) \neq (0, 0)$ and the suspension homomorphism $\sum: H^{i+1}((\Omega)X), 3)) \to H^i(X')$ is equivalent to $E_2^{i+1,0} \to E_{i+1}^{i+1,0} \underset{\approx}{\xleftarrow{d}} E_{i+1}^{0,i} \to E_2^{0,i}$. The following diagram is commutative.

$$\begin{array}{ccccc} H^{i+1}((\Omega(X), 3)) & \xrightarrow{\xi^*} & H^{i+1}(E^2M) & & \\ \downarrow \Sigma & & \downarrow \Sigma & \searrow E & \\ H^i(X') & \xrightarrow{\Omega\xi^*} & H^i(\Omega(E^2M)) & \xrightarrow{i^*} & H^i(EM). \end{array}$$

Set $\sum p^* a_{k+1} = e_k' \in H^{2k+1}(X')$ and $f' = \Omega\xi \circ i : EM \subset \Omega(E^2M) \to X'$, then

(3.4) $\qquad f'^* e_k' = (-1)^k Ee^{2k},$

by Proposition 3.6 and the commutativity of the above diagram.

Proposition 3.7. *$H^*(X')$ is the exterior algebra $\Lambda(e_1', e_2', \cdots, e_k', \cdots)$ over $\{e_k'\}$.*

Proof. $\sum p^* a_{k+1} = e_k'$ means that e_k' is transgressible, i.e., $d_i(1 \otimes e_k') = 0$ for $2 \leq i \leq 2k+1$ and $d_{2k+2}(1 \otimes e_k') = p^* a_{k+1} \otimes 1$. Construct a formal cohomological spectral sequence $('E_r^{p,q})$ by setting $'E_2 = P[p^* a_2, p^* a_3, \cdots] \otimes \Lambda(e_1', e_2', \cdots)$, $'d_i(1 \otimes e_k') = 0$ for $2 \leq i \leq 2k+1$ and $'d_{2k+2}(1 \otimes e_k') = p^* a_{k+1} \otimes 1$. Then we see that $'E_{2k+2} = P[p^* a_{k+1}, p^* a_{k+2}, \cdots] \otimes \Lambda(e_k', e_{k+1}', \cdots)$ and $'E_\infty = 0$. The natural correspondence defines a homomorphism of $'E$ into E satisfying the condition of Lemma 3.3. Thus this homomorphism is an isomorphism, in particular $H^*(X') = E_2^{0,*} \approx {}'E_2^{0,*} = \Lambda(e_1', e_2', \cdots)$.

q. e. d.

vi) *Proof of Main theorem.* Since $(\Omega(X), 3)$ is 2-connected, $X' = \Omega((\Omega(X), 3)$ is simply connected. Thus X' satisfies (U_1). Since X' is a space of loops, the condition (U_2) is satisfied. (3.4) and Proposition 3.7 show that X' satisfies the conditions (U_4) and (U_3) respectively. Consequently the proof of the main theorem is accomplished.

§4. Applications.

Let X be a space which has the properties (U_1)-(U_4).

From the definition of $X' = \Omega((\Omega(X), 3))$, we have the following isomorphism.

(4.1) $$\pi_i(X') \approx \pi_{i+1}((\Omega(X), 3)) \approx \pi_{i+1}(\Omega(X)) \approx \pi_{i+2}(X) \qquad \text{for } i+1 > 2.$$

Set $X' = X^{(1)}$ and $X^{(n)} = (X^{(n-1)})'$ inductively. Then $X^{(n)}$ has the properties (U_1)-(U_4) by the main theorem. Then, by (4.1), $0 = H_2(X^{(n)}) \approx \pi_2(X^{(n)}) \approx \pi_4(X^{(n-1)}) \approx \cdots \approx \pi_{2n+2}(X)$ and $Z \approx H_3(X^{(n)}) \approx \pi_3(X^{(n)}) \approx \pi_5(X^{(n-1)}) \approx \cdots \approx \pi_{2n+3}(X)$. Thus we have

Theorem 4.1. *If a space has the properties (U_1)-(U_4), then*

$$\pi_i(X) \begin{cases} \approx Z & \text{for odd } i \geq 3, \\ = 0 & \text{for even } i. \end{cases}$$

In particular, this is ture for $X=SU(\infty)$.

Since the dimension of $SU(\infty)-SU(m)$ is greater than $2m$, $\pi_i(SU(\infty), SU(m))=0$ for $i\leqq 2m$. From the homotopy exact sequence of the pair $(SU(\infty), SU(m))$, it follows isomorphisms

$$i_*:\pi_i(SU(m))\approx\pi_i(SU(\infty)) \qquad \textit{for} \quad i<2m\,.$$

Therefore we have

Theorem of Bott.

$$\pi_{2n}(SU(m))=0 \qquad \textit{for} \quad m>n\,,$$
$$\pi_{2n+1}(SU(m))\approx Z \qquad \textit{for} \quad m>n\geqq 1\,.$$

Define a mapping

$$\zeta_n: S^{2n+1}\longrightarrow EM_n(\subset SU(n+1)\subset SU(\infty)) \tag{4.2}$$

by the composition $\zeta\circ E^2\zeta\circ\cdots\circ E^{2(n-2)}\zeta: S^{2n+1}=E^{2n+1}M_1\rightarrow\cdots\rightarrow E^3M_{n-1}\rightarrow EM_n$. ($\zeta_1=$*identity*).

Proposition 4.2. *Let* X *be a space satisfying* (U_1)-(U_4), *then the composition* $f\circ\zeta_n: S^{2n+1}\rightarrow EM_n\rightarrow X$ *represents a generator of* $\pi_{2n+1}(X)$. *In particular,* ζ_n *represents a generator of* $\pi_{2n+1}(SU(m))$, $m>n$.

Proof. First we see that $\zeta_n=\zeta\circ E^2\zeta_{n-1}: S^{2n+1}\rightarrow E^3M_{n-1}\rightarrow EM_n$. In the case $n=1$, the proposition is proved without difficulties. Assume that the proposition is proved for $n<k$ $(k>1)$. Let $X'=\Omega((\Omega(X), 3))$, then X' satiefies (U_1)-(U_4) and $f'\circ\zeta_{n-1}: S^{2n-1}\rightarrow EM_{n-1}\rightarrow X'$ represents a generator of $\pi_{2n-1}(X')$. Then $\xi\circ E\zeta_{n-1}: S^{2n}\rightarrow E^2M_{n-1}\rightarrow(\Omega(X), 3)$ represents a generator of $\pi_{2n}((\Omega(X), 3))$ since $f'=\Omega\xi\circ i$ as in v) of §3. Also the composition $p\circ\xi\circ E\zeta_{n-1}=\xi'\circ E\zeta_{n-1}=\Omega f\circ\Omega\zeta\circ E\zeta_{n-1}$ represents a generator of $\pi_{2n}(\Omega(X))$. Finally $f\circ\zeta_n=f\circ\zeta\circ E^2\zeta_{n-1}$ represents a generator of $\pi_{2n+1}(X)$. By the induction, the proposition is proved.

q. e. d.

The fibering $p: SU(n+1)\rightarrow SU(n+1)/SU(k)$ shrinks the sub-complex EM_{k-1} of EM_n to a point. The image $p(EM_n)$ will be denoted by

$$EM_n/EM_{k-1}$$

and the composition $p\circ\zeta_n$ by

$$\zeta_{n,k}: S^{2n+1}\longrightarrow EM_n/EM_{k-1}\,.$$

Let $\{\zeta_{n,k}\}$ denote the subgroup of $\pi_{2n+1}(EM_n/EM_{k-1})$ generated by the homotopy class of $\zeta_{n,k}$.

Theorem 4.3. *We have isomorphisms*

$$\pi_{2n}(SU(k)) \approx \pi_{2n+1}(EM_n/EM_{k-1})/\{\zeta_{n,k}\} \qquad for \quad k \geqq n/2,$$
$$\pi_{2n-1}(SU(k)) \approx \pi_{2n}(EM_n/EM_{k-1}) \qquad for \quad n > k \geqq (n-1)/2.$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} H^*(EM_n) & \xleftarrow{i^*} & H^*(SU(n+1)) = \Lambda(e_1, e_2, \cdots, e_n) \\ \uparrow p^* & & \uparrow p^* \\ H^*(EM_n/EM_{k-1}) & \xrightarrow{i^*} & H^*(SU(n+1)/SU(k)) = \Lambda(e_k, \cdots, e_n). \end{array}$$

From iii) of §2, $p^*(e_i) = e_i$ for $k \leqq i \leqq n$. Since $i^*(e_i) = \pm Ee^{2i}$ in the upper homomorphism i^*, it follows easily that the lower i^* maps each e_i, $k \leqq i \leqq n$, onto a generator of $H^{2i+1}(EM_n/EM_{k-1})$. Therefore $i^*: H^t(SU(n+1)/SU(k)) \to H^t(EM_n/EM_{k-1})$ are isomorphisms for $t < (2k+1)+(2k+3) = 4k+4$. This is true for the homological case and thus we have isomorphisms $i_*: \pi_t(EM_n/EM_{k-1}) \approx \pi_t(SU(n+1)/SU(k))$ for $t < 4k+3$ by J.H.C. Whitehead's theorem. Next consider the following exact sequence.

$$\pi_{2n+1}(SU(n+1)) \xrightarrow{p_*} \pi_{2n+1}(SU(n+1)/SU(k)) \longrightarrow \pi_{2n}(SU(k))$$
$$\longrightarrow \pi_{2n}(SU(n+1)) \xrightarrow{p_*} \pi_{2n}(SU(n+1)/SU(k)) \longrightarrow \pi_{2n-1}(SU(k))$$
$$\xrightarrow{i_*} \pi_{2n-1}(SU(n+1)) \xrightarrow{p_*} \pi_{2n-1}(SU(n+1)/SU(k)).$$

By Proposition 4.2, the image $p_*\pi_{2n+1}(SU(n+1))$ is generated by the class of $p\circ\zeta_n = \zeta_{n,k}$. By Theorem 4.1, $\pi_{2n}(SU(n+1)) = 0$. Thus $\pi_{2n}(SU(k)) \approx \pi_{2n+1}(SU(n+1)/SU(k))/\{\zeta_{n,k}\}$. By the isomorphism $i_*: \pi_{2n+1}(EM_n/EM_{k-1}) \approx \pi_{2n+1}(SU(n+1)/SU(k))$ for $2n+1 \leqq 4k+2$, the first isomorphism of this theorem is established.

By (2.5) and by the definition of ζ_n, we have $\zeta_n(E^{2n-1}e^2) = n!\,Ee^{2n}$. Thus $(n \geqq k)$

(4.3) $\zeta_{n,k*}: H_{2n+1}(S^{2n+1}) \longrightarrow H_{2n+1}(EM_n/EM_{k-1})$ *is a homomorphism of degree* $n!$.

This shows that the homotopy class of $\zeta_{n,k}$ in $\pi_{2n+1}(EM_n/EM_{k-1})$ does not vanish by the natural homomorphism of $\pi_{2n+1}(EM_n/EM_{k-1})$ into $H_{2n+1}(EM_n/EM_{k-1})$. Thus the class of $\zeta_{n,k}$ has an infinite order for $n \geqq k$. By the isomorphism $i_*: \pi_{2n-1}(EM_n/EM_{k-1}) \approx \pi_{2n-1}(SU(n+1)/$

$SU(k))$ for $2n-1\leqq 4k+2$, it follows that $p_*\pi_{2n-1}(SU(n+1))=\{\zeta_{n,k}\}$ is an infinite cyclic subgroup. Then $p^{*-1}(0)=0=i_*\pi_{2n-1}(SU(k))$. Since $\pi_{2n}(SU(n+1))=0$, it follows from the exactness of the above sequence that $\pi_{2n}(EM_n/EM_{k-1})\approx\pi_{2n}(SU(n+1)/SU(k))\approx\pi_{2n-1}(SU(k))$ for $n\geqq k$ and $2n\leqq 4k+2$. This proves the second isomorphism.

q. e. d.

It follows from this theorem and from (4. 3)

Theorem of Borel-Hirzeburch.

$$\pi_{2n}(SU(n))\approx Z_{n!} \quad \textit{for} \quad n\geqq 2.$$

Finally we shall prove the following theorem as an application of our theory.

Theorem 4. 4.

$$\pi_{2n+1}(SU(n))\begin{cases}\approx Z_2 & \textit{for even } n\geqq 2,\\ =0 & \textit{for odd } n,\end{cases}$$

$$\pi_{2n+2}(SU(n))\begin{cases}\approx Z_2+Z_{(n+1)!} & \textit{for even } n\geqq 4,\\ \approx Z_{(n+1)!/2} & \textit{for odd } n\geqq 3.\end{cases}$$

Proof. First we consider the homotopy type and homotopy groups of $EM_{n+1}/EM_{n-1}=S^{n+1}\cup e^{2n+3}$. The homotopy type is determined by the homotopy class $\alpha_n\in\pi_{2n+2}(S^{2n+1})\approx Z_2(n\geqq 2)$ of attaching mapping of e^{2n+3}. It is known that $\alpha_n\neq 0$ if and only if the squaring operator Sq^2 is essential in EM_{n+1}/EM_{n-1}. By Cartan's formula, it is calculated easily that $Sq^2e^{2n}=n\cdot e^{2(n+1)}$ in M and thus $Sq^2Ee^{2n}=n\cdot Ee^{2(n+1)}$ *in* EM_{n+1}/EM_{n-1}. It follows that $\alpha_n\neq 0$ for odd n and $\alpha_n=0$ for even n.

For even n, EM_{n+1}/EM_{n-1} has the same homotopy type as the union $S^{2n+1}\vee S^{2n+3}$ of two spheres having a point in common. Thus $\pi_{2n+2}(EM_{n+1}/EM_{n-1})\approx\pi_{2n+2}(S^{2n+1})+\pi_{2n+2}(S^{2n+3})\approx Z_2$ and $\pi_{2n+3}(EM_{n+1}/EM_{n-1})\approx\pi_{2n+3}(S^{2n+1})+\pi_{2n+3}(S^{2n+3})\approx Z_2+Z$.

For odd n, we consider the following diagram.

$$\begin{array}{ccccc}\cdots\longrightarrow\pi_{i+1}(EM_{n+1}/EM_{n-1},\ S^{2n+1}) & \xrightarrow{\partial} & \pi_i(S^{2n+1}) & \longrightarrow & \pi_i(EM_{n+1}/EM_{n-1})\cdots\\ \uparrow g_* & & \uparrow (g\,|\,S^{2n+2})_* & & \\ \pi_{i+1}(E^{2n+3},\ S^{2n+2}) & \xrightarrow[\approx]{\partial} & \pi_i(S^{2n+2}), & & \end{array}$$

where g is a characteristic mapping of e^{2n+3}. Obviously $(g\,|\,S^{2n+2})_*\beta=\alpha_n\circ\beta$. g_* are isomorphisms for $i+1\leqq(2n+2)+2n=4n+2$ by [1]. Since α_n and $\alpha_n\circ E\alpha_n$ are generators of $\pi_{2n+2}(S^{2n+1})\approx Z_2$ and

$\pi_{2n+3}(S^{2n+1}) \approx Z_2$ respectively, it follows that $\partial : \pi_{i+1}(EM_{n+1}/EM_{n-1}, S^{2n+1}) \to \pi_i(S^{2n+1})$ are homomorphisms onto for $i=2n+2$ and $i=2n+3$. Then from the exact sequence in the above diagram, it follows that $\pi_{2n+2}(EM_{n+1}/EM_{n-1})=0$ and $\pi_{2n+3}(EM_{n+1}/EM_{n-1}) \approx Z$ and that Hurewicz homomorphism $\tau : \pi_{2n+3}(EM_{n+1}/EM_{n-1}) \to H_{2n+3}(EM_{n+1}/EM_{n-1}) \approx Z$ is of degree *2*.

By the second isomorphism of Theorem 4. 3, it follows from the above results that $\pi_{2n+1}(SU(n)) \approx \pi_{2n+2}(EM_{n+1}/EM_{n-1}) \approx Z_2$ for even n and $\pi_{2n+1}(SU(n))=0$ for odd n.

Next consider the mapping $\zeta_{n+1,n}$ for odd n. From (4. 3) it follows that

$$\begin{aligned}\tau\{\zeta_{n+1,n}\} &= (n+1)!\ H_{2n+3}(EM_{n+1}/EM_{n-1})\\ &= \tau(n+1)!/2\pi_{2n+3}(EM_{n+1}/EM_{n-1})\end{aligned}$$

for Hurewicz homomorphism $\tau : \pi_{2n+3}(EM_{n+1}/EM_{n-1}) \to H_{2n+3}(EM_{n+1}/EM_{n-1})$. Since this τ is an isomorphism into, it follows that $\{\zeta_{n+1,n}\}=(n+1)!/2\pi_{2n+3}(EM_{n+1}/EM_{n-1})$. Therefore by Theorem 4. 3,

$$\pi_{2n+2}(SU(n)) \approx \pi_{2n+3}(EM_{n+1}/EM_{n-1})/\{\zeta_{n+1,n}\} \approx Z_{(n+1)!/2} \quad \textit{for odd } n \geqq 3 .$$

Let n be even. In this case, we may replace EM_{n+1}/EM_{n-1} by $S^{2n+1} \vee S^{2n+3}$ in the sense of homotopy equivalence. Let $\beta_n+\gamma_n$, $\beta_n \in \pi_{2n+3}(S^{2n+1})$, $\gamma_n \in \pi_{2n+3}(S^{2n+3})$ be the class represented by $\zeta_{n+1,n}$. From (4. 3), it follows that $\gamma_n=(n+1)!\ \iota_{2n+3}$ for a generator ι_{2n+3} of $\pi_{2n+3}(S^{2n+3})$. Now assume that $n \geqq 4$ and consider a mapping $\bar{\zeta}_2 : S^{2n+1} \vee S^{2n+3} = E^4(EM_{n-1}/EM_{n-3}) \to S^{2n+1} \vee S^{2n+3} = EM_{n+1}/EM_{n-1}$ defined by $\zeta_2=\zeta \circ E^2\zeta$. By (2. 5), $\bar{\zeta}_2 | S^{2n+1}$ represents $n(n-1)\iota_{2n+1}$ and $\bar{\zeta}_2 | S^{2n+3}$ represents $\beta'_n+(n+1)n\iota_{2n+3}$ for some $\beta'_n \in \pi_{2n+3}(S^{2n+1})$. Since $\zeta_{n+1,n}=\bar{\zeta}_2 \circ E^4\zeta_{n-1,n-2}$, it follows that $\beta_n=\beta'_n \circ E^4\gamma_{n-2}+n(n-1)\iota_{2n+1} \circ E^4\beta_{n-2}=(n-1)!\ \beta'_n+n(n-1)E^4\beta_{n-2}$. Since $n \geqq 4$ and since $2\pi_{2n+3}(S^{2n+1})=0$, it follows $\beta_n=0$. Therefore $\zeta_{n+1,n}$ represents $(n+1)!\ \iota_{2n+3}$. By Theorem 4. 3,

$$\pi_{2n+2}(SU(n)) \approx \pi_{2n+3}(EM_{n+1}/EM_{n-1})/\{\zeta_{n+1,n}\} \approx Z_2+Z_{(n+1)!} \quad \textit{for even } n \geqq 4.$$

Remark. For $n=2$, $\pi_{2n+2}(SU(n))=\pi_6(SU(2))=\pi_6(S^3) \approx Z_{12}$. In this case, the isomorphism $\pi_6(SU(2)) \approx \pi_7(EM_3/EM_1)/\{\zeta_{3,2}\}$ still holds and we see that $\beta_2 \neq 0$.

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