MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES A
Vol. XXXII, Mathematics No. 1, 1959.

Note on a chain condition for prime ideals*

By

Masayoshi NAGATA

(Received Dec. 27, 1958)

We say that a ring R is of *finitely generated type* over a ring S if R is a ring of quotients of a finitely generated ring over S.

We say that the *dimension formula* holds for a local integral domain S[1] if the following formula is true for any local integral domain R which dominates S and which is of finitely generated type over S:[2]

$$\text{rank } R + \dim_{S/\mathfrak{n}} R/\mathfrak{m} = \text{rank } S + \dim_{((S))}((R)),$$

where $\mathfrak{n}$, $((S))$; $\mathfrak{m}$, $((R))$ denote the maximal ideals and the fields of quotients of S and R respectively.

On the other hand, we introduced in [C. P.][3] the *second chain condition* for prime ideals, which is stated as follows if we restrict ourselves only to integral domains:

The first chain condition holds in an integral domain R if and only if every maximal chain of prime ideals in R has length equal to rank R. The second chain condition holds in an integral domain R if and only if the first chain condition holds in any integral extension[4] of R.

It should be remarked here that if R is a Noetherian integral domain, the second chain condition for R is equivalent to each of the following conditions, as was shown in [C. P.]:

Condition C′: The first chain condition holds in every *finite*

---

* The work was supported by a research grant of National Science Foundation.

1) The same notion can be defined for general local rings, but is a trivial generalization.

2) In general, if S is Noetherian, then we have the inequality rank $R+\dim R/\mathfrak{m} \leq \text{rank } S+\dim((R))$.

3) We refer by [C. P.] the paper "On the chain problem of prime ideals" Nagoya Math. J. 10 (1956).

4) An integral extension of an integral domain R is an integral domain which is integral over R.

integral extension of R *contained in the derived normal ring of* R.

Condition C″: The first chain condition holds in the derived normal ring of R.[5]

The purpose of the present paper is to prove the following

Theorem. *If the second chain condition holds in a Noetherian integral domain* I *and if* R *is a local integral domain which is of finitely generated type over* I, *then the second chain condition and the dimension formula hold for* R.

§ 1. The second chain condition.

The definition of the second chain condition shows the validity of the following

LEMMA 1. If the second chain condition holds for an integral domain R and if $\mathfrak{p}$ is a prime ideal of R, then the second chain condition holds in both $R/\mathfrak{p}$ and $R_{\mathfrak{p}}$[6].

Under the notation in Theorem, we shall prove at first the validity of the second chain condition in R. In order to do so, by virtue of Lemma 1, we may assume that I is a local ring dominated by R. By induction on the number of generators of an integral domain over I of which R is a ring of quotients, we may assume that $R=I[x]_{\mathfrak{p}}$ with an element x of R and a prime ideal $\mathfrak{p}$ of $I[x]$. Again by virtue of Lemma 1, we may assume that x is transcendental over I and that $\mathfrak{p}$ is a maximal ideal of $I[x]$. Let I' be the derived normal ring of I. Then $I'[x]$ is the derived normal ring of $I[x]$. Therefore it is sufficient to prove that

(*) *For any maximal ideal* $\mathfrak{m}'$ *of* $I'[x]$ *containing the maximal ideal* $\mathfrak{m}$ *of* I, *then length of any maximal chain of prime ideals in* $I'[x]$ *which ends at* $\mathfrak{m}'$ *is equal to* $1+$*rank* I.

The assertion is obvious if rank $I=0$ (i.e., I is a field). Hence we assume that rank $I>0$. We shall prove the assertion by induction on rank I. By the second chain condition in I, we see that rank $(\mathfrak{m}'\cap I')=$ rank I and therefore rank $\mathfrak{m}'=$ rank $I+1$. Let $0\subset\mathfrak{p}'_1\subset\cdots\subset\mathfrak{p}'_s=\mathfrak{m}'$ be a maximal chain of prime ideals in $I'[x]$. Since rank $\mathfrak{m}'=$ rank $I+1\geq 2$, we have $s\geq 2$. Hence, if rank $I=1$, the assertion is true. Thus we assume that rank $I\geq 2$. If $\mathfrak{p}'_1\cap I'\neq 0$,

5) The condition C″ is equivalent to the second chain condition even if R is not Noetherian (see [C. P.]).

6) By virtue of the definition of the second chain condition (see [C. P.]), Lemma 1 is valid even if we omit the assumption that R is an integral domain.

then $\mathfrak{p}_1' = (\mathfrak{p}_1' \cap I')I'[x]$, and we see that $s = \text{rank } I + 1$ by induction assumption. If $\mathfrak{p}_2' \cap I' \neq \mathfrak{m}' \cap I'$, then applying the induction assumption to $I/(\mathfrak{p}_2' \cap I)$ and $I_{(\mathfrak{p}_2' \cap I)}$, we see that $s = \text{rank } I + 1$. Therefore we assume that $\mathfrak{p}_2' \cap I$ is the maximal ideal and that $\mathfrak{p}_1' \cap I' = 0$. If rank $I = 2$ and if $\mathfrak{p}_2' = (\mathfrak{p}_2' \cap I')I'[x]$, then we have $s = 3$. Therefore it is sufficient to show that each of the following situations does not occur:

(1) rank $I = 2$, $\mathfrak{p}_2'$ is maximal and $\mathfrak{p}_2' \cap I'$ is maximal.

(2) rank $I > 2$ and $\mathfrak{p}_2' \cap I'$ is maximal.

By the assumption that $\mathfrak{p}_1' \cap I' = 0$, we see that $a = (x \text{ modulo } \mathfrak{p}_1')$ is algebraic over I. Since the second chain condition is preserved under integral extensions, we may assume that a is in the field of quotients of I. Furthermore, since I' has only a finite number of maximal ideals, by the same reason as above, we may assume that I' has only one maximal ideal. If a is integral over I, we have a contradiction immediately (to each of the cases (1) and (2)).

If a is not integral over I and if $1/a$ is integral over I, then we have a contradiction by the assumption that $\mathfrak{p}_2' \cap I'$ is maximal. Thus neither a nor $1/a$ are integral over I. Therefore $\mathfrak{p}_1'$ is generated by certain number (which may be infinite) of elements of the form $ax - b$ $(a, b \in I')$ and the a's and the b's of these elements generate ideals $\mathfrak{a}$ and $\mathfrak{b}$ of purely rank 1 in I'. Therefore $\mathfrak{p}_1'$ is contained in the ideal of $I'[x]$ generated by the maximal ideal of I'. Since $\mathfrak{p}_2'$ contains the maximal ideal of I', we have $\mathfrak{p}_2' = (\mathfrak{p}_2' \cap I')I'[x]$. This shows in particular that (1) is impossible. Furthermore, in the case (2), $\mathfrak{a} + \mathfrak{b}$ must be a primary ideal belonging to the maximal ideal. Hence, under the induction assumption, we proved in particular that, if c and d are non-units in I' which are not contained in any prime ideal of rank 1, then, for a transcendental element y, the second chain condition holds in $I'(y)/(cy - d)$,[7] (whose rank is equal to rank $I' - 1$). We consider $I'(y)[x]$. Then there is no prime ideal between $\mathfrak{p}_1^* = \mathfrak{p}_1' I'(y)[x]$ and $\mathfrak{p}_2^* = \mathfrak{p}_2' I'(y)[x]$. By induction assumption, we see that $\mathfrak{a}^* = \mathfrak{a}I'(y) + (cy - d)I'(y)$ and $\mathfrak{b}^* = \mathfrak{b}I'(y) + (cy - d)I'(y)$ are of rank 2. Since $\mathfrak{a} + \mathfrak{b}$ is primary to maximal ideal of I', $\mathfrak{a}^* + \mathfrak{b}^*$ is primary to the maximal ideal of $I'(y)$. Since

7) When I is a local ring with maximal ideal $\mathfrak{m}$, for a transcendental element y over I, the ring $I(y)$ denotes the local ring $I[y]_{\mathfrak{m}I[y]}$.

rank $I'(y)=\text{rank } I'>2$, $\mathfrak{a}^*$ and $\mathfrak{b}^*$ have no common prime divisor of rank 2. Therefore the ideal generated by $\mathfrak{p}_1'$ and $cy-d$ in $I'(y)[x]$ is contained in a prime ideal, say $\mathfrak{q}^*$, such that (i) $\mathfrak{q}^*\subseteq\mathfrak{p}_2'I'(y)[x]$ and (ii) $\mathfrak{q}^*/(cy-d)$ is of rank 1. By the validity of the second chain condition in $I'(y)/(cy-d)$ and by the induction assumption, we see that $\mathfrak{q}^*\neq\mathfrak{p}_2'I'(y)[x]$, which contradicts to that there is no prime ideal between $\mathfrak{p}_1'I'(y)$ and $\mathfrak{p}_2'I'(y)$. Thus the proof is completed.

§2. The dimension formula.

Under the notation in Theorem, we have proved that the second chain condition is true for R. Let R' be any local integral domain which is of finitely generated type over R and which dominates R. Since R' is of finitely generated type over R, there is a sequence of local rings $R_0=R, R_1,\cdots,R_{n-1}, R_n=R'$ such that (i) R_i dominates R_{i-1} and (ii) there exists an element a_i of R_i such that R_i is a ring of quotients of $R_{i-1}[a_i]$. Then by (*) in §1 applied to R_{i-1}, the dimension formula between R_i and R_{i-1} holds, which proves the dimension formula between R' and R.

§3. A suplementary remark.

We shall prove the following

Proposition. Let R be a Noetherian integral domain and let R' be a finitely generated integral domain over R with transcendence degree r. If $\mathfrak{p}$ is a prime ideal of R and if $\mathfrak{p}'$ is a minimal prime divisor of $\mathfrak{p}R'$ such that $\mathfrak{p}'\cap R=\mathfrak{p}$, then the transcendence degree of $R'/\mathfrak{p}'$ over $R/\mathfrak{p}$ is at least r.

Proof. Let S be the complements of $\mathfrak{p}$ in R. Then, considering R_S and R_S', we may assume that R is a local ring and that $\mathfrak{p}$ is the maximal ideal of R. We use double induction on rank $\mathfrak{p}$ and the number of generators of R' over R.

(i) If rank $R\leq 1$, then the second chain condition holds in R. Therefore, by Theorem, we see the assertion immediately. Therefore we assume that rank $R>1$.

(ii) If rank $\mathfrak{p}'\neq 1$, then there exists a prime ideal $\mathfrak{q}'$ of rank 1 in R' such that $\mathfrak{q}'\subset\mathfrak{p}'$ and that $\mathfrak{q}'\cap R\neq 0$. Since $\mathfrak{p}'$ is a minimal prime divisor of $\mathfrak{p}R'$, $\mathfrak{q}'\cap R$ is different from $\mathfrak{p}$. Therefore, by our induction assumption applied to $\mathfrak{q}'\cap R$, we see that the transcendence degree of $R'/\mathfrak{q}'$ over $R/(\mathfrak{q}'\cap R)$ is at least r. Therefore, by our induction assumption applied to $\mathfrak{p}/(\mathfrak{q}'\cap R)$, we see that the assertion is true in this case. Therefore, we assume that rank $\mathfrak{p}'=1$.

(iii) Assume that $R'/\mathfrak{p}'$ is not algebraic over $R/\mathfrak{p}$. We may assume that x_1 modulo $\mathfrak{p}'$ is transcendental. Then, by induction assumption applied to $\mathfrak{p}'\cap R[x_1]$, we see that the assertion is true in this case. Therefore we assume that $R'/\mathfrak{p}'$ is algebraic over $R/\mathfrak{p}$.

(iv) Now we shall show that R' must be algebraic over R. Let R'^* be the derived normal ring of R' and let $\mathfrak{p}'^*$ be a prime ideal of R'^* lying over $\mathfrak{p}'$. Since rank $\mathfrak{p}'=1$, we have rank $\mathfrak{p}'^*=1$. Therefore $\mathfrak{v}=R'^*_{\mathfrak{p}'^*}$ is a discrete valuation ring (for, since R is Noetherian, R' is Noetherian and R'^* is a Krull ring)[8]. Set $A=\mathfrak{v}\cap K$, K being the field of quotients of R. Set $B=A[x_1,\cdots,x_n]$, $\mathfrak{p}^*=\mathfrak{p}'^*\mathfrak{v}\cap A$, $\mathfrak{p}''=\mathfrak{p}'^*\mathfrak{v}\cap B$ and let B^* be the derived normal ring of B. Then, obviously, $R'^*\subseteq B^*\subseteq\mathfrak{v}$. Therefore $\mathfrak{v}$ is a ring of quotients of B^* ($\mathfrak{v}=B^*_{(\mathfrak{p}'^*\mathfrak{v}\cap B^*)}$). It follows that $\mathfrak{p}^{**}=\mathfrak{p}'^*\mathfrak{v}\cap B^*$ is a minimal prime divisor of $\mathfrak{p}^*B^*$ and rank $\mathfrak{p}^*=1$. It follows that, replacing B to a finite integral extension of B contained in B^* if neccessary, rank $\mathfrak{p}''=1$. Since $R'/\mathfrak{p}'$ is algebraic over $R/\mathfrak{p}$, $B/\mathfrak{p}''$ is algebraic over $A/\mathfrak{p}^*$. Since A is of rank 1, the dimension formula is true and we see that R' is algebraic over R. Thus the proof is completed.

Corollary. If, in the proposition, there are elements $y_1,\cdots,y_s$ of R' which are algebraic over R and such that their residue classes modulo $\mathfrak{p}'$ are algebraically independent over $R/\mathfrak{p}$, then the transcendence degree of $R'/\mathfrak{p}'$ over $R/\mathfrak{p}$ is at least $r+s$.

§4. One question.

Problem. *Whether or not exists a Noetherian local integral domain R with maximal ideal* $\mathfrak{m}$ *such that* (1) *rank R is greater than* 1, (2) *the derived normal ring of R is a local ring which may not be Noetherian and* (3) *the union of all the* $\mathfrak{m}^{-n}=\{a\,;\,a\mathfrak{m}^n\subseteq R\}$ *is not finite over R.*

The reason why the writer is asking this problem is that
(一) if there is no such an example, then we can prove the following two assertions:

8) See for instance, On the derived normal rings of Noetherian integral domains, in this Journal vol. 29, No. 3 (1955), pp. 293-303.

(甲) The zero ideal of the completion of a Noetherian local integral domain has no imbedded prime ideal.

(乙) The following 3 conditions for a Noetherian local integral domain R are equivalent to each other:

(い) R is unmixed.

(ろ) The second chain condition holds in R.

(は) Any maximal ideal of the derived normal ring of R has rank equal to rank R.

(二) If there is such an example, it is nearly certain that such an example can produce an example of Noetherian local integral domain for which (甲) above is not true.

Kyoto University and Harvard University