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Geometric structure of the cohomology rings in abstract algebraic geometry

By

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In the present note we shall study the (sheaf-theoretical) cohomology rings of non-singular algebraic varieties in their geometric aspects: their relation to birational transformations, the birational invariance problem of arithmetic genus, the classes defined by divisors or subvarieties, etc. Our method is purely algebraic and independent of the field characteristic. We do not attempt to go deep into the questions related to the particular phenomena which are presented by the case of positive characteristic.

Regarding the relation to birational transformations, the fundamental theorem is Proposition 5.2: "Let V and V' be non-singular projective varieties and let T be a birational transformation from V' onto V which is regular on V'. Then $T^*: H^*(V) \rightarrow H^*(V')$ is injective".

This proposition will be proved by means of spectral sequences in the standard manner.

As for the theory of the classes corresponding to the subvarieties, our theory will be constructed on the basis of Prop. 5.2 and of Serre duality. If one admits Prop. 5.2, one can read §7 and §11 without reading the rest except §1, §2 and Prop. 9.2.

In §10 we shall study the monoidal transformation of a non-singular projective variety with a non-singular subvariety as center, and obtain a result similar to the one obtained by Denniston by topological method. In particular, we shall prove that the numbers $h^{0,q}$ are invariant under such transformations. This is a generalization of a result of Muhly-Zariski [1] concerning the birational invariance of the arithmetic genus p_a. In fact, the invariance problem of p_a was the motive of this research. In this aspect, however, we do not essentially exceed Muhly-Zariski, though their main results are all reproduced in the present note.

The result of §6 is isolated; we hope it will be of some use in future.

We shall make free use of the main results in Serre [2], and our terminologies and notations concerning sheaf theory will be borrowed from there, except that we shall use the word "variety" in the sense of irreducible variety. As for the theory of the cup-product and the Künneth relation in sheaves, refer to the book of Godement. As for the terminologies of algebraic geometry, such as local coordinates, the divisor of a differential form, a prime divisor (=a simple subvariety of codimension 1), etc., we follow the usage in Lang's book "Introduction to Algebraic Geometry". We shall say "$(\mathfrak{o}, \mathfrak{m})$ is a local ring" instead of "$\mathfrak{o}$ is a local ring and $\mathfrak{m}$ is its maximal ideal". The dimensions of varieties $U, V, W, \cdots$ will be denoted usually by the corresponding letters $u, v, w, \cdots$.

The first manuscript of the present work was written and presented to Kyoto University in February 1958. After that I heard that a good part of my results had already been obtained by others[1], such as A. Grothendieck and G. Washnitzer. As far as I know[2], my method is not entirely the same as theirs. I hope that the present work contains some new contributions.

I would like to express my heartfelt thanks to Y. Akizuki for his constant encouragement, and to S. Nakano for his precious advices and criticisms. I wish also to thank O. Zariski, whose advices and encouragement at the beginning of this research were very helpful.

Contents

1) Akizuki wrote me that A. Andreotti and J. P. Serre had kindly communicated the informations to him after reading a résumé of my work.

2) Their results have not yet been published in a form accessible for me, but H. Hironaka of Harvard University wrote me about the situation.

§1. The cohomology ring of a variety.

Let k be an algebraically closed ground field. Everything of our algebraic geometry will be defined over k except when the contrary is explicitly stated, and a variety will be considered as the set of its k-rational points. Let V be an (abstract) variety. Assuming that V is normal, we denote by Ω_V^p (or Ω^p) the sheaf of germs of differential forms of degree p without poles. Then $\Omega^0=\mathcal{O}$. More generally, when D is a divisor on V, we denote by $\Omega^p(D)$ the sheaf of germs of p-forms ω satisfying $(\omega)\geqslant -D$ locally. These sheaves are algebraic coherent (see Appendix A). If V is non-singular, then Ω^p is the sheaf of germs of regular[1] p-forms, and is locally isomorphic to $\mathcal{O}^{n_p}$, $n_p={}_vC_p$.

We set $H^{p,q}(V)=H^{p,q}=H^q(V, \Omega^p)$ and $H^*(V)=\sum_{p,q}H^{p,q}$. $H^*(V)$ is a ring having the cup-product as the multiplication law, and will be called the cohomology ring of V[2]. An element of $H^{p,q}$ will be called a class of type (p, q). The cup-product $\alpha\cup\beta$, which we shall write simply $\alpha\beta$, has the following properties[3]:

$$\alpha\in H^{p,q}\,,\ \beta\in H^{p',q'} \Longrightarrow \alpha\beta\in H^{p+p',\,q+q'}\,,\ \alpha\beta=(-1)^{pp'+qq'}\beta\alpha\,.$$

The $H^{p,q}(V)$'s are modules over $H^{0,0}(V)$ ($=k$ if V is complete), a

1) In the terminology of Lang's book, "holomorphic". We avoid this word since Zariski's holomorphic function has begun to enter into sheaf theory.

2) We shall be mainly concerned with non-singular varieties. When V has singular points, I am not certain that this definition is the adequate one; at least, it sometimes helps us to study $H^{0,q}=H^q(V, \mathcal{O})$ (cf. §5).

3) The cup-product for cochains with respect to a covering $\mathfrak{U}$ is defined as follows: $f\in C^q(\mathfrak{U}, \Omega^p)$, $g\in C^{q'}(\mathfrak{U}, \Omega^{p'}) \Longrightarrow$

$$(f\cup g)_{i_0,\cdots,i_{q+q'}}=f_{i_0,\cdots,i_q}\wedge g_{i_q,\cdots i_{q+q'}}\,.$$

Then $d(f\cup g)=df\cup g+(-1)^q f\cup dg$, and this last formula enables us to define the cup-product of cohomology classes. Passing to the limit, we obtain the cup-product in $H^*(V)$.

fortiori over k, and their dimensions over k will be denoted by $h^{p,q}(V)$ or $h^{p,q}$.

Let W be a locally closed subset (i.e. intersection of an open set and a closed set) of V, and assume that W is a normal variety such that its prime divisors are all simple on V. Then one can define the trace mapping $\mathrm{tr}_W : H^*(V) \to H^*(W)$ in an obvious way. If W is open, tr_W is nothing but the restriction mapping.

Let $T : V' \to V$ be an everywhere regular rational mapping (briefly: regular mapping) from a normal variety V' into a non-singular variety V. Then T determines in a natural way a mapping $T^* : H^*(V) \to H^*(V')$. Both tr_W and T^* are ring homomorphisms.

Let now W be an arbitrary subset of V. We say that an element α of $H^*(V)$ is *locally zero at* W if there exists an open neighborhood U of W such that $\mathrm{tr}_U(\alpha) = 0$. If is clear that the elements which are locally zero at W make up an ideal of $H^*(V)$, which we shall denote by $N(W, V)$ or simply by $N(W)$.

If the normal variety V is projective, then we have $h^{v,v} = 1$ ($v = \dim V$). (This proposition is perhaps due to Serre. For a proof, see Appendix B). If moreover V is non-singular, then Serre duality, which will be of prime importance in the sequel, holds in $H^*(V)$. Namely:
"$H^{p,q}$ and $H^{v-p,v-q}$ are dual: given $\alpha \in H^{p,q}$, $\alpha \neq 0$, one can find $\beta \in H^{v-p,v-q}$ such that $\alpha\beta \neq 0$. In particular, we have

$$h^{p,q} = h^{v-p,v-q}\text{''}.$$

(More generally, $H^q(V, \Omega^p(D))$ and $H^{v-q}(V, \Omega^{v-p}(-D))$ are dual for any divisor D of V[4]).
We shall sometimes use Serre duality in the following form:

Lemma 1.1. *Let V be a non-singular projective variety and let φ be a ring homomorphism of $H^*(V)$ into some ring R. If φ is injective on $H^{v,v}$, then φ is injective on all $H^{p,q}$.*

A remark on the field of rationality. Let V and D be as above, and let $\mathbf{K}$ be an algebraically closed field containing k. One can consider V as the set of its $\mathbf{K}$-rational points. Then

4) Cf. Serre [3]. It is reported that a more general duality theorem has been obtained by Grothendieck.

$H^q(V, \Omega^p(D))$ is a $\mathbf{K}$-module. In calculating the cohomology, however, one may use an affine covering $\mathfrak{U}=\{U_i\}$ consisting of k-open subsets, and then the modules $\Gamma(U_i, \Omega^p(D))$ are spanned over $\mathbf{K}$ by p-forms defined over k since the U_i are defined over k and since D is rational over k (see Appendix C). Therefore it is easy to see that the cohomology module constructed over $\mathbf{K}$ is obtained from the one over k by the operation of coefficient extension $\otimes_k \mathbf{K}$. This remark shows that $h^{p,q}$ are independent of the choice of k, and enables us to use, if convenient, general (or, in Weil's terminology, generic) elements over k.

§2. The class of type (1, 1) defined by a divisor.

Let V be a non-singular variety, and let D be a divisor on V. Taking a sufficiently fine affine covering $\mathfrak{U}=\{U_i\}$, we can express D by its local equations f_i:

$$D=(f_i) \text{ in } U_i .$$

Now, since f_j/f_i is regular in U_{ij}, the logarithmic differentials

$$\xi_{ij}=d(f_j/f_i)/(f_j/f_i)=df_j/f_j-df_i/f_i$$

form a 1-cocycle of the sheaf Ω^1. Hence a class of type (1, 1) of V, which we shall denote by $c(D)$. It is immediate from the definition that $c(D)$ depends only on the linear equivalence class of D and not on the choices of $\mathfrak{U}$ and of the local equations.

It is also clear that $c(D)$ is linear in D, so that $c(pD)=0$ if k is of characteristic p. If V is projective and k is the complex number field, then it is known, by analytical method, that $c(D)$ depends only on the algebraic equivalence class of D. The same is trivially true in the case of characteristic p, because we have $G_a(V)=G_l(V)+pG_a(V)$ by the divisibility of the Picard variety $G_a(V)/G_l(V)$ of V. This proof is too accidental, and an algebraic proof, depending on the theory of specialization and independent of the characteristic, is very desirable. We shall give later, in the case of projective varieties, a (rather geometric) proof which is independent of the characteristic.

Proposition 2. 1. $c(D)\cdot N(\mathrm{Supp}\,(D))=0$.

Proof. Let $\alpha \in N(\mathrm{Supp}\,(D)) \cap H^{p,q}$. Then there exists an open neighborbood U of D such that $\mathrm{tr}_U(\alpha)=0$. Let $\mathfrak{U}'=\{U'_i\}_{i\in I'}$ be an affine covering of U such that D has a local equation f_i in each U_i', and let $\mathfrak{U}''=\{U_i''\}_{i\in I''}$ be an affine covering of $V-\mathrm{Supp}\,(D)$. Then $\mathfrak{U}=\mathfrak{U}'\cup\mathfrak{U}''$ is an affine covering of V. Take a representative cocycle a of α with respect to $\mathfrak{U}$. By the assumption there exists a $(q-1)$-cochain b with respect to $\mathfrak{U}'$ such that $a_{i_0,\dots,i_q}=(db)_{i_0,\dots,i_q}$ $(i_0,\cdots,i_q\in I')$. Extending b to a cochain of $\mathfrak{U}$ by $b_{i_0,\dots,i_q}=0$ (if some $i_\nu\in I''$), and replacing a by $a-db$, we see that we can assume $a_{i_0,\dots,i_q}=0$ $(i_0,\cdots,i_q\in I')$. Put $\gamma_i=df_i/f_i$, and $\xi_{ij}=df_j/f_j-df_i/f_i$, where $f_i=1$ if $i\in I''$. Then we have $\xi\cup a=d\gamma\cup a=d(\gamma\cup a)$. Though γ is not a cochain of Ω^1, $\gamma\cup a$ is a cochain of Ω^{p+1} since $(\gamma\cup a)_{i_0,\dots,i_q}=0$ $(i_0,\cdots,i_q\in I')$. This proves $c(D)\,\alpha=0$.

Corollary. *Let* D_i $(i=1,2,\cdots,s)$ *be divisors such that* $\bigcap_i \mathrm{Supp}\,(D_i)=\emptyset$. *Then we have* $c(D_1)\cdot c(D_2)\cdots c(D_s)=0$.

Proof. We proceed by induction on s. Put $U=V-\bigcap_{i<s}\mathrm{Supp}\,(D_i)$. Then U is an open neighborhood of $\mathrm{Supp}\,(D_s)$ by the assumption. On the other hand, since $\bigcap_{i<s}(\mathrm{Supp}\,(D_i)\cap U)=\emptyset$, we have $\mathrm{tr}_U(c(D_1)\cdots c(D_{s-1}))=0$ by the induction hypothesis. Therefore $c(D_1)\cdots c(D_{s-1})$ belongs to $N(\mathrm{Supp}\,(D_s))$.

Let D be a non-singular prime divisor on a non-singular variety V of dimension v. The following well-known exact sequence

$$0 \longrightarrow \Omega^v \longrightarrow \Omega^v(D) \xrightarrow{R} \Omega_D^{v-1} \longrightarrow 0,$$

where R denotes the Poincaré residue mapping, gives rise to the cohomology sequence

$$\cdots \longrightarrow H^{v-1,v-1}(D) \xrightarrow{\delta} H^{v,v}(V) \longrightarrow H^v(V,\,\Omega^v(D)) \longrightarrow 0\,.$$

Proposition 2.2. *Let* $\alpha\in H^{v-1,v-1}(V)$. *Then we have*

$$c(D)\,\alpha=\delta(\mathrm{tr}_D(\alpha))\,.$$

Proof. Let $\mathfrak{U}=\{U_i\}$ be a sufficiently fine affine covering of V, let f_i be a local equation of D in U_i and let $a=\{a_{i_0,\dots,i_{v-1}}\}$ be a representative cocycle of α with respect to $\mathfrak{U}$. Let $\gamma=\{\gamma_i\}=\{df_i/f_i\}$ and $\xi=d\gamma$ have the same meaning as above. Then we have $\xi\cup a=d\gamma\cup a=d(\gamma\cup a)$, and $(\gamma\cup a)_{i_0,\dots,i_{v-1}}=(df_{i_0}/f_{i_0})\,a_{i_0,\dots,i_{v-1}}$. But $R((df_{i_0}/f_{i_0})\,a_{i_0,\dots,i_{v-1}})=\mathrm{tr}_D\,(a_{i_0,\dots,i_{v-1}})$. Therefore our assertion follows from the definition of δ.

If, moreover, V is projective, then Serre duality permits us to prove the following

Corollary. *If V is projective, then*

$$(1)\quad \mathrm{tr}_D(\alpha)=0 \Longleftrightarrow c(D)\cdot\alpha=0 \qquad \textit{for } \alpha\in H^{v-1,v-1}(V)\,,$$

$$(2)\quad \mathrm{tr}_D(\alpha)=0 \Longrightarrow c(D)\cdot\alpha=0 \qquad \textit{for any } \alpha\in H^*(V)\,.$$

Proof. We have $H^v(V, \Omega^v(D))\simeq H^0(V, \mathcal{O}(-D))=0$ and $h^{v,v}(V)=h^{v-1,v-1}(D)=1$ by Serre duality, so that the mapping δ is bijective. This proves (1). Let now α be a class of type (p, q) such that $c(D)\,\alpha\neq 0$. Then $c(D)\,\alpha\beta\neq 0$ for some β of type $(v-1-p, v-1-q)$. This means, by (1), that $\mathrm{tr}_D(\alpha\beta)\neq 0$. Therefore $\mathrm{tr}_D(\alpha)$ cannot be zero.

When does $c(D)=0$ hold? In order to investigate this problem, it is more appropriate to consider divisors with coefficients in k. Such divisors will be called k-divisors. If $D=\sum a_\nu D_\nu$ ($a_\nu\in k$, D_ν prime) is a k-divisor, we define $c(D)$ by linearity: $c(D)=\sum a_\nu c(D_\nu)$. If V is projective, then we can define the k-degree of D by

$$\deg_k(D)=\sum a_\nu \deg(D_\nu) \qquad (\in k)\,,$$

where the $\deg(D_\nu)$ are to be taken mod p if k is of characteristic p. The Kronecker index

$$I_k(D_1\cdot D_2\cdots D_v) \qquad (\in k)$$

of k-divisors $D_1,\cdots,D_v$ ($v=\dim V$) is defined similarly. Proposition 2.1 and its corollary hold also for k-divisors.

Now, let $D=\sum_1^s a_\nu D_\nu$ ($a_\nu\neq 0$) be a k-divisor with the prime components $D_1,\cdots,D_s$. Let $\mathfrak{U}=\{U_i\}$ be a sufficiently fine affine covering of V, and let $f_{i,\nu}$ be a local equation of D_ν in U_i. Then

$$\xi_{ij}=\sum_\nu a_\nu\, df_{j,\nu}/f_{j,\nu}-\sum_\nu a_\nu\, df_{i,\nu}/f_{i,\nu}$$

is a cocycle of $c(D)$. Therefore $c(D)=0$ is equivalent to the existence of a 1-form ω such that

$$\omega-\sum_\nu a_\nu\, df_{i,\nu}/f_{i,\nu} \textit{ is regular in } U_i \qquad (\textit{for all } i)\,.$$

Clearly $(df_{i,\nu}/f_{i,\nu})+D_\nu\geqslant 0$ in U_i. Hence we have $(\omega)+\sum D_\nu\geqslant 0$. On the other hand, if $D_\nu\cap U_i\neq 0$ and if P is a simple point of $D_\nu\cap U_i$ not lying on the other components D_μ, then $f_{i,\nu}$ is a member of a regular system of parameters of the local ring $\mathfrak{o}_P$ and hence $(\omega)_\infty=D_\nu$ locally at P. Therefore we have $(\omega)_\infty=\sum_\nu D_\nu$.

If V is a complete (non-singular) curve, then we have $\mathrm{res}_{D_\nu}(\omega)=a_\nu$ and the existence of such an ω is equivalent to $\sum a_\nu=\deg_k(D)=0$. Thus we have proved:

Proposition 2. 3. *If $c(D)=0$, then there exists a 1-form ω such that its pole consists exactly of the prime components of D (taken with multiplicity 1). If V is a complete (non-singular) curve, then we have $c(D)=0 \Longleftrightarrow \deg_k(D)=0$.*

Corollary 1. *Let V be projective, let C_0 be a k-divisor and let $|C_1|, \cdots, |C_{v-1}|$ be ample linear systems. Then we have*

$$c(C_0)\cdot c(C_1) \cdots c(C_{v-1})=0 \Longleftrightarrow I_k(C_0\cdot C_1 \cdots C_{v-1})=0 .$$

Proof. We proceed by induction on v, the case $v=1$ being the proposition above. We can choose a non-singular prime divisor C from $|C_{v-1}|$. Put $C_i\cdot C=C_i'$. Then $|C_i'|$ $(i=1, \cdots, v-2)$ are ample linear systems on C, $\mathrm{tr}_C(c(C_i))=c(C_i')$, and $I_k(C_0 \cdots C_{v-1})=I_k(C_0' \cdots C_{v-2}')$. Our assertion now follows from the cor. of prop. 2. 2 and from the induction hypothesis.

Corollary 2. *Let V be projective and let D be a k-divisor. Then*

$$\deg_k(D)\neq 0 \Longrightarrow c(D)\neq 0 .$$

Proposition 2. 4. *If V is projective, and if D is algebraically equivalent to zero on V, then $c(D)=0$.*

Proof. By Weil [3], § 1, Lemme 10, there exist a non-singular projective curve Γ, a divisor Z on $V\times\Gamma$, divisors D_i on V and points P_i on Γ $(i=1, 2)$, such that $D=D_1-D_2$ and $Z\cdot(V\times P_i)=D_i\times P_i$ $(i=1, 2)$. We shall prove $c(D_1)=c(D_2)$. By Serre duality, it is sufficient to prove that $c(D_1)\alpha=c(D_2)\alpha$ holds for any $\alpha\in H^{v-1,v-1}(V)$. Now, Künneth relation shows that the cohomology ring $H^*(V\times\Gamma)$ is the tensor product, over k, of the rings $H^*(V)$ and $H^*(\Gamma)$. Moreover, we have $c(V\times P_i)=1\otimes c(P_i)$, $c(D_i\times\Gamma)=c(D_i)\otimes 1$. On the other hand, since the divisors Z and $D_i\times\Gamma$ have the same trace $D_i\times P_i$ on $V\times P_i$, we have $\mathrm{tr}_{V\times P_i}(c(D_i\times\Gamma))=\mathrm{tr}_{V\times P_i}(c(Z))=c(D_i\times P_i)$. Hence we have

$$\begin{aligned} c(D_i)\alpha\otimes c(P_i)&=c(V\times P_i)c(D_i\times\Gamma)(\alpha\otimes 1)=\delta_i(\mathrm{tr}_{V\times P_i}(c(D_i\times\Gamma)(\alpha\otimes 1))) \\ &=\delta_i(\mathrm{tr}_{V\times P_i}(c(Z)(\alpha\otimes 1)))=c(Z)c(V\times P_i)(\alpha\otimes 1) , \end{aligned}$$

where δ_i has the same meaning as in prop. 2. 2 with respect to $V\times\Gamma$ and $V\times P_i$. But $c(P_1)=c(P_2)$ by prop. 2. 3. Thus we obtain $c(D_1)\alpha=c(D_2)\alpha$, completing the proof.

An application to birational transformations. Let $T: V' \to V$ be a regular (but not biregular) birational transformation from a non-singular variety V' onto V. We assume that V' is complete over V with respect to T. A prime divisor E of V' will be called *exceptional with respect to* T if $\dim T(E) < \dim E$. By a well-known theorem, V' has one or more (and of course finitely many) exceptional prime divisors with respect to T, and the union of their images under T is precisely the fundamental locus of T^{-1} on V. Now we have

Proposition 2.5. *Let $E_1, \cdots, E_e$ be the exceptional prime divisors of V' with respect to T. Then $c(E_1), \cdots, c(E_e)$ are linearly independent in $H^{1,1}(V')$ mod $T^*H^{1,1}(V)$.*

Proof. Suppose we have a non-trivial relation

$$a_1 c(E_1) + \cdots + a_e c(E_e) + T^*\alpha = 0\,, \quad \alpha \in H^{1,1}(V),\ a_\nu \in k,\ a_1 \neq 0.$$

Let U be an affin open subset of V having common points with $T(E_1)$, and replace V and V' by U and $T^{-1}(U)$ respectively (i.e. operate $\mathrm{tr}_{T^{-1}(U)}$ to everything). Since $\mathrm{tr}_{T^{-1}(U)}(T^*\alpha) = T^*(\mathrm{tr}_U \alpha) = 0$, we now have $c(a_1E_1 + \cdots + a_eE_e) = 0$. It follows that there exists a 1-form ω such that we have $E_1 \leqslant (\omega)_\infty \leqslant \sum_1^e E_\nu$ (on $T^{-1}(U)$). But then ω can have no polar divisor in U when considered as a differential form on U. Therefore ω must be regular on U, hence also on $T^{-1}(U)$. Contradiction.

Let $\mathfrak{U} = \{U_i\}$ be an affine covering of V, and denote by $T^{-1}\mathfrak{U}$ the covering $\{T^{-1}(U_i)\}$ of V'. Then $H^{1,1}(V) \simeq H^1(T^{-1}\mathfrak{U}, \Omega^1_{V'})$ and T^* is essentially the mapping associated with a refinement of the covering $T^{-1}\mathfrak{U}$ of V' by an affine covering (see the next section). But it is well known, and can be easily proved, that the refinement mappings of 1-dimensional cohomology groups are always injective. Therefore:

Corollary. *If $h^{1,1}(V)$ is finite, then we have*

$$h^{1,1}(V') \geqslant h^{1,1}(V) + e > h^{1,1}(V)\,.$$

In particular, an infinite descending chain of birationally equivalent complete non-singular varieties cannot exist[5].

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5) The cohomology group of an algebraic coherent sheaf over a complete variety is finite dimensional (Grothendieck [2]).

The second part of this corollary shows the "existence of a relatively minimal model", which was proved by Zariski by different methods. (Cf. Zariski [2], [3], [7].)

§ 3. The spectral sequence attached to a refinement of a covering.

Let $C=\sum C^{m,n}$ ($m\geqslant 0$, $n\geqslant 0$) be a positive double complex, with differentiations d' and d'' of degree (1, 0) and (0, 1) respectively. We prefer the commutativity $d'd''=d''d'$ to the anticommutativity of Cartan-Eilenberg [1] (p. 60), so that the total differentiation is $d=d'+(-1)^m d''$ on $C^{m,n}$. To C are attached two spectral sequences, the first and the second, which we shall denote by $\{I_r^{m,n}\}$ and $\{II_r^{m,n}\}$ (as in C.-E. [1] p. 331). Thus we have

$$I_2^{m,n}=H^m_{d'}(H^n_{d''}(C)) \underset{m}{\Longrightarrow} H(C)\,,$$
$$II_2^{m,n}=H^n_{d''}(H^m_{d'}(C)) \underset{n}{\Longrightarrow} H(C)\,.$$

Let X be a topological space, let F be a sheaf on X and let $\mathfrak{U}$, $\mathfrak{U}'$ be two open coverings of X. Let us consider the double complex

$$C=C(\mathfrak{U},\mathfrak{U}',F)$$

which was defined in Serre [2], p. 220[6]. Then we have

$$I_2^{m,n}=H^m(\mathfrak{U},H^n_{\mathfrak{U}'}F)\,,\qquad II_2^{m,n}=H^n(\mathfrak{U}',H^m_{\mathfrak{U}}F)\,,$$

where $H^n_{\mathfrak{U}'}F$ denotes the presheaf (Garbendatum) defined by

$$(H^n_{\mathfrak{U}'}F)(U)=H^n(\mathfrak{U}'_U,F)$$

$\mathfrak{U}'_U$ being the covering of U induced by $\mathfrak{U}'$. $H^m_{\mathfrak{U}}F$ is defined similarly. Since $(H^0_{\mathfrak{U}'}F)(U)=\Gamma(U,F)$, we have $I_2^{m,0}=H^m(\mathfrak{U},F)$, and similarly $II_2^{0,n}=H^n(\mathfrak{U}',F)$.

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6) Let $\mathfrak{U}=\{U_i\}_{i\in I}$ and $\mathfrak{U}'=\{U'_j\}_{j\in J}$. We set $C(\mathfrak{U},\mathfrak{U}',F)=\sum_{m,n}C^{m,n}(\mathfrak{U},\mathfrak{U}',F)$, $C^{m,n}(\mathfrak{U},\mathfrak{U}',F)=\prod\Gamma(U_s\cap U'_{s'},F)$, where the product is extended to all the pairs (s, s'), where s is an m-dimensional simplex of the nerve $S(I)$ of $\mathfrak{U}$ and s' is an n-dimensional simplex of the nerve $S(J)$ of $\mathfrak{U}'$. Thus an element of $C^{m,n}$ is a system $f=\{f_{i_0\cdots i_m,j_0\cdots j_n}\}$ of sections $f_{i_0\cdots i_m,j_0\cdots j_n}\in\Gamma(U_{i_0\cdots i_m}\cap U'_{j_0\cdots j_n},F)$. The differentiations are defined by

$$(d'f)_{i_0\cdots i_{m+1},j_0\cdots j_n}=\sum_{r=0}^{m+1}(-1)^r f_{i_0\cdots\hat{i}_r\cdots i_{m+1},j_0\cdots j_n}\,,$$
$$(d''f)_{i_0\cdots i_m,j_0\cdots j_{n+1}}=\sum_{r=0}^{n+1}(-1)^r f_{i_0\cdots i_m,j_0\cdots\hat{j}_r\cdots j_{n+1}}\,.$$

If $\mathfrak{U}'$ is a refinement of $\mathfrak{U}$, then $II_2^{m,n}=0$ for $m>0$, hence $H^n(C)\simeq II_2^{0,n}=H^n(\mathfrak{U}', F)$, and we have

$$I_2^{m,n}=H^m(\mathfrak{U}, H^n_{\mathfrak{U}'}F) \underset{m}{\Longrightarrow} \sum_q H^q(\mathfrak{U}', F).$$

Moreover, the edge-homomorphism

$$I_2^{m,0}=H^m(\mathfrak{U}, F) \to H^m(C)\cong H^m(\mathfrak{U}', F)$$

is precisely the homomorphism induced by the refinement (see Serre [2], N°29). If this homomorphism is injective, then $d_r: I_r^{m-r,r-1}\to I_r^{m,0}$ are zero and $I_2^{m,0}=I_r^{m,0}$ for all $r\geqslant 2$.

Still under the assumption that $\mathfrak{U}'$ is a refinement of $\mathfrak{U}$, the exact sequence for terms of low degree (C.-E. [1], p. 332) reads:

$$0\to H^1(\mathfrak{U}, F)\to H^1(\mathfrak{U}', F)\to H^0(\mathfrak{U}, H^1_{\mathfrak{U}'}F)\to H^2(\mathfrak{U}, F)\to H^2(\mathfrak{U}', F).$$

If F is a sheaf of rings, then one can define in C the cup-product by

$$a\in C^{m,n},\ b\in C^{s,t} \Longrightarrow a\cup b\in C^{m+s,n+t},$$

$$(a\cup b)_{i_0,\dots,i_{m+s};\,j_0,\dots,j_{n+t}}=a_{i_0,\dots,i_m;\,j_0,\dots,j_n}\cdot b_{i_m,\dots,i_{m+s};\,j_n,\dots,j_{n+t}}.$$

Then $d'(a\cup b)=d'a\cup b+(-1)^m a\cup d'b$, $d''(a\cup b)=d''a\cup b+(-1)^n a\cup d''b$. We introduce a new multiplication ab by $ab=(-1)^{ns}a\cup b$. Then C remains to be an associative ring, and this time we have

$$d(ab)=(da)b+(-1)^{m+n}a(db).$$

This ring structure of C induces ring structures on I_r and on $H(C)$. In I_r we have

$$d_r(ab)=(d_r a)b+(-1)^{m+n}a(d_r b) \qquad (a\in I_r^{m,n},\ b\in I_r^{s,t}).$$

An application of the multiplicative structure. Suppose that the following conditions are satisfied:

(A) $d_2: I_2^{0,1}\to I_2^{2,0}$ is zero,

(B) $I_2^{m,n}=I_2^{m,0}(I_2^{0,1})^n$ (for a fixed pair (m,n)).

$I_2^{m,n}$ is a residue class module of a submodule of $\sum_{i=0}^{n} C^{m+i,n-i}$, and from the conditions (A) and (B) it follows that each class of $I_2^{m,n}$ contains d-cocycles of $C^{m,n}$. Hence there is a natural homomorphism from $I_2^{m,n}$ onto $I_\infty^{m,n}$, and $d_r=0$ on $I_r^{m,n}$ ($r\geqslant 2$). If (B) is valid for all (m,n), then the spectral sequence is trivial in the sence that we have $d_r=0$ ($r\geqslant 2$), $I_2\simeq I_\infty$.

§ 4. Application to the study of T^*.

Lemma 4.1. *Let V and V' be algebraic varieties; let T be a rational mapping from V' onto V; let U (resp. U') be an affine open subset of V (resp. V') with affine ring A (resp. A'). Assume that T is regular in $T^{-1}(U)\cap U'$. Then $T^{-1}(U)\cap U'$ is an affine open subset of V', and (after identifying the function field $k(V)$ of V with a subfield of $k(V')$ by T) its affine ring is $A[A']$.*

Proof. Let P' be a point of $T^{-1}(U)\cap U'$ and let $(\mathfrak{o}', \mathfrak{m}')$ be the local ring of P' on V'. By assumption the point $T(P')=P$ is uniquely determined and belongs to U. We denote the local ring of P on V by $(\mathfrak{o}, \mathfrak{m})$. Then $\mathfrak{o}'$ dominates $\mathfrak{o}$ (i.e. $\mathfrak{o}'\supseteq\mathfrak{o}$ and $\mathfrak{m}'\cap\mathfrak{o}=\mathfrak{m}$). Since $\mathfrak{o}$ contains A and since $\mathfrak{o}'$ is a quotient ring of A', $\mathfrak{o}'$ is a quotient ring of $A[A']$ with respect to the prime ideal $A[A']\cap\mathfrak{m}'$.

Conversely, let $(\mathfrak{o}'', \mathfrak{m}'')$ be a quotient ring of $A[A']$ with respect to a prime ideal. Then $\mathfrak{o}''$ dominates the local rings $(\mathfrak{o}', \mathfrak{m}')$, $(\mathfrak{o}, \mathfrak{m})$ of points P', P of U' and U respectively. ($\mathfrak{o}'=A'_{(A'\cap\mathfrak{m}'')}$, $\mathfrak{o}=A_{(A\cap\mathfrak{m}'')}$). But then P and P' correspond under T, so that $P'\in T^{-1}(U)\cap U'$. Hence $\mathfrak{o}'$ is itself a quotient ring of $A[A']$ with respect to a prime ideal by what was just proved. Therefore $\mathfrak{o}''$ must coincide with $\mathfrak{o}'$.

Lemma 4.2. *Let V be an algebraic variety and let f be a regular function on V. Put $V_f=\{P\in V|f(P)\neq 0\}$. Let F be an algebraic coherent sheaf on V. Then:*

(1) *if $a'\in H^q(V_f, F)$, then there exist a natural number n and an element a of $H^q(V, F)$ such that $\mathrm{tr}_{V_f}(a)=f^n a'$,*

(2) *if $a\in H^q(V, F)$, and if $\mathrm{tr}_{V_f}(a)=0$, then there exists a natural number n such that $f^n a=0$.*

Proof. We begin with the case $q=0$. In this case (2) is proved in Serre [2], N°43, prop. 6, while (1) is proved in N°55, Lemme 1 of the same paper of Serre under the additional condition that V is affine. If V is not affine, we cover V by a finite number of affines U_i. Then there exist a natural number m and sections $a_i\in\Gamma(U_i, F)$ such that $a_i=f^m a'$ on $V_f\cap U_i$. Applying (2) to $U_i\cap U_j$, we have $f^t a_i=f^t a_j$ on $U_i\cap U_j$ for large t. Therefore the section $a\in\Gamma(V, F)$ defined by $a=f^t a_i$ on U_i satisfies our requirement: $a=f^{m+t}a'$ on V_f.

The general case. Let $\mathfrak{U}=\{U_i\}$ be an affine covering of V. It suffices to prove our assertion for the cohomology groups with respect to $\mathfrak{U}$. (1) Let $\{a'_{i_0,\dots,i_q}\}$ be a cocycle of a'. Then $a'_{i_0,\dots,i_q} \in \Gamma(U_{i_0,\dots,i_q} \cap V_f, F)$, therefore there exist a natural number n and a cochain a on V such that $a=f^n a'$ on V_f. Since $da=0$ holds on V_f, we can assume that $da=0$ on the whole V by augmenting the value of n if necessary. Then the cohomology class determined by the cocycle a satisfies the requirement. (2) can be proved similarly.

Let V, V' be algebraic varieties and let $T: V' \to V$ be a regular mapping from V' onto V. Let F' be an algebraic coherent sheaf on V', and let $\mathfrak{U}=\{U_i\}$ and $\mathfrak{U}'=\{U'_j\}$ be affine coverings of V and V' respectively such that $\mathfrak{U}'$ is a refinement of $T^{-1}\mathfrak{U}$. We now apply the spectral sequence method to the double complex $C(T^{-1}\mathfrak{U}, \mathfrak{U}', F')$.

Let us define presheaves H^qF' and TH^qF' $(q=0, 1, 2, \cdots)$ on V and V' by

$(H^qF')(U')=H^q(U', F')$ (U': open subset in V')

and by $(TH^qF')(U)=H^q(T^{-1}(U), F')$ (U: open subset in V)

and by the natural restriction mappings. Then it is clear that

$$H^m(\mathfrak{U}, TH^nF')=H^m(T^{-1}\mathfrak{U}, H^nF').$$

On the other hand, Lemma 4.1 implies that we can identify $H^m(T^{-1}\mathfrak{U}, H^nF')$ with $H^m(T^{-1}\mathfrak{U}, H^n_{\mathfrak{U}'}F')$. Thus the first spectral sequence takes the following form:

$$I_2^{m,n}=H^m(\mathfrak{U}, TH^nF') \underset{m}{\Longrightarrow} \sum_q H^q(V', F').$$

It is evident that TH^0F' is a sheaf[7], which can be denoted also by TF' according to a general rule. We denote temporarily by ^nF the sheaf on V associated with the presheaf TH^nF'. The ^nF are certainly algebraic sheaves (i.e. sheaves of $\mathcal{O}$-modules). They enjoy some of the properties of algebraic coherent sheaves[8].

Proposition 4.1. *Let U be an affine open subset of V. Then the canonical mappings $(TH^nF')(U)=H^n(T^{-1}(U), F') \to \Gamma(U, {}^nF)$ are bijective for all n.*

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7) We identify a sheaf F with the presheaf $U \to \Gamma(U, F)$ as usual.

8) Grauert-Remmert [1] denotes them by $T_n(F')$, and Grothendieck [2] by $R^qT_*(F')$. Grothendieck ([2]) has proved that they are coherent if V' is complete over V.

Proof. Let us denote by φ the mappings in question. Let a be an element of $H^n(T^{-1}(U), F')$ such that $\varphi a=0$. This amounts to say that there is a (finite) covering $\{U_i\}$ of U such that $\mathrm{tr}_{T^{-1}U_i}(a)=0$ for each i. Taking a refinement if necessary, we can assume that U_i have the form U_{f_i}, f_i being regular functions on U. By Lemma 4.2 we have $f_i^m a=0$ for large m[9]. Since the functions f_i^m have no common zero points, there exist regular functions g_i on U such that $\sum_i g_i f_i^m=1$ (Hilbert Nullstellensatz applied to the affine ring of U). Hence $a=\sum_i g_i f_i^m a=0$. The proof of the surjectivity is similar. We have to prove that, given an open covering $\{U_i\}$ of U and a system $\{a_i'\}$ of cohomology classes $a_i'\in H^n(T^{-1}(U_i), F')$ such that $a_i'=a_j'$ in $T^{-1}(U_{ij})$, we can find an element a of $H^n(T^{-1}(U), F')$ satisfying the relations $\mathrm{tr}_{T^{-1}(U_i)}(a)=a_i'$. We can assume, as above, that $U_i=U_{f_i}$ for some regular functions f_i. Then, by Lemma 4.2, there exist a natural number m and elements a_i of $H^n(T^{-1}(U), F')$ satisfying $a_i=f_i^m a_i'$ in $T^{-1}(U_i)$. Since $a_j=f_j^m a_j'=f_j^m a_i'$ holds in $T^{-1}(U_{ij})$, $f_j^{m'} a_j=f_j^{m+m'} a_i'$ holds in $T^{-1}(U_i)$ for sufficiently large m'. Let g_i be regular functions on U such that $\sum_i g_i f_i^{m+m'}=1$. Put $\sum_i g_i f_i^{m'} a_i=a$. Then we have $\mathrm{tr}_{T^{-1}(U_i)}(a)=\sum_j g_j f_j^{m+m'} a_i'=a_i'$.

Proposition 4.2. *With the same notations as above, let $\mathfrak{W}$ be an affine covering of the affine variety U. Then*

$$H^q(\mathfrak{W}, {}^nF)=0 \qquad (q>0,\ n \text{ arbitrary}).$$[10]

Proof. We prove the proposition in the case $\mathfrak{W}=\{U_i\}_{i\in I}$, $U_i=U_{f_i}$, f_i being regular functions on U. If this case is settled, then the proof of Serre [2], N°47, prop. 8 can be used, mutatis mutandis, to prove the general case. Now, let $\alpha=\{\alpha_{i_0,\dots,i_q}\}$ be a q-cocycle of ${}^nF$ with respect to $\mathfrak{W}$. By Prop. 4.1 and by Lemma 4.2, there exist a $(q-1)$-cochain $\beta^{(i)}$ for each $i\in I$, and a natural number t, with the following property:

$$\beta^{(i)}_{i_1,\dots,i_q}=f_i^t\alpha_{i,i_1,\dots,i_q} \qquad \text{in} \quad U_{i,i_1,\dots,i_q}.$$

On the other hand, it follows from $d\alpha=0$ that

9) Note that, identifying f_i with $f_i\circ T$, we have $T^{-1}(U)_{f_i}=T^{-1}(U_{f_i})$.

10) In particular, $H^q(V, {}^nF)=0$ $(q>0)$ if V is affine. This implies that our Čech cohomology of ${}^nF$ coincides with the Grothendieck cohomology of F.

$$\alpha_{i_0, i_1, \dots, i_q} = \sum_r^{q=0} (-1)^r \alpha_{i, i_0, \dots, i_r, \dots, i_q} \quad \text{in} \quad U_{i, i_0, \dots, i_q}$$

for any $i \in I$. Hence we have

$$f_i^{t+t'} \alpha = d(f_i^{t'} \beta^{(i)})$$

provided that t' is large enough. Let g_i be regular functions on U such that $\sum g_i f_i^{t+t'} = 1$. Then $\alpha = d(\sum_i g_i f_i^{t'} \beta^{(i)})$, which was to be proved.

Corollary. *Let $\mathfrak{V}$ be an arbitrary affine covering of V. Then the canonical homomorphism $H^q(\mathfrak{V}, {}^nF) \to H^q(V, {}^nF)$ is an isomorphism (for any q, n).*

Therefore our spectral sequence can be written as

$$I_2^{m,n} = H^m(V, {}^nF) \underset{m}{\Longrightarrow} \sum H^q(V', F') .$$

From this it follows, by a theorem of the theory of spectral sequences, that our spectral sequence $\{I_r | r \geqslant 2\}$ is independent of the choice of the *affine* coverings $\mathfrak{U}$, $\mathfrak{U}'$.

Example. Suppose that V is normal, that V' is complete over V and that the function field $k(V)$ of V maximally algebraic in $k(V')$. Under these conditions we have

$$\Gamma(U, \mathcal{O}_V) = \Gamma(T^{-1}(U), \mathcal{O}_{V'})$$

for any open subset U of V[11], so that $T\mathcal{O}_{V'} = \mathcal{O}_V$. Therefore, if we put $F' = \mathcal{O}_{V'}$ in the spectral sequence considered above, we have

$$I_2^{m,0} = H^m(V, \mathcal{O}_V)$$

and the edge-homomorphism $I_2^{m,0} \to H^m(V', \mathcal{O}_{V'})$ is precisely the homomorphism T^*.

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11) If we assume only the condition that V' is complete over V, then for each point P of V we have

$$\mathfrak{o}_P \subseteq \bigcap_{P' \in T^{-1}(P)} \mathfrak{o}_{P'} \subseteq \tilde{\mathfrak{o}}_P ,$$

where $\tilde{\mathfrak{o}}_P$ denotes the integral closure of $\mathfrak{o}_P$ in $k(V')$. Similarly we see that $\Gamma(T^{-1}(U), \mathcal{O}_{V'})$ contains $\Gamma(U, \mathcal{O}_V) (= \bigcap_{P \in U} \mathfrak{o}_P)$ and is contained in the integral closure in $k(V')$ of the latter ring. These are direct consequences of the following general theorem of Krull: if K is a field and if R is a subring of K, then the integral closure of R in K is the intersection of the valuation rings of K containing R. (See e.g. Akizuki-Nagata [1] or Weil [1]. Cf. also Zariski [3], p. 49.) Also it follows that, if V' is normal instead of V, then we have $I_2^{m,0} = H^m(\tilde{V}, \mathcal{O}_{\tilde{V}})$ for $F' = \mathcal{O}_{V'}$, where $\tilde{V}$ denotes the normalization of V in $k(V')$.

From now on we assume the following conditions:

(1) V' is complete over V,

(2) V' can be embedded in a projective space (as a locally closed subset).

Put $W_i=\{P \mid P\in V, \dim T^{-1}(P)\geqslant i\}$. The W_i are closed subsets of V (for a proof see Samuel [1], p. 36).

Proposition 4.3. *${}^nF$ is zero outside of W_n.*

Proof. Let P be a point of V outside of W_n. Let $\bar{V}'$ be the closure of V' in the ambient projective space, and let $D_1, \cdots, D_n$ be suitable hyperplane sections of $\bar{V}'$ such that $D_1\cap\cdots\cap D_n\cap T^{-1}(P)=\emptyset$. Then $T(V'\cap D_1\cap\cdots\cap D_n)$ does not contain P, hence there is an affine neighborhood U of P such that $T(V'\cap D_1\cap\cdots\cap D_n)\cap U=\emptyset$, or equivalently $D_1\cap\cdots\cap D_n\cap T^{-1}(U)=\emptyset$. By Lemma 4.1 $T^{-1}(U)$ is thus covered by n affine subsets, hence we have $H^n(T^{-1}(U), F')$ $=TH^nF'(U)=0$. Since the same holds for any affine neighborhood of P contained in U, the stalk of ${}^nF$ at P is zero, q.e.d.

(Remark. Our assumption (1) is essential to this proposition, while we do not know whether or not (2) is indispensable.)

Therefore, ${}^nF$ can be considered as a sheaf on W_n. The cohomological dimension of W_n is $\leqslant\dim W_n$ by a theorem of Grothendieck[12]. Hence:

Proposition 4.4. *We have $I_2^{m,n}=H^m(V, {}^nF)=0$ if $m>\dim W_n$.*

Corollary. *If $W_1=\emptyset$, i.e. if $T^{-1}(P)$ consists of a finite number of points for every point P of V, then the homomorphism $T^*: H^m(V, TF')\rightarrow H^m(V', F')$ is an isomorphism. If, moreover, V is affine, then also V' is affine.*

Proof. The first part is an immediate consequence of the proposition, since we have $I_2^{m,n}=0$ $(n>0)$. If V is affine, then it follows from the first part and from Prop. 4.2 that we have $H^m(V', F')=0$ $(m>0)$ for any algebraic coherent sheaf F'. By a theorem of Serre [4] this means that V' is affine.

Let F', G', H' be algebraic coherent sheaves on V' and let φ: $F'\otimes G'\rightarrow H'$ be a homomorphism. Then φ defines a cup-product $H^q(V', F')\times H^{q'}(V', G')\rightarrow H^{q+q'}(V', H')$ in the usual manner.

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12) If V is projective, then W_n can be covered by $1+\dim W_n$ affines and hence we can do without Grothendieck's theorem.

Proposition 4. 5. *Let U be an open subset of V containing W_1, and let α be an element of $H^q(V, TF')$ such that* $\mathrm{tr}_U(\alpha)=0$. *Then for each $\alpha' \in H^{q'}(V', G')$ there exists an element β of $H^{q+q'}(V, TH')$ satisfying* $\mathrm{tr}_U(\beta)=0$ *and* $\alpha \cup \alpha' = T^*\beta$.

Proof. Let $\mathfrak{U}_1=\{U_i\}_{i\in A}$ be an affine covering of U and let $\mathfrak{U}_2=\{U_i\}_{i\in B}$ be an affine covering of $V-W_1$. Put $\mathfrak{U}=\mathfrak{U}_1\cup\mathfrak{U}_2$. Let $\mathfrak{U}'=\{U'_j\}$ be an affine refinement of $T^{-1}\mathfrak{U}$. Let a (resp. a') be a representative cocycle of α (resp. α') with respect to $\mathfrak{U}$ (resp. $\mathfrak{U}'$). We can choose a in such a way that we have

$$a_{i_0\cdots i_q}=0 \quad (i_0, \cdots, i_q \in A).$$

Now consider the double complex $C=C(T^{-1}\mathfrak{U}, \mathfrak{U}' ; H')$. Setting

$$a^{(0)}_{i_0\cdots i_q, j_0\cdots j_{q'}}=\varphi(a_{(i)}\otimes a'_{(j)}) \quad (\in \Gamma(T^{-1}(U_{(i)})\cap U'_{(j)}, H')),$$

$a^{(0)}$ is a d-cocycle of $C^{q,q'}$, and its cohomology class in $H^{q+q'}(C)$ corresponds to $\alpha\cup\alpha'$ under the canonical isomorphism between $H(C)$ and $H(V', H')$. We shall construct successively d-cocycles $a^{(r)}\in C^{q+r, q'-r}$ $(r=1, 2, \cdots q')$ satisfying the following conditions:

(1) $a^{(r)}$ is chomologous to $a^{(r-1)}$,

(2) $a^{(r)}_{i_0\cdots i_{q+r}, j_0\cdots j_{q'-r}}=0 \quad (i_0, \cdots, i_{q+r}\in A)$.

Then $a^{(q')}$ will determine a cohomology class $\beta\in H^{q+q'}(V, TH')$ satisfying the requirements of the proposition.

Assume that $a^{(r-1)}$ is already constructed. Now, since $U_i\cap W_1 =\emptyset$ for $i\in B$, $T^{-1}(U_{i_0\cdots i_{q+r-1}})$ is affine if some i_s is in B. In this case, therefore, we can find $(q'-r)$-cochain $b_{i_0\cdots i_{q+r-1}}=\{b_{(i), j_0\cdots j_{q'-r}}\}_{(j)}$ of H' over $T^{-1}(U_{i_0\cdots i_{q+r-1}})$ such that its coboundary (with respect to the indices (j)) is $\{a^{(r-1)}_{i_q\cdots q+r-1, (j)}\}_{(j)}$. We set $b_{i_0\cdots i_{q+r-1}(j)}=0$ if $i_0, \cdots, i_{q+r-1}\in A$. Thus we get an element $b=\{b_{(i), (j)}\}$ of $C^{q+r-1, q'-r}$ satisfying $d''b=a^{(r-1)}$. Setting $a^{(r)}=(-1)^{q+r}d'b$, we have $d'a^{(r)} =d''a^{(r)}=0$, $a^{(r-1)}-a^{(r)}=d((-1)^{q+r-1}b)$. Also it is clear that $a^{(r)}$ satisfies our condition (2). Thus the proof is completed.

§ 5. The birational case.

With the same notations as in the preceding section, we now assume, besides the conditions (1) and (2) of p. 15, that T is birational. Then it is clear that we have $n+\dim W_n \leqslant v-1$ for $n\geqslant 1$ (principle of counting constants). Therefore it follows from Prop. 4. 4 that $I_2^{m,n}=0$ for $m+n\geqslant v (n>0)$. Thus we get the following proposition.

Proposition 5.1. *The homomorphism* $T^*: H^v(V, TF')\ (=I_2^{v,0}) \to H^v(V', F')$ *is surjective.*

Example. Suppose that T^{-1} has only a finite number of fundmental points (on V), say $P_1, \cdots, P_s$. Then $I_2^{m,n}=0$ except $I_2^{0,0}, \cdots, I_2^{v,0}$ and $I_2^{0,1}, \cdots, I_2^{0,v-1}$, and hence we have the following exact sequence:

$$\begin{array}{l} 0 \longrightarrow H^0(V, TF') \longrightarrow H^0(V', F') \longrightarrow 0 \\ \longrightarrow H^1(V, TF') \longrightarrow H^1(V', F') \longrightarrow I_2^{0,1} \\ \xrightarrow{d_2} H^2(V, TF') \longrightarrow H^2(V', F') \longrightarrow I_2^{0,2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \xrightarrow{d_v} H^v(V, TF') \longrightarrow H^v(V', F') \longrightarrow 0, \end{array}$$

where we know $I_2^{0,q}=\Pi_{i=1}^s H^q(T^{-1}(U_i), F')$, U_i being an affine neighborhood of P_i such that $U_i \not\ni P_j\ (i \neq j)$. If V is a normal surface then our hypothesis holds. If moreover all the modules in the exact sequence hove finite dimensions over k (which is the case when V and V' are projective varieties and TF' is algebraic coherent), then we have

$$\chi(V', F') = \chi(V, TF') - \dim I_2^{0,1} \leqslant \chi(V, TF').$$

If we set $F'=\mathcal{O}_{V'}$, then $TF'=\mathcal{O}_V$ and the above inequality reduces to the inequality $p_a(V') \leqslant p_a(V)$ which was proved by Muhly-Zariski for normal projective surafaces by a different method.

If V is non-singular, then $T\Omega_{V'}^p = \Omega_V^p$ holds. In fact, a differential form ω of $k(V)$ is regular at a simple point P of V if and only if it is regular along every prime divisor D of V passing through P[13]. But if ω has no poles which intersect $T^{-1}(P)$, then it must be regular along D since $T^{-1}[D] \cap T^{-1}(P) \neq \emptyset$. Hence ω is regular at P if and only if it has no poles which intersect $T^{-1}(P)$, and it follows that

$$\Gamma(T^{-1}(U), \Omega_{V'}^p) = \Gamma(U, \Omega_V^p)$$

for any open set U of V provided that V is non-singular.

From now on we assume that V and V' are normal projective varieties and that V is even non-singular. Then it follows from

13) If P is a simple point, if $(t_1, \dots, t_v)$ is a local coordinate system at P and if $\omega = \sum_{i_1 < \cdots < i_p} f_{i_1 \cdots i_p}\, dt_{i_1} \dots dt_{i_p}$, then it is easy to see that ω is regular at P (resp. along D) if and only if all the coefficients $f_{(i)}$ are in $\mathfrak{o}_P$ (resp. in $\mathfrak{o}_D$). But $\mathfrak{o}_P = \bigcap_{D \ni P} \mathfrak{o}_D$, hence our assertion. (For a detail see Koizumi [1] or Lang [1]).

Prop. 5. 1 that T^*: $H^{v,v}(V) \to H^{v,v}(V')$ is bijective since both modules have dimension one. Then we see by Serre duality (Lemma 1. 1.) that T^* is injective on the whole ring $H^*(V)$.

Proposition 5. 2. *Let V and V' be normal projectiev varieties and let T be a birational transformation from V' onto V which is regular on V'. If V is non-singular, then the homomorphism T^*: $H^*(V) \to H^*(V')$ is injective. In particular, we have $h^{p,q}(V) \leqslant h^{p,q}(V')$.*

Remark. It is conjectured that under the conditions of the proposition T^* is bijective on $H^{0,q}(V)$ at least when also V' is non-singular. This is certainly true in the case of characteristic zero by virtue of the equality $h^{0,q} = h^{q,0}$. We shall see later that this is true also in the case where T^{-1} is a monoidal transformation with a non-singular subvariety as center. As the problem of reduction of singularities is not yet solved in a satisfactory manner, it is desirable to prove the above conjecture without the assumption that V' is non-singular[14); if this is possible, then the birational invariance of arithmetic genus (for non-singular projective models) will be established most satisfactorily.

By virtue of the proposition we can identify $H^*(V)$ with the subring $T^*H^*(V)$ of $H^*(V')$. Assuming henceforth that also V' is non-singular, denote by $M^{p,q}$ or $M^{pq}(V, V')$ the subspace of $H^{p,q}(V')$ orthogonal to $H^{v-p,v-q}(V)$. Comparing the dimensions we have the direct decomposition

$$H^{p,q}(V') = H^{p,q}(V) + M^{p,q}\,.$$

Setting $M = \sum M^{p,q}$, we can easily see that $H^*(V) \cdot M \subseteq M$, though M is not an ideal of $H^*(V')$. Further we have the following propositions.

Proposition 5. 3. *Let W be a closed subset of V containing the fundamental locus W_1 of T^{-1}. Then $N(W, V) \cdot M = 0$. In particular, $H^{p,q}(V) \cdot M = 0$ for $q > \dim W_1$.*

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14) The normality of V' may perhaps be necessary. It should be noticed that we do not remove the assumption that V is non-singular. Therefore the conjecture implies that the $h^{0,q}$ of a normal projective model of $k(V)$ which dominates a non-singular projective model of $k(V)$ are numerical invariants of the function field $k(V)$, but not that all the normal models of $k(V)$ have the same $h^{0,q}$, which is evidently false. In the case of dimension 2 our conjecture holds, since the normal surface V' is dominated by a non-singular model V'', since $P_a(V) \geqslant P_a(V') \geqslant P_a(V'')$ and since V'' is obtainad from V by quadratic transformations.

Proof. Let us recall that $N(W, V)$ is the ideal of $H^*(V)$ formed by the classes of V locally zero at W. Now prop. 4.5 shows that $N(W, V)$ is also an ideal of $H^*(V')$. On the other hand we have $N(W, V)\cdot M\subseteq M$ by what was just remarked. Therefore $N(W, V)\cdot M\subseteq M\cap H^*(V)=0$.

Proposition 5.4. *If E is an exceptional prime divisor of V' with respect to T, then $c(E)\in M^{1,1}$.*

Proof. Since $T(E)$ is at most $(v-2)$-dimensional, we can cover it by $v-1$ affine open subsets of V. Denoting their union by U, we have $\mathrm{tr}_{T^{-1}(U)}(T^*H^{v-1,v-1}(V))=T^*(\mathrm{tr}_U(H^{v-1,v-1}(V)))=0$. Hence $c(E)\cdot T^*H^{v-1,v-1}(V)=0$ by prop. 2.1, as was to be proved.

§6. Apalication to the study of tr_U (U open).

Proposition 6.1. *Let V be a normal variety (complete or not), and let W be closed subset of codimension w' of V. Put $U=V-W$. Let F be an algebraic coherent sheaf on V locally isomorphic to $\mathcal{O}^n$. If the unmixedness theorem (Ungemischtheitssatz) holds in every local ring of V, then the restriction mapping*

$$\mathrm{tr}_U\colon H^q(V, F) \longrightarrow H^q(U, F)$$

is bijective for $q<w'-1$, and injective for $q=w'-1$.
Before proving this theorem we note the following

Corollary. *Let V be a non-singular variety (complete or not), let V' be a variety and let T be a regular birational transformation from V' onto V. Assvme that V' is complete over V. Then T^*: $H^{p,q}(V)\rightarrow H^{p,q}(V')$ is injective for $q<$codim W, where W denotes the fundamental locus of T^{-1}.*

Proof of the corollary. Put $U=V-W$, $U'=V'-T^{-1}(W)$. Then U and U' correspond biregularly under T. Identifying them, we have $\mathrm{tr}_U=(\mathrm{tr}_{U'})\circ T^*$. Therefore our assertion follows from the proposition[15].

For the proof of Prop. 6.1 we need a theorem of de Rham (de Rham [1]). We formulate it in the following slightly generalized form, this generalization being necessary later in §10.

15) The unmixedness theorem holds in any regular local ring (Cohen). See Akizuki-Nagata [1], pp. 138–139 or Nagata: "The theory of multiplicity in general local rings" (Proc. Intern. Symp. Algebraic Number Theory, Tokyo-Nikko 1955), where the question is discussed in detail.

Theorem of de Rham. Let R be a commutative ring and let M be an R-module. Let $y_1, \cdots, y_n$ be elements of R and $M_0, \cdots, M_{n-1}$ be subgroups of M satisfying the following conditions:

(1) $\sum_i y_i M_j \subseteq M_{j+1} \quad (0 \leqslant j < n)$,

(2) $m \in M_j,\ y_1 m = 0 \implies m = 0 \quad (0 \leqslant j < n)$,

(3) $m \in M_j,\ y_i m \in \sum_{s<i} y_s M_j \implies m \in \sum_{s<i} y_s M_{j-1}$
$(1 < i \leqslant n,\ 0 < j < n)$.

We make the convention $M_n = M$. For each q, $q = 0, 1, \cdots, n$, let now N_q be the additive group of the exterior q-forms in n indeterminates $X_1, \cdots, X_n$ with coefficients in M_q. Thus an element α of N_q can be expressed as

$$\alpha = \sum_{i_1 < \cdots < i_q} m_{i_1 \cdots i_q} X_{i_1} \wedge \cdots \wedge X_{i_q} \quad (m_{(i)} \in M_q),$$

and we have $N_0 = M_0$. Put $\omega = \sum y_i X_i$. Then the exterior product

$$\omega \wedge \alpha = \sum_{i_1 < \cdots < i_{q+1}} \Big(\sum_{r=1}^{q+1} (-1)^{r-1} y_{i_r} m_{i_1 \cdots \hat{i}_r \cdots i_{q+1}}\Big) X_{i_1} \wedge \cdots \wedge X_{i_{q+1}}$$

is well defined and belongs to N_{q+1}. Now de Rham's theorem asserts that, *if* $\alpha \in N_q$, $0 < q < n$ *and if* $\omega \wedge \alpha = 0$, *then there exists an element* β *of* N_{q-1} *such that* $\alpha = \omega \wedge \beta$.

The following proof differs little from de Rham's and is given here only for the sake of completeness.

We proceed by induction on n, the case $n=1$ being trivial. First we treat the case $q < n-1$. Put $\alpha = \alpha_1 + \alpha_2 \wedge X_n$, $\omega = \omega_1 + y_n X_n$, where the forms α_1, α_2 and ω_1 do not contain X_n. Then we have

$$\omega \wedge \alpha = \omega_1 \wedge \alpha_1 + (\omega_1 \wedge \alpha_2 + (-1)^q y_n \alpha_1) \wedge X_n = 0.$$

Hence $\omega_1 \wedge \alpha_1 = 0$, $\omega_1 \wedge \alpha_2 + (-1)^q y_n \alpha_1 = 0$. Since $q < n-1$ there exists, by the induction hypothesis, a $(q-1)$-form β_1 in $X_1, \cdots, X_{n-1}$ with coefficients in M_{q-1} such that $\alpha_1 = \omega_1 \wedge \beta_1$. Therefore we have

$$(*) \qquad \omega_1 \wedge (\alpha_2 + (-1)^q y_n \beta_1) = 0.$$

Here $\alpha_2 + (-1)^q y_n \beta_1$ is a $(q-1)$-form in $X_1, \cdots, X_{n-1}$ with coefficients in M_q. If $q > 1$, the induction hypothesis (with n and $M_0, \cdots, M_{n-1}$ replaced by $n-1$ and $M_1, \cdots, M_{n-1}$) shows the existence of a $(q-2)$-form β_2 in $X_1, \cdots, X_{n-1}$ with coefficients in M_{q-1} such that $\omega_1 \wedge \beta_2 = \alpha_2 + (-1)^q y_n \beta_1$. Setting $\beta = \beta_1 + \beta_2 \wedge X_n$, we have

$$\begin{aligned} \omega \wedge \beta &= \omega_1 \wedge \beta_1 + \omega_1 \wedge \beta_2 \wedge X_n + (-1)^{q-1} y_n \beta_1 \wedge X_n \\ &= \alpha_1 + \alpha_2 \wedge X_n = \alpha. \end{aligned}$$

If $q=1$, then (*) implies $y_1(\alpha_2-y_n\beta_1)=0$, hence $\alpha_2-y_n\beta_1=0$. Therefore we have $\alpha=\alpha_1+\alpha_2\wedge X_n=\omega_1\wedge\beta_1+y_nX_n\wedge\beta_1=\omega\wedge\beta_1$. The case $q=n-1$ requires another procedure. For simplicity, let us introduce the following notation (of "adjoint" forms):

$$*(X_{i_1}\wedge\cdots\wedge X_{i_p})=\varepsilon\begin{pmatrix}1\,2\cdots & & n\\ i_1\cdots i_p & j_1\cdots j_{n-p}\end{pmatrix}X_{j_1}\wedge\cdots\wedge X_{j_{n-p}}$$

where $\{j\}$ is the complementary set of $\{i\}$ in $\{1,\cdots,n\}$. Then an $(n-1)$-form α and an $(n-2)$-from β can be written as

$$\alpha=\sum_i a_i{*X_i}\quad(a_i\in M_{n-1}),\qquad \beta=\sum_{i<j}b_{ij}{*(X_i\wedge X_j)}\quad(b_{ij}\in M_{n-2}).$$

By easy calculations we have

$$\omega\wedge\alpha=(\sum_i y_ia_i)X_1\wedge\cdots\wedge X_n,\quad \omega\wedge\beta=\sum_i(\sum_j y_jb_{ij}){*X_i}$$

$(b_{ii}=0,\ b_{ij}+b_{ji}=0)$. Therefore we have only to prove the following statement: *if the elements* $a_i\in M_{n-1}$ *satisfy* $\sum y_ia_i=0$, *then there exist elements* $b_{ij}\in M_{n-2}$ *satisfying the relations* $b_{ii}=0$, $b_{ij}+b_{ji}=0$, $a_i=\sum_j y_jb_{ij}$. We prove this statement again by induction on n. If $n=1$, then the hypothesis implies $a_i=0$ and we can take $b_{11}=0$. Suppose $n>1$. Since $\sum y_ia_i=0$ implies $y_na_n\in\sum_{i<n}y_iM_{n-1}$, there exist elements $b_{nj}\in M_{n-2}(1\leqslant j<n)$ such that $a_n=\sum_{j<n}y_jb_{nj}$. Set $b_{nn}=0$, $b_{jn}=-b_{nj}$. Then we have $a_n=\sum y_jb_{nj}$, and the preceding relation $\sum_j y_ia_i=0$ is transformed into $\sum_{i<n}y_i(a_i-y_nb_{in})=0$. By the induction hypothesis, there exist elements $b_{ij}(1\leqslant i<n,\ 1\leqslant j<n)$ such that $b_{ii}=0$, $b_{ij}+b_{ji}=0$ and such that $a_i-y_nb_{in}=\sum_{j<n}y_jb_{ij}$. Thus we have obtained all the required elements b_{ij}.

Using this theorem of de Rham we prove the following lemma, which is a special case of the proposition (except that we need not the assumption of normality here).

Lemm. *Let V be an affine variety with affine ring A. Assume that the unmixedness theorem holds in A*[16]. *Let $f_1,\cdots,f_n$ be elements of A such that we have* $\operatorname{rank}(f_1,\cdots,f_i)A=i$ $(1\leqslant i\leqslant n)$, *and let W be the closed subset of V (of codimension n) defined by the ideal* $(f_1,\cdots,f_n)A$. *Then* $H^q(V-W,\mathcal{O})=0$ *for* $q\neq0,\ \neq n-1$.

16) It is clear that the unmixedness theorem holds in A if and only if it holds in every quotient ring (local ring) of A.

Proof of the Lemma. Put $U_i=\{P|P\in V, f_i(P)\neq 0\}$ for $i=1,2,\cdots,n$. Then $\mathfrak{U}=\{U_i\}$ is an affine covering of $V-W$, and the affine ring of U_i is $A[1/f_i]$. Let us calculate $H^q(\mathfrak{U}, \mathcal{O})$. Note that we can identify $\Gamma(U_{i_0\cdots i_q}, \mathcal{O})$ with the affine ring $A[1/f_{i_0},\cdots, 1/f_{i_q}]$. Let now a be an alternating q-cocycle. Then for sufficiently large integer t the functions $F_{i_0\cdots i_q}=(f_{i_0}\cdots f_{i_q})^t a_{i_0\cdots i_q}$ belong to A, and we have the relations $\sum_{r=0}^{q+1}(-1)^r f_{i_r}^t F_{i_0\cdots \hat{i}_r\cdots i_{q+1}}=0$. Since rank $(f_1^t,\cdots,f_i^t)A=$ rank $(f_1,\cdots,f_i)A=i$ and since the unmixedness theorem holds in A, $(f_1^t,\cdots,f_i^t)A : f_{i+1}^t=(f_1^t,\cdots,f_i^t)A$ for $1\leqslant i<n$. Setting therefore $R=A$, $M_0=M_1=\cdots=M_n=A$ and $y_i=f_i^t$ $(1\leqslant i\leqslant n)$ in the theorem of de Rham, the $(q+1)$-form

$$\alpha=\sum_{i_0<\cdots<i_q} F_{i_0\cdots i_q}X_{i_0}\wedge\cdots\wedge X_{i_q}$$

satisfies $\omega\wedge\alpha=0$, and consequently there exists a q-form

$$\beta=\sum_{i_0<\cdots<i_{q-1}} G_{i_0\cdots i_{q-1}}X_{i_0}\wedge\cdots\wedge X_{i_{q-1}}$$

such that $\omega\wedge\beta=\alpha$, provided that $q<n-1$. If $0<q<n-1$ and if we set $b_{i_0\cdots i_{q-1}}=(f_{i_0}\cdots f_{i_{q-1}})^{-t}G_{i_0\cdots i_{q-1}}$, then we obtain a $(q-1)$-cochain b such that $db=a$. This proves $H^q(\mathfrak{U}, \mathcal{O})=0$ $(0<q<n-1)$, whence follows the lemma.

Proof of Proposition 6.1. Since two sections of $\mathcal{O}$ which coincide at a point coincide everywhere, tr_U is injective at $q=0$. Therefore the proposition is trivial if $w'=1$. We shall assume that $w'>1$. In that case tr_U is even bijective, for a rational function which is not regular at a normal point P of V would admit a polar divisor passing through P.

Let $\mathfrak{U}=\{U_i\}$ be an affine covering of V and let $\bar{\mathfrak{U}}=\{U_i\cap U\}$ be the (non-affine) open covering of U induced by $\mathfrak{U}$. Let $\mathfrak{U}'=\{U'_j\}$ be an affine refinement of $\bar{\mathfrak{U}}$. Then $\Gamma(U_i, F)=\Gamma(U_i\cap U, F)$ by what has just been remarked (substituting U_i for V). Therefore we have $H^q(V, F)\simeq H^q(\bar{\mathfrak{U}}, F)=H^q(\mathfrak{U}, F)$, and the mapping tr_U reduces again to the mapping $H^q(\bar{\mathfrak{U}}, F)\to H^q(\mathfrak{U}', F)\simeq H^q(U, F)$ induced by the refinement of the covering, to which the method of §3 applies.

We proceed by induction on q (w' fixed). Let $q\leqslant w'-1$, and assume that the bijectivity of the restriction mappings is proved for any $q'<q$ (and for any V and W satisfying the conditions).

Consider the spectral sequence $\{I_r^{s,t}\}$ of the double complex $C(\mathfrak{U}, \mathfrak{U}', F)$. We choose $\mathfrak{U}$ so fine that F may be isomorphic to $\mathcal{O}^n$ in each U_i. Now we contend that $I_1^{s,t}=0$ for $0<t<q$. Since $\mathfrak{U}'$ induces on each $U_{i_0,\dots,i_s}-W$ an affine covering, we have

$$I_1^{s,t} = \prod_{i_0,\cdots,i_s} H^t(U_{i_0,\dots,i_s} - W, F)$$

and our contention will be proved if we can prove $H^t(V'-W, \mathcal{O})=0$ for an arbitrary affine open subset V' of V. Let A be the affine ring of V' and let $\mathfrak{a}$ be the ideal of W in A. Then rank $\mathfrak{a}\geqslant w'$. Therefore we can choose elements $f_1, \cdots, f_{w'}$ of $\mathfrak{a}$ such that $(f_1, \cdots, f_i)A$ has rank i (hence is unmixed) for $1\leqslant i\leqslant w'$. Let W' be the closed subset of V' defined by $f_1=\cdots=f_{w'}=0$. Then $W'\supseteq W\cap V'$, and the restriction mapping $H^t(V'-W, \mathcal{O})\to H^t(V'-W', \mathcal{O})$ is bijective (though we need only the injectivity) by the induction hypothesis. But $H^t(V'-W', \mathcal{O})=0$ by the lemma. Hence we have $H^t(V'-W, \mathcal{O})=0$, and our assertion $I_1^{s,t}=0$ $(0<t<q)$ is proved. It follows that $\mathrm{tr}_U\colon H^q(V, F)\to H^q(U, F)$ is injective. If $q<w'-1$, then, using the injectivity just proved for dimension q, we can repeat the same argument to prove $I_1^{s,q}=0$. From this we see that tr_U is bijective for dimension q. Thus the proposition is proved completely.

Examples. If V is an affine plane and W is a point, then it is easy to see $H^1(U, \mathcal{O})\neq 0$ (in fact, this module is isomorphic to the quotient module $k[x, y] \bmod k[x, y, 1/x]+k[x, y, 1/y]$, where x and y are independent variables). Therefore tr_U is not surjective for $q=1(=w'-1)$ in this case.

If V is a non-singular projective variety and U is a proper open subset of V, then $H^{v,v}(V)\to H^{v,v}(U)$ is not injective (see §7). If V is an abelian variety, then tr_U is not injective also on $H^{p,v}(V)$ for all p since $H^{p,q}(V)=H^{p,0}(V)\cdot H^{0,q}(V)$.

§7. The class of type (v, v) defined by a point.

Let V be a non-singular projective variety of dimension v, and let P be a point of V. We now propose to attach to P a cohomology class of V of type (v, v). In order to get a natural and useful definition, however, we can not confine ourselves to the consideration of the single model V, contrary to the case of divisors.

Let T^{-1} be the quadratic transformation of V with center P. Put $V'=T^{-1}(V)$, $E=T^{-1}(P)$. Then V' is again non-singular and E is (isomorphic to) a projective space of dimension $v-1$ (see Zariski [1]). It is well known, and can be easily proved by the consideration of local equations, that $E\cdot E=-H$, where H denotes the linear class of hyperplanes of the projective space E. Therefore $I(E \cdots E)=(-1)^{v-1}$.

Since $\mathrm{tr}_E(c(E))=c(E\cdot E)=-c(H)$, we have, by the cor. of prop. 2.3,

$$\mathrm{tr}_E(c(E)^{v-1}) = \pm\, c(H)^{v-1} \neq 0\,.$$

Hence, by the cor. of prop. 2.2, we see that $c(E)^v \neq 0$. On the other hand, we know $T^*: H^{v,v}(V) \rightarrow H^{v,v}(V')$ is bijective. These observations lead us to the following definition: $c(P)=(-1)^{v-1}(T^*)^{-1}(c(E)^v)$. Since $c(P)$ is not zero, it is a generator of the 1-dimensional vector space $H^{v,v}(V)$. Sometimes we shall write c_V instead of $c(P)$, which is justified by the following

Proposition 7.1. *The class $c(P)$ is independent of the choice of the point P on V.*

Proof. First we remark that, if $v=1$, then T is biregular and the class $c(P)$ as defined here coincides with the class attached to the divisor P in § 2. Therefore our assertion follows from prop. 2.3 in this case. We proceed by induction on v. Let P and Q be two points of V. Assuming $v \geqslant 2$, one can find a non-singular prime divisor S of V passing through both P and Q[17]. By the induction hypothesis, P and Q determine the same class c_S of S of type $(v-1, v-1)$. We shall prove $c(P)=c(Q)$ by showing the following formula:

17) Let L be the linear system cut out on V by the hypersurfaces of order three passing through P and Q. Then L defines the monoidal transformation of V with center $P+Q$, which we denote by T_0. Let S be the general member of L. Then P and Q are simple points S, because L contains divisors of the form $C+C'+C''$, where C (resp. C') is a hyperplane section of V containing P (resp. Q) as a simple point and not containing Q (resp. P) and C'' is hyperplane section of V passing through neither P nor Q. On the other hand, the proper transform $T_0[S]$ is non-singular since it is the general hyperplane section of the non-singular variety $T_0(V)$. But the birational correspondence between S and $T_0[S]$ is biregular at each point of S except at P and Q. Therefore S is non-singular. By the standard specialization argument, almost all (k-rational) members of L are irreducible and non-singular and hence satisfy our requirements.

$$c(P) = \delta(c_S),$$

where δ has the same meaning as in prop. 2. 2.

Let T^{-1} be, as above, the quadratic transformation of V with center P, and set $S' = T^{-1}[S]$. Then T induces a regular birational transformation $T_1 : S' \to S$, and T_1^{-1} is the quadratic transformation of S with center P. Setting $E_1 = T_1^{-1}(P)$, we have $E_1 = S' \cdot E$, and hence $c(E_1) = \mathrm{tr}_{S'}(c(E))$. Moreover, since $S' + E = T^{-1}(S)$ holds and since there is a divisor S_1 such that $S \sim S_1 \not\ni P$, it holds

$$(c(S') + c(E)) \cdot c(E) = c(T^{-1}(S_1)) \cdot c(E) = 0$$

by the cor. of prop. 2. 1. Hence we have, by prop. 2. 2,

$$\begin{aligned} T^*(\delta(c_S)) &= \delta_1(T_1^*(c_S)) = \delta_1((-1)^{v-2}c(E_1)^{v-1}) = (-1)^{v-2}c(S')c(E)^{v-1} \\ &= (-1)^{v-1}c(E)^v = T^*c(P), \end{aligned}$$

where δ_1 has the same meaning for V' and S' as δ has for V and S. It follows $c(P) = \delta(c_S)$. Similarly we have $c(Q) = \delta(c_S)$, hence $c(P) = c(Q)$ as wanted[18].

In the course of this proof we have incidentally proved the following

Proposition 7. 2. *Let D be a non-singular prime divisor of V. Then*

$$c_V = \delta(c_D),$$

where δ has the same meaning as in prop. 2. 2.

Remark. When V reduces to a point, we make the convention $c_V = 1 (\in k = H^{0,0}(V))$. Then it is easy to see that this proposition holds also in the case $v = 1$.

Let us now introduce a new definition. Let V and W be non-singular projective varieties. We shall denote by $\delta_{V,W}$ the isomorphism between the k-modules $H^{w,w}(W)$ and $H^{v,v}(V)$ which maps c_W to c_V. Then the proposition above implies that $\delta_{V,D}$ coincides with the connecting homomorphism obtained from the Poincaré residue exact sequence. This and prop. 2. 2 show $c(D) \cdot \alpha = \delta_{V,D}(\mathrm{tr}_D \alpha)$ $(\alpha \in H^{v-1,v-1}(V))$, a relation which will be generalized to non-singular subvarieties of any dimension in § 11.

Proposition 7. 3. *Let U and V be non-singular projective varieties. Then it holds*

18) For the geometric properties of quadratic transformations used in this proof, see Zariski [1], [2].

$$c_{U\times V}=c_U\otimes c_V\,.$$

Proof. The statemant of the proposition depends on the Künneth relation $H^*(U\times V)=H^*(U)\otimes H^*(V)$, which implies in particular $H^{u+v,u+v}(U\times V)=H^{u,u}(U)\otimes H^{v,v}(V)$. Now, if V reduces to a point, the proposition is trivial by our convention $c_V=1$. We proceed by induction on v. Let D be a non-singular prime divisor of V. Then the connecting homomorphism property shows $\delta_{U\times V,U\times D}=1\otimes\delta_{V,D}$. It follows from this and the induction hypothesis that

$$c_{U\times V}=\delta_{U\times V,U\times D}(c_{U\times D})=c_U\otimes\delta_{V,D}(c_D)=c_U\otimes c_V\,.$$

Proposition 7.4. *Let $D_1,\cdots,D_v$ be k-divisors on V, and let $I_k(D_1,\cdots,D_v)=a$. Then $c(D_1)\cdots c(D_v)=a\cdot c_V$.*

Proof. When $v=1$, this is nothing but a restatement of prop. 7.1 (or prop. 2.3). We proceed by induction on v. Assume $v\geqslant 2$. By linearity, one can assume D_1 is a prime divisor. Then one can find two non-singular prime divisor S_i $(i=1,2)$ such that $D_1\sim S_1-S_2$[19]. Again by linearity, therefore, one may assume D_1 is non-singular. Then we have, by prop. 2.2, by the induction hypothesis and by prop. 7.2,

$$c(D_1)\cdots c(D_v)=\delta_{V,D_1}(c(D_1D_2)\cdots c(D_1D_v))=\delta_{V,D_1}(a\cdot c_{D_1})=a\cdot c_V\,.$$

Proposition 7.5. *Let T be a regular mapping from a non-singular projective variety V' onto V such that $[V':V]=n<\infty$. Then we have*

$$T^*(c_V)=n\cdot c_{V'}\,.$$

As a consequence, $T^*:H^*(V)\to H^*(V')$ is injective if and only if n is not divisible by the field characteristic[20].

Proof. If T is inseparable, then any v-fold differential form of V vanishes when considered as a form on V'. In fact, any v functions $x_1,\cdots,x_v$ of $k(V)$ cannot form a separating transcendental

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19) This is well known, and follows from the fact that, if $|C_m|$ denotes the complete linear system cut out on V by the hypersurfaces of sufficiently high order m, the complete linear system $|C_m+D_1|$ is ample. (See Matsusaka [1]).

20) J.P. Serre constructed an interesting example in his paper [6]. There k is of characteristic p, $p\geqslant 5$; V' is a non-singular surface in a projective 3-space (hence $h^{1,0}(V')=h^{0,1}(V')=0$); $n=p$, but T is separable (V' is even unramified over V); and V has the pathological property $h^{1,0}(V)=0\neq h^{0,1}(V)=1$. In this example T^* is not injective, not only on $H^{2,2}(V)$, but also on $H^{0,1}(V)$.

base of $k(V')$ and hence their differentials in $k(V')$ are not linearly independent over $k(V')$. Therefore T^* is zero on $H^{v,q}(V)$ for any q, and in particular $T^*(c_V)=0$.

Now assume T is separable. Then for almost every point P of V, $T^{-1}(P)$ consists of exactly n distinct points. Let P be such a point and let $P'_1, \cdots, P'_n$ be the points of $T^{-1}(P)$. Denote by $(\mathfrak{o}, \mathfrak{m})$ and by $(\mathfrak{o}_i, \mathfrak{m}_i)$ the local rings of P and P'_i respectively. Then it is well known that we have $\mathfrak{m}\mathfrak{o}_i=\mathfrak{m}_i$ (see e.g. Abhyankar [1]). It follows easily that, if $S^{-1}: V \to V_1$ is the quadratic transformation of V with center P and if we denote by V'_1 the graph of the algebraic correspondence $S^{-1}\circ T$ between V' and V_1, then V'_1 is the monoidal transform of V' with center $T^{-1}(P)=P'_1+\cdots+P'_n$. Let us denote the projections $V'_1 \to V'$ and $V'_1 \to V_1$ by S' and T_1 respectively, so that the following diagram is commutative:

$$\begin{tikzcd}[column sep=huge, row sep=large] V' \arrow[d, "T"'] & V'_1 \arrow[l, "S'"', "\text{center } P'_1+\cdots+P'_n"] \arrow[d, "T_1"] \\ V & V_1 \arrow[l, "S"', "\text{center } P"] \end{tikzcd}$$

Set $S^{-1}(P)=E$, $S'^{-1}(P'_i)=E'_i$. Then $T_1^{-1}(E)=\sum E'_1$. Since the prime divisors E'_i are pairwise disjoint, we have, by the cor. of prop. 2.1 and by the definitions,

$$\begin{aligned} S'^*T^*(c_V) &= T_1^*S^*(c_V) = (-1)^{v-1}(c(E'_1)+\cdots+c(E'_n))^v \\ &= (-1)^{v-1}(c(E'_1)^v+\cdots+c(E'_n)^v) \\ &= S'^*(n\cdot c_{V'})\,. \end{aligned}$$

Since S'^* is injective, this proves the first assertion. The second assertion follows from the first and from Lemma 1.1.

At this juncture, we note the following elementary

Proposition 7.6. *Let V be a normal variety (complete or not), let K' be an algebraic extension field of the function field $k(V)$ of V, and let V' be the normalization of V in K'. If $[V': V](=[K': k(V)])$ is not divisible by the field characteristic, then*

$$T^*: H^q(V, \mathcal{O}) \to H^q(V', \mathcal{O}_{V'})$$

(where T denotes the natural regular mapping from V' onto V) is injective for all q.

Proof. Let $\mathfrak{U}$ be an affine covering of V. Then $T^{-1}(\mathfrak{U})$ is an affine covering of V'. Let $f=\{f_{i_0\cdots i_q}\}$ be a cocycle of $\mathcal{O}_V$ with respect to $\mathfrak{U}$, and assume that f, considered as a cocycle of $\mathcal{O}_{V'}$ with respect to $T^{-1}\mathfrak{U}$, is a coboundary: $f=dg$. Then $g_{i_0\cdots i_{q-1}}$ is an element of the affine ring $A'_{i_0\cdots i_{q-1}}$ of $T^{-1}(U_{i_0\cdots i_{q-1}})$, which is the integral closure, in K', of the normal affine ring $A_{i_0\cdots i_{q-1}}$ of $U_{i_0\cdots i_{q-1}}$. Taking trace from K' to $k(V)$, we obtain a $(q-1)$-cochain $Sp(g)$ of $\mathcal{O}_V$ with reapect to $\mathfrak{U}$ such that $f=d\left(\frac{1}{n}Sp(g)\right)(n=[V':V])$. This proves our proposition.

§8. Projective space.

Let L^r be an r-dimensional projective space, and let C^{r-1} be a hyperplane of L.

Proposition 8.1. *In L^r we have the following exact sequences:*

$$0\to\Omega^p(m)\to\mathcal{O}(m-p)^{\binom{r}{p}}\to\Omega_C^{p-1}(m)\to 0\quad(0<p\leqslant r)\,.$$

Proof. Let $x_1,\cdots,x_r$ be inhomogeneous coordinates of L, and let P_0 and C be, respectively, the origin and the plane at infinity with respect to the coordinate system (x). We shall identify $\Omega^p(m)$ with $\Omega^p(mC)$. Let K be the function field of L, and let G be the constant sheaf on L determined by the Grassmann algebra over the vector space $KX_1+\cdots+KX_r$, where $X_1,\cdots,X_r$ are indeterminates. We denote by $G(m,p)$ the subsheaf $\sum_{(j)}\mathcal{O}(m)X_{j_1}\wedge\cdots\wedge X_{j_{r-p}}$ of G. $G(m,p)$ is isomorphic to $\mathcal{O}(m)^{\binom{r}{p}}$. Put $\Theta=\sum_1^r x_iX_i$, and denote by θ the left multiplication by Θ. Then the sequence of sheaves

$$\begin{aligned}(1)\quad 0&\longrightarrow G(m-r,r)=\mathcal{O}(m-r)\xrightarrow{\theta}G(m-r+1,r-1)\xrightarrow{\theta}\cdots\xrightarrow{\theta}\\ &G(m-p,p)\xrightarrow{\theta}G(m-p+1,p-1)\xrightarrow{\theta}\cdots\xrightarrow{\theta}G(m,0)\xrightarrow{\theta}0\end{aligned}$$

is exact on $L-P_0$. For, if P is a point of L with local coordinates $y_1,\cdots,y_r(y_i=x_i/x_\alpha(i\neq\alpha),\ y_\alpha=1/x_\alpha)$, then $\Theta=y_\alpha^{-1}(\sum_{i\neq\alpha}y_ix_i+X_\alpha)$, and y_α is a local equation of C in a neighborhood of P, while $\sum_{i\neq\alpha}y_iX_i+X_\alpha$ is a member of a basis of the $\mathfrak{o}_P$-module $\sum_i\mathfrak{o}_PX_i$. Hence the sequence is exact at P.

Let ω be a p-fold differential form on L. Expressing ω by $dx_1,\cdots,dx_r$ as

$$\omega = \sum_{i_1<\cdots<i_p} f_{i_1\cdots i_p} dx_{i_1} \cdots dx_{i_p},$$

we define $\psi(\omega)$ by

$$\psi(\omega) = \sum_{i_1<\cdots<i_p} f_{i_1\cdots i_p} \quad *(X_{i_1} \wedge \cdots \wedge X_{i_p}),$$

where the star denotes the adjoint operator defined in § 6.

Let P be, as above, a point of L with local coordinates $y_1, \cdots, y_r$. Let the expression of ω in terms of $dy_1, \cdots, dy_r$ be

$$\omega = \sum_{i_1<\cdots<i_p} g_{i_1\cdots i_p} dy_{i_1} \cdots dy_{i_p} \text{ [21]}.$$

Then, using the relations $dx_i = (1/y_\alpha)dy_i - (x_i/y_\alpha)dy_\alpha$ $(i \neq \alpha)$ and $dx_\alpha = (-x_\alpha/y_\alpha)dy_\alpha$, we have

$$\text{(A)} \qquad \begin{cases} g_{\alpha i_2\cdots i_p} = -y_\alpha^{-p} \sum_{\beta=1}^{r} x_\beta f_{\beta i_2\cdots i_p} & (i_2, \cdots, i_p \neq \alpha) \\ g_{i_1\cdots i_p} = y_\alpha^{-p} f_{i_1\cdots i_p} & (i_1, \cdots, i_p \neq \alpha) \end{cases}$$

Solving these equations in f, we obtain

$$\text{(B)} \qquad \begin{cases} f_{\alpha i_2\cdots i_p} = -y_\alpha^{n} (\sum_{1}^{r} y_\beta g_{\beta i_2\cdots i_p}) & (i_2, \cdots, i_p \neq \alpha) \\ f_{i_1\cdots i_p} = y_\alpha^{n} g_{i_1\cdots i_p} & (i_1, \cdots, i_p \neq \alpha). \end{cases}$$

Finally, if $p \geqslant 2$, we have for $i_3, \cdots, i_p \neq \alpha$

$$\begin{aligned} \sum_{\beta=1}^{r} x_\beta f_{\beta\alpha i_3\cdots i_p} &= -\sum_{\beta\neq\alpha} x_\beta f_{\alpha\beta i_3\cdots i_p} \\ &= -\sum_{\beta\neq\alpha} x_\beta \Big(-\sum_{\gamma\neq\alpha} y_\alpha^{n} y_\gamma g_{\gamma\beta i_3\cdots i_p} - y_\alpha^{n+1} g_{\alpha\beta i_3\cdots i_p}\Big) \\ &= y_\alpha^{n+1} \sum_{\beta\neq\alpha}\sum_{\gamma\neq\alpha} x_\beta x_\gamma g_{\gamma\beta i_3\cdots i_p} + y_\alpha^{n} \sum_{\beta\neq\alpha} y_\beta g_{\alpha\beta i_3\cdots i_p} \\ \text{(C)} \qquad &= y_\alpha^{n} \sum_{\beta=1}^{r} y_\beta g_{\alpha\beta i_3\cdots i_p}. \end{aligned}$$

From these relations, it follows that

$$\begin{aligned} \omega \in \Omega^p(m)_P &\Longleftrightarrow \text{all } g_{i_1\cdots i_p} \in y_\alpha^{n-m}\mathfrak{o}_P \\ &\Longleftrightarrow \begin{cases} \text{all } f_{i_1\cdots i_p} \in y_\alpha^{n-m}\mathfrak{o}_P \\ \sum_{\beta=1}^{r} x_\beta f_{\beta i_2\cdots i_p} \in y_\alpha^{n-m}\mathfrak{o}_P \end{cases} \\ &\Longleftrightarrow \begin{cases} \psi(\omega) \in G(m-p, p)_P \\ \theta\psi(\omega) \in G(m-p, p-1)_P. \end{cases} \end{aligned}$$

21) In the following calculations, the f's and the g's are supposed to be alternating in the indices.

The first and the last conditions are equivalent also when P is the origin P_0 (proof is trivial). Thus we have shown:

(2) ψ maps $\Omega^p(m)$ into $G(m-p, p)$,

(3) $\psi(\Omega^p(m))$ coincides with the kernel of the homomorphism

$$G(m-p, p) \to G(m-p+1, p-1)/G(m-p, p-1)$$

induced by θ.

Moreover, $\theta\psi(\omega)=0$ implies $g_{\alpha i_2 \cdots i_p}=0$ (for all $i_2, \cdots, i_p$) (by (A)), and conversely (by (A) and (C)). This proves

(4) $G(m-p, p) \cap \operatorname{Ker}\theta = \psi(\Omega'^p(m))$,

where $\Omega'^p(m)$ denotes the subsheaf of $\Omega^p(m)$ consisting of the p-forms which are independent of the differential dy_α of the local equation y_α of C when expressed by $dy_1, \cdots, dy_r$. It may be remarked that, by the convention $\Omega'^0(m)=\Omega^0(m)=\mathcal{O}(m)$, (4) holds also for $p=0$.

From (2) and (3) it follows

$$G(m-p, p)/\psi(\Omega^p(m)) \simeq \theta G(m-p, p)/[G(m-p, p-1) \cap \theta G(m-p, p)].$$

Since $G(m-p, p)/\psi(\Omega^p(m))$ is zero outside C, we may henceforth confine ourselves to $L-P_0$. Then, by (1) and (4), we have

$$\begin{aligned} \theta G(m-p, p) &= G(m-p+1, p-1) \cap \operatorname{Ker}\theta = \psi(\Omega'^{p-1})(m)), \\ G(m-p, p-1) \cap \theta G(m-p, p) &= G(m-p, p-1) \cap \operatorname{Ker}\theta \\ &= \psi(\Omega'^{p-1}(m-1)). \end{aligned}$$

Therefore

$$\begin{aligned} G(m-p, p)/\psi(\Omega^p(m)) &\simeq \psi(\Omega'^{p-1}(m))/\psi(\Omega'^{p-1}(m-1)) \\ &\simeq \Omega'^{p-1}(m)/\Omega'^{p-1}(m-1) \simeq \Omega_C^{p-1}(m). \end{aligned}$$

This proves our proposition.

The exact sequences of this proposition, together with Serre duality, enable us to compute $\dim H^q(L^r, \Omega^p(m))$ rapidly, as we shall show in the next proposition.

Proposition 8.2. *We have*

(1) $\dim H^p(L^r, \Omega^p)=1 \quad (0 \leqslant p \leqslant r)$,

(2) $\dim H^0(L^r, \Omega^p(m)) = \binom{m+r-p}{r-p}\binom{m-1}{p} \qquad (m>p)$,

(3) $\dim H^r(L^r, \Omega^p(m)) = \binom{-m+p}{p}\binom{-m-1}{r-p} \qquad (p-r>m)$,

(4) $H^q(L^r, \Omega^p(m))=0$ *in all other cases.*

Proof. We begin with the case $p=0$. It is easy to see that the elements of $H^0(L^r, \mathcal{O}(m))$ $(m\geqslant 0)$ are in one-to-one correspondence with the forms of degree m in the homogeneous coordinates. Hence

$$\dim H^0(L^r,\mathcal{O}(m)) = \binom{m+r}{r} \quad (m\geqslant 0), \quad = 0 (m<0)\,.$$

By Serre duality and by the fact that a canonical divisor of L^r is $-r-1$ times hyperplane, we obtain
$\dim H^r(L^r, \mathcal{O}(m)) = \dim H^0(L^r, \Omega^r(-m)) = \dim H^0(L^r, \mathcal{O}(-m-r-1))$
$=\binom{-m-1}{r}(-r>m)$, $=0$ (otherwise). $H^q(L^r, \mathcal{O}(m))=0$ $(0<q<r)$ can be proved by the method used in the proof of the lemma of §6, i.e. essentially by the theorem of de Rham (cf. Serre [2] No. 78). Thus (1)$\sim$(4) hold for $p=0$, and, by Serre duality, for $p=r$.

Therefore our proposition holds for $r=0$, 1. We proceed by induction on r, assuming henceforth that $r\geqslant 2$.

Since $H^q(L^r, \mathcal{O}(-p))=0$ $(0\leqslant p\leqslant r,\ 0\leqslant p\leqslant r)$ by what was already proved, the preceding proposition shows $H^p(L^r, \Omega^p) \simeq H^{p-1}(L^{r-1}, \Omega^{p-1})$. This and the induction hypothesis prove (1).

Next we prove $H^q(L^r, \Omega^p(m))=0$ $(0<q<r,\ m\neq 0)$. If $q>1$, or if $q=1$ and $m<p$, then $H^{q-1}(L^{r-1}, \Omega^{p-1}(m))=0$ by the induction hypothesis. Since also $H^q(L^r, \mathcal{O}(m-p))=0$ holds, the preceding proposition shows $H^q(L^r, \Omega^p(m))=0$. The remaining case ($q=1$ and $m\geqslant p$) can be settled by Serre duality.

Now (2) follows from the exact sequence

$$0 \longrightarrow H^0(L^r, \Omega^p(m)) \longrightarrow H^0(L^r, \mathcal{O}(m-p))^{\binom{r}{p}}$$
$$\longrightarrow H^0(L^{r-1}, \Omega^{p-1}(m)) \longrightarrow 0$$

by a straightforward calculation. (3) is proved similarly (or can be reduced to (2) by Serre duality). Thus our proof is completed.

Remark 1. When $p=r$, the exact sequence of prop. 8. 1 is nothing but the exact sequence of the Poincaré residue.

Remark 2. Prop. 8. 2 is, of course, not new (see Hirzebruch [1]).

Remark 3. From $h^{p,q}=\delta^{p,q}(p,\ q\leqslant r)$ it follows that $H^*(L^r)$ is generated by $c(C)$, C being the linear equivalence class of hyperplanes, and is isomorphic to $k[X]/(X^{r+1})$, where $k[X]$ is a polynomial ring of one variable. In particular, the classes of $H^*(L)$ are invariant under the projective transformations of L.

§ 9. Projective bundles.

Let V^v be a non-singular variety, and let E be an algebraic bundle, with base V, fibre L^r, and structure group $GP(r)$. According to a recent theorem of Grothendieck[22], the bundle structure of E can be derived from a bundle with structure group $GL(r+1)$. In other words E can be considered as the "projective realization" of a vector bundle. Let $\pi: E \to V$ be the projection, and let $\mathfrak{U} = \{U_\alpha\}$ be a sufficiently fine affine covering of V. Then $\pi^{-1}(U_\alpha) \simeq U_\alpha \times L^r$, and we can introduce homogeneous coordinates $Y_{0\alpha}, Y_{1\alpha}, \cdots, Y_{r\alpha}$ in each $\pi^{-1}(U_\alpha)$ in such a way that we have

$$\rho Y_{i\alpha} = \sum_0^r g_{ij}^{\alpha\beta}(x) Y_{j\beta} \quad (\rho: \text{constant of proportionality})$$

over $x \in U_\alpha \cap U_\beta$, where the g's are regular functions on $U_\alpha \cap U_\beta$ such that the matrices $G^{\alpha\beta} = (g_{ij}^{\alpha\beta})$ satisfy the usual cocycle conditions. Denote by $U_{i\alpha}$ the affine open subset of $\pi^{-1}(U_\alpha)$ defined by $Y_{i\alpha} \neq 0$. Put $f_{j\beta, i\alpha} = \sum_h g_{ih}^{\alpha\beta}(Y_{h\beta}/Y_{j\beta})$. Then the f's define a line bundle over E, hence a divisor class D_0 of E.[23] D_0 is uniquely determined by the bundle structure of E. If resticted to $\pi^{-1}(U_\alpha)$, it is nothing but the class of $U_\alpha \times C$ in the product representation of $\pi^{-1}(U_\alpha)$, where C denotes the class of hyperplanes of L.

Now it is easy to see that $H^q(\pi^{-1}(U_\alpha), \Omega^p) = \Gamma(U_\alpha, \Omega_V^{p-q}) c(D_0)^q = \pi^* H^0(U_\alpha, \Omega_V^{p-q}) c(D_0)^q$. For, the Künneth relation and prop. 8.2 show

$$\begin{aligned} H^q(U_\alpha \times L, \Omega^p) &= \sum_{p'} H^0(U_\alpha, \Omega_V^{p-p'}) \otimes H^q(L, \Omega_L^{p'}) \\ &= H^0(U_\alpha, \Omega_V^{p-q}) \otimes H^q(L, \Omega_L^q) = H^0(U_\alpha, \Omega_V^{p-q}) \otimes c(C)^q, \end{aligned}$$

and $c(D_0)$ corresponds to $1 \otimes c(C)$.

Let $\mathfrak{U}'$ be an affine refinement of $\pi^{-1}\mathfrak{U}$, and consider the spectral sequence $\{I_s^{m,n}\}$ of $C(\pi^{-1}\mathfrak{U}, \mathfrak{U}', \Omega)$ $(\Omega = \sum_p \Omega^p)$. A representative cocycle of $c(D_0)$ can be considered as a d-cocycle γ of $C^{0,1}$, that is, an element γ of $Z^{0,1}$. Then we have

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22) Grothendieck [1], Proposition 3.4.1. We do not use this theorem in the sequel, since the fact asserted by the theorem is evident for the particular cases which we encounter.

23) Here we summarized a part of Washnitzer's lucid exposition in his paper [1], to which the reader is refferred for a detail. But, since the correspondeuce between line bundles and divisor classes given in his paper, differs by sign from ours which is the usual one, our $f_{i\beta, i\alpha}$ is his $f_{i\alpha, i\beta}$

$$I_2^{m,n}=\begin{cases}\sum_{p=0}^{r} H^m(V,\Omega_V^p)\cdot\gamma^n & (r\geqslant n)\\ 0 & (n>r)\,.\end{cases}$$

Hence follows $I_2^{m,n}=I_2^{m,0}\cdot\gamma^n$. Thus we find ourselves in the situation described at the end of § 3, so that the spectral sequence $\{I_s\}$ is trivial: $I_\infty\simeq I_2$. From this one can easily conclude the following

Proposition 9.1. *Let V, E, π, D_0 be as above. Then we have*

$$H^{p,q}(E)=\sum_{i=0}^{r}\pi^*H^{p-i,q-i}(V)\cdot c(D_0)^i \qquad (\textit{direct sum}).$$

Moreover, $H^*(V)\to\pi^*H^*(V)\cdot c(D_0)^i$ is bijective for $i\leqslant r$. In other words π^* is injective and $1, c(D_0),\cdots, c(D_0)^r$ are linearly independent over $\pi^*H^*(V)$, while $1, c(D_0),\cdots, c(D_0)^{r+1}$ are not.

From this proposition follows $h^{p,q}(E)=\sum_{i=0}^{r}h^{p-i,q-i}(V)$, in particular $h^{p,0}(E)=h^{p,0}(V)$, $h^{0,q}(E)=h^{0,q}(V)$. If V has the property that $h^{p,q}=h^{q,p}$ for all (p,q), then E enjoys the same property.

If V is projective, then also E is projective, as was proved by Washnitzer. In fact, it is not difficult to verify that the complete linear system $|D_0+H_m|$, where H_m denotes the inverse image, under π, of the linear system cut out on V by the hypersurfaces of sufficiently high order m, satisfies the conditions of an ample linear system given in Weil [2][24].

Proposition 9.2. *If V is projective, then we have*

$$c_E=c(D_0)^r\cdot\pi^*(c_V)\,.$$

Proof. Let P be a point of V and let T^{-1} be the quadratic transformation of V with center P. Set $T^{-1}(V)=V'$, $T^{-1}(P)=D_1$, and let E' be the induced bundle $T^{-1}E$ over V'. Denote by T' and by π', the induced map $E'\to E$ and the projection $E'\to V'$ respectively.

$$\begin{array}{ccc} E & \xleftarrow{T'} & E' \\ \pi\downarrow & & \downarrow\pi' \\ V & \xleftarrow{T} & V' \end{array}$$

Let U_P be a sufficiently small neighborhood of P. Put $D_1'=\pi'^{-1}(D_1)$,

24) As regards these conditions, see also Nakai-Nagata [1], p. 166, Th. 6.32.

$D_0' = T'^{-1}(D_0)$. Then $D_1' = D_1 \times L^r$ and $D_0' = T^{-1}(U_P) \times C$ in a product representation $\pi'^{-1}(T^{-1}(U_P)) = T^{-1}(U_P) \times L^r$. Since D_1' is contained in $\pi'^{-1}(T^{-1}(U_P))$ and since $I(D_1 \cdots D_1) = (-1)^{v-1}$, we have $I(\underbrace{D_0' \cdots D_0'}_{r} \cdot \underbrace{D_1' \cdots D_1'}_{v}) = (-1)^{v-1}$. Therefore it follows from the definition of c_V and from prop. 7.4 that

$$T'^*(c(D_0)^r \cdot \pi^*(c_V)) = c(D_0')^r (-1)^{v-1} c(D_1')^v = c_{E'} = T'^*(c_E)\,.$$

This proves the proposition since T'^* is injective.

This proposition provides us a second proof (for the case when V is projective) of the fact that $H^*(V) \to \pi^* H^*(V) \cdot c(D_0)^i$ $(i \leqslant r)$ are bijective.

§ 10. Non-singular monoidal transformations.

Let V^v be a non-singular variety, and let W be a non-singular subvariety of V of codimension $w' > 1$. Let T^{-1} be the monoidal transformation (sometimes called dilatation) of V with center W. (Such a monoidal transformation will be called non-singular.) Put $T^{-1}(V) = V'$, $T^{-1}(W) = E$. Then E is a projective bundle with base W and fibre $L^{w'-1}$, as we shall show presently. We shall investigate the structure of $H^*(V')$ by reducing the problem to that of E.

First we treat the problem locally. Let P be a point of W. Since P is simple on W, the prime ideal of W in the local ring $\mathfrak{o}_P$ of P on V is generated by w' elements, say $x_1, \cdots, x_{w'}$, and one can complete them to a regular system of parameters $\{x_1, \cdots, x_{w'}, \cdots, x_v\}$ of $\mathfrak{o}_P$. Let U be an affine neighborhood of P with affine ring A. Taking U sufficiently small, one may assume

a) $x_i \in A$ $(1 \leqslant i \leqslant v)$,

b) $\{x_1, \cdots, x_v\}$ is a local coordinate system in U (i.e. $\{x_1 - x_1(Q), \cdots, x_v - x_v(Q)\}$ is a regular system of parameters of $\mathfrak{o}_Q$ at each point Q of U),

c) $(x_1, \cdots, x_{w'})A = \mathfrak{p}$ is the prime ideal of W in A.

Then $T^{-1}(U)$ is covered by w' affines U_i' $(1 \leqslant i \leqslant w')$ with affine rings $A_i' = A[x_1/x_i, \cdots, x_{w'}/x_i]$. Put $t_{ji} = x_j/x_i$ $(j \neq i)$, $t_{ii} = x_i$. Then $\{t_{1i}, \cdots, t_{w'i}, x_{w'+1}, \cdots, x_v\}$ is a local coordinate system in U_i', and $t_{ii} (= x_i) = 0$ is a local equation of E in U_i'. The local ring $\mathfrak{o}_E = (A_i')_{x_i A_i'}$ of E on V' is the valution ring of the $\mathfrak{m}_W$-adic valuation

$v_E(\ )$ of $k(V)$, where $(\mathfrak{o}_W, \mathfrak{m}_W)$ is the local ring of W on V. Let $u_{ji} = \mathrm{tr}_E(t_{j_i})$ be the image of t_{j_i} in $k(E) = \mathfrak{o}_E/\mathfrak{m}_E =$ (the quotient field of $A'_i/x_i A'_i$). $k(E)$ contains $k(W) =$ (the quotient field of $A/\mathfrak{p}$) as a subfield, and $u_{1i}, \cdots, \hat{u}_{ii}, \cdots, u_{w'i}$ are algebraically independent over $k(W)$. Let X_1 be an indeterminate over $k(E)$, and put $X_j = u_{j_i} X_1$ $(2 \leqslant i \leqslant w')$. Then $u_{j_i} = X_j/X_i$ for $j \neq i$. Since the affine ring of $E \cap U'_i$ is $A'_i/x_i A'_i = (A/\mathfrak{p})[u_{1i}, \cdots, \hat{u}_{ii}, \cdots, u_{w'i}]$, $i = 1, \cdots, w'$, we see that $E \cap T^{-1}(U)$ is the product variety $(W \cap U) \times L^{w'-1}$ where $L^{w'-1}$ is the projective space with the homogenepus coordinates $X_1, \cdots, X_{w'}$.

If $\{y_1, \cdots, y_{w'}\}$ is another set of generators of $\mathfrak{p}$, then there exist functions $g_{ij} \in A$ and $f_{ij} \in A$ such that

$$y_i = \sum g_{ij} x_j, \quad x_i = \sum f_{ij} y_j .$$

Since the images of $y_1, \cdots, y_{w'}$ in $\mathfrak{m}_Q/\mathfrak{m}_Q^2$ are linearly independent over $k = \mathfrak{o}_Q/\mathfrak{m}_Q$ for each $Q \in U \cap W$, the regular functions $\bar{g}_{ij}$ induced on $W \cap U$ by g_{ij} are uniquely determined. For a similar reason, we have $(\bar{g}_{ij})(\bar{f}_{jk}) =$ the unit matrix. Let u'_{ji} and $(Y) = (Y_1, \cdots, Y_{w'})$, with $Y_1 = X_1$, have the same meanings for (y) as u_{ji} and (X) have for (x). Then

$$\begin{aligned} Y_i &= \mathrm{tr}_E(y_i/y_1) X_1 \\ &= \mathrm{tr}_E(x_1/y_1)\, \mathrm{tr}_E(y_i/x_1) X_1 \\ &= \mathrm{tr}_E(x_1/y_1) \sum_j \bar{g}_{ij} X_j . \end{aligned}$$

Thus the homogeneous coordinate system (X) and (Y) are related by the projective transformation induced by the linear transformation $(\bar{g}_{ij}) \in GL(w')$. Evidently, this local observation brings the global conclusion that E is a projective bundle with base W and fibre $L^{w'-1}$, and that the bundle structure is induced by a bundle structure with group $GL(w')$. Moreover, it is easy to see, by inducing on E the line bundle defined by E over V', that $E \cdot E = -D_0$, where the divisor class D_0 on E is defined as in §9.

Resticting our consideration again to $T^{-1}(U)$, we denote by $\Omega_p(mE)$ the subsheaf of $\Omega^p(mE)$ consisting of the p-forms which, when expressed by $dx_1, \cdots, dx_v$, contain only $dx_1, \cdots, dx_{w'}$. Then we can write

$$\Omega^p(mE) = \sum_{p'=0}^{p} \sum_{w' < j_1 < \cdots < j_{p-p'}} \Omega_{p'}(mE) dx_{j_1} \cdots dx_{j_{p-p'}} .$$

Further, we denote by $\Omega_{p,E}(mD_0)$ the subsheaf of $\Omega_E^p(mD_0)$ consisting of the forms which contain only the differentials of the local coordinates of L in the product representation $E\cap T^{-1}(U)=U\times L$ given above.

Proposition 10.1. *In $T^{-1}(U)$ we have the following exact sequences.*

$$0\to\Omega_p(mE)\to\mathcal{O}((m+p)E)^{\binom{w'}{p}}\to\Omega_{p-1,E}(-mD_0)\to 0\quad(0<p\leqslant w')\,.$$

Proof. Let $\omega=\sum_{j_1<\cdots<j_p\leqslant w'} f_{j_1\cdots j_p}\,dx_{j_1}\cdots dx_{j_p}$ be a p-form on V' and let P' be a point of U_i'. Let the expression of ω in terms of $dt_{1i},\cdots,dt_{w'i}$ ($t_{ji}=x_j/x_i$ $(i\neq j)$, $t_{ii}=x_i$) be

$$\omega=\sum_{j_1<\cdots<j_p} g_{j_1\cdots j_p}\,dt_{j_1i}\cdots dt_{j_pi}\,.$$

Then an easy calculation shows

$$\begin{cases} g_{ij_2\cdots j_p}=x_i^{p-2}\sum_{k=1}^{w'}x_kf_{kj_2\cdots j_p} & (j_2,\cdots,j_p\neq i)\\ g_{j_1\cdots j_p}\;=x_i^pf_{j_1\cdots j_p} & (j_1,\cdots,j_p\neq i)\end{cases}$$

$$\begin{cases} f_{ij_2\cdots j_p}=x_i^{-p}(-\sum_{k\neq i}t_{ki}g_{kj_2\cdots j_p}+x_ig_{ij_2\cdots j_p}) & (j_2,\cdots,j_p\neq i)\\ f_{j_1\cdots j_p}\;=x_i^{-p}g_{j_1\cdots j_p} & (j_1,\cdots,j_p\neq i)\end{cases}$$

and
$$\sum_{k=1}^{w'}x_kf_{kij_3\cdots j_p}=x_i^{2-p}\sum_{k=1}^{w'}t_{ki}g_{kij_3\cdots j_p}\qquad(p\geqslant 2)\,.$$

Consequently we have

$$\omega\in\Omega_p(mE)_{P'}\iff\begin{cases} f_{j_1\cdots j_p}\in\mathcal{O}(m+p)_{P'}\\ \sum_{k=1}^{w'}x_kf_{kj_2\cdots j_p}\in\mathcal{O}(m+p-2)_{P'}\,.\end{cases}$$

This conclusion, in which the subscript i plays no particular role, holds for any point P' of $T^{-1}(U)$. Now we may omit the rest of the proof, since it is entirely similar to that of prop. 8.1.

Next we must determine $H^q(T^{-1}(U),\ \mathcal{O}(mE))$.

Proposition 10.2. *With the same notations as above, we have*

$$H^q(T^{-1}(U),\ \mathcal{O}(mE))=0\qquad \textit{for}\ \ (0<q<w'-1,\ \textit{any}\ m)$$
$$\textit{and for}\ \ (q=w'-1,\ m<w')\,,$$
$$H^0(T^{-1}(U),\ \mathcal{O}(mE))=\begin{cases} H^0(U,\ \mathcal{O}_V)=A & (m\geqslant 0)\\ \mathfrak{p}^{-m} & (m<0)\,.\end{cases}$$

For the proof we need the following

Lemma. *Let t be an arbitrary but fixed natural number, and put*

$$\mathfrak{q}_i = (x_1^t, \cdots, x_i^t)A, \quad \mathfrak{p}_i = (x_1, \cdots, x_i)A \qquad (1 \leqslant i \leqslant w').$$

Then $\mathfrak{q}_i$ is a primary ideal belonging to $\mathfrak{p}_i$. Moreover, we have for $i < j \leqslant w'$

$$\mathfrak{q}_i\mathfrak{p}_j^k : x_{i+1}^t = \mathfrak{q}_i\mathfrak{p}_j^{k-t} = \mathfrak{q}_i \cap \mathfrak{p}_j^k \qquad (k \geqslant t)$$

and

$$\mathfrak{q}_i\mathfrak{p}_j^k : x_{i+1}^t = \mathfrak{q}_i \qquad (0 \leqslant k < t).$$

Proof of the lemma. By the condition b) imposed on U at the beginning, $x_1, \cdots, x_i$ are a part of a regular system of parameters of $\mathfrak{o}_Q$ for every point Q of U satisfying $x_1(Q) = \cdots = x_i(Q) = 0$. This implies that $\mathfrak{p}_i A_{\mathfrak{m}}$ is a prime ideal of rank i for every maximal ideal $\mathfrak{m}$ of A containing $\mathfrak{p}_i$. Therefore $\mathfrak{p}_i$ is a prime ideal of rank i. But $\mathfrak{p}_i \supseteq \mathfrak{q}_i \supseteq \mathfrak{p}_i^n$ for large n. Consequently rank $\mathfrak{q}_i =$ rank $\mathfrak{p}_i = i$, and $\mathfrak{p}_i$ is the only minimal prime of $\mathfrak{q}_i$. On the other hand, the unmixedness theorem holds in the affine ring A of U since U is non-singular. Therefore $\mathfrak{q}_i$ has no associated primes other than $\mathfrak{p}_i$. This proves our first assertion. It follows $\mathfrak{q}_i : x_{i+1}^t = \mathfrak{q}_i$, so that $\mathfrak{q}_i\mathfrak{p}_j^k : x_{i+1}^t = \mathfrak{q}_i$ $(0 \leqslant k < t)$ is trivial. In order to settle the remaining case $k \geqslant t$, we shall prove

(1) $\mathfrak{q}_i \cap \mathfrak{p}_j^{k+t} = \mathfrak{q}_i\mathfrak{p}_j^k$ and (2) $\mathfrak{p}_j^{k+t} : x_{i+1}^t = \mathfrak{p}_j^k$ for $k \geqslant 0$. Then it will follow

$$\begin{aligned}\mathfrak{q}_i\mathfrak{p}_j^k : x_{i+1}^t &= (\mathfrak{q}_i \cap \mathfrak{p}_j^{k+t}) : x_{i+1}^t = (\mathfrak{q}_i : x_{i+1}^t) \cap (\mathfrak{p}_j^{k+t} : x_{i+1}^t) \\ &= \mathfrak{q}_i \cap \mathfrak{p}_j^k = \mathfrak{q}_i\mathfrak{p}_j^{k-t}\end{aligned}$$

for $k \geqslant t$. Our proofs of (1) and (2) are modelled on Zariski's proof in his paper [4],

Proof of (1): we proceed by induction on k, the case $k = 0$ being trivial. Let us assume $\mathfrak{q}_i \cap \mathfrak{p}_j^{k+t-1} = \mathfrak{q}_i\mathfrak{p}_j^{k-1}$. Let $f \in \mathfrak{q}_i \cap \mathfrak{p}_j^{k+t}$. Then, by the induction hypothesis, f belongs to $\mathfrak{q}_i\mathfrak{p}_j^{k-1}$. Therefore we can write

$$f = \sum_\nu a_\nu M_\nu \qquad (a_\nu \in A),$$

where M_ν are the power products of $x_1, \cdots, x_j$ of degree $k+t-1$ in which at least one of $x_1, \cdots, x_i$ appears with exponent $\geqslant t$. Set $A\mathfrak{p}_j = \mathfrak{o}$, $\mathfrak{p}_j\mathfrak{o} = \mathfrak{m}$. Them $(\mathfrak{o}, \mathfrak{m})$ is a regular local ring, and $\{x_1, \cdots, x_j\}$ is a regular system of parameters of $\mathfrak{o}$. By a basic property of the regular local rings, therefore, $f = \sum a_\nu M_\nu \in \mathfrak{p}_j^{k+t} \subseteq \mathfrak{m}^{k+t}$ implies that all the coefficients a_ν belong to $\mathfrak{m}$, hence to $\mathfrak{m} \cap A = \mathfrak{p}_j$.

Consequently $f \in \mathfrak{q}_i \mathfrak{p}_j^k$. Also the proof of (2) is easy by a similar method (induction on k).

Proof of the proposition.

The case $q=0$. Let $f \in H^0(T^{-1}(U), \mathcal{O}(mE))$. The rational function f has no pole in $T^{-1}(U)$ except possibly at E. Hence it has no pole in U, so that it is regular everywhere in U (and consequently in $T^{-1}(U)$). Therefore $f \in A$. If $m=-m'<0$, then $f \in \mathfrak{p}^{m'}A_{\mathfrak{p}} \cap A = \mathfrak{p}^{m'}$ (this last equality is known and can be proved by the same method as used in the proof of the lemma).

The case $q>0$. We use the affine covering $\{U'_i\}_{1 \leqslant i \leqslant w'}$ of $T^{-1}(U)$. Then $\Gamma(U'_{i_0 \cdots i_q}, \mathcal{O}(mE)) = x_{i_0}^{-m} A[x_1/x_{i_0}, \cdots, x_{w'}/x_{i_0}, \cdots, x_{w'}/x_{i_q}]$. Let $f=\{f_{i_0 \cdots i_q}\}$ be an alternating cocycle of $\mathcal{O}(mE)$ with respect to the covering. Each $f_{(i)}$ is a homogeneous function in $x_1, \cdots, x_{w'}$ of degree $-m$ with coefficients in A. Taking a sufficiently large integer t (we assume, in particular, $t>m$), and setting $F_{i_0 \cdots i_q} = (x_{i_0}, \cdots, x_{i_q})^t f_{i_0 \cdots i_q}$, we have $F_{(i)} \in \mathfrak{p}^{t(q+1)-m}$ and $\sum_{r=0}^{q+1} (-1)^r x_{i_r} F_{i_0 \cdots \hat{i}_r \cdots i_{q+1}} = 0$. Now, the lemma above implies, among others, the following: setting $R=A$, $M=A$, $n=w'$, $y_i = x_i^t$ $(k \leqslant i \leqslant w')$, and $M_0 = A$, $M_i = \mathfrak{p}^{ti-m}$ $(1 \leqslant i < w')$, the assumptions of the theorem of de Rham given in §6 are satisfied. It follows easily that f is a coboundary provided that $1 \leqslant q \leqslant w'-2$. On the other hand, if $q=w'-1$ and $m<w'$, then $F_{12 \cdots w'} \in \mathfrak{p}^{tw'-m} = (x_1^t, \cdots, x_{w'}^t)\mathfrak{p}^{t(w'-1)-m}$ since $(t-1)w' < tw'-m$. This implies that f is a coboundary also in this case. Thus our proof is completed.

Now put $m=-1$ in prop. 10.1. Then, if $p>1$, the associated cohomology sequence shows $H^q(T^{-1}(U), \Omega_p(-E))=0$ $(q>0)$, since $H^q(T^{-1}(U), \mathcal{O}(-E))=0$ $(q>0)$ by prop. 10.2 and $H^q(T^{-1}(U) \cap E, \Omega_{p-1,E}(D_0)) \simeq H^0(U, \mathcal{O}_V) \otimes H^q(L, \Omega_L^{p-1}(C))=0$ $(q \geqslant 0, p>1)$ by prop. 8.2. When $p=1$, a similar consideration shows $H^q(T^{-1}(U), \Omega_1(-E))=0$ for $q>1$. For $q=1$, we obtain the following exact sequence:

$$
\begin{aligned}
0 \longrightarrow H^0(T^{-1}(U), \Omega_1(-E)) &\longrightarrow H^0(T^{-1}(U), \mathcal{O})^{w'} \\
\xrightarrow{\varphi} H^0(T^{-1}(U) \cap E, \mathcal{O}_E(D_0)) &\longrightarrow H^1(T^{-1}(U), \Omega_1(-E)) \longrightarrow 0 .
\end{aligned}
$$

It is not difficult to see that the mapping which is denoted here by φ is surjective. For that purpose, let us recall that φ is induced by the surjective homomorphism of sheaves

$$\mathcal{O}^{w'} \ni (f_1, \cdots, f_{w'}) \longrightarrow \sum_1^{w'} x_i f_i \in \mathcal{O}(-E)$$

followed by the isomorphism

$$\mathcal{O}(-E)/\mathcal{O}(-2E) \simeq \mathcal{O}_E(-E\cdot E) = \mathcal{O}_E(D_0)\,. \quad \text{(See p. 69 and p. 63).}$$

The mapping $H^0(T^{-1}(U), \mathcal{O}^{w'}) = A^{w'} \ni (f_1, \cdots, f_{w'}) \to \sum x_i f_i \in \mathfrak{p} = H^0(T^{-1}(U), \mathcal{O}(-E))$ is evidently surjective, and also the mapping $H^0(T^{-1}(U), \mathcal{O}(-E)) \to H^0(T^{-1}(U), \mathcal{O}_E(D_0))$ is surjective since $H^1(T^{-1}(U), \mathcal{O}(-2E))=0$. Therefore φ, which is the composition of these two mappings, is surjective. Consequently we have $H^1(T^{-1}(U), \Omega_1(-E))=0$. Thus we have seen $H^q(T^{-1}(U), \Omega_p(-E))=0$ for all $q>0$ and for all $p\geqslant 0$. It follows $H^q(T^{-1}(U), \Omega^p(-E))=0$ $(q>0,\ p\geqslant 0)$.

Consider now the following well known exact sequences (of Kodaira-Spencer):

$$\begin{cases} 0 \to \Omega'^p \to \Omega^p \to \Omega^p_E \longrightarrow 0 \\ 0 \to \Omega^p(-E) \to \Omega'^p \to \Omega^{p-1}_E(-E\cdot E) \to 0\,. \end{cases}$$

The result above, combined with $H^q(T^{-1}(U) \wedge E,\ \Omega^{p-1}_E(-E\cdot E)) \simeq H^q((U \wedge W) \times L^{w'-1},\ \Omega^{p-1}(W\times C)) = \sum_{s=0}^{p-1} H^0(U \wedge W,\ \Omega^s_W) \otimes H^q(L,\ \Omega^{p-1-s}_L(C))=0$ $(q>0)$, shows $H^q(T^{-1}(U),\ \Omega'^p)=0$ for $q>0$. Hence

Proposition 10. 3. *The mappings*

$$\mathrm{tr}_E : H^q(T^{-1}(U),\ \Omega^p) \longrightarrow H^q(T^{-1}(U) \wedge E,\ \Omega^p_E)$$

are bijective for $q>0$.

Now we are prepared for the global study. Let $\mathfrak{U}=\{U_\lambda\}$ be a sufficiently fine affine covering of V, and let $\mathfrak{U}'$ be an affine refinement of $T^{-1}\mathfrak{U}$. Besides the double complex $C=C(T^{-1}\mathfrak{U}, \mathfrak{U}', \Omega)$, we consider another, $C(E)=C(T^{-1}\mathfrak{U}, \mathfrak{U}', \Omega_E)$. Let $\{I^{m,n}_r\}$ and $\{I^{m,n}_r(E)\}$ be the associated spectral sequences. The homomorphism $\mathrm{tr}_E : \Omega \to \Omega_E$ induces homomorphisms $C \to C(E)$, $I_r \to I_r(E)$ which commute with the differential operators d, d_r. Now, prop. 10. 3 implies that $I^{m,n}_1 \to I^{m,n}_1(E)$ is bijective for $n>0$. Then, also $I^{m,n}_2 \to I^{m,n}_2(E)$ is bijective for $n>0$ since d_1 does not change the complementary degree, and $I^{m,n}_3 \to I^{m,n}_3(E)$ is bijective for $n\geqslant 1$ since d_2 diminishes the complementary degree by one, and so on.

From now on, we suppose that $T^* : H^*(V) \to H^*(V')$ is injective (which we could prove only in the case when V is projective).

Then $d_r: I_r^{m-r,r-1} \to I_r^{m,0}$ is zero for any $r \geqslant 2$. Then same holds for $I_r(E)$, as was seen in the preceding section. Therefore $I_r^{m,n} \to I_r^{m,n}(E)$ is bijective for $n>0$ and for all r. But the spectral sequence $\{I_r(E)\}$ was trivial. Hence also $\{I_r\}$ is trivial: $I_\infty \simeq I_2$. On the other hand we have $I_2^{m,n} \simeq I_2^{m,n}(E) \simeq \sum_p H^{p,m}(W)$ $0<n \leqslant w'-1)$, $=0$ $(n>w')$, and $I_2^{m,0} = H^m(V, \Omega)$. If we replace Ω^p for Ω, we obtain $I_2^{m,n} \simeq H^{p-n,m}(W)$ $(0<n \leqslant w'-1)$, $=0$ $(n \geqslant w')$, and $I_2^{m,0} = H^m(V, \Omega^p)$. Thus we have proved the following proposition.

Proposition 10. 4. *Let $T^{-1}: V \to V'$ be a non-singular monoidal transformation with center W, and let $w' = \mathrm{codim}\, W$. Then, assuming that $T^*: H^*(V) \to H^*(V')$ is injective, we have*

$$h^{p,q}(V') = h^{p,q}(V) + \sum_{i=1}^{w'-1} h^{p-i,q-i}(W)\,.$$

Remark 1. In fact, we have obtained a little more than this formula. Among other things we have

Kernel of $\{\mathrm{tr}_E: H^*(V') \to H^*(E)\} = T^*[$Kernel of $\{\mathrm{tr}_W: H^*(V) \to H^*(W)\}]$.

Remark 2. As a special case of this proposition, we have $h^{0,q}(V') = h^{0,q}(V)$. More precisely, $T^*: H^q(V, \mathcal{O}) \to H^q(V', \mathcal{O}_{V'})$ is bijective for all q. This result, which generalizes one of the main results of Muhly-Zariski [1] concerning the arithmetic genera of non-singular projective varieties, is of course a direct consequence of our prop. 10. 2 and Serre [2] No. 29 prop. 5, and so we need for its proof neither the spectral sequences nor the assumption that T^* is injective. It may be remarked that, if $v=2$ (and hence $W=$ *a* point), $H^1(T^{-1}(U), \mathcal{O})=0$ becomes almost trivial by the simple fact $A[t, 1/t] = A[t] + A[1/t]$.

Remark 3. The vanishing of $H^q(T^{-1}(U), \mathcal{O}_{V'})$ for $q>0$ (prop. 10. 2) implies, not only the bijectivity of $T^*: H^q(V, \mathcal{O}) \to H^q(V', \mathcal{O}_{V'})$ which was just remarked, but also the bijectivity of

$$T^*: H^q(V, \mathfrak{L}(D)) \longrightarrow H^q(V', \mathfrak{L}(T^{-1}(D)))$$

for any divisor D of V. Similar invariance theorems hold also for Witt vectors. Let $\boldsymbol{W}_n$ (resp. $\boldsymbol{W}'_n$) be the sheaf of the germs of regular Witt vectors of length n on V (resp. on V') (cf. Serre [6]). From the exact sequences $0 \to \mathcal{O}_{V'} \to \boldsymbol{W}'_n \to \boldsymbol{W}'_{n-1} \to 0$ it follows, by induction on n, that $H^q(T^{-1}(U), \boldsymbol{W}'_n) = 0$ for $q>0$. On the other hand, we have $T\boldsymbol{W}'_n = \boldsymbol{W}_n$ since $\boldsymbol{W}'_n$ and $\boldsymbol{W}_n$ are isomorphic to $(\mathcal{O}_{V'})^n$ and $(\mathcal{O}_V)^n$ respectively as far as the structure of sheaves of *sets* is concerned. Hence we have natural isomorphisms

$$T^*: H^q(V, \boldsymbol{W}_n) \longrightarrow H^q(V', \boldsymbol{W}'_n)$$

for any q and n. Taking the projective limit with respect to n, we obtain $H^q(V, \boldsymbol{W}) \simeq H^q(V', \boldsymbol{W}')$.

§11. The class of type (w', w') defined by a non-singular subvariety of condimension w'.

Let V^v be a non-singular projective variety, and let W be a non-singular subvariety of V of codimension w'. Let T^{-1} be the monoidal transformation of V with center W, and let $T^{-1}(V)=V'$, $T^{-1}(W)=E$. Then $c(E)^{w'} \in H^{w',w'}(V')$. Using the direct decomposition $H^{w',w'}(V')=T^*H^{w',w'}(V)+M^{w',w'}$ obtained in §5, we define a class $c(W)$ of V of type (w', w') by the following formula: $c(E)^{w'} =(-1)^{w'-1}T^*(c(W))+m$, $m \in M^{w',w'}$. Identifying $H^*(V)$ with $T^*H^*(V)$, we can characterize the element $c(W)$ in $H^{w',w'}(V)$ by the following property: $c(E)^{w'}+(-1)^{w'}c(W)$ is orthogonal to $H^{w,w}(V)$, where $w=v-w'=\dim W$. Another characterization is given by the following

Proposition 11. 1. *Let α be a cohomology class of V of type (w, w). Then*

$$c(W)\cdot\alpha = \delta_{V,W}(\mathrm{tr}_W(\alpha)) .$$

Proof. Let T, V', E be as above, and let $\pi: E \to W$ be the projection of the projective bundle E, namely the regular mapping of E induced by $T: V' \to V$. It is easy to see $\pi^*(\mathrm{tr}_W\alpha)=\mathrm{tr}_E(T^*\alpha)$. Put $\mathrm{tr}_W\alpha = a\cdot c_W$, $a \in k$. Then we have

$$\begin{aligned}
&T^*(c(W)\,\alpha)\\
&\quad = (-1)^{w'-1}\,c(E)^{w'}\,T^*\alpha \qquad \text{(by the definition)}\\
&\quad = (-1)^{w'-1}\,\delta_{V',E}(c(E\cdot E)^{w'-1}\,\mathrm{tr}_E\,T^*\alpha) \quad \text{(by prop. 2. 2 \& 7. 2)}\\
&\quad = (-1)^{w'-1}\,\delta_{V',E}((-1)^{w'-1}c(D_0)^{w'-1}\pi^*(\mathrm{tr}_W\alpha))\\
&\quad = \delta_{V',E}(c(D_0)^{w'-1}\pi^*(a\cdot c_W))\\
&\quad = \delta_{V',E}(a\cdot c_E) \qquad \text{(by prop. 9. 2)}\\
&\quad = a\cdot c_{V'}
\end{aligned}$$

Hence $c(W)\cdot\alpha = a\cdot c_V = \delta_{V,W}(ac_W) = \delta_{V,W}(\mathrm{tr}_W\alpha)$.

Corollary 1.

(1) $\operatorname{tr}_W \alpha = 0 \Longleftrightarrow c(W)\cdot\alpha = 0$ *for* $\alpha \in H^{w,w}(V)$,

(2) $\operatorname{tr}_W \alpha = 0 \Longrightarrow c(W)\cdot\alpha = 0$ *for any* $\alpha \in H^*(V)$.

Proof. (1) is an immediate consequence of the proposition, while (1) follows from (1) as in the cor. of Prop. 2. 2.

Corollary 2. *If* $\operatorname{tr}_W : H^*(V) \longrightarrow H^*(W)$ *is surjective, then*

$$\operatorname{tr}_W \alpha = 0 \quad \Longleftrightarrow c(W)\cdot\alpha = 0$$

holds for any $\alpha \in H^*(V)$.

Proof. If $\operatorname{tr}_W \alpha \neq 0$, then, by Serre duality applied to W, one can find $\beta \in H^*(V)$ such that $\operatorname{tr}_W(\alpha\beta) \in H^{w,w}(W)$, $\operatorname{tr}_W(\alpha\beta) \neq 0$. By Cor. 1 we have $c(W)\,\alpha\beta \neq 0$, hence $c(W)\,\alpha \neq 0$.

As an application of Cor. 2, we have the following

Praposition 11. 2. *Let V and V' be non-singular projective varieties. Let T be a regular mapping from V' into V and let Γ be its graph. Then the homomorphism $T^* : H^*(V) \to H^*(V')$ is determined completely by the class $c(\Gamma) \in H^{v,v}(V' \times V)$ of Γ.*

Proof. We have $H^*(V' \times V) = H^*(V') \otimes H^*(V)$ be Künneth relation, and the biregularity of the correspondence between V' and Γ implies that every class of $H^*(\Gamma)$ is of the form $\operatorname{tr}_\Gamma(\gamma \otimes 1)$, $\gamma \in H^*(V')$. On the other hand, if $\alpha \in H^*(V)$ and $\beta \in H^*(V')$, then $T^*(\alpha) = \beta$ is equivalent to $\operatorname{tr}_\Gamma(1 \otimes \alpha - \beta \otimes 1) = 0$. Applying the Cor. 2, we see that the last condition is equivalent to $c(\Gamma)\cdot(1 \otimes \alpha - \beta \otimes 1) = 0$.

More precisely, we can prove directly the following

Proposition 11. 2 bis. *Let $\{f_i^{(p,q)}\ (1 \leqslant i \leqslant h^{p,q}(V))\}$ and $\{g_j^{(p,q)}\ (1 \leqslant j \leqslant h^{p,q}(V')\}$ be bases of the k-modules $H^{p,q}(V)$ and $H^{p,q}(V')$ respectively, and let*

$$T^* f_i^{(p,q)} = \sum_j a_{ij}^{(p,q)} g_j^{(p,q)}$$

with $a_{ij}^{(p,q)} \in k$. Let $\varphi_i^{(v-p,v-q)}$ be the dual basis of $H^{v-p,v-q}(V)$ defined by

$$f_i^{(p,q)} \cdot \varphi_m^{(v-p,v-q)} = \delta_{im} \cdot c_V\,.$$

Tnen we have

$$c(\Gamma) = \sum_{p=0}^{v} \sum_{q=0}^{v} (-1)^{p+q} \sum_i \sum_j a_{ij}^{(p,q)} g_j^{(p,q)} \otimes \varphi_i^{(v-p,v-q)}\,.$$

Proof. By Künneth relation, $g_j^{(p,q)} \otimes \varphi_i^{(v-p,v-q)}$ $(0 \leqslant p \leqslant v,$ $0 \leqslant q \leqslant v,$ $1 \leqslant i \leqslant h^{p,q}(V),$ $1 \leqslant j \leqslant h^{p,q}(V'))$ constitute a basis of $H^{v,v}(V' \times V)$. Therefore we can write

$$c(\Gamma) = \sum_{p=0}^{v} \sum_{q=0}^{v} \sum_{i} \sum_{j} b_{ij}^{(p,q)} g_j^{(p,q)} \otimes \varphi_i^{(v-p,v-q)}$$

with $b_{ij}^{(p,q)} \in k$. Now, put $p' = v' - p$, $q' = v' - q$, and let $\{\psi_j^{(p',q')}$ $(1 \leqslant j \leqslant h^{p,q}(V'))\}$ be the dual basis of $H^{p',q'}(V')$ defined by

$$g_j^{(p,q)} \cdot \psi_n^{(p,q)} = \delta_{jn} \cdot c_V' .$$

Then

$$\begin{aligned}
&c(\Gamma) \cdot (\psi_j^{(p',q')} \otimes f_i^{(p,q)}) \\
&= b_{ij}^{(p,q)} (g_j^{(p,q)} \otimes \varphi_i^{(v-p,v-q)}) (\psi_j^{(p',q')} \otimes f_i^{(p,q)}) \\
&= (-1)^{v'(p+q)} b_{ij}^{(p,q)} g_j^{(p,q)} \psi_j^{(p',q')} \otimes f_i^{(p,q)} \varphi_i^{(v-p,v-p)} \\
&= (-1)^{v'(p+q)} b_{ij}^{(p,q)} c_{V'} \otimes c_V \\
&= (-1)^{v'(p+q)} b_{ij}^{(p,q)} c_{V' \times V} .
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
c(\Gamma) \cdot (\psi_j^{(p',q')} \otimes f_i^{(p,q)}) &= \delta_{V' \times V, \Gamma}(\mathrm{tr}_\Gamma(\psi_j^{(p',q')} \otimes f_i^{(p,q)})) \\
&= \delta_{V' \times V, \Gamma}(\mathrm{tr}_\Gamma(a_{ij}^{(p,q)} \psi_j^{(p',q')} g_j^{(p,q)} \otimes 1)) \\
&= (-1)^{pp'+qq'} a_{ij}^{(p,q)} \delta_{V' \times V, \Gamma}(\mathrm{tr}_\Gamma(c_{V'} \otimes 1)) \\
&= (-1)^{v'(p+q)+p+q} a_{ij}^{(p,q)} c_{V' \times V} .
\end{aligned}$$

It follows $b_{ij}^{(p,q)} = (-1)^{p+q} a_{ij}^{(p,q)}$, and the proof is completed.

Remark. With the same notations as above, let $V' = V$, and let Δ be the diagonal of $V \times V$. If we replace $f_i^{(p,q)}$ with $\varphi_i^{(p,q)}$ in the proposition, then we must use $(-1)^{(v+1)(p+q)} f_i^{(v-p,v-q)}$ in place of $\varphi_i^{(v-p,v-q)}$ since $\varphi_i^{(p,q)} f_m^{(v-p,v-q)} = (-1)^{(v+1)(p+q)} f_m^{(v-p,v-q)} \varphi_i^{(p,q)}$. Thus we obtain

$$c(\Delta) = \sum_{p=0}^{v} \sum_{q=0}^{v} (-1)^{v(p+q)} \sum_{i} \varphi_i^{(p,q)} \otimes f_i^{(v-p,v-q)} .$$

Let T be a regular mapping from V into itself with graph Γ, such that

$$T^* f_i^{(p,q)} = \sum_{j} a_{ij}^{(p,q)} f_j^{(p,q)} .$$

Denote the square matrices $(a_{ij}^{(p,q)})$ by $A(p, q)$. Then it follows from the proposition that

$$c(\Delta)\cdot c(\Gamma)=\{\sum_{p=0}^{v}\sum_{q=0}^{v}(-1)^{p+q}\operatorname{Sp}A(p,\ q)\}\ c_{V\times V}\,.$$

In the classical case, where $c(\Delta)\,c(\Gamma)=I(\Delta\cdot\Gamma)\cdot c_{V\times V}$, this formula is nothing but the fixed point formula of Lefschetz. In the case of characteristic p, one is led to the following problem: to find a cohomology ring of characteristic zero in which the geometric properties considered in the present note remain valid so that the Lefschetz fixed point theorem holds. (Cf. Serre [6]). As is well known, the investigation of this problem is a natural approach to Weil's conjecture on the number of rational points and the zeta function of a variety defined over a finite field.

Proposition 11. 3. *Let V be a non-singular projective variety, and let $W_1, \cdots, W_r$ be non-singular subvarieties of V, of the same codimension s, which we assume to be pairwise disjoint. Let $D_1, \cdots, D_s$ be positive disivisors of V such that $D_1\cdots D_s=W_1+\cdots+W_r$. Then we have*

$$c(D_1)\cdots c(D_s)=c(W_1)+\cdots+c(W_r)\,.$$

Proof. Let T^{-1} be the monoidal transformation of V with center $W=W_1\cup\cdots\cup W_r$, and put $T^{-1}(V)=V'$, $T^{-1}(W_i)=E_i$, $E_1+\cdots+E_r=E$. Then the exceptional prime divisors $E_1,\cdots,E_r$ are mutually disjoint, and hence we have $c(E)^s=c(E_1)^s+\cdots+c(E_r)^s$ by the cor. of prop. 2. 1. On the other hand, if T_1^{-1} is the monoidal transformation of V with center W_1 and if we denote $T_1^{-1}(W_1)=E_1'$, then $c(W_1)\alpha=(-1)^{s-1}c(E_1')^s\alpha$ for any $\alpha\in H^{w,w}(V)$ $(w=v-s=\dim W_1)$. Since E_1' is transformed to E_1 on the model V', we have $c(W_1)\alpha=(-1)^{s-1}c(E_1)^s$ in $H^*(V')$ (we identify $H^*(V)$ with $T^*H^*(V)$). Therefore we have $(c(W_1)+\cdots+c(W_r))\alpha=(-1)^{s-1}c(E)^s$ for any $\alpha\in H^{w,w}(V)$. Since, by Serre duality, $c(D_1)\cdots c(D_s)$ is equal to $\sum c(W_i)$ if and only if $c(D_1)\cdots c(D_s)\alpha=(\sum c(W_i))\alpha$ holds for any $\alpha\in H^{w,w}(V)$, we have only to prove the following: $c(D_1)\cdots c(D_s)\alpha=(-1)^{s-1}c(E)^s\alpha$ for any $\alpha\in H^{w,w}(V)$.

Let D_j' be the proper transform $T^{-1}[D_j]$ of D_j, the proper transform of a divisor being defined by linearity from the proper transforms of the components. Then $T^{-1}(D_j)=D_j'+E$ $(1\leqslant j\leqslant s)$

and $D'_1 \cap \cdots \cap D'_s = \emptyset$[25]. Put $c(D_j) = d_j$, $c(D'_j) = d'_j$, $c(E) = e$. Then we have $d'_1 \cdots d'_s = 0$ by the cor. of prop. 2.1. Let α be an arbitrary element of $H^{w,w}(V)$. Then $ed_j\alpha = 0$ by prop. 5.3 and prop. 5.4. Therefore

$$\begin{aligned} 0 = d'_1 \cdots d'_s\, \alpha &= (d_1 - e) \cdots (d_s - e)\alpha \\ &= d_1 \cdots d_s \alpha + (-1)^s e^s \alpha . \end{aligned}$$

Hence $d_1 \cdots d_s \alpha = (-1)^{s-1} e^s \alpha = \sum c(W_i)\alpha$.

Corollary. *Let V be a non-singular projective variety and let $U_1, \cdots, U_t$ be non-singular subvarieties of V of the same condimension u'. Let $W_1, \cdots, W_r$ be non-singular subvarieties of V, of the same codimension $s+u'$, which we assume to be pairwise disjoint. Let $D_1, \cdots, D_s$ be positive divisors of V such that $(\sum_1^t U_j) D_1 \cdots D_s = \sum_1^r W_i$. Then we have*

$$(\sum_{j=1}^t c(U_j)) c(D_1) \cdots c(D_s) = \sum_{i=1}^r c(W_i) .$$

Proof. Since each W_i is contained in one and only one U_j, we may assume, by linearity, that $t = 1$. Put $U = U_1$. Let $\bar{c}_i \in H^{s,s}(U)$ be the cohomology class of U corresponding to the subvariety W_i, $i = 1, 2, \cdots, r$, and put $\bar{D}_k = U \cdot D_k$, $k = 1, 2, \cdots, s$. Then we have for any $\alpha \in H^{w,w}(V)$ ($w = \dim W_i$)

$$\begin{aligned} c(U)c(D_1) \cdots c(D_s)\alpha &= \delta_{V,U}(\mathrm{tr}_U(c(D_1) \cdots c(D_s)\alpha)) \\ &= \delta_{V,U}(c(\bar{D}_1) \cdots c(\bar{D}_s)\, \mathrm{tr}_U \alpha) \\ &= \delta_{V,U}(\sum_{i=1}^r \bar{c}_i\, \mathrm{tr}_U \alpha) \\ &= \delta_{V,U}(\sum \delta_{UW_i}(\mathrm{tr}_{W_i}\alpha)) \\ &= \sum \delta_{V,W_i}(\mathrm{tr}_{W_i}\alpha) = \sum c(W_i)\alpha . \end{aligned}$$

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25) These assertions being of local nature, we shall prove them locally. Let P be a point of $W = W_1 \cup \ldots \cup W_r$, say of W_1. Let U be an affine neighborhood of P with affine ring A. We can take U so small that (1) $U \cap W_i = \emptyset$, $i = 2, \ldots, r$: (2) each D_j has a local equation $f_i = 0$ in U, and $f_j A$ is a prime ideal of A. Then $(f_1, \ldots, f_s) A$ is the prime ideal of W_1 in A, and $T^{-1}(U)$ is covered by s affines $U'_j (1 \leqslant j \leqslant s)$ with affine rings $A'_j = A[f_1/f_j, \ldots, f_s/f_j]$. $f_j = 0$ is a local equation of E (or, what is the same thing, of E_1) in U'_j. The local ring of D'_j (or, strictly speaking, of the unique component of D'_j which intersects with $T^{-1}(U)$) on V' is the local ring $A_{f_j A}$ of D_j on V. Therefore $D'_j \cap U'_j = \emptyset$ since $A_{f_j A} \not\supseteq A'_j$. Hence $D'_1 \cap \ldots \cap D'_s \cap U'_j = 0$ for any j. On the other hand, if $j \neq k$, $(f_j/f_k)A'_k$ is a prime ideal of A'_k, and the quotient ring of A'_k with respect to it coincides with $A_{f_j} A$. This shows that $f_j/f_k = 0$ is a local equation of D'_j in U'_k. Consequently, the local equation $f_j = 0$ of D_j in U becomes the local equation $(f_j/f_k)f_k = 0$ of $D'_j + E$ in U'_k, proving the relation $T^{-1}(D_j) = D'_j + E$ in U'_k.

This proves our assertion.

Discussion. Our theory is very incomplete as it stands. A complete theory would contain the following propositions:

(1) orthogonality formula: if $W_1 \cap W_2 = \emptyset$, $c(W_1)c(W_2) = 0$;

(2) carrier formula: $\mathrm{tr}_{V-W}(c(W)) = 0$;

(3) trace formula: if W_1, W_2, W_3 are non-singular subvarieties of V such that $W_1 \cdot W_2 = W_3$, then $\mathrm{tr}_{W_1}(c(W_2)) = c_1(W_3)$, where $c_1(W_3)$ is the cohomology class of W_1 corresponding to the subvariety W_3;

(4) intersection formula: with the same notations as above, $c(W_1)c(W_2) = c(W_3)$;

(5) transformation formula: if W_1 and W_2 are non-singular subvarieties of V intersecting properly on V, if T^{-1} is the monoidal transformation of V with center W_2, and if $W_1' = T^{-1}(W_1)$ is non-singular, then $c(W_1') = T^*(c(W_1))$.

By our cor. of prop. 11.1, the orthogonality formula is an immediate consequence of the carrier formula. The intersection formula follows easily from the trace formula or from the transformation formula. Since we could not prove these formulae in the general cases, and since it has been reported the Grothendieck succeeded in establishing a satisfactory theory, we will not enter into the detail. The main defect of our theory lies in the fact that it depends too much on the global property of the cohomology rings, i. e. on Serre duality. Is it possible to define $c(W)$ on non-complete varieties as we did in the case of divisors? Is it possible to define $c(W)$ also when W has singularities?

It may be remarked that, if W_1 is such that $c(W_1) = (-1)^{w'} c(D_1)^{w'}$, where $w' =$ codim W_1 and E_1 is the image of W_1 under the monoidal transformation of V with center W_1, then the transformation formula holds. This is the case, in particular, when the projective bundle E_1 is the product bundle. For example, if V is the product of two non-singular projective varieties V_1 and V_2, and if $W_1 = V_1 \times P$, $P \in V_2$, then E_1 is the product bundle, so that the transformation formula (and hence the intersection formula) is applicable. In this case $c(W_1) = 1 \otimes c(P)$ is clearly independent of the choice of P on V_2, so that the following proposition holds:

Proposition 11.4. *Let V_1 and V_2 be non-singular projective* *varieties and let Z be a non-singular subvariety of $V_1 \times V_2$. Let W_1 and W_2 be non-singular subvarieties of V_1, and P_1 and P_2 be points of V_2, such that $W_i \times P_i = Z(V_1 \times P_i)$, $i=1,2$. Then $c(W_1)=c(W_2)$.*

Corollary 1. *Every member of an algebraic family of regular mappings induce the same homomorphism between the cohomology rings. More precisely, let V_1, V_2, V_3 be non-singular projective varieties, let Z be a subvariety of $V_1 \times V_2 \times V_3$ such that $Z \cdot (V_1 \times V_2 \times P) = Z_P \times P$ is defined for any point P of V_3, and suppose that Z_P is the graph of a regular mapping T_P from V_2 into V_1 for any P. Then the homomorphism $T_P^*: H^*(V_1) \rightarrow H^*(V_2)$ is independent of P.*

Proof. Since Z_P is irreducible and non-singular for any P, Z is itself non-singular. By the proposition, therefore, $c(Z_P \times P) = c(Z_P) \otimes c_{V_3}$ is independent of P. Now our assertion follows from prop. 11.2.

This corollary is a kind of "homotopy theorem" (two continuous mappings which are homotopic induce the same homomorphism between the cohomology rings). Though we have proved this homotopy theorem only under a very restrictive condition, we can apply it to abelian varieties:

Corollary 2. *The elements of the cohomology ring of an abelian variety are invariant under the translations.*

Finally, we add the following

Proposition 11.5. *Let V be a non-singular projective variety, let W be a non-singular subvariety of V and let T^{-1} be the monoidal transformation of V with center W. Put $T^{-1}(W)=E$. In order that the projective bundle E is the product bundle, it is necessary that we have* $\mathrm{tr}_W(c(W))=0$.

Proof. If E is the product bundle $W \times L^{a-1}$, $a=$ codim W, then $\mathrm{tr}_E(c(E))^a = (1 \otimes c(H))^a = 0$, where H denotes the linear class of hyperplanes of L^{a-1}. By the remark 1 of prop. 10.4, this implies $c(E)^a \in T^* H^{a,a}(V)$, so that $c(E)^a = (-1)^{a-1} T^*(c(W))$ and $\mathrm{tr}_W(c(W)) = 0$.

Example. If V is the projective r-space L^r and W is a linear subspace L^s such that $2s \geqslant r$, then $\mathrm{tr}_W(c(W)) \neq 0$ by prop. 11.3, so that E is not the product bundle.

Appendix.

A. Let V be a normal variety and let D be a divisor on V. We want to prove that the algebraic sheaf $\Omega^p(D)$ is coherent. Since the question is local, one may assume that V is affine. Setting $M=\Gamma(V, \Omega^p(D))$, $A=\Gamma(V, \mathcal{O})$, we have to prove (i) M is a finte A-module and (ii) $\Omega^p(D)_x=M\otimes_A\mathcal{O}_x$ for every point x of V. Let $\omega\in\Omega^p(D)_x$, and let $D_1, \cdots, D_s$ be the prime divisors which appear in $-(\omega)-D$ with positive coefficients. Since $D_i \not\ni x$, the prime ideal $\mathfrak{p}_i$ of D_i in A contains a function s_i such that $s_i(x)\neq 0$. Set $s=(\Pi s_i)^N$. Then we have $s\omega\in M$ for sufficiently large N. Hence $\omega\in M\otimes\mathcal{O}_x$, and (ii) is proved. Now, let $f_1, \cdots, f_v$ be a fixed separating transcendental base of the function field $k(V)$ of V. Let ω be a p-form belonging to M, and write

$$\omega=\sum_{i_1<\cdots<i_p} g_{i_1\cdots i_p}df_{i_1}\wedge\cdots\wedge df_{i_p}=\sum_{(i)}g_{(i)}df_{(i)}\,.$$

We shall show that the coefficients $g_{(i)}$ belong to a fixed finite A-module. Then M is a submodule of a finite A-module, hence is itself finite over A. Now, if $\{j_1\cdots, j_{v-p}\}=\{1, \cdots, v\}-\{i_1, \cdots, i_p\}$, then $\omega\wedge df_{(j)}=\pm g_{(i)}df_1\wedge\cdots\wedge df_v$ and hence we have

$$(g_{(i)})\geqslant(df_{(j)})-D-(df_1\wedge\cdots\wedge df_v)\,.$$

Thus the problem is reduced to the case $p=0$, that is to say, it suffices to prove $\Gamma(V, \mathcal{O}(D))$ is finite over A. Let $D_+=\sum n_iD_i$ $(n_i>0)$ be the zero part of D and D_i be its components. It is easy to prove that there exists $f\in A$ such that $\operatorname{ord}_{D_i}(f)=n_i$ (Lang [1], p. 157, prop. 5). Then $\Gamma(V, \mathcal{O}(D))$ is a submodule of the finite A-module Af^{-1}, therefore is itself finite.

B. Let V be a normal projective variety. We shall show $h^{v,v}(V)=1$ by induction on $v(=\dim V)$. When $v=1$, V is a non-singular curve. In this case one may begin with showing the equality $i(D)=\dim H^1(V, \mathcal{O}(D))$ and then apply the Riemann-Roch theorem (cf. Serre [7] Ch. II). Another method is as follows. Let $F=P_1, \cdots, P_s$ be a finite subset of V and put $U=V-F$, $U_i=U\cup\{P_i\}$. Then $\mathfrak{U}=\{U_i\}$ is an open covering of V and such a covering is arbitrary fine. On the other hand, if P is a point of V, then the class $c(P)$ defined in §2 is not zero, so that we have $h^{1,1}\geqslant 1$. Therefore it suffices to show $\dim H^1(\mathfrak{U}, \Omega^1)=1$. If $\alpha=\{\omega_{ij}\}$ is an (alternating) 1-cocycle, then a necessary and sufficient condition for α to be coboundary is the existence of a 1-form $\omega_1 \in \Gamma(U_1, \Omega^1)$ such that $\omega_{j1} - \omega_1$ is regular at $P_j (j > 1)$. This is equivalent (by the residue theorem and R.-R. theorem) with $\sum_{j>1} \mathrm{Res}_{P_j}(\omega_{j1}) = 0$.

When $v > 1$, let C be a general hypersurface section of degree m of V. Then C is irreducible and normal. If ω is a v-form on V such that $\mathrm{ord}_C(\omega) = -1$, the "Poincaré residue" $R(\omega)$ of ω on C is defined. If P is a point of C and if $\omega \in \Omega^v(C)_P$ then $R(\omega) \in (\Omega_C^{v-1})_P$ since every prime divisor of C is simple on V. The sequence of sheaves

$$(1) \qquad 0 \longrightarrow \Omega^v \longrightarrow \Omega^v(C) \xrightarrow{R} \Omega_C^{v-1}$$

is exact, and R is onto on $V-S$, where S denotes the singular locus of C. Therefore, denoting the image of R in (1) by F, we have an exact sequence $0 \to F \to \Omega_C^{v-1} \to F'' \to 0$ and dim Supp (F'') $\leqslant n-3$. Hence $H^{v-1}(V, F) \simeq H^{v-1, v-1}(C)$. From the exact sequence

$$0 \longrightarrow \Omega^v \longrightarrow \Omega^v(C) \longrightarrow F \longrightarrow 0$$

we obtain an exact sequence $H^{v-1}(V, \Omega^v(C)) \to H^{v-1}(V, F) \to H^{v,v}(V)$ $\to H^v(V, \Omega^v(C))$, but if $\deg C = m$ is sufficiently large the extreme terms vanish, so that we have $H^{v,v}(V) \simeq H^{v-1,v-1}(C)$. This completes our induction step.

C. Let V be a variety defined over a field k_0, and let D be a k_0-rational divisor on V. Let ω be a p-form on V such that $(\omega) \geqslant -D$. Then ω can be written in the following form

$$\omega = \sum_\lambda c_\lambda \omega_\lambda ,$$

where ω_λ are p-forms defined over k_0 and the c are constants linearly independent over k_0. When such expression is given, the ω_λ's satisfy $(\omega_\lambda) \geqslant -D$.

In the case $p=0$, this proposition is "the last theorem of Weil's Foundations" (see also Lang [1], pp. 170-178). It can easily be generalized to the case $p > 0$ as follows. Take a separating transcendental base $f_1, \cdots, f_n$ of $k_0(V)$, and write $\omega = \sum_{(i)} g_{(i)} df_{(i)}$. Then, as in Appendix A, the $g_{(i)}$ satisfy relations of the form $(g_{(i)}) \geqslant -D'$, where D' is a k_0-rational divisor. Therefore, applying the case $p=0$ to the $g_{(i)}$, we obtain the first half of the proposition. For the second assertion, let Δ be an arbitrary divisor (rational over k_0 or not), and recall the fact one can always take a local coordinate system $(t_1, \cdots, t_v)$ at Δ consisting of functions defined over k_0 (in fact, one can choose the t_i's from the affine coordinates at Δ). If $\omega_\lambda = \sum_{(i)} g_{(i),\lambda} dt_{(i)}$, then we have $\mathrm{ord}_\Delta (\sum_\lambda c_\lambda g_{(i),\lambda}) \geqslant \mathrm{ord}_\Delta(-D)$. If follows $\mathrm{ord}_\Delta(g_{(i),\lambda}) \geqslant \mathrm{ord}_\Delta(-D)$ since $g_{(i),\lambda}$ are defined over k_0, and hence $(\omega_\lambda) \geqslant -D$.

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