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On a characterization of a Jacobian variety[1),2)]

By

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Let J^g be a Jacobian variety of a complete non-singular curve Γ, φ be a canonical mapping of Γ into J and let Θ be a canonical divisor corresponding to $\varphi(\Gamma)$. It is known, in the classical case, that $\Theta_{u_1}\cdots\Theta_{u_{g-1}}$ is numerically equivalent to $(g-1)!\ \varphi(\Gamma)$ and the self-intersection number of Θ is $g!$. Originally these are due to Poincaré and later Castelnuovo gave an algebro-geometric proof for the first (cf. Castelnuovo [1]). Castelnuovo's idea is very simple but the proof depends upon a rather difficult result, the irreducibility of the variety of moduli of curves of the given genus. First, we shall prove them by using the theorem of Riemann-Roch and an equivalence criterion for numerical equivalence we shall discuss. Later in the Appendix, we shall prove them using Weil's idea, which was communicated to the writer by him. Next let A be an Abelian variety of dimension n, X be an irreducible subvariety of dimension $n-1$ on A such that the self-intersection number of X is $n!$ and that $X_{u_1}\cdots X_{u_{n-1}}$ is numerically equivalent to $(n-1)!\,C$, where C is a positive 1-cycle on A. Then we shall show that C is irreducible, non-singular, A is the Jacobian variety of C, C is canonically embedded into A and that X is a canonical divisor corresponding to C. Therefore, we can say that the two numerical relations, together with the irreducibility of the divisor, characterizes a canonically polarized Jacobian variety completely. In §1, we define an endomorphism $\alpha(X, Y)$ relative to a pair (X, Y) of

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2) We shall follow the terminology and conventions of Weil [6], [8].

cycles of complementary dimensions on an Abelian A. We shall see that it is bilinear and depends only on classes of cycles modulo numerical equivalence. Mappings $X \rightarrow \alpha(X, Y)$, $Y \rightarrow \alpha(X, Y)$, when Y is kept fixed in the former case and X is kept fixed in the latter case, define homomorphisms of groups of classes of cycles, modulo numerical equivalence, into the ring $\mathfrak{A}$ of endomorphisms of A. When Y (resp. X) is either a positive non-degenerate divisor or a complete intersection of positive non-degenerate divisor, the former (resp. latter) will be shown to be an isomorphism.

§ 1. Endomorphisms attached to cycles

The writer assumes that the reader is familiar with the theory of Picard varieties. Those results which will be needed in this paper could be found in writer's another paper (cf. Matsusaka [2]) with references. Let $G^r(A)$ be the additive group of r-cycles on A and let $G^r_n(A)$ be the set of r-cycles X on A such that $\deg(X \cdot Y^{n-r})=0$ for all Y^{n-r} on A, whenever $X \cdot Y$ is defined. $G^r_n(A)$ forms a subgroup of $G^r(A)$, and any r-cycle X in $G^r_n(A)$ is said to be *numerically equivalent* to 0. When X and X' are two r-cycles such that $X-X'$ is numerically equivalent to 0, we say that X and X' are numerically equivalent to each other. When $X-X' \in G^r_n(A)$, we write $X-X' \equiv 0 \bmod G^r_n(A)$. We shall omit r, when there is no danger of confusion.

We are going to define three types of homomorphisms attached to cycles. Let Z be a cycle on a product $A \times B$ of two Abelian varieties such that $\dim Z = \dim A$. Z defines a homomorphism α of A into B by $S(Z(u)) = \alpha(u)+c$, where c is a constant (cf. Weil [8], Th. 1, Th. 9). In particular, when U and V are A-cycles of complementary dimensions, there is a cycle Z of the same dimension as A on $A \times A$ such that

$$u \times U \cdot V_u = Z \cdot (u \times A) .$$

Then the endomorphism α of A defined by this Z is such that $S(U \cdot V_u) = \alpha(u)+c$. We shall denote this α by $\alpha(U, V)$.

Let W be a divisor on $A \times B$ and β be the canonical homomorphism of the group $G_a(B)$ (of B-divisors algebraically equivalent to 0) onto $P(B)$, the Picard variety of B. Then W defines a homomorphism γ of A into $P(B)$ such that

$$\gamma(u) = \beta(W(u) - W(u_0)) + c ,$$

where u_0 is a fixed point such that $W(u_0)$ is defined and c is a constant. Let us consider a special case when $A=B$ and let X be a positive non-degenerate divisor on A. Let T be the transform of $Z\times X$ on $A\times A\times A$ by the automorphism $(x, y, z)\to(x, y, y-z)$. When we put $W=pr_{13}T$, we have

$$pr_1(Z\cdot(A\times X_u))\times u = W\cdot(A\times u)\,;$$
$$W\cdot(w\times A) = w\times\sum m_i X^{-}{}_{v_i}\,,$$

where $Z\cdot(w\times A)=w\times\sum m_i(v_i)$, and X^- is the transform of X by the automorphism $-\delta$ of A (cf. Weil [8], Th. 4, Prop. 2). Let γ be the homomorphism of A into $P(A)$ determined by W, then we shall denote it by $\gamma(Z\,;\,X)$. If Z is the cycle on $A\times A$ such that $u\times U\cdot V_u=Z(u\times A)$, where U and V are complementary dimensional A-cycles, then we shall denote γ by $\gamma(U,\,V\,;\,X)$.

Finally, since X is non-degenerate, X^- is also non-degenerate; therefore the homomorphism β_{X^-} of A into $P(A)$, defined by $\beta_{X^-}(u)=\beta(X^{-}{}_u-X^-)$, is surjective.

Lemma 1. *Let U and V be two A-cycles of complementary dimensions on A and let X be a positive non-degenerate A-divisor. Then we have*

$$\gamma(U,\,V\,;\,X)=\beta_{X^-}\cdot\alpha(U,\,V)\,.$$

Proof. Let Z be the transform of $U\times V$ by the automorphism $(x, y)\to(y-x, y)$ of $A\times A$ and T be the transform of $Z\times X$ by the automorphism $(x, y, z)\to(x, y, y-z)$ of $A\times A\times A$. If we put $W=pr_{13}T$, we have

$$pr_1(Z\cdot(A\times X_u))\times u = W\cdot(A\times u)\,,$$

with $W\cdot(w\times A)=w\times\sum m_i X^-_{v_i}$, $Z\cdot(w\times A)=w\times\sum m_i(v_i)=w\times U\cdot V_w$, whenever every intersection-product involved is defined. Then we have

$$\alpha(U,\,V)(w)=S(U\cdot V_w)+c=S(Z(w))+c\,,$$

where c is a constant. Putting $\alpha(U,\,V)=\alpha$, $d=\deg(Z(w))=\deg(U\cdot V_w)$, we have

$$\sum m_i X^{-}{}_{v_i}-d\cdot X^-\sim X^{-}{}_{\alpha(w)+c}-X^- \quad \text{(cf. Weil [8], Cor. 2, Th. 30).}$$

On the other hand, the homomorphism $\gamma=\gamma(U,\,V\,;\,X)$ of A into $P(A)$ determined by W is such that $\gamma(w)=\beta(W(w)-W(t))+c'=$

$\beta(X^{-}_{\alpha(w)+c}-X^{-})-\beta(X^{-}_{\alpha(t)+c}-X^{-})+c'$, where t, c, c', are constants. Hence $\gamma(w)=B_{X^{-}}\cdot\alpha(w)+c''$ with a constant c''. Our lemma is thereby proved.

Lemma 2. *Let W and W' be two divisors on a product of two Abelian varieties A and B^s. Let β be the canonical homomorphism of $G_a^{s-1}(B)$ on the Picard variety $P(B)$ of B. Let γ (resp. γ') be the homomorphism of A into $P(B)$ determined by W (resp. W'). If W and W' are numerically equivalent to each other on $A\times B$, we have $\gamma=\gamma'$.*

Proof. We shall show that we may assume W and W' to be both non-degenerate and algebraically equivalent on $A\times B$. There is a positive integer m such that $m(W-W')\equiv 0 \mod G_a(A\times B)$ (cf. Matsusaka [3]); it is easy to see that mW and mW' define $m\gamma$ and $m\gamma'$ respectively. If $m\gamma=m\gamma'$, then we have $\gamma=\gamma'$. Therefore, we may assume that W and W' are algebraically equivalent to each other. It is also easy to see that there is a positive divisor T such that $W+T$ and $W'+T$ are non-degenerate. Let γ^* be the homomorphism of A into $P(B)$ determined by T. $W+T$ and $W'+T$ determine $\gamma+\gamma^*$ and $\gamma'+\gamma^*$ respectively. Hence we may assume that W and W' are non-degenerate.

There is a point $t=(u, v)$ on $A\times B$ such that $W'\sim W_t$. We have $(w\times B)\cdot W_t=w\times W_t(w)\sim w\times W'(w)$ on $w\times B$ by Weil [6], VIII, Cor. 1, Th. 4. On the other hand, we have $W_t(w)=W(w-u)_v$. Hence

$$W'(w)-W'(w_0)\sim W(w-u)_v-W(w_0-u)_v\sim W(w-u)-W(w_0-u)\,,$$

since $W'(w)-W'(w_0)\equiv 0$ in the sense of Weil (cf. Weil [8], Cor. 1, Prop. 3), where w_0 is a constant. Thus we have $\gamma'(w)+c=\gamma(w)-\gamma(w_0)+c'$. Since u, c, c' are constants, our lemma is proved.

Theorem 1. *Let U, V, U', V' be four cycles on an Abelian variety A such that $\dim U+\dim V=\dim A$, $U\equiv U' \mod G_n(A)$, $V\equiv V' \mod G_n(A)$. Then we have*

$$\alpha(U, V)=\alpha(U', V')\,.$$

Proof. Let X be a positive non-degenerate A-divisor. By lemma 1, we have $\gamma(U, V; X)=\beta_{X^{-}}\cdot\alpha(U, V)$, $\gamma(U', V'; X)=\beta_{X^{-}}\cdot\alpha(U', V')$. By our assumption, we have $U\times V\equiv U'\times V' \mod G_n(A\times A)$. Let Z and Z' be the transforms of $U\times V$ and $U'\times V'$ by the automorphism of $A\times A$ defined by $(x, y)\rightarrow(y-x, y)$. Then

$Z \equiv Z' \bmod G_n(A\times A)$ and consequently $Z\times X \equiv Z'\times X \bmod G_n(A\times A \times A)$. Thus, the transforms T, T' of $Z\times X$, $Z'\times X$ by the automorphism of $A\times A\times A$ determined by $(x, y, z)\rightarrow(x, y, y-z)$ are such that $T\equiv T' \bmod G_n(A\times A\times A)$. By lemma 2, we have $\gamma(U, V; X)=\gamma(U', V'; X)$, since $W=pr_{13}T\equiv pr_{13}T'=W' \bmod G_n(A\times A)$. Since X is non-degenerate, β_{X^-} is an isogeny, i.e. a surjective homomorphism with a finite kernel. Our theorem is thereby proved.

Corollary. *Let B and B' be two Abelian subvarieties of A such that $B\equiv B' \bmod G_n(A)$. Then $B=B'$.*

Proof. Let X be a positive A-cycle such that $B\cdot X$ and $B'\cdot X$ are both defined. Then since B and B' are numerically equivalent to each other, we have $\deg(B\cdot X)=\deg(B'\cdot X)$. $\alpha(B, X)$ is an endomorphism of A, mapping A onto B; $\alpha(B', X)$ is also an endomorphism of A, mapping A onto B' (cf. Weil [8], Prop. 25). Since we have $\alpha(B, X)=\alpha(B', X)$ by our theorem, it follows that $B=B'$.

Thus we have seen that $\alpha(X^r, Y^{n-r})$ *is a bilinear mapping of $G^r(A)\times G^{n-r}(A)$ into the ring $\mathfrak{A}$ of endomorphisms of A, and it is* 0 *on $G^r_n(A)\times G^{n-r}(A)$ as well as on $G^r(A)\times G^{n-r}_n(A)$.* We shall prove a few formulas for $\alpha(X, Y)$ which we shall need later (these formulas are not new and are originally due to Morikawa, cf. Morikawa [4]).

Proposition 1. (i) *Let X and Y be two A-cycles of complementray dimensions on A, then we have $\alpha(X, Y)+\alpha(Y, X)=I(X, Y)\cdot\delta$, where $I(X, Y)$ denotes the intersection number of X and Y.* (ii) *Let $T, Z_1, \cdots, Z_r$ be $r+1$ cycles on A such that $T\cdot Z_1\cdots Z_r$ is defined on A and is a zero-cycle, then we have*

$$\alpha(T, Z_1\cdots Z_r)=\textstyle\sum_i \alpha(T\cdot Z_1\cdots Z_{i-1}\cdot Z_{i+1}\cdots Z_r, Z_i).$$

(iii) *Let $X_1, \cdots, X_r$ be r cycles on A such that $X_1\cdots X_r$ is defined and is a zero-cycle, then we have*

$$\textstyle\sum_i \alpha(X_i, X_1\cdots X_{i-1}\cdot X_{i+1}\cdots X_r)=(r-1)\, I(X_1, \cdots, X_r)\cdot\delta,$$

where $I(X_1, \cdots, X_r)$ denotes the intersection number of our r cycles. (iv) *Let $X_1, \cdots, X_n$ be divisors on A such that $X_1\cdots X_n$ is defined and that $X_i\equiv X_1 \bmod G_n(A)$ for all i, then we have*

$$\alpha(X_1\cdots X_r, X_{r+1}\cdots X_n)=((n-r)/n)\, I(X_1, \cdots, X_n)\cdot\delta.$$

Proof. To see the first formula, first we replace Y by a translation of it such that the intersection-product $X\cdot Y$ is defined (cf. Weil [8], Th. 3). Then $\alpha(X, Y)(t)=S(X\cdot(Y_t-Y))=S(X\cdot Y_t)-S(X\cdot Y)=S((X\cdot Y_t)_{-t})+I(X, Y)\cdot t-S(X\cdot Y)$ and our formula (i) is an immediate consequence of this.

To prove our (ii), we proceed as follows. Let T, X, Y be three A-cycles such that $T\cdot X\cdot Y$ is defined and is a zero-cycle. $\alpha(T, X\cdot Y)(u)=S(T\cdot(X_u\cdot X_u-X\cdot Y))=S(T\cdot(X_u\cdot Y_u-X_u\cdot Y)+T\cdot(X_u\cdot Y-X\cdot Y))=\alpha(T\cdot X, Y)(u)+\alpha(T\cdot Y, X)(u)$ by Theorem 1. Hence $\alpha(T, X\cdot Y)=\alpha(T\cdot X, Y)+\alpha(T\cdot Y, X)$ and our case follows from this by induction.

When $r=2$, (iii) coincides with (i). Hence we assume that (iii) is true for $r-1$ A-cycles. Putting $Y=X_{r-1}\cdot X_r$, we then get

$$\textstyle\sum_1^{r-2}\alpha(X_i, X_1\cdots X_{i-1}\cdot X_{i+1}\cdots X_{r-2}\cdot Y)+\alpha(Y, X_1\cdots X_{r-2})=(r-2)I(X_1, \cdots, X_r)\delta\,.$$

Using (ii), we have

$$\begin{aligned}\alpha(Y, X_1\cdots X_{r-2}) &= I(X_1, \cdots, X_r)\cdot\delta-\alpha(X_1\cdots X_{r-2}, X_{r-1}\cdot X_r)\\ &= I(X_1, \cdots, X_r)\cdot\delta-\alpha(X_1\cdots X_{r-2}\cdot X_{r-1}, X_r)-\alpha(X_1\cdots X_{r-2}\cdot X_r, X_{r-1})\\ &= -I(X_1, \cdots, X_r)\cdot\delta+\alpha(X_r, X_1\cdots X_{r-1})+\alpha(X_{r-1}, X_1\cdots X_{r-2}\cdot X_r).\end{aligned}$$

(iii) is thereby proved. (iv) is an immediate consequence of (iii) and Theorem 1.

§ 2. A criterion for numerical equivalence.

In this section, we limit our discussions to the case of divisors and 1-cycles on A^n. We say that a curve C on A is a *generating curve* of A, when any point of A can be written as a sum of n points on C. In other words, let c be a point on C, then C is a generating curve of A if and only if A is the smallest Abelian subvariety of A containing C_{-c}. Let X be a divisor on A and C be a generating curve of A, then we shall show that X is numerically equivalent to 0 if and only if $\alpha(C, X)=0$ or $\alpha(X, C)=0$. Thus we get a faithful repersentation of $G^{n-1}(A)/G_n^{n-1}(A)$ in the ring $\mathfrak{A}$ by $X\rightarrow\alpha(C, X)$ or by $X\rightarrow\alpha(X, C)$. Next, let X be a positive non-degenerate divisor on A and Z be a 1-cycle on A. We shall show that $\alpha(X, Z)=0$ or $\alpha(Z, X)=0$ if and only if Z is numerically equivalent to 0. Hence we have a faithful representation of $G^1(A)/G_n^1(A)$ into the ring $\mathfrak{A}$ of endomorphisms of A, by the correspondence $Z\rightarrow\alpha(X, Z)$ or by $Z\rightarrow\alpha(Z, X)$.

We are going to introduce some conventions and notations, which will be kept fixed throughout this section. Let $C_1, \cdots, C_m$ be m curves on A; let the J_i be the Jacobian varieties of the C_i, the f_i be birational correspondences between the complete non-singular models C_i^* of C_i and the C_i, the φ_i be canonical mappings of the C_i^* into the J_i and finally let the Γ_i be the graphs of the f_i. f_i can be extended to a homomorphism β_i of J_i into A such that

$$f_i = \beta_i \cdot \varphi_i + c_i ,$$

where c_i is a constant (cf. Weil [8], Th. 21). Put $B = J_1 \times \cdots \times J_m$ and $C = \sum_1^m a_i C_i$, where the coefficients a_i are either $+1$ or -1. Then we can define a homomorphism β of B into A by $\beta(x_1, \cdots, x_m) = \sum_1^m a_i \beta_i(x_i)$. We fix also a common field k of definition for A, and for $C_i, C_i^*, f_i, \varphi_i, J_i$ $(1 \leqslant i \leqslant m)$. Then the β_i and also β are defined over k (cf. Weil [8], Th. 3).

Let now X be a divisor on A and K be a field containing k over which X is rational. Put

$$x_i = S[\varphi_i(pr_{C_i^*}(\Gamma_i \cdot (C_i^* \times X_u)))] .$$

The point $(x_1, \cdots, x_m)$ is rational over $K(u)$ (cf. Weil [8], Th. 1), and hence there is a homomorphism ${}^t\beta_X$ of A into B such that

$${}^t\beta_X(u) = (x_1, \cdots, x_m) + c ,$$

where c is a constant (cf. Weil [8], Th. 9).

Lemma 3. *We have* $\alpha \cdot {}^t\beta_X = \alpha(C, X)$.

Proof. Let K be a field containing k, over which X is rational, and u be a generic point of A over K. It is easy to see that every component of $\Gamma_i \cdot (C_i^* \times X_u)$ is a generic point of Γ_i over K (cf. Weil [8], Th. 3). Hence

$$\Gamma_i \cdot (C_i^* \times X_u) = (\Gamma_i \cdot (C_i^* \times C_i \cdot X_u))_{C_i^* \times C_i} ,$$

by Weil [6], Chap. VII, Th. 18. From this and from our definitions, our lemma follows immediately.

Weil defined the symbol $d(\beta_i, X)$ by $d(\beta_i, X) = \deg(\Gamma_i \cdot (C_i^* \times X_u))$ (cf. Weil [8], no. 44). Using a similar remark to the one in the preceding proof, we see also that

$$d(\beta_i, X) = I(C_i, X) = (1/2) \cdot Tr(\alpha(C_i, X)) ,$$

by Weil [8], Th. 31. and Cor. 1 to Th. 36. From this we see that $I(C, X)=(1/2)\cdot Tr(\alpha(C, X))$, and hence we have the following

Corollary. $Tr(\alpha(C, X))=2\cdot I(C, X)$

and $Tr(\alpha(X, C))=(2n-2)\cdot I(C, X)$.

The latter half of our Corollary follows from (i), Prop. 1.

Proposition 2. *Let φ be the canonical homomorphism of $G_a^{n-1}(A)$ onto the Picard variety $P(A)$ of A and φ_X be the homomorphism of A into $P(A)$ defined by $\varphi(X_u - X)=\varphi_X(u)$, in terms of a divisor X on A. Then there is a homomorphism λ_X of $P(A)$ into B such that ${}^t\beta_X=\lambda_X\cdot\varphi_X$. When X is non-degenerate, the image of A by ${}^t\beta_X$ contains the image of ${}^t\beta_Y$, for any A-divisor Y.*

Proof. Let K be a field containing k over which X is rational. Let u and v be independent generic points of A over K and put $\mathfrak{m}_i(u)=pr_{C_i^*}(\Gamma_i\cdot(C_i^*\times X_u))$, then we have

$$ {}^t\beta_X(u)=(S(\varphi_1(\mathfrak{m}_1(u+v)-\mathfrak{m}_1(v))), \cdots, S(\varphi_m(\mathfrak{m}_m(u+v)-\mathfrak{m}_m(v)))) . $$

We can find a rational divisor Z in $G_a(A)$ over $K(\varphi_X(u))$ such that $\varphi(Z)=\varphi(X_u-X)=\varphi_X(u)$. Let T be a non-degenerate A-divisor, then $Z\sim T_s-T$ for some s on A, and consequently $Z_t\sim Z$ for any t on A (cf. Weil [8], Cor. 2, Th. 30). Therefore we can find a point t on A such that $\mathfrak{n}_i=pr_{C_i^*}(\Gamma_i\cdot(C_i^*\times Z_t))$ is defined and that

$$ {}^t\beta_X(u)=(\cdots, S(\varphi_i(\mathfrak{n}_i)), \cdots) , $$

by Weil [8], Th. 3, Th. 19, Th. 21 and Cor. 2 to Th. 30. Taking K to be algebraically closed, if necessary, we may assume that t is rational over K. Then since every $\mathfrak{n}_i$ is rational over $K(\varphi_X(u))$, it follows that ${}^t\beta_X(u)$ is rational over the same field (cf. Weil [8], Th. 1). Thus the first part of our assertion is proved.

Now let us assume that X is non-degenerate and Y is an arbitrary A-divisor. For any given point u on A, there is a point v on A such that

$$ Y_u - Y \sim X_v - X . $$

This implies ${}^t\beta_Y(u)={}^t\beta_X(v)$ and the second assertion is proved.

Theorem 2. *Let X be a divisor and C be a 1-cycle on A. (i) When X is positive and non-degenerate, $\alpha(C, X)=0$ or $\alpha(X, C)=0$ is a necessary and sufficient condition that C is numerically equivalent to* 0. (ii) *When C is a generating curve of A, $\alpha(C, X)=0$ or*

$\alpha(X, C)=0$ *is a necessary and sufficient condition that* X *is numerically equivalent to* 0.

Proof. First, let us observe that $\alpha(C,X)=0$ and $\alpha(X,C)=0$ imply each other. If $\alpha(C,X)=0$, then $Tr(\alpha(C, X))=0$ and $I(C,X)=0$ by our Corollary to Lemma 3. Hence $\alpha(C, X)+\alpha(X, C)=I(C, X)\cdot\delta$ implies $\alpha(X, C)=0$. Conversely, if $\alpha(X, C)=0$, then $Tr(\alpha(X, C))=(2n-2)\cdot I(C, X)=0$ (cf. Cor. to Lemma 3) implies $I(C, X)=0$ and we have $\alpha(C, X)=0$ by the same reason.

Therefore, let us assume that $\alpha(C, X)=0$, assuming that X is a positive non-degenerate divisor on A. Let $C=\sum_1^m a_i C_i$ be the reduced expression for C, with $a_i=\pm 1$ and let us use the same notations and conventions explained in the beginning of this §. Let D be the image of A by ${}^t\beta_X$. Since $\alpha(C, X)=\beta\cdot{}^t\beta_X$ by Lemma 3, it follows that D is contained in the kernel of β. Let Y be an arbitrary divisor on A, then the image of A by ${}^t\beta_Y$ is contained in D by Proposition 2. Since $\alpha(C, Y)=\beta\cdot{}^t\beta_Y$ again by Lemma 3, we have $\alpha(C, Y)=0$. Thus we have shown that $\alpha(C, X)=0$ implies $\alpha(C, Y)=\alpha(Y, C)=0$ for any A-divisor Y and hence $Tr(\alpha(C, Y))=2I(C, Y)=0$ for any Y, which proves $C\equiv 0 \bmod G_n^1(A)$. Our first assertion follows from this and from Theorem 1.

Let $P(A)$ be the Picard variety of A and φ be the canonical rational homomorphism of $G_a^{n-1}(A)$ onto $P(A)$. Let us assume now that C is a generating curve of A; from the universal mapping property of φ, we have

$$S(pr_J(\Gamma\cdot(C^*\times Z)))=\gamma\cdot\varphi(Z),\ Z\in G_a^{n-1}(A),$$

where γ is a homomorphism of $P(A)$ into $J=J_1$ (putting $C=C_1$, $C_1^*=C^*$, $\Gamma_1=\Gamma$). Therefore, we have

$${}^t\beta_X=\gamma\cdot\varphi_X,$$

and γ is independent of X. $\alpha(C, X)=0$ implies $\beta\cdot{}^t\beta_X=0$ and we are going to show that ${}^t\beta_X=0$ in such a case.

Assume, for a moment, that there is a positive non-degenerate A-divisor T such that $\alpha(C, T)$ is a surjective endomorphism of A. Then from the relation $\alpha(C, T)=\beta\cdot{}^t\beta_T$ and the fact φ_T is an isogeny, it follows that γ is an isogeny and β induces on $\gamma(P(A))$ an isogeny. Therefore $\beta\cdot{}^t\beta_X=0$ implies ${}^t\beta_X=0$ and (ii) follows from Theorem 1 and from Weil [8], Th. 30. As to the existence of a positive non-degenerate T such that $\alpha(C, T)$ is surjective, it is easy to see that the subvariety of A of dimension $n-1$, consisting of sums of $n-1$ points on C, has the required property.

Corollary. *Let C be a positive* 1*-cycle on A, which is a complete intersection of positive non-degenerate divisors. Then a divisor X on A is numerically equivalent to* 0 *if and only if $\alpha(C, X) =0$ or $\alpha(X, C)=0$.*

This is an easy consequence of our Theorem 2 and Weil [9].

§ 3. A Characterization of a Jacobian variety.

Proposition 3. *Let J^g be the Jacobian variety of a complete non-singular curve Γ, φ be the canonical mapping of Γ into J, and Θ be the corresponding canonical divisor on J. Then we have*

$$deg(\Theta_{u_1} \cdots \Theta_{u_g}) = g\,!\,,$$
$$\Theta_{u_1} \cdots \Theta_{u_{g-1}} \equiv (g-1)\,!\ \varphi(\Gamma) \bmod G_n^1(J)\,.$$

Proof. The first equality is an immediate consequence of the theorem of Riemann-Roch (cf. Nishi [5] and $l(\Theta)=1$ (cf. Weil [10], Th. 1). By Weil [8], Th. 20, we have $\alpha(\varphi(\Gamma), \Theta) = \delta$; on the other hand, we have $\alpha(\Theta_{u_1} \cdots \Theta_{u_{g-1}}, \Theta)=(g-1)!\,\delta$ by (iv), Proposition 1. Our second relation follows then from Theorem 2.

Theorem 3. *Let A^n be an Abelian variety and X be an irreducible divisor on A such that $X_{u_1} \cdots X_{u_{n-1}} \equiv (n-1)\,!\ C \bmod G_n^1(A)$, $C>0$, and $deg(X_{u_1} \cdots X_{u_n})=n\,!$, then C is irreducible, A is the Jacobian variety of C, C is canonically embedded into A and X is a corresponding canonical divisor on A.*

Proof. From (iv), Proposition 1, we have $\alpha(C, X)=\delta$. We have also $I(C, X)=n$. Hence, in the reduced expression for C, every component has the coefficient 1. Let $C=\sum_1^m C_i$ be the reduced expression for C, the J_i be the Jacobian varieties of the C_i, the C_i^* be non-singular models of the C_i, the f_i be birational transformations of the C_i^* onto the C_i, the φ_i be canonical mappings of the C_i^* into the J_i and the β_i be linear extensions of the f_i (cf. Weil [8], Th. 21). Put $B=J_1 \times \cdots \times J_m$ and define β, ${}^t\beta_X$ as we did in § 2. By lemma 2, we have $\beta \cdot {}^t\beta_X=\alpha(C, X)=\delta$, and so ${}^t\beta_X$ is an injective isomorphism of A to B and β induces a surjective isomorphism on ${}^t\beta_X(A)$. We are going to show that ${}^t\beta_X$ is actually surjective. In order to do so, it is sufficient to show that ${}^t\beta_{iX}$ is surjective for $1\leqslant i\leqslant m$, since we have ${}^t\beta_X=({}^t\beta_{1X},\cdots,{}^t\beta_{mX})$. Let K be a common field of definition for A and for all subvarieties of A, varieties and mappings we have introduced in the proof. Let u be a generic point of A over K, then from $\alpha(C,X)=\delta$, $I(C,X)=n$, $\dim A=n$ and from Weil [8], Th. 1, it follows that $C_i\cdot X_u$ consists of independent generic points of C_i over K, each being counted once. The relation

$$f_i^{-1}((X_{u+v}-X_u-X_v+X)\cdot C_i)\sim 0$$

determines, on the symmetric product of C_i of order $I(C_i,X)$, a law of composition which makes it birationally equivalent to an Abelian variety (cf. Weil [8], Th. 16, Cor. 2, Th. 30). Since J_i is the Albanese variety of C_i (cf. Weil [8], Th. 21), it follows that $I(C_i,X)\geqslant$ genus $(C_i)=\dim J_i$. Hence ${}^t\beta_{iX}$ is surjective and consequently ${}^t\beta_X$ is surjective.

Let us put $\Gamma_i=O_1\times\cdots O_{i-1}\times\varphi_i(C_i^*)\times O_{i+1}\times\cdots\times O_m$, where the O_i are neutral elements of the J_i, and also put $R_i=J_1\times\cdots\times J_{i-1}\times\Theta_i\times J_{i+1}\times\cdots\times J_m$, where Θ_i is a canonical divisor on J_i corresponding to $\varphi_i(C_i^*)$. Since $\beta(x_1,\cdots,x_m)=\sum_1^m\beta_i(x_i)$, we have $\beta(\Gamma_i)\equiv C_i \bmod G_n^1(A)$. By Proposition 3, we have $\alpha(\varphi_i(C_i^*),\Theta_i)=\delta_i$, where δ_i is the identity automorphism of J_i. Therefore we have $\alpha(\sum_1^m\Gamma_i,\sum_1^m\Theta_i)=\delta_B$. When Z is a 1-cycle and T is a divisor on A, we have $\alpha(\beta^{-1}(Z),\beta^{-1}(T))=\beta^{-1}\cdot\alpha(Z,T)\cdot\beta$. Hence $\alpha(\beta^{-1}(C),\beta^{-1}(X))=\delta_B=\alpha(\sum_1^m\Gamma_i,\beta^{-1}(X))$ by Theorem 1. From this we see that $\beta^{-1}(X)\equiv\sum_1^mR_i \bmod G_n^{n-1}(B)$. Since $\sum_1^mR_i$ is positive and non-degenerate on B, there is a point b on B such that $\beta^{-1}(X)\sim(\sum_1^mR_i)_b$, which can be proved in the following way. Let $(u_1,\cdots,u_m)$ be a set of independent generic points of $J_1,\cdots,J_m$ over K and put $V_i=u_1\times\cdots\times u_{i-1}\times J_i\times u_{i+1}\times\cdots\times u_m$. Denoting by pr_i the operation of algebraic projection on the i-th factor J_i of B, we have $pr_i(\beta^{-1}(X)\cdot V_i)\equiv pr_i((\sum_1^mR_i)\cdot V_i)\bmod G_n(J_i)$. Hence there is a point b_i on J_i such that $pr_i(\beta^{-1}(X)\cdot V_i)\sim pr_i((\sum_1^mR_i)_{b_i}\cdot V_i)$ by Weil [8], Th. 32, Cor. 2. Putting $b=(b_1,\cdots,b_m)$, we see that $pr_i(\beta^{-1}(X)\cdot V_i)\sim pr_i((\sum_1^mR_i)_b\cdot V_i)$ for any choice of i, which proves our assertion (cf. Weil [6], Chap. VIII, Th. 4, Cor. 1 and Chap. VII, th. 12 (ii)). Since $\beta^{-1}(X)$ and $\sum_1^mR_i$ are positive, and since $l(\beta^{-1}(X))=l(\sum_1^mR_i)=1$, it follows that $\beta^{-1}(X)=(\sum_1^mR_i)_b$. This is possible if and only if $m=1$. Our theorem is thereby proved.

APPENDIX

Let J^g be the Jacobian variety of a complete non-singular curve Γ. We may assume, without loss of generality, that a canonical mapping of Γ into J is the injection (cf. Weil [8], Prop. 16, Prop. 18). Let W^r be the subvariety of J, consisting of points $\sum_1^r x_i$, where the x_i are points on Γ, then W^{g-1} is a canonical divisor Θ on J. We shall denote by W the variety W^{g-2}. Let $\mathfrak{k}$ be a canonical divisor on Γ, then the mapping $u \rightarrow S(\mathfrak{k})-u$ is an automorphism of the underlying variety of our Abelian variety J. When U is a cycle on J, we shall denote by U^* the transform of U by the automorphism mentioned above. In this Appendix, we are going to show that

$$\Theta_{u_1} \cdots \Theta_{u_{g-r}} \equiv (g-r)!\, W^r \bmod G_n^r(J)\,,$$
$$\deg(\Theta_{u_1} \cdots \Theta_{u_g}) = g!\,,$$

where $\Theta_{u_1}, \cdots, \Theta_{u_g}$ are g translations of Θ on J such that $\Theta_{u_1} \cdots \theta_{u_g}$ is defined. The idea of our proof is based upon that of Weil, which was communicated to the writer by him. Originally, what we needed was $\Theta_{u_1} \cdots \Theta_{u_{g-1}} \equiv (g-1)!\, \Gamma \bmod G_n^1(J)$, and as we have seen, it is an easy consequence of our criterion for numerical equivalence. But since the general formula is sometimes usuful, and since there is no existing proof for it except for the classical case, we are going to include it here.

Throughout this Appendix, we shall fix a common field k of definition for J and Γ, and all fields shall be assumed to contain k. We shall fix also a positive rational canonical divisor $\mathfrak{k}$ on Γ.

1. Let $x_1, \cdots, x_{g-r}$ be $g-r$ independent generic points of Γ over k. Put $\mathfrak{m} = \sum_1^{g-r}(x_i)$, which is a positive Γ-divisor of degree $g-r$, and also put $t = \sum_1^{g-r} x_i$. Let γ be the Γ-differential of the first kind such that $(\gamma) = \mathfrak{k}$. Let $\mathfrak{M}$ be the module of the Γ-differentials γ' of the first kind such that $(\gamma') \succ \mathfrak{m}$, then $\dim \mathfrak{M} = r$ and we can find a basis $\gamma_1, \cdots, \gamma_r$ of $\mathfrak{M}$ such that the γ_i are defined over $k(x_1, \cdots, x_{g-r})$ (cf. Weil [7], § II, no. 8). Let $y_1, \cdots, y_r$ be r independent generic points of Γ over $k(x_1, \cdots, x_{g-r})$ and let F be the function on the product U of r curves equal to Γ such that

$$F(y_1, \cdots, y_r) = \det(f_i(y_j)), \quad 1 \leqslant i,\ j \leqslant r\,,$$

where $f_i = \gamma_i/\gamma$. Since $(f_i) \succ -\mathfrak{k} + \mathfrak{m}$, it follows that

$$(F) = T + \sum_{i,j(i\neq j)}^{r} \Delta_{ij} - \sum_1^r D_i(\mathfrak{k}-\mathfrak{m}), \quad T > 0,$$

where Δ_{12} is the locus of $(y_1, y_1, y_3, \cdots, y_r)$ over k, Δ_{ij} is the transform of Δ_{12} by the permutation of factors of U, which interchanges the i-th factor with the first factor and the j-th factor with the second factor; $D_i(\mathfrak{k}-\mathfrak{m})$ denotes the divisor on U which we obtain by replacing the i-th factor of U by $\mathfrak{k}-\mathfrak{m}$ (cf. Weil [7], § I, § II of Première Partie and § I, nos. 12–13 of Deuxième Partie). We shall point out here two properties of T; (i) *T is symmetric*, i.e. T is invariant under any permutation of factors of U, and (ii) *every component of T has the coefficient* 1 *in the reduced expression for it.* In fact, permutations of factors of U either leave F invariant or change its sign. Hence (F) is invariant by any permutation of factors of U. Next, put $u = \sum_1^{r-1} y_i$. We see that $T(y_1, \cdots, y_{r-1}) + \sum_1^{r-1}(y_i) + \mathfrak{m} \sim \mathfrak{k}$ by Weil [6], Chap. VIII, Th. 4, Cor. 1. Since T is symmetric, if we show that every component of $T(y_1, \cdots, y_{r-1})$ has the coefficient 1 in the reduced expression for it, the same follows for every component of T (cf. [6], Chap. VI, Th. 12). We have $S(T(y_1, \cdots, y_{r-1})) = S(\mathfrak{k}) - u - t$ and $\deg(T(y_1, \cdots, y_{r-1})) = g-1$. Since $\Theta^* = \Theta$, $S(\mathfrak{k}) - u - t$ is a generic point of Θ over k (cf. Weil [8], Prop. 19). Then every component of $T(y_1, \cdots, y_{r-1})$ has the coefficient 1 in the reduced expression for it (cf. Weil [8], Prop. 16). Our assertions are thereby proved.

2. Let L^r be the graph of the mapping $(y_1, \cdots, y_r) \to \sum_1^r y_i$, which maps U into J. The projection of L on U is everywhere defined (cf. Weil [8], Th. 6), and L is non-singular (cf. Weil [6], Chap. IV, Th. 15). Let x be a generic point of Γ over $k(x_1, \cdots, x_{g-r})$. We claim that the intersection-product

$$L \cdot (U \times \Theta_{x-t})$$

is defined on $U \times J$. If W^r is contained in Θ_{x-t}, then W^r_{t-x} is contained in Θ; since W^r_{t-x} contains a generic point of J over k, it cannot be contained in Θ. This implies that $L \cdot (U \times \Theta_{x-t})$ is defined. We define Z to be the L-divisor such that $pr_J Z = 0$, $L \cdot (U \times \Theta_{x-t}) - Z > 0$ and that every component of $L \cdot (U \times \Theta_{x-t}) - Z$ has a non-zero algebraic projection on J.

Lemma 1. *Let $D_i(x)$ be the subvariety of U, which we get by replacing the i-th factor Γ by x. Then we have*

$$|pr_J(L \cdot (U \times \Theta_{x-t}) - Z)| \subset |T| + |\textstyle\sum_i D_i(x)| \subset |pr_J(L \cdot (U \times \Theta_{x-t}))|.$$

Any point $(z_1, \cdots, z_r)$ *on* $|T|$ *is such that* $(z_1, \cdots, z_r, \sum_1^r z_i) \in L \cap U \times \Theta_{x-t}$, *for all* x *on* Γ.

Proof. Put $K = k(x_1, \cdots, x_{g-r}, x)$. Assume first that $(z_1, \cdots, z_r, v)$ is a generic point of a component of $L \cap U \times \Theta_{x-t}$ over the algebraic closure of K, such that $v+t$ is not a point of W^*. Then $v+t-x$ is a point of Θ if and only if x is a component of $\sum_1^r(z_i)$ (cf. Weil [8], Prop. 16). Therefore $(z_1, \cdots, z_r)$ is a point of a component of $\sum_i D_i(x)$. Conversely, let $(z_1, \cdots, z_r)$ be a generic point of $U_i(x)$ over K. Then $v = \sum_1^r z_i$ is a point of Θ_{x-t} and $(z_1, \cdots, z_r, v)$ is contained in the intersection of L and $U \times \Theta_{x-t}$.

Let $(z_1, \cdots, z_r)$ be a generic point of a component of T over the algebraic closure of K. Without loss of generality, we may assume that $\dim_K(z_1, \cdots, z_{r-1}) = r-1$. Then we have

$$T(z_1, \cdots, z_{r-1}) + \textstyle\sum_1^{r-1}(z_i) + \mathfrak{m} \sim \mathfrak{k}, \qquad z_r \in T(z_1, \cdots, z_{r-1}).$$

Hence $\sum_1^r(z_i) + \mathfrak{m} \sim \mathfrak{k} - \mathfrak{s}$, where $\mathfrak{s}$ is a positive Γ-divisor of degree $g-2$. We have $\sum_1^r z_i = S(\mathfrak{k}) - S(\mathfrak{s}) - t \in W^*_{-t} \subset \Theta_{y-t}$ for any point y on Γ, since $W^*_{-y} \subset \Theta$. Conversely, let $(z_1, \cdots, z_r, v)$ be a generic point of a component of $L \cdot (U \times \Theta_{x-t}) - Z$ over the algebraic closure of K, such that $(z_1, \cdots, z_r)$ is not contained in $|\sum_i D_i(x)|$. Then $v+t$ is a point of W^*. We claim that none of the z_i can be algebraic over k. If z_1, for instance, is algebraic over k, the locus of v over the algebraic closure of K is $W_{z_1}^{r-1}$. Since $v+t$ is a point of W^*, we have $W_{z_1+t}^{r-1} \subset W^*$, which implies $\Theta_{z_1} \subset W^*$. Therefore none of the z_i can be algebraic over k and can be a component of $\mathfrak{k}$. There is a positive Γ-divisor $\mathfrak{s}$ of degree $g-2$ such that $\sum_1^r(z_i) + \mathfrak{m} + \mathfrak{s} \sim \mathfrak{k}$. Let γ' be the Γ-differential of the first kind such that $(\gamma') = \sum_1^r(z_i) + \mathfrak{m} + \mathfrak{s}$ and put $f' = \gamma'/\gamma$. There is a set of constants $(c_1, \cdots, c_r)$, not all zero, such that $f' = \sum_1^r c_i f_i$. Since $f'(z_j) = 0$ for all j and since f_i is defined at z_j for any pair (i, j) of indices, it follows that $F(z_1, \cdots, z_r)$ is defined and vanishes. If $(z_1, \cdots, z_r)$ is a point, hence a generic point over K, of Δ_{ij}, then $z_i = z_j$; but $x_1, \cdots, x_{g-r}$, together with $r-1$ points from $(z_1, \cdots, z_r)$, form a set of $g-1$ independent generic points of Γ over k and hence $\sum_1^r(z_i) + \mathfrak{m} + \mathfrak{s}$ is a generic divisor of the complete linear system $\mathfrak{L}(\mathfrak{k})$ determined by $\mathfrak{k}$, which cannot have a multiple point. Thus $(z_1, \cdots, z_r)$ cannot be contained in Δ_{ij} and this proves that it is a point of $|T|$. Our lemma is thereby proved.

Lemma 2. *Every component of* $L \cdot (U \times \Theta_{x-t}) - Z$ *has the coefficient*

1 *in the reduced expression for it.*

Proof. Let V^{r-1} be a component of $L\cdot(U\times\Theta_{x-t})-Z$, then the projection V' of V on J has the dimension $r-1$ by our definition of Z. Let $(z_1, \cdots, z_r, v)$ be a generic point of V over the algebraic closure $\bar{K}$ of K. In order to prove our lemma, we are going to show that L and $U\times\Theta_{x-t}$ are transversal to each other at $(z_1, \cdots, z_r, v)$ on $U\times J$. The point is simple on L. We have to show that the point is also simple on $U\times\Theta_{x-t}$.

Assume first that $v+t$ is not a point of W^*. Since $v+t-x\in\Theta$, one of the z_i, say z_1, must coincide with x (cf. Weil [8], Prop. 16) and $(z_2, \cdots, z_r, x_1, \cdots, x_{g-r})$ is a set of $g-1$ independent generic points of Γ over $k(x)$. Hence $v+t-x$ is a generic point of Θ over k, and v is a simple point of Θ_{x-t}. Next assume that $v+t$ is a point of W^*. There is a positive Γ-divisor $\mathfrak{s}$ of degree $g-2$ such that $\mathfrak{k}'=\sum_1^r(z_i)+\mathfrak{m}+\mathfrak{s}\sim\mathfrak{k}$. The points $x_1, \cdots, x_{g-r}$, together with suitably chosen $r-1$ points from $(z_1, \cdots, z_r)$, form a set of $g-1$ independent generic points of Γ over $k(x)$. Since $l(\mathfrak{k})=g$, it follows then that $\mathfrak{k}'$ is a generic divisor of the complete linear system $\mathfrak{L}(\mathfrak{k})$ over $k(x)$. Let us assume, for the sake of simplicity, that $\dim_{k(x)}(z_2, \cdots, z_r, x_1, \cdots, x_{g-r})=g-1$. Then $\sum_2^r z_i+t$ is a generic point of Θ over $k(x)$ and hence $S(\mathfrak{k})-z_1-S(\mathfrak{s})$ is also over $k(x)$. This implies $(z_r)+\mathfrak{s}$ is a positive Γ-divisor of degree $g-1$, consisting of independent generic points over $k(x)$ (cf. Weil [8], Th. 20), and $\mathfrak{s}$ is a positive Γ-divisor of degree $g-2$, consisting of $g-2$ independent generic points of Γ over $k(x)$. From this, it follows that $v+t$ is a generic point of W^* over $k(x)$ and $v+t-x$ is also such over k. Thus v is a simple point on Θ_{x-t}.

Let us assume, for simplicity, that $z_1, \cdots, z_{r-1}$ are $r-1$ independent generic points of Γ over K. Put $C=L\cdot((z_1, \cdots, z_{r-1})\times\Gamma)$, then C is a simple curve on L, and $(z_1, \cdots, z_r, v)$ is a simple point of C (cf. Weil [6] Chap. IV, Th. 15). Putting $z_1+\cdots+z_{r-1}=u$, Γ_u is the projection of C on J and v is a point of Γ_u. By Weil [6], Chap. IV, Prop. 24 and Chap. VI, Th. 6, in order to prove that L and $U\times\Theta_{x-t}$ are transversal to each other at $(z_1, \cdots, z_r, v)$ on $U\times J$, it is sufficient to show that C and $U\times\Theta_{x-t}$ are transversal to each other at the point on $U\times J$. Since $z_1, \cdots, z_{r-1}$, $x_1, \cdots, x_{g-r}$ are $g-1$ independent generic points of Γ over $k(x)$, $u+x-t$ is a generic point of J over k; therefore Γ_u is not contained in Θ_{x-t}. On the other hand, the projection from C to Γ_u

is an isomorphism. Thus $C\cdot(U\times\Theta_{x-t})$ and $\Gamma_u.\Theta_{x-t}$ are defined on $U\times J$ and J respectively, and $i(\Gamma_u\cdot\Theta_{x-t},\ v;\ J)=1$ implies $i(C\cdot(U\times\Theta_{x-t}),(z_1,\cdots,z_r,v);\ U\times J)=1$ (cf. Weil [6], Chap. VIII, Th. 16). Since $x-t-u$ is a generic point of J over k, "$i(\Gamma_u\cdot\Theta_{x-t},\ v;\ J)=1$" follows from Weil [8], Th. 20 and Prop. 16.

3.

Lemma 3. *Let α be a surjective homomorphism of an Abelian variety A^n to an Abelian variety B^n. Let X be a subvariety of A, G be the graph of α and G' be the graph of the rational mapping of X into B, which is induced by α on X, and Y be the projection of G' on B. Then we have*

$$\alpha^{-1}(Y)\equiv(\nu(\alpha)/[G':\ Y])\cdot X \bmod G_a(A)\,.$$

Proof. It is easy to see that the point set $|\alpha^{-1}(Y)|$ is $\bigcup|X_a|$ with $\alpha(a)=0$. We have $\alpha^{-1}(Y)=pr_A(G\cdot(A\times Y))$. Let X' be a subvariety of G having the projection X on A. Then X' is determined uniquely and the coefficient of X in the reduced expression for $\alpha^{-1}(Y)$ is the same as the coefficient of X' in the reduced expression for $G\cdot(A\times Y)$ (cf. Weil [6], Chap. VII, Th. 17, Cor. 3). On the other hand, G is invariant by translations $T_{(a,0)}$ corresponding to the points a on A with $\alpha(a)=0$. Therefore every component of $G\cdot(A\times Y)$ is of the form $X'_{(a,0)}$ and has the same coefficient as X' in the reduced expression for $G\cdot(A\times Y)$. Let $\mathfrak{g}$ be the group of points a on A such that $\alpha(a)=0$ and $\mathfrak{g}'$ be the subgroup of $\mathfrak{g}$, consisting of those points a' such that $X_{a'}=X$. Then $G\cdot(A\times Y)$ $=m\cdot\sum X'_{(a',0)}$, where the summation is extended over a set of complete representatives of $\mathfrak{g}$ modulo $\mathfrak{g}'$. Since $pr_B(G\cdot(A\times Y))=$ $\nu(\alpha)\cdot Y$ (cf. Weil [6], Chap. VII, Th. 16), it follows that $m[\mathfrak{g}:\ \mathfrak{g}']\cdot$ $[G':\ Y]=\nu(\alpha)$. From $\alpha^{-1}(Y)=m\cdot\sum X_{a'}$, we see that $\alpha^{-1}(Y)\equiv$ $m\cdot[\mathfrak{g}:\ \mathfrak{g}']\cdot X \bmod G_a(A)$ and our lemma is thereby proved.

Lemma 4. *Let C be a curve on an Abelian variety A^n. Let m be a positive integer and let G be the graph of the rational mapping of C into A induced by $m\delta=\alpha$. Denote by C' the image of C by α, then we have*

$$[G:\ C']C'\equiv m^2C \bmod G_n^1(A)\,.$$

Proof. It is easy to see that $\alpha^{-1}(X\cdot Y)=\alpha^{-1}(X)\cdot\alpha^{-1}(Y)$ whenever X and Y are positive A-cycles such that $X\cdot Y$ is defined (cf. Weil [6], Chap. VII, Th. 18, Cor.). Let now X be any positive

A-divisor such that $X\cdot C'$ is defined. We have $\alpha^{-1}(X\cdot C')=\alpha^{-1}(X)\cdot\alpha^{-1}(C')$. We have also $\alpha^{-1}(X)\equiv m^2X \bmod G_n^{n-1}(A)$ by Weil [8] Prop. 31 and $\alpha^{-1}(C')\equiv(m^{2n}/[G:C'])\cdot C \bmod G_n^1(A)$ by lemma 3 and by Weil [8], Th. 33, Cor. 1. Therefore $\deg(\alpha^{-1}(X\cdot C'))=m^{2n}\deg(X\cdot C')=(m^{2n+2}/[G:C'])\cdot\deg(X\cdot C)$. Consequently $\deg(X\cdot[G:C']\cdot C')=\deg(X\cdot m^2C)$ and our lemma is thereby proved.

Lemma 5. *Let J^g be the Jacobian variety of a complete non-singular curve Γ and we assume that Γ is embedded canonically into J. Let $y_1,\cdots,y_{r-m},y$ be $r-m+1$ independent generic points of Γ over a common field k of definition for J and Γ. Let V be the locus of $\sum_1^{r-m}y_i+my$ over k and let G be the graph of the mapping of Γ into J, which is induced by $m\delta$. When Γ' is the image of Γ by $m\delta$, we have*

$$[G:\Gamma']\cdot V\equiv m^2(r-m+1)\cdot W^{r-m+1} \bmod G_n^{r-m+1}(J)\,,$$

provided $l(my)=1$, $g>r\geqslant m$.

Proof. Γ' is the projection of G into J, or the locus of my over k, which satisfies $[G:\Gamma']\cdot\Gamma'\equiv m^2\Gamma \bmod G_n^1(J)$ by lemma 4. Let $U=\overbrace{\Gamma\times\cdots\times\Gamma}^{r-1}\times J$ and let β be a rational mapping of U into J defined by $(z_1,\cdots,z_{r-1},u)\to\sum z_i+u$. Let T be the graph of β and denote by pr^* the operation of algebraic projection to J on $U\times J$. Then we have

$$pr^*((\Gamma\times\cdots\times\Gamma\times m^2\cdot\Gamma\times J)\cdot T)=(r-m+1)!\,m^2\cdot W^{r-m+1}\,.$$

On the other hand, since $l(my)=1$, we have $l((y_1)+\cdots+(y_{r-m})+(y))=1$ and

$$pr^*((\Gamma\times\cdots\times\Gamma\times[G:\Gamma']\cdot\Gamma'\times J)\cdot T)=(r-m)![G:\Gamma']\cdot V\,.^{3)}$$

3) This can be seen as follows. $(\Gamma\times\cdots\times\Gamma\times\Gamma'\times J).T$ is irreducible and is defined over k. Let $(y_1,\cdots,y_{r-m},my,v)$ be a generic point of it over k. Then $v=\sum_1^{r-m}y_i+my$. Let us assume, for the sake of simplicity, that 0 is on Γ and that k is large enough so that we can find a rational Γ-divisor of degree zero over $k(v)$ such that its class with respect to linear equivalence is v. Since the class of $\sum_1^{r-m}(y_i)+m(y)-r(0)$ with respect to linear equivalence is v, and since $l(\sum_1^{r-m}(y_i)+m(y))=1$, it follows that $\sum_1^{r-m}(y_i)+m(y)$ itself is rational over $k(v)$ (cf. Weil [6], Chap. VIII, Th. 10). Hence $k(y_1,\cdots,y_{r-m},my)$ is a separable algebraic extension of $k(v)$, and since my is not a point of Γ (because of $l(m(y))=1$), it follows that my must be rational over $k(v)$. Our formula follows from this easily.

Since $\Gamma\times\cdots\times\Gamma\times m^2\Gamma\times J$ and $\Gamma\times\cdots\times\Gamma\times[G:\Gamma']\cdot\Gamma'\times J$ are numerically equivalent, we see that

$$m^2(r-m+1)\cdot W^{r-m+1}\equiv[G:\Gamma']\cdot V \quad \bmod G_n(J)\,.$$

Our lemma is thereby proved.

4. *Proof of the main theorem.*

Theorem. We have

$$\Theta_{u_1}\cdots\Theta_{u_{g-r}}\equiv(g-r)\,!\,W^r \quad \bmod G_n(J)\,.$$

Proof. We have $pr_J(L\cdot(U\times\Theta_{x-t}))=r\cdot W^r\cdot\Theta_{x-t}$ by Weil [6], Chap. VII, Th. 16. By lemma 1 and lemma 2, we have also the following relations:

$$pr_J(L\cdot(U\times\Theta_{x-t})-Z)=pr_J(L\cdot((T+\textstyle\sum_i D_i(x))\times J)=pr_J(L\cdot(U\times\Theta_{x-t}))\,.$$

By a result in no. 1, we know that $T\times\sum\Delta_{ij}\sim\sum_i D_i(\mathfrak{k}-\mathfrak{m})$. Hence

$$r\,!\,W^r\cdot\Theta_{x-t}\equiv pr_J(L\cdot((\textstyle\sum_i D_i(x)+\sum_i D_i(\mathfrak{k}-\mathfrak{m})-\sum\Delta_{ij})\times J) \quad \bmod G_n(J)\,.$$

We have

$$pr_J(L\cdot\textstyle\sum_i D_i(x)\times J))\equiv r!W^r \quad \bmod G_n(J)\,,$$
$$pr_J(L\cdot(\textstyle\sum_i D_i(\mathfrak{k}-\mathfrak{m})\times J)\equiv r\,!\,(g+r-2)\cdot W^r \quad \bmod G_n(J)\,,$$
$$pr_J(L\cdot(\textstyle\sum_{i,j(i\neq j)}\Delta_{ij}\times J)=\binom{r}{2}\cdot(r-2)\,!\,[G:\Gamma']\cdot V\,,$$

where V consists of points $\sum_1^{r-2}y_i+2y$ with the y_i, y on Γ, and where G is the graph of the rational mapping of Γ into J, which is induced by 2δ. By our lemma 5, we have

$$[G:\Gamma']\cdot V\equiv 2^2(r-1)\cdot W^{r-1} \quad \bmod G_n(J)\,.$$

Hence

$$\Theta_{x-t}\cdot W^r\equiv(g-r+1)\cdot W^{r-1} \quad \bmod G_n(J)\,,$$

and our theorem follows from this immediately.

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