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The Frobenius theorem and the duality theorem on an Abelian variety[1)]

By

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The main purpose of this note is to establish the following theorem on an abstract abelian variety:

Let A be an abelian variety of dimension n, and let X be a divisor on it; then the degree $\nu(\varphi_X)$ of the homomorphism φ_X of A into its dual $\hat{A}$[2)] *is equal to $[(X^{(n)})/n!]^2$, where $(X^{(n)})$ means the n-fold intersection number of X.*

If X is positive and non-degenerate, then the dimension $l(X)$ of the complete linear system $|X|$ is given by $(X^{(n)})/n!$ (cf. Nishi [6], Th. 3). Therefore, our theorem extends the classical Frobenius Theorem. The method used in this note is purely algebraic and is valid not only for the classical case but also for the modular case. The so-called Duality Theorem "the double dual $\hat{\hat{A}}$ of A is isomorphic to A" can be obtained as a simple corollary of the Frobenius Theorem.

I have received kind advices from Matsusaka and also from Nakai to whom I wish to express here my hearty thanks.

§ 1. Preliminaries.

Let A^n be an abelian variety and let X be a divisor on it; the set $\mathfrak{G}_X$ of all points t of A such that $X_t \sim X$ is a subgroup of A. By $\hat{A}$ we shall denote the dual of A (i.e. the Picard variety of A). Then it is well known that the two abelian varieties A and $\hat{A}$ are isogenous. The mapping $\varphi_X: u \rightarrow \hat{u}$, where $\hat{u}$ is the

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1) We shall use freely the notations and the results in Weil [9]. Numbers in brackets refer to the bibliography at the end.

2) See the definitions in § 1.

point of $\hat{A}$ corresponding to the linear class of $X_u - X$, defines a rational homomorphism of A into $\hat{A}$; we can readily see that the kernel of φ_X is exactly the subgroup $\mathfrak{G}_X$. We shall say that X is non-degenerate, following Morikawa [5], if the homomorphism φ_X is surjective, in other words, if the subgroup $\mathfrak{G}_X$ is finite. If the subgroup $\mathfrak{G}_X$ consists only of the unit element of A, then $\mathfrak{G}_X$ will be called *trivial*. Let now B be another abelian variety and Y be a divisor on it. Then the dual of the product $A \times B$ is $\hat{A} \times \hat{B}$; moreover we can represent the homomorphism $\varphi_{X\times B + A\times Y}$ by a matrix,

$$\varphi_{X\times B + A\times Y} = \begin{pmatrix} \varphi_X & 0 \\ 0 & \varphi_Y \end{pmatrix}.$$

Thus we can see that $\nu(\varphi_{X\times B + A\times Y}) = \nu(\varphi_X)\nu(\varphi_Y)$.

Let $\mathfrak{G}(A)$ be the additive group of all divisors on A. Then the set $\mathfrak{G}_a(A)$ of all divisors algebraically equivalent to zero is a subroup of $\mathfrak{G}(A)$. Recently it was shown that the quotient group $\mathfrak{G}(A)/\mathfrak{G}_a(A)$ is a free group of finite type; this implies that the three equivalences—the algebraic equivalence, the numerical equivalence and the equivalence $\equiv$ in Weil's sense—coincide. In what follows, we shall denote by $\equiv$ these equivalences. Let $X, Y, Z \cdots$ be divisors on A^n; we define the intersection number $(X^{(i)}Y^{(j)}Z^{(k)}\cdots)$, where $i+j+k+\cdots = n$, as follows: There exist n points $a_1, \cdots, a_i, b_1, \cdots, b_j, c_1, \cdots, c_k, \cdots$ of A such that the intersection product $X_{a_1}\cdots X_{a_i}\cdot Y_{b_1}\cdots Y_{b_j}\cdot Z_{c_1}\cdots Z_{c_k}\cdots$ is defined; then $\deg(X_{a_1}\cdot \cdots \cdot X_{a_i}\cdot Y_{b_1} \cdot \cdots \cdot Y_{b_j}\cdot Z_{c_1}\cdot \cdots \cdot Z_{c_k}\cdot \cdots)$ is independent of the choice of the points $a_1, \cdots, a_i, b_1, \cdots, b_j, c_1, \cdots, c_k, \cdots$, and we denote it by $(X^{(i)}Y^{(j)}Z^{(k)}\cdots)$. If $X', Y', Z', \cdots$ are divisors on A such that $X \equiv X'$, $Y \equiv Y'$, $Z \equiv Z', \cdots$, then we have $(X^{(i)}Y^{(j)}Z^{(k)}\cdots) = (X'^{(i)}Y'^{(j)}Z'^{(k)}\cdots)$.

Let λ be a homomorphism of A into B; let Y be a divisor on B. If λ is not surjective, then $\lambda^{-1}(Y)$ may not be defined. But there exists a point b of B such that $\lambda^{-1}(Y_b)$ is defined; furthermore if b and c are such points, then $\lambda^{-1}(Y_b) \equiv \lambda^{-1}(Y_c)$ and in addition, when $Y \equiv 0$ on B, then $\lambda^{-1}(Y_b) \sim \lambda^{-1}(Y_c) \equiv 0$ on A. Thus we know that λ induces homomorphisms $\lambda^{-1}: \mathfrak{G}(B)/\mathfrak{G}_a(B) \to \mathfrak{G}(A)/\mathfrak{G}_a(A)$ and $\lambda^{-1}: \mathfrak{G}_a(B)/\mathfrak{G}_l(B) \to \mathfrak{G}_a(A)/\mathfrak{G}_l(A)$. The latter defines a rational homomorphism of $\hat{B}$ into $\hat{A}$ which will be called the transpose of λ and will be denoted by ${}^t\lambda$. The divisor Y being

3) This result is originally due to Barsotti [1]. Serre [7] gives another proof.

arbitrary and the point b being as above, we denote by $\lambda^{-1}(Y)$ a representative of the class of $\lambda^{-1}(Y_b)$ in $\mathfrak{G}(A)/\mathfrak{G}_a(A)$. Throughout this paper we shall use this convention. Notations being as above we have the following formula[4]:

$$\varphi_{\lambda^{-1}(Y)} = {}^t\lambda\varphi_Y\lambda .$$

Finally, let $\lambda: A_1\times \cdots \times A_r \to B_1\times \cdots B_s$ be a homomorphism of a product of r abelian varieties $A_1, \cdots, A_r$ into a product of s abelian varieties $B_1, \cdots, B_s$. Then λ can be represented by a matrix,

$$\lambda = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1r} \\ \cdots & \cdots & \cdots \\ \lambda_{s1} & \cdots & \lambda_{sr} \end{pmatrix},$$

where λ_{ij} is a homomorphism of A_j into B_i for each pair of i and j. We can see that, by straight-forward computations, the transpose ${}^t\lambda$ of λ can be represented as follows:

$${}^t\lambda = \begin{pmatrix} {}^t\lambda_{11} & \cdots & {}^t\lambda_{s1} \\ \cdots & \cdots & \cdots \\ {}^t\lambda_{1r} & \cdots & {}^t\lambda_{sr} \end{pmatrix}.$$

§ 2. Frobenius Theorem on Jacobian varieties.

For any Jacobian variety J, we may assume that J is self-dual, i.e. $J=\hat{J}$ and that φ_X, where X is a divisor on J, coincides with the endomorphism δ'_X; in particular, if Θ is the canonical divisor on J, then φ_Θ is nothing else but the identity endomorphism δ_J of J. From now on we shall go on with these assumtions.

Let X be a positive non-degenerate divisor on J. Then the Frobenius Theorem on J can be stated as follows: $\nu(\delta'_X)=l(X)^2$. Morikawa's idea in his paper [5] is very usefull. First we shall sketch the outline of a proof based on his idea. There exists a natural number c such that $c\delta'_X=\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2$, where each λ_i is a symmetric endomorphism and $\lambda_i\lambda_j=\lambda_j\lambda_i$ for each pair of i and j. We put

$$\lambda = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ -\lambda_2 & \lambda_1 & -\lambda_4 & \lambda_3 \\ \lambda_3 & -\lambda_4 & -\lambda_1 & \lambda_2 \\ \lambda_4 & \lambda_3 & -\lambda_2 & -\lambda_1 \end{pmatrix}.$$

4) Note the fact that $\varphi_X=0$ if and anly if $X\equiv 0$.

Then λ is an endomorphism of $J\times J\times J\times J$. By the statement in §1, and by the fact that each λ_i is symmetric, we have[5]

$$
{}^t\lambda = \begin{pmatrix} \lambda_1 & -\lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_2 & \lambda_1 & -\lambda_4 & \lambda_3 \\ \lambda_3 & -\lambda_4 & -\lambda_1 & -\lambda_2 \\ \lambda_4 & \lambda_3 & \lambda_2 & -\lambda_1 \end{pmatrix}.
$$

Therefore the diagonal matrix

$$
\begin{pmatrix} c\delta'_X & & & 0 \\ & c\delta'_X & & \\ & & c\delta'_X & \\ 0 & & & c\delta'_X \end{pmatrix}
$$

is equal to the product ${}^t\lambda\lambda$. On the one hand the above diagonal matrix represents the endomorphism $\varphi_{cX^{(4)}}$ of $J\times J\times J\times J$, where $X^{(4)}=X\times J\times J\times J+J\times X\times J\times J+J\times J\times X\times J+J\times J\times J\times X$; on the other hand, we have ${}^t\lambda\lambda=\varphi_{\lambda^{-1}(\Theta^{(4)})}$, where $\Theta^{(4)}$ has the same meaning as in the case of $X^{(4)}$. Hence $cX^{(4)}$ is algebraically equivalent to $\lambda^{-1}(\Theta^{(4)})$ on $J\times J\times J\times J$. According to Nishi [6], we have

$$
l(cX^{(4)}) = l(\lambda^{-1}(\Theta^{(4)}),
$$

and

$$
\begin{aligned}
l(cX^{(4)}) &= l(cX)^4 \\
&= c^{4g}l(X)^4 \qquad (g=\dim J) \\
l(\lambda^{-1}(\Theta^{(4)})) &= \nu(\lambda)l(\Theta^{(4)}) = \nu(\lambda) \\
&= \sqrt{\nu({}^t\lambda\lambda)} = \nu(c\delta'_X)^2 \\
&= c^{4g}\cdot\nu(\delta'_X)^2 .
\end{aligned}
$$

It follows from these relations that $\nu(\delta'_X)=l(X)^2$ as asserted.

In the rest of this §, we shall give another proof of our Theorem which is more elementary than the preceding one.

PROPOSITION 1. *Let X be a divisor and Y be a non-degenerate divisor on an abelian variety A^n. Then $mY+X$, where m is an integer, is degenerate only for a finite number of values of m. In addition, if Y is positive, then the complete linear system $|mY+X|$ exists, i.e. $l(mY+X)>0$, whenever m is sufficiently large.*

5) Note the fact that the involution coincides with the transpose operation.

PROOF. We know that $mY+X$ is non-degenerate if and only if the homomorphism $\varphi_{mY+X}: A\rightarrow\hat{A}$ is surjective, i.e. $\nu(\varphi_{mY+X})>0$. By linearity we have $\varphi_{mY+X}=m\varphi_Y+\varphi_X$; $\nu(m\varphi_Y+\varphi_X)$ is a polynomial in m of degree $2n$, and the coefficient of the leading term is $\nu(\varphi_Y)$ (cf. Weil [9], Th. 33). Since Y is non-degenerate, $\nu(\varphi_Y)$ is not zero. Therefore this polynomial is not identically zero; this implies the first assertion.

The latter half in our Proposition follows immediately from the fact that the complete linear system $|mY|$ is ample for sufficiently large values of m (cf. Weil [8]).

PROPOSITION 2. *Let J be a Jacobian variety and let X be a divisor on it. Let Θ be the canonical divisor on J. Then, for any integer n, we have*

$$\delta'^{-1}_{n\Theta+X}(\Theta)\equiv n^2\Theta+2nX+\xi^{-1}(\Theta)$$

and

$$\delta'^{-1}_{2nX+\xi^{-1}(\Theta)}(\Theta)\equiv 4n^2\xi^{-1}(\Theta)+4n\xi^{-1}(X)+(\xi^2)^{-1}(\Theta)\,,$$

where ξ denotes the endomorphism δ'_X.

PROOF. Let λ be an endomorphism of J and let Z be a divisor on J. Then, by Weil [9], Th. 25, we have $\delta'_{\lambda^{-1}(Z)}=\lambda'\delta'_Z\lambda$; and the necessary and sufficient condition that $Z\equiv 0$ is $\delta'_Z=0$.

As to the first assertion, let us put $Z_1=\delta'^{-1}_{n\Theta+X}(\Theta)$ and $Z_2=n^2\Theta+2nX+\xi^{-1}(\Theta)$. Then $\delta'_{Z_1}=\delta'_{n\Theta+X}\delta'_{n\Theta+X}=n^2\delta+2n\xi+\xi^2$, and $\delta'_{Z_2}=n^2\delta+2n\xi+\xi^2$. This shows that $Z_1\equiv Z_2$. As to the second assertion we can prove it quite similarly.

Now we can state the Frobenius Theorem on J^g in the following form:

THEOREM 1. *Let X be a divisor on a Jacobian variety J^g. Then we have*

$$g!\,(\delta'^{-1}_X(\Theta)^{(g)})=[(X^{(g)})]^2\,,$$

where Θ is the canonical divisor on J. In particular, if X is positive and non-degenerate, then we have $\nu(\delta'_X)=l(X)^2$.

PROOF. First we show that if X is a divisor of the type $\lambda^{-1}(\Theta)$ where λ is an endomorphism such that $\nu(\lambda)>0$ and that $\lambda'=\lambda$, then the formula in our Theorem holds. In fact, since $(\Theta^{(g)})=g!$ (cf. Matsusaka [4]) and $(\lambda^{-1}(\Theta)^{(g)})=\nu(\lambda)(\Theta^{(g)})$, we have

$(\lambda^{-1}(\Theta)^{(g)})=g!\,\nu(\lambda)$. On the other hand, since $\delta'_{\lambda^{-1}(\Theta)}=\lambda'\lambda$ (cf. Weil [9], Th. 25), we have

$$\begin{aligned}(\delta'^{-1}_{\lambda^{-1}(\Theta)}(\Theta)^{(g)}) &= ((\lambda^2)^{-1}(\Theta)^{(g)})\\ &= \nu(\lambda^2)(\Theta^{(g)})\\ &= g!\,\nu(\lambda)^2\,.\end{aligned}$$

These relations settle our assertion.

Let now X be any divisor on J; we denote δ'_X by ξ. If we put $f(n\,;X)=g!\,(\delta'^{-1}_{n\Theta+X}(\Theta)^{(g)})-((n\Theta+X)^{(g)})^2$, where n is an integer, then we know from Proposition 2 that

$$f(n\,;X) = g!\,((n^2\Theta+2nX+\xi^{-1}(\Theta))^{(g)})-((n\Theta+X)^{(g)})^2\,.$$

We can consider $f(n\,;X)$ as a polynomial in n as follows:

$$f(n\,;X) = \sum_{r=2g}^{0} A_r(X)n^r\,,$$

where

$$\begin{aligned}A_r(X) = g!\sum_{\substack{2i+j=r\\0\leqslant i,j,i+j\leqslant g}} 2^j\binom{g}{i}\binom{g-i}{j}(\Theta^{(i)}X^{(j)}\xi^{-1}(\Theta)^{(g-i-j)})\\ -\sum_{\substack{i+j=r\\0\leqslant i,j\leqslant g}}\binom{g}{i}\binom{g}{j}(\Theta^{(i)}X^{(g-i)})(\Theta^{(j)}X^{(g-j)})\,.\end{aligned}$$

In what follows, we shall show that each coefficient $A_r(X)$ must vanish. First we can see immediately that $A_{2g}(X)=g!\,(\Theta^{(g)})-(\Theta^{(g)})(\Theta^{(g)})=0$ and $A_{2g-1}(X)=2g\cdot g!\,(\Theta^{(g-1)}X)-2g\cdot g!\,(\Theta^{(g-1)}X)=0$. Consider now a function $g(n\,;X)$ of n:

$$g(n\,;X) = g!\,(\delta'^{-1}_{n^2\Theta+2nX+\xi^{-1}(\Theta)}(\Theta)^{(g)})-((n^2\Theta+2nX+\xi^{-1}(\Theta))^{(g)})^2\,.$$

Then similarly $g(n\,;X)$ also becomes a polynomial in n. In fact, since $g(n\,;X)=f(n^2\,;2nX+\xi^{-1}(\Theta))$, we have

$$g(n\,;X) = \sum_{r=2g-2}^{0} A_r(2nX+\xi^{-1}(\Theta))n^{2r}\,,$$

and each coefficient $A_r(2nX+\xi^{-1}(\Theta))$ is also a polynomial in n:

$$\begin{aligned}&A_r(2nX+\xi^{-1}(\Theta))\\ &= g!\sum_{2i+j=r}2^j\binom{g}{i}\binom{g-i}{j}(\Theta^{(i)}(2nX+\xi^{-1}(\Theta))^{(j)}\delta'^{-1}_{2nX+\xi^{-1}(\Theta)}(\Theta)^{(g-i-j)})\\ &\quad-\sum_{i+j=r}\binom{g}{i}\binom{g}{j}(\Theta^{(i)}(2nX+\xi^{-1}(\Theta))^{(g-i)})(\Theta^{(j)}(2nX+\xi^{-1}(\Theta))^{(g-j)})\,.\end{aligned}$$

and, using the second formula in Proposition 2, we have

$$A_r(2nX+\xi^{-1}(\Theta)) = 2^{2g-r}A_r(X)n^{2g-r}+h_r(n)\,,$$

where $h_r(n)$ is a polynomial of degree $\leqslant 2g-r-1$, with coefficients of intersection numbers of divisors Θ, $\xi^{-1}(\Theta)$, X, $\xi^{-1}(X)$ and $(\xi^2)^{-1}(\Theta)$; therefore we have

$$g(n\,;X) = 2^2A_{2g-2}(X)n^{4g-2}+\cdots.$$

Since $\delta'^{-1}_{n\Theta+X}(\Theta)\equiv n^2\Theta+2nX+\xi^{-1}(\Theta)$ by Proposition 2 and $\delta'_{n\Theta+X}$ is symmetric, it follows from Proposition 1 and the statement at the beginning of this proof that $g(n\,;X)=0$ for almost all values of n. Therefore the polynomial $g(n\,;X)$ must be identically zero; this implies that $A_{2g-2}(X)=0$.

The X being arbitrary, $A_{2g-2}(2nX+\xi^{-1}(\Theta))=0$ for all integers n. Thus $g(n\,;X)$ can be witten in the form,

$$g(n\,;X) = \sum_{r=2g-3}^{0} A_r(2nX+\xi^{-1}(\Theta))n^{2r}.$$

Continuing the same process, we can get

$$A_{2g-3}(X) = \cdots = A_0(X) = 0\,.$$

This completes the proof of the first half.

Since $g!\,\nu(\delta'_X)=(\delta'^{-1}_X(\Theta)^{(g)})$, the second half follows immediately from the first half combined with the fact that $l(X)=(X^{(g)})/g!$ (cf. Nishi [6]).

§ 3. The constant $f(A, B)$.

Let A^n and B^n are isogenous abelian varieties; let $\lambda: A\rightarrow B$ be an isogeny. It is well known that there exists an abelian variety C such that $A\times C$ and $B\times C$ are both isogenous to a Jacobian variety J. Let $\mu: J\rightarrow A\times C$ be an isogeny; we consider an isogeny

$$\begin{pmatrix}\lambda & 0\\ 0 & \delta_C\end{pmatrix}: A\times C\rightarrow B\times C\,,$$

where δ_C is the identity endomorphism of C. Then

$$\alpha = \begin{pmatrix}\lambda & 0\\ 0 & \delta_C\end{pmatrix}\mu$$

is a homomorphism of J onto $B\times C$. Furthermore, if Y is a positive non-degenerate divisor on $B\times C$, then we have $\delta'_{\alpha^{-1}(Y)}={}^t\alpha\varphi_Y\alpha$; hence $\nu(\delta'_{\alpha^{-1}(Y)})=\nu({}^t\alpha)\nu(\varphi_Y)\nu(\alpha)$. Since $\alpha^{-1}(Y)$ is positive and non-degenerate, we have, by using Theorem 1, $\nu(\delta'_{\alpha^{-1}(Y)})=l(\alpha^{-1}(Y))^2$. According to Nishi [6], Th. 4, $l(\alpha^{-1}(Y))=\nu(\alpha)l(Y)$ and hence $\nu(\delta'_{\alpha^{-1}(Y)})=\nu(\alpha)^2 l(Y)^2$. Now on the one hand, ${}^t\alpha$ is equal to the composition map of the transpose of

$$\begin{pmatrix} \lambda & 0 \\ 0 & \delta_C \end{pmatrix}$$

and ${}^t\mu$; therefore we can easily see that $\nu({}^t\alpha)=\nu({}^t\lambda)\cdot\nu({}^t\mu)$. On the other hand, $\nu(\alpha)=\nu(\lambda)\nu(\mu)$. Hence we have

$$\frac{\nu({}^t\lambda)}{\nu(\lambda)}=\frac{l(Y)^2}{\nu(\varphi_Y)}\frac{\nu(\mu)}{\nu({}^t\mu)}\,.$$

This implies that the ratio $\nu({}^t\lambda)/\nu(\lambda)$ is independent of the choice of λ.

Next we shall prove that the above ratio is a power of p. There exists a homomorphism $\gamma: B\rightarrow A$ such that $\gamma\lambda=\nu(\lambda)\delta_A$. (cf. Weil [9], Th. 27). Since the transpose of $\nu(\lambda)\delta_A$ is $\nu(\lambda)\delta_{\hat{A}}$ by linearity, and since ${}^t(\gamma\lambda)={}^t\lambda{}^t\gamma$, we get $\nu({}^t\lambda)\nu({}^t\gamma)=\nu(\lambda)^{2n}$.

First suppose that $\nu(\lambda)$ is not divisible by p. Then we can see that, by using Morikawa's idea in the proof of Th. 4, Morikawa [5], $\nu({}^t\lambda)\geqslant\nu(\lambda)$; the proof as follows. We may assume that $\nu(\lambda)=l$ is a prime number different from p. Let Y be a non-degenerate divisor on B; then we have $E_l(\lambda^{-1}(Y))={}^tM_l(\lambda)\,E_l(Y)M_l(\lambda)$. There exist l-adic vectors $y_1,\cdots,y_l$ modulo 1 such that ${}^tM_l(\lambda)y_i\equiv 0$ (mod 1) and $y_i\not\equiv 0$, $y_i\not\equiv y_j$ $(i\neq j)$, (mod 1) $(i,j=1,\cdots,l)$; let v_i be points of $\mathfrak{g}_l(B)$ such that the corresponding l-adic vectors are congruent to $E_l(Y)^{-1}y_i$ modulo 1. Since λ is a surjective homomorphism, there exist l points u_i of $\mathfrak{g}_l(A)$ such that $\lambda(u_i)=v_i$. If x_i are l-adic vectors modulo 1 corresponding to the points u_i, then we can easily see that $E_l(\lambda^{-1}(Y))x_i\equiv 0$ (mod 1). This implies that $\lambda^{-1}(Y_{v_i})=\lambda^{-1}(Y)_{u_i}\sim\lambda^{-1}(Y)$ for $i=1,\cdots,l$; furthermore it is clear, by definitions, that any two of the divisors $Y_{v_1},\cdots,Y_{v_l}$ are not linearly equivalent to each other. From these we have $\nu({}^t\lambda)\geqslant\nu(\lambda)$; since $\nu(\gamma)$ is also not divisible by p, $\nu({}^t\gamma)\geqslant\nu(\gamma)$. We can now conclude that, in this case, $\nu({}^t\lambda)=\nu(\lambda)$.

Next we consider the case where $\nu(\lambda)$ is a power of p. From the preceding formula: $\nu({}^t\lambda)\nu({}^t\gamma)=\nu(\lambda)^{2n}$, $v({}^t\lambda)$ is also a power of p. Hence, in this case, $\nu({}^t\lambda)/\nu(\lambda)$ is obviously a power of p.

Now we proceed to the general case. We know that λ can be written as $\lambda=\lambda_1\lambda_2$, where λ_1 is such that $\nu(\lambda_1)$ is not divisible by p and λ_2 is such that $\nu(\lambda_2)$ is a power of p; then we have

$$\frac{\nu({}^t\lambda)}{\nu(\lambda)}=\frac{\nu({}^t\lambda_2)\nu({}^t\lambda_1)}{\nu(\lambda_1)\ \nu(\lambda_2)}$$
$$=\frac{\nu({}^t\lambda_2)}{\nu(\lambda_2)}\ .$$

Thus we have proved that $\nu({}^t\lambda)/\nu(\lambda)$ is a power of p.

If we put $\nu({}^t\lambda)=p^{f(A,B)}\nu(\lambda)$, then $f(A, B)$ is an integer not depending on the choice of the homomorphism λ of A onto B. In the last part of this paper we shall prove that $f(A, B)=0$ for any A and B.

§ 4. The constant $f(A)$.

LEMMA 1. *Let X be a divisor on an abelian variety A^n. If a point a of A is of order q (q may be divisible by p) and $X_a\sim X$, then there exists a divisor Y on A such that $Y\sim X$ and $Y_a=Y$; in addition, when X is positive, Y can be chosen to be positive.*

PROOF. We put $(\varphi)=X_a-X$; we may assume that φ is not a constant. Consider[6] q functions $\varphi, \varphi^{T_a}, \cdots, \varphi^{T_{(q-1)a}}$; then we have $(\varphi^{T_{ia}})=X_{(i+1)a}-X_{ia}$, and $(\varphi\varphi^{T_a}\cdots\varphi^{T_{ia}})=X_{(i+1)a}-X$. Since $(\varphi\varphi^{T_a}\cdots\varphi^{T_{(q-1)}})=0$, the function $\varphi\varphi^{T_a}\cdots\varphi^{T_{(q-1)a}}$ is a constant; we may assume that, without loss of generality, $\varphi\varphi^{T_a}\cdots\varphi^{T_{(q-1)a}}=1$.

If $1, \varphi, \varphi\varphi^{T_a}\cdots\varphi^{T_{(r+1)a}}$ are linearly dependent and $1, \varphi, \varphi\varphi^{T_a}, \cdots, \varphi\varphi^{T_a}\cdots\varphi^{T_{ra}}$ are linearly independent over an algebraically closed field k over which A, φ and a are defined, where $0\leqslant r\leqslant q-2$, then there exists a non-trivial relation:

$$c_0+c_1\varphi+c_2\varphi\varphi^{T_a}+\cdots+c_{r+2}\varphi\varphi^{T_a}\cdots\varphi^{T_{(r+1)a}}=0$$

with coefficients c_i in k. Here c_0 must not be zero; for otherwise $\varphi(c_1+c_2\varphi^{T_a}+\cdots+c_{r+2}\varphi^{T_a}\cdots\varphi^{T_{(r+1)a}})=0$ and this implies a relation

6) In general, for any function φ on A and for any point a of A, we define a function φ^{T_a} in the following way. Let k be a field over which φ is defined and a is rational; let x be a generic point of A over k. If we put $\varphi^{T_a}(x)=\varphi(x-a)$, then φ^{T_a} is a function defined over k and clearly $(\varphi^{T_a})=(\varphi)_a$.

$c_1+c_2\varphi+\cdots+c_{r+2}\varphi\varphi^{Ta}\cdots\varphi^{Tra}=0$, which is a contradiction. We may thus assume that $c_0=1$.

We can find out constants $b_1,\cdots,b_{r+1}$ and b which satisfy the following condition:

$$\frac{1+b_1\varphi+b_2\varphi\varphi^{Ta}+\cdots+b_{r+1}\varphi\varphi^{Ta}\cdots\varphi^{Tra}}{(1+b_1\varphi+b_2\varphi\varphi^{Ta}+\cdots+b_{r+1}\varphi\varphi^{Ta}\cdots\varphi^{Tra})^{Ta}}=b\varphi .$$

In fact, by the principle of undetermined coefficients, we have only to solve the following system of equations:

$$b_1-b=c_1,\ b_2-b_1b=c_2,\ \cdots,\ b_{r+1}-b_rb=c_{r+1}$$
$$-b_{r+1}b=c_{r+2};$$

this system is equivalent to

$$b_1=b+c_1,\ b_2=b^2+c_1b+c_2,\ \cdots,\ b_{r+1}=b^{r+1}+c_1b^r+$$
$$\cdots+c_rb+c_{r+1},\ b^{r+2}+c_1b^{r+1}+\cdots+c_{r+1}b+c_{r+2}=0,$$

and the last equation in b has a non-zero root; if such a root is determined, the other constants $b_1,\cdots,b_{r+1}$ are also determined. For such constants $b_1,\cdots,b_{r+1}$ and b, we put $(1+b_1\varphi+\cdots+b_{r+1}\varphi\varphi^{Ta}\cdots\varphi^{Tra})=Y-X$; here notice that, if X is positive, then Y is also positive. It follows from the preceding relation that $(Y-X)-(Y-X)_a=X_a-X$, and hence we have $Y_a=Y$.

LEMMA 2. *Let A be an abelian variety. Then there exists an abelian variety B, isogenous to A, such that there is a positive non-degenerate divisor Y on B for which the group $\mathfrak{G}_Y$ is trivial.*

PROOF. Let X be a positive non-degenerate divisor on A. Suppose that the order of the subgroup $\mathfrak{G}_X$ is greater than 1. Let a be a point of $\mathfrak{G}_X$; let q be the order of $a(q>1)$. Then, by definition, $X_a\sim X$; Lemma 1 shows that there exists a positive divisor Y, linearly equivalent to X, such that $Y_{ia}=Y$ for $i=1,2,\cdots,q-1$. Consider the quotient variety $A_1=A/\{a\}$, where $\{a\}$ denotes the subgroup of A generated by a, and consider the natural separable homomorphism λ_1 of A onto A_1. Then, by Weil [9], Prop. 33, there exists a positive divisor X_1 on A_1 such that $Y=\lambda_1^{-1}(X_1)$; clealy X_1 is non-degenerate. It follows from Nishi [6], Th. 4, that $l(X)=\nu(\lambda_1)l(X_1)$; and so $l(X)=ql(X_1)$. If the order of $\mathfrak{G}_{X_1}$ is greater than 1, then we continue the same process as above;

by the condition that $l(X) > l(X_1)$, after a finite number of steps, we can get an abelian variety asserted in our Lemma.

LEMMA 3. *Let A be an abelian variety and let X be a non-degenerate divisor on it. If the group $\mathfrak{G}_X$ is trivial, then $\nu(\varphi_X)$ is a power of p; furthermore, if X is positive, then $l(X)$ is also a power of p.*

PROOF. The first assertion follows immediately from the fact that the kernel of φ_X is exactly the group $\mathfrak{G}_X$.

Now we suppose that X is positive and non-degenerate. There exists an abelian variety B such that $A \times B$ is isogenous to a Jacobian variety J; moreover, by Lemma 2, we may assume that there is a positive non-degenerate divisor Y on B such that $\mathfrak{G}_Y$ is trivial. Consider the divisor $X \times B + A \times Y$ on $A \times B$ and an isogeny $\lambda : J \to A \times B$; then we have

$$\delta'_{\lambda^{-1}(X \times B + A \times Y)} = {}^t\lambda \begin{pmatrix} \varphi_X & 0 \\ 0 & \varphi_Y \end{pmatrix} \lambda .$$

Using the second half of Theorem 1, we can compute the degree of the left-hand side as follows:

$$\begin{aligned} \nu(\delta'_{\lambda^{-1}(X \times B + A \times Y)}) &= l(\lambda^{-1}(X \times B + A \times Y))^2 \\ &= \nu(\lambda)^2 l(X \times B + A \times Y)^2 \\ &= \nu(\lambda)^2 l(X)^2 l(Y)^2 . \end{aligned}$$

On the other hand, the degree of the right-hand side is equal to $\nu({}^t\lambda)\,\nu(\lambda)\,\nu(\varphi_X)\,\nu(\varphi_Y)$; therefore we have

$$l(X)^2 l(Y)^2 = p^{f(J, A \times B)}\,\nu(\varphi_X)\,\nu(\varphi_Y) ;$$

since the groups $\mathfrak{G}_X$ and $\mathfrak{G}_Y$ are both trivial, $\nu(\varphi_X)$ and $\nu(\varphi_Y)$ are powers of p. This implies that $l(X)$ is a power of p.

Now we can introduce a constant $f(A)$ attached to a given abelian variety A". Let X be a positive non-degenerate divisor on A. There exists an abelian variety B such that $A \times B$ is isogenous to a Jacobian variety J; here we may assume that, by Lemma 2 and Lemma 3, there is a positive non-degenerate divisor Y on B such that $l(Y)$ and $\nu(\varphi_Y)$ are powers of p. Let $\lambda : J \to A \times B$ be an isogeny; then similarly as in the proof of Lemma 3, we have

$$l(X)^2 l(Y)^2 = p^{f(J, A \times B)}\,\nu(\varphi_X)\,\nu(\varphi_Y) ,$$

hence

$$\nu(\varphi_X)/l(X)^2 = p^{-f(J,A\times B)}l(Y)^2/\nu(\varphi_Y).$$

It follows from the above relation that the ratio $\nu(\varphi_X)/l(X)^2$ is independent of the choice of X. Thus, if we denote this ratio by $p^{f(A)}$, then we know that, whenever a divisor X is positive and non-degenerate, $\nu(\varphi_X)=p^{f(A)}l(X)^2$. Here we need the following

PROPOSITION 3. *Let X be any divisor on A^n. Then $(X^{(n)})$ is a multiple of $n!$.*

PROOF. When X is positive and non-degenerate, our assertion is already known. Now let X be arbitrary. Let Y be a positive non-degenerate divisor on A. Then, by Proposition 1, $mY+X$ is linearly equivalent to a positive non-degenerate divisor, if m is sufficiently large. Hence the polynomial $((mY+X)^{(n)})/n!$ in m takes an integer value, whenever m is sufficiently large. It is well known that such a polynomial in m takes an integer value for arbitrary integer m; in particular the constant term $(X^{(n)})/n!$ is an integer.

Now let X and Y be as in Proposition 3. Then we have $\nu(\varphi_{mY+X})=p^{f(A)}l(mY+X)^2$, if m is sufficiently large. This implies that two polynomials $\nu(\varphi_{mY+X})$ and $p^{f(A)}[((mY+X)^{(n)})/n!]^2$ in m are equal if m is sufficiently large. Hence they coincide identically. Comparing the costants terms of the both sides, we can get the following formula:

$$(1) \qquad \nu(\varphi_X)=p^{f(A)}[(X^{(n)})/n!]^2$$

Here notice that the formula (1) is valid for any divisor X on A^n, in other words, $f(A)$ is independent of the choice of divisors X. In particular we have the following

COROLLARY. *A divisor X on A^n is non-degenerate if and only if the integer $(X^{(n)})/n!$ is not zero.*

In the rest of this §, we shall see several relations between the constants $f(A, B)$, $f(A)$ and $f(B)$. Let A^n and B^n be isogenous abelian varieties and let $\lambda: A\to B$ be an isogeny. Let Y be a non-degenerate divisor on B. Then we have $\nu(\varphi_{\lambda^{-1}(Y)})=\nu({}^t\lambda)\nu(\varphi_Y)\nu(\lambda)$. Applying the results in §3 and the formula (1), we have

$$p^{f(A)}[(\lambda^{-1}(Y)^{(n)})/n!]^2 = p^{f(A,B)+f(B)}\nu(\lambda)^2[(Y^{(n)})/n!]^2;$$

since $(\lambda^{-1}(Y)^{(n)})=\nu(\lambda)(Y^{(n)})$, we can get the following formula:

$$(2) \qquad f(A, B) = f(A) - f(B) .$$

Next let A^m and B^n are two abelian varieties; let X and Y be non-degenerate divisors on A and on B respectively. Then we have $\nu(\varphi_{X\times B+A\times Y}) = \nu(\varphi_X)\nu(\varphi_Y)$; applying the formula (1), we have

$$\begin{aligned}\nu(\varphi_{X\times B+A\times Y}) &= p^{f(A\times B)}[((X\times B+A\times Y)^{(m+n)})/(m+n)!]^2 \\ &= p^{f(A\times B)}[(X^{(m)})/m!]^2[(Y^{(n)})/n!]^2 ;\end{aligned}$$

and similarly

$$\nu(\varphi_X)\nu(\varphi_Y) = p^{f(A)+f(B)}[(X^{(m)})/m!]^2[(Y^{(n)})/n!]^2 .$$

Consequently we can get the following formula:

$$(3) \qquad f(A\times B) = f(A)+f(B) .$$

§ 5. Preliminary step of the proof of the Main Theorem.

In this §, we shall show that the constant $f(A)$ is not greater than zero.

Let A be an abelian variety and let $\hat{A}$ be its dual; hereafter we shall denote by o and ô the unit elements of A and $\hat{A}$ respectively. There is a divisor P on $A\times\hat{A}$, rational over a suitable common field of definition k for A and $\hat{A}$, with the following properties: The point o×ô is not contained in any component of P and therefore $P(\mathrm{o})$, ${}^tP(\hat{\mathrm{o}})$ are both defined; moreover if a point $\hat{u}$ of $\hat{A}$ is generic over k, then the linear class of ${}^tP(\hat{u})-{}^tP(\hat{\mathrm{o}})$ corresponds to the point $\hat{u}$ of $\hat{A}$. Let now u be a generic point of A over k. The correspondence: $u\to$ the linear class of $P(u)-P(\mathrm{o})$ defines a homomorphism κ_A of A onto $\hat{\hat{A}}$, where $\hat{\hat{A}}$ means the double dual of A. If we put $T = P - {}^tP(\hat{\mathrm{o}})\times\hat{A} - A\times P(\mathrm{o})$, then we have the following

PROPOSITION 4. *The homomorphism* $\varphi_T : A\times\hat{A}\to\hat{A}\times\hat{\hat{A}}$ *can be witten as follows*:

$$\varphi_T = \begin{pmatrix} 0 & -\delta_{\hat{A}} \\ -\kappa_A & 0 \end{pmatrix}.$$

PROOF. Let $u\times\hat{u}$ be a generic point of $A\times\hat{A}$ over k. It is well known that $T_{u\times\hat{u}} - T \sim X\times\hat{A}+A\times Y$, where X is a divisor on A such that $X\equiv 0$ and Y a divisor on $\hat{A}$ such that $Y\equiv 0$ (cf. Lang [3], Prop. 6, IV). We can determine the linear classes of X

and Y explicitly. Since $T(\mathrm{o})=0$, $T_{u\times\hat{u}}(\mathrm{o})$ is also defined. We may assume ahat, by making a suitable translation if necessary, any component of X does not contain the unit o; then $(X\times\hat{A}+A\times Y)\cdot(\mathrm{o}\times\hat{A})$ is defined and equal to $\mathrm{o}\times Y$. Therefore $T_{u\times\hat{u}}(\mathrm{o})\sim Y$ on $\hat{A}$. On the other hand, making a translation to the intersection, we can compute as follows:

$$\begin{aligned} T_{u\times\hat{u}}\cdot(\mathrm{o}\times\hat{A}) &= [T\cdot((-u)\times\hat{A})]_{u\times\hat{u}} \\ &= [(-u)\times T(-u)]_{u\times\hat{u}} \\ &= \mathrm{o}\times T(-u)_{\hat{u}}\,. \end{aligned}$$

This shows that $Y\sim T(-u)_{\hat{u}}\sim T(-u)$; by the theorem of the square, $T(-u)\sim -T(u)$. Thus we see that $Y\sim -T(u)$. Similarly we have $X\sim -{}^tT(\hat{u})$. This completes the proof.

Let now X be a divisor on A^n; the homomorphism $\kappa_A: A\rightarrow\hat{\hat{A}}$ being as above, we can see that $\varphi_X={}^t\varphi_X\kappa_A$.[7] If we assume that X is non-degenerate, then it follows from the above formula that $\nu(\varphi_X)=\nu({}^t\varphi_X)\nu(\kappa_A)$, and therefore that $\nu(\kappa_A)=p^{-f(A,\hat{A})}$. Using the formula (2), we get

$$\nu(\kappa_A) = p^{f(\hat{A})-f(A)}\,. \tag{4}$$

On the other hand, Proposition 4 shows that $\nu(\varphi_T)=\nu(\kappa_A)$; since $\nu(\varphi_T)=p^{f(A\times\hat{A})}[(T^{(2n)})/(2n)!]^2$, we have $p^{f(A\times\hat{A})}[(T^{(2n)})/(2n)!]^2=p^{f(\hat{A})-f(A)}$. It follows from the formula (3) that

$$[(T^{(2n)})/(2n)!]^2 = p^{-2f(A)}\,. \tag{5}$$

Since $(T^{(2n)})/(2n)!$ is an integer, this shows that $f(A)\leqslant 0$.

§ 6. The proof of the Main Theorem.

In this §, we shall prove the final result: $f(A)=0$.

LEMMA 4. *Let A and $B_1\times B_2$ be isogenous abelian varieties and let $\mu: A\rightarrow B_1\times B_2$ be an isogeny. Let k be a field of definition for μ; let x be a generic point of A over k. Putting $\mu(x)=y_1\times y_2$, we define surjective homomorphisms $\lambda_i: A\rightarrow B_i$ by $\lambda_i(x)=y_i$ $(i=1, 2)$. Then the intersection product $\lambda_1^{-1}(\mathrm{o}_1)\cdot\lambda_2^{-1}(\mathrm{o}_2)$ is defined and $\nu(\mu)=$* deg $(\lambda_1^{-1}(\mathrm{o}_1)\cdot\lambda_2^{-1}(\mathrm{o}_2))$, *where o_1 and o_2 are the unit elements of B_1 and B_2 respectively.*

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7) Cf. Lang [3], Prop. 10, V.

PROOF. Since $\mu(a)=\lambda_1(a)\times\lambda_2(a)$ for any point a of A, the finiteness of the kernel of μ implies that $\lambda_1^{-1}(o_1)\cap\lambda_2^{-1}(o_2)$ is finite; consequently $\lambda_1^{-1}(o_1)\cdot\lambda_2^{-1}(o_2)$ is defined. As to the second assertion, we define homomorphisms $\mu_i : B_1\times B_2\to B_i$ by $\mu_i(y_1\times y_2)=y_i (i=1, 2)$. Then clearly we have $\mu_i\mu=\lambda_i$. Since $\lambda_i^{-1}(o_i)=\mu^{-1}(\mu_i^{-1}(o_i))$, we have

$$\begin{aligned}\lambda_1^{-1}(o_1)\cdot\lambda_2^{-1}(o_2) &= \mu^{-1}(\mu_1^{-1}(o_1))\cdot\mu^{-1}(\mu_2^{-1}(o_2))\\ &= \mu^{-1}(o_1\times B_2)\cdot\mu^{-1}(B_1\times o_2)\\ &= \mu^{-1}(o_1\times B_2\cdot B_1\times o_2)\\ &= \mu^{-1}(o_1\times o_2)\,.\end{aligned}$$

This completes the proof.

LEMMA 5. *Notations being as in Lemma 4, let $\hat{y}_1\times\hat{y}_2$ be a generic point of $\hat{B}_1\times\hat{B}_2$. Then we have ${}^t\mu(\hat{y}_1\times\hat{y}_2)={}^t\lambda_1(\hat{y}_1)+{}^t\lambda_2(\hat{y}_2)$.*

This is a special case of the statement at the end of § 1.

LEMMA 6. *Let $A\times B$ be a product of two abelian varieties A and B. Then the canonical homomorphism $\kappa_{A\times B} : A\times B\to\hat{\hat{A}}\times\hat{\hat{B}}$ can be written by a matrix,*

$$\kappa_{A\times B}=\begin{pmatrix}\kappa_A & 0\\ 0 & \kappa_B\end{pmatrix}.$$

The proof is easy and is omitted.

The following Theorem, due to Chow [2], plays an essential rôle in our proof.

> *For any given abelian variety A, we can find a product of Jacobian varieties $J_1, J_2, \cdots, J_r$, such that there is a regular homomorphism*[8] *λ of $J_1\times J_2\times\cdots\times J_r$ onto A.*

We shall, in what follows, denote by ΠJ the product $J_1\times J_2\times\cdots\times J_r$. Let k be an algebraically closed field of definition for λ. It is not so difficult to see that ${}^t\lambda : \hat{A}\to\widehat{\Pi J}=\Pi J$ is a regular injection, i.e. $k(\hat{x})=k({}^t\lambda(\hat{x}))$, where $\hat{x}$ is a generic point of $\hat{A}$ over k (In fact, we can prove it by using the results in Nishi [6], § 3; a detailed

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8) Let λ be a homomorphism of an abelian variety A into another abelian variety B. Then λ is called a regular homomorphism, if $k(x)$ is a regular extension over $k(\lambda(x))$. Obviously such a definition is independent of the choice of the field k.

proof is also given in Lang [3], Th. 10, VIII). We put ${}^t\lambda(\hat{A})=C$; then C is an abelian subvariety of ΠJ. We consider the quotient variety $\Pi J/C=A_1$ and the natural regular homomorphism $\lambda_1:\Pi J\to A_1$.

Since λ is regular, $\lambda^{-1}(o)$ is an abelian subvariety of ΠJ, where o is the unit element of A; we set $\lambda^{-1}(o)=B$. We consider the exact sequence:

$$0\longrightarrow B\xrightarrow{\ \alpha\ }\Pi J\xrightarrow{\ \lambda\ }A\longrightarrow 0,$$

where α is a regular injection. We consider also the following sequence

$$0\longleftarrow \hat{B}\xleftarrow{\ {}^t\alpha\ }\Pi J\xleftarrow{\ {}^t\lambda\ }\hat{A}\longleftarrow 0.$$

Then, since ${}^t\alpha\,{}^t\lambda={}^t(\lambda\alpha)$, we have ${}^t\alpha\,{}^t\lambda=0$; this implies[9] that C is contained in the kernel of ${}^t\alpha$. We can readily see that ${}^t\alpha$ is surjective (cf. Lang [3], Prop. 2, V); consequently C must be a component, containing the unit element, of the kernel of ${}^t\alpha$. Since ${}^t\alpha\alpha: B\to\hat{B}$ is surjective[10], the subgroup $B\cap C$ is finite.

The homomorphism ${}^t\alpha$ can be decomposed as follows:

$$\begin{array}{ccccc} 0\longleftarrow & \hat{B} & \xleftarrow{\quad {}^t\alpha \quad} & \Pi J & \longleftarrow\cdots \\ & {\scriptstyle\beta}\nwarrow & & \swarrow{\scriptstyle\lambda_1} & \\ & & A_1 & & \end{array}$$

where β is an isogeny. Consider also the following sequence:

$$\begin{array}{ccccc} 0\longrightarrow & \hat{\hat{B}} & \xrightarrow{\quad {}^{tt}\alpha \quad} & \Pi J & \longrightarrow\cdots \\ & {\scriptstyle {}^t\beta}\searrow & & \nearrow{\scriptstyle {}^t\lambda_1} & \\ & & \hat{A}_1 & & \end{array}$$

Then, as in the case of ${}^t\lambda$, ${}^t\lambda_1$ is a regular injection and ${}^{tt}\alpha={}^t\lambda_1\,{}^t\beta$. It is well known that the following diagram

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9) The regularity of ${}^t\alpha$ is not proved at present. But after all observations in this §, the regularity will follow.

10) Let Θ_i be the canonical divisors on J_i for $i=1,\cdots,r$ and put $\Theta^{(r)}=\Theta_1\times J_2\times\cdots\times J_r+J_1\times\Theta_2\times\cdots\times J_r+\cdots+J_1\times\cdots\times J_{r-1}\times\Theta_r$. Then, since $\varphi_{\Theta^{(r)}}=\delta_{\Pi J}$, we have ${}^t\alpha\alpha=\varphi_{\alpha^{-1}(\Theta^{(r)})}$. Now we can easily see that $\alpha^{-1}(\Theta^{(r)})$ is non-degenerate; this implies our assertion.

$$\begin{array}{ccc} B & \xrightarrow{\alpha} & \Pi J \\ {\scriptstyle \kappa_B}\downarrow & & \downarrow{\scriptstyle \kappa_{\Pi J}} \\ \hat{\hat{B}} & \xrightarrow{{}^{tt}\alpha} & \Pi J \end{array}$$

is commutative; since $\kappa_{\Pi J}$ is equal to the identity endomorphism $\delta_{\Pi J}$ of ΠJ by Lemma 6 (note that κ_J coincides with δ_J for any Jacobian variety J), we have $\alpha = {}^{tt}\alpha\kappa_B$. Consequently $\alpha = {}^t\lambda_1\, {}^t\beta\kappa_B$. Combining this with the fact that α is a regular homomorphism, we know that κ_B and ${}^t\beta$ are both regular isomorphisms and that B is the isomorphic image of $\hat{A}_1$ by the regular injection ${}^t\lambda_1$, i.e. ${}^t\lambda_1(\hat{A}_1) = B$.

Assume now that all the varieties and the maps appearing above are defined over a field K containing k; let x be a generic point of ΠJ over K. If we define a homomorphism $\mu: \Pi J \rightarrow A \times A_1$ by $\mu(x) = \lambda(x) \times \lambda_1(x)$, then μ is an isogeny; for otherwise there are infinitely many points a of ΠJ such that $\mu(a) = \lambda(a) \times \lambda_1(a) = 0$ and, since $B \cap C$ is finite, this is a contradiction. Lemma 4 shows that $\nu(\mu) = \deg(B \cdot C)$.

On the other hand, let $\hat{u} \times \hat{v}$ be a generic point of $\hat{A} \times \hat{A}_1$ over K; then, according to the regularity of ${}^t\lambda$ and that of ${}^t\lambda_1$, we have $K(\hat{u}, \hat{v}) = K({}^t\lambda(\hat{u}), {}^t\lambda_1(\hat{v}))$; Lemma 5 shows that ${}^t\mu(\hat{u} \times \hat{v}) = {}^t\lambda(\hat{u}) + {}^t\lambda_1(\hat{v})$. By Weil [9], Th. 4, Cor. 2, we have $[K({}^t\lambda(\hat{u}), {}^t\lambda_1(\hat{v})) : K({}^t\lambda(\hat{u}) + {}^t\lambda_1(\hat{v}))] = \deg(B \cdot C)$. Since $\nu({}^t\mu) = [K(\hat{u} \times \hat{v}) : K({}^t\lambda(\hat{u}) + {}^t\lambda_1(\hat{v}))]$, $\nu({}^t\mu)$ is equal to $\deg(B \cdot C)$. Thus we can get $\nu({}^t\mu) = \nu(\mu)$.

Using the notations in § 3, this shows that $f(\Pi J, A \times A_1) = 0$. It follows from the formula (2) that $f(\Pi J) = f(A \times A_1)$, and from the formula (3), that $\sum_{j=1}^{r} f(J_j) = f(A) + f(A_1)$. Now Theorem 1 implies that each $f(J_j)$ must vanish; consequently $f(A) + f(A_1) = 0$. Since $f(A) \leqslant 0$ and $f(A_1) \leqslant 0$, we can obtain the final result: $f(A) = 0$.

We can thus state the following

MAIN THEOREM (FROBENIUS). *Let A^n be an abelian variety and let X be a divisor on it. Then we have*

$$\nu(\varphi_X) = [(X^{(n)})/n!]^2$$

COROLLARY 1. *Notations being the same as in the above Theorem, assume that X is a positive non-degenerate divisor. Then we have*

$$\nu(\varphi_X) = l(X)^2 .$$

Furthermore let q be a prime number different from p; then q^e divides $l(X)$ if and only if q^{2e} divides $|E_q(X)|$, where e is a non-negative integer.

PROOF. The first assertion follows from our Theorem and from Nishi [6]. We proceed to the second assertion. By the statement in Weil [9], XI, the highest power of q which divides the order of $\mathfrak{G}_X$ is equal to the highest power of q which divides $|E_q(X)|$. Since the kernel of φ_X is $\mathfrak{G}_X$, our assertion follows immediately from the first assertion.

COROLLARY 2. (DUALITY THEOREM) *Let λ be a homomorphism of an abelian variety A^n onto another abelian variety B^n. Then we have* $\nu({}^t\lambda)=\nu(\lambda)$.

Added in the proof. I hear that the duality theorem has already been proved by Cartier in the Bourbaki Seminar, 1958.

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