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p-primary components of homotopy groups

IV. Compositions and toric constructions

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Here we shall use the following notations. $G_k=\pi_k(\mathfrak{S})$ is the k-th stable homotopy group of the sphere and $\pi_k(\mathfrak{S}\,;p)$ is its p-primary component. $\alpha\circ\beta\in G_{h+k}$ indicates the composition of $\alpha\in G_h$ and $\beta\in G_k$. The toric construction

$$\{\alpha,\ \beta,\ \gamma\}\in G_{h+k+l+1}/(\alpha\circ G_{k+l+1}+\gamma\circ G_{h+k+1})$$

will be defined if $\alpha\in G_h$, $\beta\in G_k$ and $\gamma\in G_l$ satisfy the condition $\alpha\circ\beta=\beta\circ\gamma=0$. This is different only in sign to one given in Chapter 5 of [4]. Denote by $\alpha_1\in G_{2p-3}$ a generator of $\pi_{2p-3}(\mathfrak{S}\,;p)=Z_p$ and choose elements α_t of $G_{2t(p-1)-1}$ inductively such that $\alpha_t\in\{\alpha_{t-1},\ p\iota,\ \alpha_1\}$. Denote by β_s, $1\leqq s<p$, a generator of $\pi_{2(sp+s-1)(p-1)-2}(\mathfrak{S}\,;p)=Z_p$, and denote by β_1^r the r-fold iterated composition $\beta_1\circ\cdots\circ\beta_1$ of β_1. There exist elements α'_{rp}, $1\leqq r<p$, such that $p\alpha'_{rp}=\alpha_{rp}$ for $1\leqq r<p-1$ and $p\alpha'_{(p-1)p}=\alpha_{(p-1)p}+x\alpha_1\circ\beta_1^{p-1}$ for some integer x. Then these elements and their compositions generate the p-components $\pi_k(\mathfrak{S}\,;p)$ of the stable groups G_k for $k\leqq 2p^2(p-1)-3$.

Theorem 4.15. (*cf.* Theorem 3.13 of [6]).

$$\begin{aligned}
\pi_{2rp(p-1)-1}(\mathfrak{S}\,;p) &= Z_{p^2} = \{\alpha'_{rp}\} && \textit{for } 1\leqq r<p-1\,,\\
&= Z_{p^2}+Z_p = \{\alpha'_{(p-1)p}\}+\{\alpha_1\circ\beta_1^{p-1}\} && \textit{for } r=p-1\,,\\
\pi_{2t(p-1)-1}(\mathfrak{S}\,;p) &= Z_p = \{\alpha_t\} && \textit{for } 1\leqq t<p^2 \textit{ and } t\not\equiv 0 \pmod p\,,\\
\pi_{2(rp+s)(p-1)-2(r-s)}(\mathfrak{S}\,;p) &= Z_p = \{\beta_1^{r-s-1}\circ\beta_{s+1}\} && \textit{for } 0\leqq s<r\leqq p-1\,,\\
\pi_{2(rp+s+1)(p-1)-2(r-s)-1}(\mathfrak{S}\,;p) &= Z_p = \{\alpha_1\circ\beta_1^{r-s-1}\circ\beta_{s+1}\} &&\\
& && \textit{for } 0\leqq s<r\leqq p-1 \textit{ and } r-s\neq p-1\,,
\end{aligned}$$

$\pi_{2p^2(p-1)-2p}(\mathfrak{S};p) = Z_p = \{\beta_1^p\}$,
$\pi_k(\mathfrak{S};p) = 0$ *otherwise for* $k<2p^2(p-1)-3$,
$\pi_{2p^2(p-1)-3}(\mathfrak{S};p) = \{\alpha_1\circ\beta_1^p\} = Z_p$ *or* 0 .

Concerning the fact $\beta_1^p \neq 0$, we have

Corollary 4.12. *For an arbitrary positive integer r, there exists an element* β *of* G_k *for some k such that the iterated r-fold composition* $\beta^r = \beta\circ\cdots\circ\beta$ *does not vanish.*

Many other compositions and toric constructions for the above generators are computed after giving several properties of the toric construction. As an application, we calculate stable homotopy groups of some elementary complexes in the last §.

The results of this section IV will be powerful in computation of unstable homotopy groups of spheres. See forthcoming section V.

Notations and results, such as (3.10), Theorem 3.13, etc., refer to the preceding sections [6].

§ Toric constructions.

A suspension SX of a space X, with respect to a base point x_0, means the image of an identification

$$d_X : X\times[-1,1] \longrightarrow SX$$

which shrinks $X\times(-1)\cup x_0\times[-1,1]\cup X\times(1)$ to a single point. In particular, the unit $(n+1)$-sphere S^{n+1} is a suspension of S^n under the identification

$$d_n : S^n\times[-1,1] \longrightarrow S^{n+1}$$

of [7] $(x_0=(-1,0,\cdots,0)=e_0)$. *A suspension* $Ef : SX\to SY$ of a mapping $f:(X,x_0)\to(Y,y_0)$ is defined by the formula

$$Ef(d_X(x,t)) = d_Y(f(x),t)\,, \qquad x\in X,\ t\in[-1,1]\,.$$

If α is the homotopy class of the above mapping f, then $E\alpha$ denotes the homotopy class of Ef. If α and β are the classes of mappings $f:(Z,z_0)\to(Y,y_0)$ and $g:(Y,y_0)\to(X,x_0)$ respectively, then $\beta\circ\alpha$ denotes the class of the composition $g\circ f:(Z,z_0)\to(X,x_0)$.

A null-homotopy $f_t:(X,x_0)\to(Y,y_0)$ of f means a homotopy such that $f_0=f$ and $f_1(X)=y_0$. For two null-homotopies f_t and $g_t:(S^n,e_0)\to(X,x_0)$ of $f_0=g_0$, we define *a difference element*

$$\mathfrak{d}(f_t,\ g_t) \in \pi_{n+1}(X)$$

by the class of the mapping $h:(S^{n+1},\ e_0) \to (X,\ x_0)$ given by

$$h(d_n(y,\ t)) = \begin{cases} f_t(y)\,, & 0 \leqq t \leqq 1\,, \quad y \in S^n\,, \\ g_{-t}(y)\,, & -1 \leqq t \leqq 0\,, \quad y \in S^n\,. \end{cases}$$

The following properties of $\mathfrak{d}(f_t,\ g_t)$ are verified easily.

(4. 1). i). $\mathfrak{d}(f_t,\ g_t) + \mathfrak{d}(g_t,\ h_t) = \mathfrak{d}(f_t,\ h_t)$, *and in particular,* $\mathfrak{d}(f_t,\ f_t) = 0$ *and* $\mathfrak{d}(g_t,\ f_t) = -\mathfrak{d}(f_t,\ g_t)$.

ii). *Let* $f_{t,s}$, $g_{t,s}$ *be homotopies which depend continuously on the both of* t *and* s. *Assume that* $f_{0,s} = g_{0,s}$ *and* $f_{1,s}(S^n) = g_{1,s}(S^n) = x_0$, *then* $\mathfrak{d}(f_{t,0},\ g_{t,0}) = \mathfrak{d}(f_{t,1},\ g_{t,1})$.

iii). *Let* α *and* $\alpha' \in \pi_q(S^p)$ *be the homotopy classes of mappings* $h:(X,\ x_0) \to (Y,\ y_0)$ *and* $h':(S^q,\ e_0) \to (S^p,\ e_0)$ *respectively. Then* $\mathfrak{d}(h\circ f_t,\ h\circ g_t) = \alpha\circ\mathfrak{d}(f_t,\ g_t) = h_*\mathfrak{d}(f_t,\ g_t)$ *and* $\mathfrak{d}(f_t\circ h',\ g_t\circ h') = \mathfrak{d}(f_t,\ g_t)\circ E\alpha'$.

iv). *For an arbitrary null-homotopy* f_t *and an arbitrary element* α *of* $\pi_{n+1}(X)$, *there exists a null-homotopy* g_t *of* f_0 *such that* $\alpha = \mathfrak{d}(f_t,\ g_t)$.

v). $E\mathfrak{d}(f_t,\ g_t) = -\mathfrak{d}(Ef_t,\ Eg_t)$.

Consider the homotopy classes α, β and γ of mappings $a:(Y,\ y_0) \to (X,\ x_0)$, $b:(S^q,\ e_0) \to (Y,\ y_0)$ and $c:(S^p,\ e_0) \to (S^q,\ e_0)$ respectively.

Assume that

(4. 2). $\qquad \alpha\circ\beta = 0 \quad$ *and* $\quad \beta\circ\gamma = 0$.

Then there exist null-homotopies $A_t:(S^q,\ e_0) \to (X,\ x_0)$ and $B_t:(S^p,\ e_0) \to (Y,\ y_0)$ of $a\circ b$ and $b\circ c$ respectively. Denote by

$$\{\alpha,\ \beta,\ \gamma\} \subset \pi_{p+1}(X)$$

the set of all $\mathfrak{d}(a\circ B_t,\ A_t\circ c)$ for arbitrary choices of a, b, c, A_t and B_t. By the definition, $\{\alpha,\ \beta,\ \gamma\}$ depends only on α, β and γ.

Lemma 4. 1. $\{\alpha,\ \beta,\ \gamma\}$ *is a coset of the subgroup* $\alpha\circ\pi_{p+1}(Y) + \pi_{q+1}(X)\circ E\gamma$ *in* $\pi_{p+1}(X)$.

Proof. Let a_s, b_s and c_s be homotopies from $a_0 = a$, $b_0 = b$ and $c_0 = c$, then there are null-homotopies $A_{t,s}$ and $B_{t,s}$ of $a_s\circ b_s$ and

$b_s \circ c_s$ which depend continuously on s. Then it follows from ii) of (4.1) that any element of $\{\alpha, \beta, \gamma\}$ has a form $\mathfrak{d}(a \circ B_t, A_t \circ c)$ for fixed representatives a, b and c. Next for null-homotopies A_t, A_t' of $a \circ b$ and for B_t, B_t' of $b \circ c$, it follows from (4.1)

$$\begin{aligned}\mathfrak{d}(a \circ B_t, A_t \circ c) - \mathfrak{d}(a \circ B_t', A_t' \circ c) &= \mathfrak{d}(a \circ B_t, a \circ B_t') + \mathfrak{d}(A_t' \circ c, A_t \circ c) \\ &= \alpha \circ \mathfrak{d}(B_t, B_t') + \mathfrak{d}(A_t', A_t) \circ E\gamma .\end{aligned}$$

Thus the difference of two elements of $\{\alpha, \beta, \gamma\}$ belongs to the subgroup $\alpha \circ \pi_{p+1}(Y) + \pi_{q+1}(X) \circ E\gamma$. Conversely any element of the subgroup has a form $\alpha \circ \mathfrak{d}(B_t, B_t') + \mathfrak{d}(A_t', A_t) \circ E\gamma$ by iv) of (4.1). Therefore $\{\alpha, \beta, \gamma\}$ is a coset of the subgroup $\alpha \circ \pi_{p+1}(Y) + \pi_{q+1}(X)^\circ E\gamma$. q. e. d.

Now we call

$$\{\alpha, \beta, \gamma\} \in \pi_{p+1}(X)/(\alpha \circ \pi_{p+1}(Y) + \pi_{q+1}(X) \circ E\gamma)$$

the toric construction of α, β *and* γ.

The following properties are verified without difficulties.

(4.3). i). $\{\alpha, \beta, \gamma_1\} \pm \{\alpha, \beta, \gamma_2\} \supset \{\alpha, \beta, \gamma_1 \pm \gamma_2\}$,
$\{\alpha, \beta_1, \gamma\} \pm \{\alpha, \beta_2, \gamma\} = \{\alpha, \beta_1 \pm \beta_2, \gamma\}$ *if* $\gamma = E\gamma'$.
$\{\alpha_1, \beta, \gamma\} \pm \{\alpha_2, \beta, \gamma\} \supset \{\alpha_1 \pm \alpha_2, \beta, \gamma\}$
if $\beta = E\beta'$, $\gamma = E\gamma'$, $Y = S^r$.

ii). $\{\alpha, \beta, \gamma \circ \delta\} \supset \{\alpha, \beta, \gamma\} \circ E\delta$,
$\{\alpha, \beta \circ \gamma, \delta\} \supset \{\alpha, \beta, \gamma \circ \delta\}$,
$\{\alpha \circ \beta, \gamma, \delta\} \subset \{\alpha, \beta \circ \gamma, \delta\}$,
$\alpha \circ \{\beta, \gamma, \delta\} \subset \{\alpha \circ \beta, \gamma, \delta\}$.

iii). $-E\{\alpha, \beta, \gamma\} \subset \{E\alpha, E\beta, E\gamma\}$.

For a mapping $a : (Y, y_0) \rightarrow (X, x_0)$, we construct a space

$$X \bigvee_a \hat{Y}$$

identifying X and $Y \times [0, 1]$ by the relations $x_0 = (y_0, t) = (y, 1)$, $t \in [0, 1]$, $y \in Y$, and $a(y) = (y, 0)$. Since the homotopy type of the space $X \bigvee_a \hat{Y}$ depends on the homotopy class α of a, we denote the space by

$$X \bigvee_\alpha \hat{Y}.$$

If $Y=S^r$, then we write $X\bigcup_\alpha \hat{Y}$ by $X\bigcup_\alpha e^{r+1}$ as usual. If $\beta\in\pi_q(Y)$ is the class of a mapping $b:(S^q, e_0)\to(Y, y_0)$ satisfying (4.2), then we denote by

$$\tilde{\beta}\in\pi_{q+1}(X\bigcup_\alpha \hat{Y})$$

the class of the mapping $\tilde{b}:S^{q+1}\to X\bigcup_\alpha \hat{Y}$ given by $\tilde{b}(d_q(z, t)) =(b(z), t)$ for $0\leqq t\leqq 1$ and $\tilde{b}(d_q(z, t))=A_{-t}(z)$ for $-1\leqq t\leqq 0$, where A_t is a null-homotopy of $a\circ b$. $\tilde{\beta}$ depends not only on β but on A_t. The formula $p(y, t)=d_Y(y, 2t-1)$ defines a shrinking map

$$p:(X\bigcup_\alpha \hat{Y}, X) \longrightarrow (SY, y_0).$$

Proposition 4.2. **i).** *Let* $\gamma\in\pi_p(S^q)$ *be an element satisfying* (4.2), *then the set of all* $\tilde{\beta}\circ E\gamma$ *coincides with the image* $i_*\{\alpha, \beta, \gamma\}$, *where* $i_*:\pi_{p+1}(X)\to\pi_{p+1}(X\bigcup_\alpha \hat{Y})$ *is the injection homomorphism.*

ii). *Let* $\tilde{\gamma}\in\pi_{p+1}(Y\bigcup_\beta e^{q+1})$ *be defined as above. Let* $\tilde{a}:Y\bigcup_\beta e^{q+1}\to X$ *be an extension of* $a\in\alpha$. *Then the set of all* $\tilde{a}_*(-\tilde{\gamma})$ *coincides with* $\{\alpha, \beta, \gamma\}$. *Remark that* $p_*(\tilde{\gamma})=E\gamma$, *and if* Y *is* $(p-q+1)$-*connected,* $p+1>q$, *then this characterizes* $\tilde{\gamma}$.

Proof. Define a homotopy $H_s:(S^{p+1}, e_0)\to(X\bigcup_\alpha \hat{Y}, x_0)$ by

$$H_s(d_p(x, t)) = \begin{cases} (B_{st}(x), (1-s)t), & 0\leqq t\leqq 1, \\ A_{-t}(c(x)), & -1\leqq t\leqq 0. \end{cases}$$

Then $H_0=\tilde{b}\circ Ec$ represents $\tilde{\beta}\circ E\gamma$ and H_1 represents $i_*\mathfrak{d}(a\circ B_t, A_t\circ c)$ of $i_*\{\alpha, \beta, \gamma\}$, thus we have i). It follows directly from the definition of the mappings $\tilde{c}$ and $\tilde{a}$ that $\tilde{a}_*(\tilde{\gamma})=\mathfrak{d}(A_t\circ c, a\circ B_t)=-\mathfrak{d}(a\circ B_t, A_t\circ c)\in-\{\alpha, \beta, \gamma\}$. The last assertion follows from Theorem II of [2]. q. e. d.

It is seen that we may take ii) as a definition of the toric construction.

Next we shall prove

Theorem 4.3. **i).** *Let* α, β *and* γ *be elements of* (4.2) *and assume that* $\gamma\circ\delta=0$ *for an element* $\delta\in\pi_n(S^p)$. *Then*

$$\alpha\circ\{\beta, \gamma, \delta\} = \{\alpha, \beta, \gamma\}\circ(-E\delta).$$

ii). *Further assume that* $\delta\circ\varepsilon=0$ *for an element* $\varepsilon\in\pi_m(S^n)$ *and that* $\alpha\circ\{\beta,\gamma,\delta\}$ *and* $\{\beta,\gamma,\delta\}\circ E\varepsilon$ *contain the zero elements. Then there exist* $\lambda\in\{\alpha,\beta,\gamma\}$, $\mu\in\{\beta,\gamma,\delta\}$ *and* $\nu\in\{\gamma,\delta,\varepsilon\}$ *such that* $\lambda\circ E\delta=\alpha\circ\mu=0$, $\beta\circ\nu=\mu\circ E\varepsilon=0$ *and the sum*

$$\{\lambda, E\delta, E\varepsilon\}+\{\alpha,\mu,E\varepsilon\}+\{\alpha,\beta,\nu\}$$

contains the zero element.

It is convenient to remember the result ii) *as follows*:

$$\{\{\alpha,\beta,\gamma\}, E\delta, E\varepsilon\}+\{a,\{\beta,\gamma,\delta\},E\varepsilon\}+\{\alpha,\beta,\{\gamma,\delta,\varepsilon\}\}\equiv 0\,.$$

Proof. i). Let a, b, c and d be representatives of α, β, γ and δ respectively and let A_t, B_t and C_t be null-homotopies of $a\circ b$, $b\circ c$ and $c\circ d$ respectively. By i), ii) and iii) of (4.1),

$$\begin{aligned}\alpha\circ\mathfrak{d}(b\circ C_t,\ B_t\circ d) &= \mathfrak{d}(a\circ b\circ C_t,\ a\circ B_t\circ d) = \mathfrak{d}(A_{st}\circ C_t,\ a\circ B_t\circ d)\\ &= \mathfrak{d}(A_t\circ C_t,\ a\circ B_t\circ d) = \mathfrak{d}(A_t\circ C_{st},\ a\circ B_t\circ d)\\ &= \mathfrak{d}(A_t\circ c\circ d,\ a\circ B_t\circ d) = \mathfrak{d}(A_t\circ c,\ a\circ B_t)\circ E\delta\\ &= -\mathfrak{d}(a\circ B_t,\ A_t\circ c)\circ E\delta\,.\end{aligned}$$

Thus $\alpha\circ\{\beta,\gamma,\delta\}$ and $\{\alpha,\beta,\gamma\}\circ(-E\delta)$ have a common element. Since $\alpha\circ\{\beta,\gamma,\delta\}$ and $\{\alpha,\beta,\gamma\}\circ(-E\delta)$ are cosets of the same subgroup $\alpha\circ(\beta\circ\pi_{n+1}(S^q)+\pi_{p+1}(Y)\circ E\delta)=\alpha\circ\pi_{p+1}(Y)\circ E\delta=(\alpha\circ\pi_{p+1}(Y)+\pi_{q+1}(X)\circ E\gamma)\circ E\delta$, then it follows the equality of i).

ii). From the assumption, there are elements μ_1 and μ_2 of $\{\beta,\gamma,\delta\}$ such that $\alpha\circ\mu_1=0$ and $\mu_2\circ E\varepsilon=0$. By Lemma 4.1, $\mu_1-\mu_2=\beta\circ\zeta+\xi\circ E\delta$ for some $\zeta\in\pi_{n+1}(S^q)$ and $\xi\in\pi_{p+1}(Y)$. By setting $\mu=\mu_1-\xi\circ E\delta=\mu_2+\beta\circ\zeta$, we have that $\alpha\circ\mu=\mu\circ E\varepsilon=0$.

Let a, b, c, d and e be representatives of α, β, γ, δ and ε respectively and let A_t and D_t be null-homotopies of $a\circ b$ and $d\circ e$ respectively. By Lemma 4.1, there are null-homotopies B_t and C_t of $b\circ c$ and $c\circ d$ such that $\mu=\mathfrak{d}(b\circ C_t, B_t\circ d)$. Denote that $\lambda=\mathfrak{d}(a\circ B_t, A_t\circ c)$ and $\nu=\mathfrak{d}(c\circ D_t, C_t\circ e)$, then $\lambda\circ E\delta=-\alpha\circ\mu=0$ and $\beta\circ\nu=-\mu\circ E\varepsilon=0$ by the above proof of i).

λ, μ and ν are represented by mappings L, M and N, respectively, given by

$$L(d_p(x,t))=\begin{cases}a(B_t(x)), & 0\leqq t\leqq 1,\\ A_{-t}(c(x)), & -1\leqq t\leqq 0,\end{cases}\quad M(d_n(y,t))=\begin{cases}b(C_t(y)), & 0\leqq t\leqq 1,\\ B_{-t}(d(y)), & -1\leqq t\leqq 0,\end{cases}$$

$$N(d_m(z,t))=\begin{cases}c(D_t(z)), & 0\leqq t\leqq 1,\\ C_{-t}(e(z)), & -1\leqq t\leqq 0.\end{cases}$$

Since $\lambda\circ E\delta=0$ and $\beta\circ\nu=0$, there exist null-homotopies $Q_s: S^{n+1}\to X$ of $L\circ Ed$ and $R_s: S^{m+1}\to Y$ of $b\circ N$. Define mappings $U, W: S^{m+2}\to X$ by

$$U(d_{m+1}(x, s)) = \begin{cases} L(ED_s(x)), & 0\leqq s\leqq 1, \\ Q_{-s}(Ee(x)), & -1\leqq s\leqq 0, \end{cases}$$

$$W(d_{m+1}(x, s)) = \begin{cases} a(R_s(x)), & 0\leqq s\leqq 1, \\ A_{-s}(N(x)), & -1\leqq s\leqq 0, \end{cases}$$

then U and W represent $\mathfrak{d}(L\circ ED_s, Q_s\circ Ee)\in\{\lambda, E\delta, E\varepsilon\}$ and $\mathfrak{d}(a\circ R_s, A_s\circ N)\in\{\alpha, \beta, \nu\}$ respectively.

For $z\in S^m$, $0\leqq r\leqq 1$ and $0\leqq\theta\leqq 2\pi$, we use the following notation:

$$(z, r, \theta) = d_{m+1}(d_m(z, r\cos\theta/\mathrm{Max}(|\cos\theta|, |\sin\theta|)), \\ r\sin\theta/\mathrm{Max}(|\cos\theta|, |\sin\theta|)).$$

By deformations of θ, U and W are homotopic to mappings U' and W', respectively, given by

$$U'(z, r, \theta) = \begin{cases} U(z, r, \theta), & 0\leqq\theta\leqq\pi/2, \\ U(z, r, \theta/2+\pi/4), & \pi/2\leqq\theta\leqq 3\pi/2, \\ U(z, r, 2\theta-2\pi), & 3\pi/2\leqq\theta\leqq 2\pi, \end{cases}$$

$$W'(z, r, \theta) = \begin{cases} W(z, r, (\theta/2-\pi/4), & -\pi/2\leqq\theta\leqq\pi/2, \\ W(z, r, 2\theta-\pi), & \pi/2\leqq\theta\leqq\pi, \\ W(z, r, \theta), & \pi\leqq\theta\leqq 3\pi/2. \end{cases}$$

Then U' and W' have the following properties. For $\pi/2\leqq\theta\leqq 3\pi/2$, $U'(z, r, \theta)=W'(z, r, \pi-\theta)$, and thus a mapping $V: S^{m+2}\to X$ given by the formula

$$V(z, r, \theta) = \begin{cases} U'(z, r, \theta), & -\pi/2\leqq\theta\leqq\pi/2, \\ W'(z, r, \theta), & \pi/2\leqq\theta\leqq 3\pi/2, \end{cases}$$

represents the sum $\mathfrak{d}(L\circ ED_s, Q_s\circ Ee)+\mathfrak{d}(a\circ R_s, A_s\circ N)$. Further, it is verified directly that for some homotopies $S_s: S^{m+1}\to Y$ and $T_s: S^{n+1}\to X$,

$$V(d_{m+1}(d_m(z, t), s)) = \begin{cases} a(S_s(d_m(z, t))), & 0\leqq s\leqq 1, \\ a(M(d_n(e(z), -t))) & s=0, \\ T_{-s}(Ee(d_m(z, t))), & -1\leqq s\leqq 0. \end{cases}$$

This indicates that V represents an element $\mathfrak{d}(a\circ S_s,\ T_s\circ Ee)$ of $\{\alpha,\ -\mu,\ E\varepsilon\} = -\{\alpha,\ \mu,\ E\varepsilon\}$. Consequently

$$0 = \mathfrak{d}(L\circ ED_s,\ Q_s\circ Ee) - \mathfrak{d}(a\circ S_s,\ T_s\circ Ee) + \mathfrak{d}(a\circ R_s,\ A_s\circ N)$$
$$\in \{\lambda,\ E\delta,\ E\varepsilon\} + \{\alpha,\ \mu,\ E\varepsilon\} + \{\alpha,\ \beta,\ \nu\}\ ,$$

and ii) is proved. q. e. d.

§ Stable toric construction.

Denote by $G_k = \pi_k(\mathfrak{S})$ the stable homotopy group of the sphere, with respect to the suspension homomorphisms $E: \pi_{N+k}(S^N) \to \pi_{N+k+1}(S^{N+1})$. G_k may be identified with $\pi_{N+k}(S^N)$ for large $N > k+1$. From the relation $E(\alpha'\circ\beta') = E\alpha'\circ E\beta'$, the composition $\alpha\circ\beta \in G_{k+h}$ is defined naturally for $\alpha \in G_h$, $\beta \in G_k$, and this operation $\circ$ is bilinear. The anti-commutativity $\beta\circ\alpha = (-1)^{hk}\alpha\circ\beta$ holds [1]. Thus $\alpha\circ G_k = G_k\circ\alpha$.

Let $\alpha \in G_h$, $\beta \in G_k$ and $\gamma \in G_l$ satisfy (4. 2). Let $\alpha' \in \pi_{N+h}(S^N)$, $\beta' \in \pi_{N+h+k}(S^{N+h})$ and $\gamma' \in \pi_{N+h+k+l}(S^{N+h+k})$ be representatives of α, β and γ. For sufficiently large N, the elements α', β' and γ' satisfy (4. 2), and the toric construction $\{\alpha',\ \beta',\ \gamma'\} \in \pi_{N+h+k+l+1}(S^N)/(\alpha'\circ\pi_{N+h+k+l+1}(S^{N+h}) + \pi_{N+h+k+1}(S^N)\circ E\gamma')$ is defined. Then *the stable toric construction*

$$\{\alpha,\ \beta,\ \gamma\} \in G_{h+k+l+1}/(\alpha\circ G_{k+l+1} + \gamma\circ G_{h+k+1})$$

is defined as a limit of $(-1)^{N-1}\{\alpha',\ \beta',\ \gamma'\}$ in virtue of iii) of (4. 3).

The properties of i), ii) of (4. 3) are valid for the stable case, by omitting the conditions in i) and the letter E in the first inclusion of ii) and taking the sign $(-1)^h$ in the last of ii).

Theorem 4. 3 is translated to the following

(4. 4). i). $\alpha\circ\{\beta,\ \gamma,\ \delta\} = (-1)^{h+1}\{\alpha,\ \beta,\ \gamma\}\circ\delta$.

ii). $\{\{\alpha,\ \beta,\ \gamma\},\ \delta,\ \varepsilon\} + (-1)^h\{\alpha,\ \{\beta,\ \gamma,\ \delta\},\ \varepsilon\} + (-1)^{h+k}\{\alpha,\ \beta,\ \{\gamma,\ \delta,\ \varepsilon\}\} \equiv 0$.

Next we shall prove the following anti-commutativity and Jacobi identity for stable toric constructions.

Theorem 4. 4. i). $\{\alpha,\ \beta,\ \gamma\} = (-1)^{hk+kl+lh+1}\{\gamma,\ \beta,\ \alpha\}$.

ii). $(-1)^{hl}\{\alpha,\ \beta,\ \gamma\} + (-1)^{kh}\{\beta,\ \gamma,\ \alpha\} + (-1)^{lk}\{\gamma,\ \alpha,\ \beta\} \equiv 0 \bmod \alpha\circ G_{k+l+1} + \beta\circ G_{h+l+1} + \gamma\circ G_{h+k+1}$.

Denote by $\iota \in G_0$ the fundamental class, the class of the identity of S^N.

Assume that $\alpha \in G_h$ has an order $r>0$. Then $r\alpha=\alpha\circ r\iota=r\iota\circ\alpha=0$, and the toric construction $\{r\iota, \alpha, r\iota\} \in G_{h+1}/rG_{h+1}$ is defined. By i) of the above theorem, $\{r\iota, \alpha, r\iota\}=-\{r\iota, \alpha, r\iota\}$ and thus $2\{r\iota, \alpha, r\iota\}\equiv 0 \bmod rG_{h+1}$. We have

Corollary 4.5. *If $r\alpha=0$ for $\alpha\in G_h$ and for an odd integer r, then*

$$\{r\iota, \alpha, r\iota\} = rG_{h+1}.$$

According to [5], we use the following notations and properties of reduced joins. Denote by $a\divideontimes b:(S^{p+h+q+k}, e_0)\to(S^{p+q}, e_0)$ the reduced join of mappings $a:(S^{p+h}, e_0)\to(S^p, e_0)$ and $b:(S^{q+k}, e_0)\to(S^q, e_0)$ and by $\sigma_{p,q}:(S^{p+q}, e_0)\to(S^{p+q}, e_0)$ a homeomorphism of degree $(-1)^{pq}$ such that

$$b\divideontimes a = \sigma_{p,q}\circ(a\divideontimes b)\circ\sigma_{q+k,p+h} \quad\text{and}\quad \sigma_{q,p}=\sigma_{p,q}^{-1}.$$

The following relation holds:

$$(a\divideontimes b)\circ(a'\divideontimes b') = (a\circ a')\divideontimes(b\circ b').$$

Denote by i_q the identity of S^q, then $a\divideontimes i_q=E^q a$ is the q-fold iterated suspension of a. Let α' be the class of a, then $a\divideontimes i_q$ represents $E^q\alpha'$ and $i_q\divideontimes a=\sigma_{p,q}\circ(a\divideontimes i_q)\circ\sigma_{q,p+h}$ represents $(-1)^{hq}E^q\alpha'$. Let β' be the class of b, then the reduced join $\alpha'\divideontimes\beta'$ is the class of $a\divideontimes b$. From the equalities

$$\begin{aligned} a\divideontimes b &= (a\circ i_{p+h})\divideontimes(i_q\circ b) = (a\divideontimes i_q)\circ(i_{p+h}\divideontimes b) \\ &= (i_p\circ a)\divideontimes(b\circ i_{q+k}) = (i_p\divideontimes b)\circ(a\divideontimes i_{q+k}), \end{aligned}$$

we have the equality

$$\alpha'\divideontimes\beta' = (-1)^{(p+h)k}E^q\alpha'\circ E^{p+h}\beta' = (-1)^{pk}E^p\beta'\circ E^{q+k}\alpha'$$

and thus the anti-commutativity of the composition in the stable groups.

Now Theorem 4.4 is a direct consequence of the following

Proposition 4.6. i). *Assume that $\alpha'\in\pi_{p+h}(S^p)$, $\beta'\in\pi_{q+k}(S^q)$ and $\gamma'\in\pi_{r+l}(S^r)$ satisfy the conditions $\alpha'\divideontimes\beta'=0$ and $\beta'\divideontimes\gamma'=0$, then $\{E^{q+r}\alpha', E^{p+h+r}\beta', E^{p+h+q+k}\gamma'\}$ and $(-1)^{hk+kl+lh+1}\{E^{p+q}\gamma', E^{p+r+l}\beta', E^{q+k+r+l}\alpha'\}$ have a common element.*

ii). *Further assume that* $\alpha' \divideontimes \gamma' = 0$, *then* $(-1)^{hl}\{E^{q+r}\alpha', E^{p+h+r}\beta', E^{p+h+q+k}\gamma'\} + (-1)^{kh}\{E^{p+r}\beta', E^{p+q+k}\gamma' E^{q+k+r+l}\alpha'\} + (-1)^{lk}\{E^{p+q}\gamma', E^{q+r+l}\alpha', E^{p+h+r+l}\beta'\}$ *contains the zero element.*

Proof. i). Let a, b and c be representatives of α', β' and γ' respectively, and let A_t and B_t be null-homotopies of $a \divideontimes b$ and $b \divideontimes c$ respectively. Since $a \divideontimes i_q \divideontimes i_r$, $i_{p+h} \divideontimes b \divideontimes i_r$ and $i_{p+h} \divideontimes i_{q+k} \divideontimes c$ represent $E^{q+r}\alpha'$, $(-1)^{(p+h)k}E^{p+h+r}\beta'$ and $(-1)^{(p+h+q+k)l}E^{p+h+q+k}\gamma'$ respectively and since $A_t \divideontimes i_r$ and $i_{p+h} \divideontimes B_t$ are null-homotopies of $(a \divideontimes i_q \divideontimes i_r)\circ(i_{p+h} \divideontimes b \divideontimes i_r)$ and $(i_{p+h} \divideontimes b \divideontimes i_r)\circ(i_{p+h} \divideontimes i_{q+k} \divideontimes c)$ respectively, it follows that $\mathfrak{d}(a \divideontimes B_t, A_t \divideontimes c) = \mathfrak{d}((a \divideontimes i_{q+r})\circ(i_{p+h} \divideontimes B_t), (A_t \divideontimes i_r)\circ(i_{p+h+q+k} \divideontimes c))$ belongs to $\{E^{q+r}\alpha', (-1)^{(p+h)k}E^{p+h+r}\beta', (-1)^{(p+h+q+k)l}E^{p+h+q+k}\gamma'\}$. Similarly, $\mathfrak{d}(a \divideontimes B_t, A_t \divideontimes c) = -\mathfrak{d}(A_t \divideontimes c, a \divideontimes B_t) = -\mathfrak{d}((i_{p+q} \divideontimes c)\circ(A_t \divideontimes i_{r+l}), (i_p \divideontimes B_t)\circ(a \divideontimes i_{q+k+r+l}))$ belongs to $-\{(-1)^{(p+q)l}E^{p+q}\gamma', (-1)^{pk}E^{p+r+l}\beta', E^{q+k+r+l}\alpha'\}$. Therefore $\{E^{q+r}\alpha', E^{p+h+r}\beta', E^{p+h+q+k}\gamma'\}$ and $(-1)^{\varepsilon}\{E^{p+q}\gamma', E^{p+r+l}\beta', E^{q+k+r+l}\alpha'\}$ have a common element for $\varepsilon = (p+h)k + (p+h+q+k)l + 1 + (p+q)l + pk \equiv hk + kl + lh + 1 \pmod 2$.

ii). Let C_t be a null-homotopy of $a \divideontimes c$. Then $C_t' = (i_p \divideontimes \sigma_{r,q})\circ(C_t \divideontimes i_q)\circ(i_{p+h} \divideontimes \sigma_{q,r+l})$ and $C_t'' = (i_p \divideontimes \sigma_{r,q+k})\circ(C_t \divideontimes i_{q+k})\circ(i_{p+h} \divideontimes \sigma_{q+k,r+l})$ are null-homotopies of $a \divideontimes i_q \divideontimes c = (i_p \divideontimes i_q \divideontimes c)\circ(a \divideontimes i_q \divideontimes i_{r+l})$ and $a \divideontimes i_{q+k} \divideontimes c = (i_p \divideontimes i_{q+k} \divideontimes c)\circ(a \divideontimes i_{q+k} \divideontimes i_{r+l})$ respectively. It follows that $\mathfrak{d}(A_t \divideontimes c, C_t'\circ(i_{p+h} \divideontimes b \divideontimes i_{r+l}))$ and $\mathfrak{d}((i_p \divideontimes b \divideontimes i_r)\circ C_t'', a \divideontimes B_t)$ belong to $\{(-1)^{(p+q)l}E^{p+q}\gamma', E^{q+r+l}\alpha', (-1)^{(p+h)k}E^{p+h+r+l}\beta'\}$ and $\{(-1)^{pk}E^{p+r}\beta', (-1)^{(p+q+k)l}E^{p+q+k}\gamma', E^{q+r+k+l}\alpha'\}$ respectively. Since $C_t'\circ(i_{p+h} \divideontimes b \divideontimes i_{r+l}) = (i_p \divideontimes \sigma_{r,q})\circ(C_t \divideontimes b)\circ(i_{p+h} \divideontimes \sigma_{q+k,r+l}) = (i_p \divideontimes b \divideontimes i_r)\circ C_t''$, then $\mathfrak{d}(a \divideontimes B_t, A_t \divideontimes c) + \mathfrak{d}(A_t \divideontimes c, C_t'\circ(i_{p+h} \divideontimes b \divideontimes i_{r+l})) + \mathfrak{d}((i_p \divideontimes b \divideontimes i_r)\circ C_t'', a \divideontimes B_t) = 0$ by i) of (4.1) and this element is contained in $(-1)^{(p+q+k)l+(p+h)k}[(-1)^{hl}\{E^{q+r}\alpha', E^{p+h+r}\beta', E^{p+h+q+k}\gamma'\} + (-1)^{kl}\{E^{p+q}\gamma', E^{q+r+l}\alpha', E^{p+h+r+l}\beta'\} + (-1)^{hk}\{E^{p+r}\beta', E^{p+q+k}\gamma', E^{q+k+r+l}\alpha'\}]$. Thus ii) is proved. q. e. d.

We have also

Proposition 4.7. *Assume that* $\alpha \in \pi_{p+h}(S^p)$, $\beta \in \pi_{p+h+k}(S^{p+h})$ *and* $\gamma \in \pi_{r+l}(S^r)$ *satisfy the conditions* $\alpha\circ\beta = 0$, $\alpha \divideontimes \gamma = 0$ *and* $\beta \divideontimes \gamma = 0$, *then* $\{E^r\alpha, E^r\beta, E^{p+h+k}\gamma\} - (-1)^{kl}\{E^r\alpha, E^{p+h}\gamma, E^{r+l}\beta\} + (-1)^{(h+k)l}\{E^p\gamma, E^{r+l}\alpha, E^{r+l}\beta\}$ *contains the zero element.*

Proof. Let a, b and c be representatives of α, β and γ respectively, and let A_t, B_t and C_t be null-homotopies of $a\circ b$, $b\times c$ and $a\times c$ respectively. By (4.1), $\mathfrak{d}((a\times i_r)\circ B_t,\ A_t\times c)+\mathfrak{d}(A_t\times c,\ C_t\circ(b\times i_{r+l}))=\mathfrak{d}((a\times i_r)\circ B_t,\ C_t\circ(b\times i_{r+l}))$. As is seen in the previous proof, the each terms of the equality represent $\{\alpha\times\iota_r,\ \beta\times\iota_r,\ \iota_{p+h+k}\times\gamma\}=(-1)^{(p+h+k)l}\{E^r\alpha,\ E^r\beta,\ E^{p+h+k}\gamma\}$, $\{\iota_p\times\gamma,\ \alpha\times\iota_{r+l},\ \beta\times\iota_{r+l}\}=(-1)^{pl}\{E^p\gamma,\ E^{r+l}\alpha,\ E^{r+l}\beta\}$ and $\{\alpha\times\iota_r,\ \iota_{p+h}\times\gamma,\ \beta\times\iota_{r+l}\}=(-1)^{(p+h)l}\{E^r\alpha,\ E^{p+h}\gamma,\ E^{r+l}\beta\}$ respectively. Then the proposition is proved. q. e. d.

Consider an element $\alpha\in G_h$. Let $a: S^{N+h}\to S^N$ be a representative of α with respect to given orientations of the spheres. Let E^{N+h+1} be a cube bounded by S^{N+h} and it is oriented coherently with S^{N+h}. Let $K=S^N\cup e^{N+h+1}$ be a cell complex having a characteristic map $A:(E^{N+h+1},\ S^{N+h})\to(K,\ S^N)$ of e^{N+h+1} such that $A|S^{N+h}=a$. We orient e^{N+h+1} such that A preserves the orientations. Now we call the element α *the characteristic class* of K with respect to the given orientations. α depends on the homotopy type, by orientation preserving equivalences, of K.

Lemma 4.8. i). *Let $\alpha\in G_h$ and $\beta\in G_k$. The composition $\alpha\circ\beta$ is zero if and only if, for a complex $K=S^N\cup e^{N+k+1}$ having β as the characteristic class, there exists a mapping F of K into S^{N-h} such that $F|S^N$ represents α.*

ii). *Let $\alpha\in G_h$, $\beta\in G_k$ and $\gamma\in G_l$. $\alpha\circ\beta=0$, $\beta\circ\gamma=0$ and $\{\alpha,\beta,\gamma\}\equiv 0$ (mod $\alpha\circ G_{k+l+1}+\gamma\circ G_{h+k+1}$) if and only if there exist a cell complex $K=S^N\cup e^{N+k+1}\cup e^{N+k+l+2}$ and a mapping F of K into S^{N-h} satisfying the following conditions (N: large). $F|S^N$ represents α. The characteristic class of the subcomplex $S^N\cup e^{N+k+1}$ is β. By shrinking S^N to a point, we have a complex $K/S^N=S^{N+k+1}\cup e^{N+k+l+2}$ having the characteristic class γ.*

iii). *Let $\alpha\in G_h$, $\beta\in G_k$, $\gamma\in G_l$, $\beta'\in G_{k'}$, $\gamma'\in G_{l'}$ and $k+l=k'+l'$. Assume that $\beta\circ\gamma=\beta'\circ\gamma'=0$. Then $\alpha\circ\beta=0, \alpha\circ\beta'=0$ and $\{\alpha,\beta,\gamma\}+\{\alpha,\beta',\gamma'\}\equiv 0$ (mod $\alpha\circ G_{k+l+1}+\gamma\circ G_{h+k+1}+\gamma'\circ G_{h+k'+1}$) if and only if there exist a cell complex $K=S^N\cup e^{N+k+1}\cup e^{N+k'+1}\cup e^{N+k+l+2}$ and a mapping F of K into S^{N-h} satisfying the following conditions. $F|S^N$ represents α. $L=S^N\cup e^{N+k+1}$ and $L'=S^N\cup e^{N+k'+1}$ are subcomplexes having the characteristic classes β and β' respectively. By shrinking L or L', we have complexes $K/L=S^{N+k'+1}\cup$*

$e^{N+k+l+2}$ and $K/L'=S^{N+k+1}\cup e^{N+k+l+2}$ having the characteristic classes γ' and γ respectively.

Proof. i) is obvious. ii) follows from ii) of Proposition 4.2 easily.

We prove iii). Assume that $\{\alpha, \beta, \gamma\}+\{\alpha, \beta', \gamma'\}\equiv 0$, then it follows from Lemma 4.1 that there exist elements $\lambda\in\{\alpha, \beta, \gamma\}$ and $\mu\in\{\alpha, \beta', \gamma'\}$ such that $\lambda+\mu=0$. By Proposition 4.2, ii), $-\lambda$ is represented by a composition $A\circ C: S^{N+k+l+1}\rightarrow L\rightarrow S^{N-h}$ such that $A|S^N$ represents α and, by shrinking S^N of L to a point, C represents γ. Similarly $-\mu$ is represented by a composition $A'\circ C': S^{N+k+l+1}\rightarrow L'\rightarrow S^{N-h}$ such that $A|S^N=A'|S^N$ and, by shrinking S^N of L', C' respresents γ'. By setting $F|L=A$ and $F|L'=A'$, we have a mapping F of $L\cup L'$ into S^{N-h}. Construct a complex $K=L\cup L'\cup e^{N+k+l+2}$ by attaching a cell by a mapping which represents the sum of the classes of C and C'. Since the composition of this attaching map and F represents $-\lambda-\mu=0$, it follows that the mapping F can be extended over K. The complex K and the mapping F satisfy the conditions of the lemma. Then the existence of K and F is proved.

Conversely, assume the existence of such a complex K and a mapping F. Let $B: S^{N+k+l+1}\rightarrow L\cup L'$ be the attaching map of $e^{N+k+l+2}$, and let $\{B\}$ be the homotopy class of B. Since $\beta\circ\gamma=\beta'\circ\gamma'=0$, there exist mappings C and C' as above. For sufficiently large N, by shrinking S^N to a point, we have an isomorphism of $\pi_{N+k+l+1}(L\cup L', S^N)$ onto $\pi_{N+k+l+1}(L/S^N)+\pi_{N+k+l+1}(L'/S^N)$. Then it follows from this that $j_*\{B\}=j_*(\{C\}+\{C'\})$ for the injection homomorphism $j_*: \pi_{N+k+l+1}(L\cup L')\rightarrow\pi_{N+k+l+1}(L\cup L', S^N)$. By the exactness of the homotopy sequence of the pair $(L\cup L', S^N)$, it follows that $\{B\}=\{C\}+\{C'\}+\{D\}$ for a mapping $D: S^{N+k+l+1}\rightarrow S^N$. Consider $F_*: \pi_{N+k+l+1}(L\cup L')\rightarrow\pi_{N+k+l+1}(S^N)$. $F|e^{N+k+l+2}$ gives a null-homotopy of $F\circ B$. Thus $F_*\{B\}=0$. By Proposition 4.2, ii), $F_*\{C\}\in-\{\alpha, \beta, \gamma\}$ and $F_*\{C'\}\in-\{\alpha, \beta', \gamma'\}$. Obviously $F_*\{D\}$ represents an element of $\alpha\circ G_{k+l+1}$. Therefore the relation $\{\alpha, \beta, \gamma\}+\{\alpha, \beta', \gamma'\}\equiv 0$ is obtained. q.e.d.

Consider a complex $K=S^N\cup e^{N+h+1}$ having a characteristic class $\alpha\in G_h$. Let $EK=S^{N+1}\cup e^{N+h+2}$ be a suspension of K and orient the cells of EK by the suspension classes of the orientations in K. Then the characteristic class of EK with respect to these orientations is $-\alpha$, because of the relation $E\partial=-\partial E$. The mod p Hopf invariant H_p of II [6] will be modified as follows for the stable case. For an element α of G_{2p-3}, the mod p Hopf invariant

$$H_p(\alpha)\in Z_p$$

is defined by the formula $\mathcal{P}^1 u=(-1)^{N-1}H_p(\alpha)\cdot u'$, where $u\in H^N(K, Z_p)$ and $u'\in H^{N+2p-2}(K, Z_p)$ are given by the orientations of a complex $K=S^N\cup e^{N+2p-2}$ having α as the characteristic class with respect to these orientations.

By Lemma 2.1 and Corollary 2.3, the homomophism $H_p: G_{2p-3}\to Z_p$ is onto. Since G_{2p-3} has Z_p as its p-component (Theorem 3.13), H_p is an isomorphism of the p-components. Therefore an element of order p

$$\alpha_1\in G_{2p-3}$$

is determined uniquely by the following condition

(4.5). $\quad H_p(\alpha_1)\equiv 1 \pmod p$ *and* $p\cdot\alpha_1=0$.

If $\mathcal{P}^1 u=x\cdot u'$ in a complex $K=S^N\cup e^{N+2p-2}$ of a characteristic class α, then $\alpha\equiv(-1)^{N-1}x\cdot\alpha_1 \pmod p$. Similar discussion shows that if $\Delta_r u=x\cdot u'$ in a complex $K=S^N\cup e^{N+1}$ of a characteristic class α, then $\alpha\equiv(-1)^N x\cdot p^r\iota \pmod{p^{r+1}}$, where $\iota\in G_0$ is the class of the identity of S^N.

§ Auxiliary results from III.

S. Mukohda has pointed out that in the proof of Theorem 3.10, we dropped a possibility of

$$\mathcal{P}^1 b^{(p-2)}_{p-1}=x\cdot b^{(0)}_p$$

for some coefficient x and it seems that the coefficient x cannot be determined by purely cohomological methods as in III. Then we may calculate the following two possibilities.

(4.6). *If* $\mathcal{P}^1 b^{(p-2)}_{p-1}=0$, *then Theorem* 3.10 *and Theorem* 3.13 *are true. If* $\mathcal{P}^1 b^{(p-2)}_{p-1}\neq 0$, *then* $\pi_{2(p^2-1)(p-1)-3}(\mathfrak{S}\,;p)=\pi_{2(p^2-1)(p-1)-2}(\mathfrak{S}\,;p)=0$.

It will be proved, however, in the next § that $\pi_{2(p^2-1)(p-1)-2}(\mathfrak{S};p)$ contains an element of order $p:\beta_1\circ\cdots\circ\beta_1$, using the results on $A^*(K_k, Z_p)$ for $k\leq 2(p-1)p(p-1)-1$ (Theorem 3.10) which do not depend on the value of x. Therefore the case $\mathcal{P}^1 b^{(p-2)}_{p-1} \neq 0$ fails and we conclude that

$$\mathcal{P}^1 b^{(p-2)}_{p-1} = 0$$

and Theorem 3.10 and Theorem 3.13 are true entirely.

Let N be a sufficiently large integer. Consider the sequence $K_1 \supset \cdots \supset K_{k-1} \supset K_k \supset \cdots \supset S^N$ of III, p. 192, and let X_k be the space of paths in K_k starting from a point and ending in S^N. Let $p_k: X_k \to S^N$ be the projection assigning to each path the end point, then it is easily seen that X_k is a $(N+k-1)$-connective fibre space over S^N. Obviously, $X_k \subset X_j$ and $p_k = p_j | X_k$ if $k > j$. Let S' be the space of paths starting in K_k and ending in S^N, then S' has S^N as a deformation retract and S' is a fibre space over K_k of a fibre X_k. It is verified easily that the transgression

$$(4.7). \qquad \tau = p^{*-1} \circ \delta : H^i(X_k, Z_p) \xrightarrow{\delta} H^{i+1}(S', X_k, Z_p) \xleftarrow{p^*} H^{i+1}(K_k, Z_p)$$

is an isomorphism for $N \leqq i \leqq 2N+k-1$ and that the following diagram is commutative.

$$(4.8). \qquad \begin{array}{ccc} H^i(X_j, Z_p) & \xrightarrow{\tau} & H^{i+1}(K_j, Z_p) \\ i^* \downarrow & & \downarrow i^* \\ H^i(X_k, Z_p) & \xrightarrow{\tau} & H^{i+1}(K_k, Z_p), \end{array} \qquad k > j.$$

We identify $H^{N+i}(X_k, Z_p)$ with $H^{N+i+1}(K_k, Z_p) = A^{i+1}(K_k, Z_p)$ by the isomorphism (4.7), and we denote the corresponding element by the same symbol: $a_t = \tau^{-1}(a_t)$, $b^{(s)}_r = \tau^{-1}(b^{(s)}_r)$, etc., and further $a_1 = \tau^{-1}(\mathcal{P}^1 a_0)$. Since the transgression commutes with the cohomological operations $\mathcal{P}^r$ and Δ_r, the relations in $H^*(X_k, Z_p)$ are carried from those of $H^*(K_k, Z_p)$. We recall some necessary results from Theorem 3.10 as follows.

(4.9). i). *Let* $k = 2(rp+s)(p-1) - 2(r-s)$ *and* $0 \leqq s < r < p$, *then* $H^{N+k}(X_k, Z_p) = Z_p = \{b^{(s)}_r\}$ *and* $\mathcal{P}^1 b^{(s)}_r = 0$, *where the relation* $\mathcal{P}^1 b^{(p-2)}_{p-1} = 0$ *will be valid after proved* $\pi_{2(p^2-1)(p-1)-2}(\mathfrak{S}; p) \neq 0$.

ii). *Let* $k = 2(rp+s+1)(p-1) - 2(r-s) - 1$, $0 \leqq s < r < p$ *and* $(r, s) \neq (p-1, 0)$, *then* $H^{N+k}(X_k, Z_p) = Z_p = \{c^{(s)}_r\}$ *and* $\mathcal{P}^{p-1} c^{(s)}_r = 0$.

iii). *Let* $k=2t(p-1)-1$, $0<t<p^2$ *and* $t\not\equiv 0 \pmod p$, *then*

$$H^{N+k+i}(X_k,\ Z_p)=\begin{cases} Z_p=\{a_t\} & \text{for } i=0\,,\\ Z_p=\{\Delta a_t\} & \text{for } i=1 \text{ and } t\neq(p-1)p-1\,,\\ Z_p+Z_p=\{a_t,\ b^{(0)}_{p-1}\} & \text{for } i=1 \text{ and } t=(p-1)p-1\,,\\ Z_p=\{\mathcal{P}^1 a_t\} & \text{for } i=2p-2\,,\\ Z_p=\{\mathcal{P}^1\Delta a_t\} & \text{for } i=2p-1\,,\end{cases}$$

$R_t a_t=(t+1)\mathcal{P}^1\Delta a_t-t\Delta\mathcal{P}^1 a_t=0$ *and* $\mathcal{P}^1 b^{(0)}_{p-1}=0$.

iv). *Let* $k=2rp(p-1)-1$ *and* $0<r<p-1$, *then*

$$H^{N+k+i}(X_k,\ Z_p)=\begin{cases} Z_p=\{a_{rp}\} & \text{for } i=0\,,\\ Z_p=\{a'_{rp}\} & \text{for } i=1\,,\\ Z_p=\{\mathcal{P}^1 a_{rp}\} & \text{for } i=2p-2\,,\\ Z_p=\{\mathcal{P}^1 a'_{rp}\} & \text{for } i=2p-1\,,\end{cases}$$

$\Delta a_{rp}=\Delta a'_{rp}=0$ *and* $\Delta\mathcal{P}^1 a_{rp}=\mathcal{P}^1 a'_{rp}$.

v). *Let* $k=2(p-1)p(p-1)-1$, *then*

$$H^{N+k+i}(X_k,\ Z_p)=\begin{cases} Z_p+Z_p=\{a_{(p-1)p},\ c^{(0)}_{p-1}\} & \text{for } i=0\,,\\ Z_p+Z_p=\{a'_{(p-1)p},\ \Delta c^{(0)}_{p-1}\} & \text{for } i=1\,,\\ Z_p+Z_p=\{\mathcal{P}^1 a_{(p-1)p},\ \mathcal{P}^1 c^{(0)}_{p-1}\} & \text{for } i=2p-2\,,\\ Z_p+Z_p+Z_p=\{\mathcal{P}^1 a'_{(p-1)p},\ \Delta\mathcal{P}^1 c^{(0)}_{p-1},\ \mathcal{P}^1\Delta c^{(0)}_{p-1}\} & \text{for } i=2p-1\,,\end{cases}$$

$\Delta a_{(p-1)p}=\Delta a'_{(p-1)p}=0$ *and* $\Delta\mathcal{P}^1 a_{(p-1)p}=\mathcal{P}^1 a'_{(p-1)p}$.

Next we add the following

Lemma 4.9. *Let* $1\leqq r<p$, *then* $\Delta_2(\mathcal{P}^1 a_{rp-1})=-r(\mathcal{P}^1\Delta a_{rp-1})$ *and* $\Delta_2(a_{rp})=r(a'_{rp})$.

Proof. By Lemma 3.12 and (3.14), for $2\leqq t\leqq rp$,

$$\Delta_2(c\,\mathcal{P}^{rp-t}a_t)=\begin{cases} x_{r,t}c\mathcal{P}^{rp-t}\Delta a_t\,, & t\not\equiv 0 \pmod p\,,\\ x_{r,t}c\mathcal{P}^{rp-t}a'_t\,, & t\equiv 0 \pmod p\,,\end{cases}$$

for some coefficients $x_{r,t}\neq 0$. According to the proof of Lemma 3.12, we see that $\Delta_2(c\,\mathcal{P}^{rp-2}a_2)=\Delta_2(rc\,\mathcal{P}^{rp}a_0)=(1/2)c\,\mathcal{P}^{rp-2}\Delta a_2$, where there is a misprint of the above first equality in the proof. Then $x_{r,2}=r/2$. Now compare $x_{r,t}$ and $x_{r,t+1}$ using Lemma 3.5, iii). First consider the case $t\not\equiv -1, 0 \pmod p$. It is calculated directly that $c\mathcal{P}^{rp-t-1}R_t = t\Delta c\mathcal{P}^{rp-t}$ and $c\mathcal{P}^{rp-t-1}\Delta R_t = -(t+1)\Delta c\mathcal{P}^{rp-t}\Delta$. It follows from $\delta^*(a_{t-1}) = R_t j^{*-1}(a_t)$ that $\Delta(c\mathcal{P}^{rp-t}j^{*-1}(a_t)) = \delta^*((1/t)c\mathcal{P}^{rp-t-1}a_{t+1})$ and $\Delta(c\mathcal{P}^{rp-t}\Delta j^{*-1}(a_t)) = \delta^*(-(1/(t+1))c\mathcal{P}^{rp-t-1}\Delta a_{t+1})$. Applying Lemma 3.5, iii), we have the equality $(1/t)x_{r,t+1} = (-1/(t+1))x_{r,t}$. Thus $tx_{r,t} = -(t+1)x_{r,t+1}$ for the case $t \not\equiv 0, -1 (\mathrm{mod}\ p)$. For the case $t \equiv -1 (\mathrm{mod}\ p)$, we have $\Delta(c\mathcal{P}^{rp-t}j^{*-1}(a_t)) = \delta^*(-c\mathcal{P}^{rp-t-1}a_{t+1})$ and $\Delta(c\mathcal{P}^{rp-t}\Delta j^{*-1}(a_t)) = \delta^*(c\mathcal{P}^{rp-t-1}a'_{t+1})$. Then, by Lemma 3.5, iii), $x_{r,t} = -x_{r,t+1}$ for $t \equiv -1 (\mathrm{mod}\ p)$. Similarly, from $\Delta(c\mathcal{P}^{rp-t}j^{*-1}(a_t)) = \delta^*(-c\mathcal{P}^{rp-t-1}a_{t+1})$ and $\Delta(c\mathcal{P}^{rp-t}j^{*-1}(a'_t)) = \delta^*(c\mathcal{P}^{rp-t-1}\Delta a_{t+1})$, $t \equiv 0 (\mathrm{mod}\ p)$, it follows that $x_{r,t} = -x_{r,t+1}$ for $t \equiv 0 (\mathrm{mod}\ p)$. Consequently,

$$r = 2x_{r,2} = (-3x_{r,3} =) \cdots = (p-1)x_{r,p-1} = x_{r,p} = -x_{r,p+1} = 2x_{r,p+2} = \cdots$$
$$= (p-1)x_{r,rp-1} = x_{r,rp}. \qquad \text{q. e. d.}$$

§ Generators of stable groups given by compositions.

By Theorem 3.13, the p-component $\pi_{2(sp+s-1)(p-1)-2}(\mathfrak{S}; p)$ of $G_{2(sp+s-1)(p-1)-2}$ is Z_p for $1 \leqq s \leqq p-1$. Denote by

$$\beta_s \in G_{2(sp+s-1)(p-1)-2}, \quad s = 1, 2, \cdots, p-1,$$

an element of order p.

Lemma 4.10. i). *For $1 \leqq k \leqq p-1$ and for sufficiently large N, there exists a cell complex $L_k^N = S^N \cup e^{N+2(p-1)} \cup \cdots \cup e^{N+2k(p-1)}$ such that $\mathcal{P}^k H^N(L_k^N, Z_p) = H^{N+2k(p-1)}(L_k^N, Z_p)$ and the class of the attaching map of each cell has an order of a power of p.*

ii). *Let $A: L_{p-2}^{N+2p-3} \to S^N$ be an extension of a representative $A|S^{N+2p-3}$ of α_1 and let $B: S^{N+2p(p-1)-2} \to L_{p-2}^{N+2p-3}$ be the attaching map of $e^{N+2p(p-1)-1} = L_{p-1}^{N+2p-3} - L_{p-2}^{N+2p-3}$. Then the composition $A \circ B: S^{N+2p(p-1)-2} \to S^N$ represents an element of order p (N: large): $x\beta_1$, $x \not\equiv 0$ (mod p).*

Proof. i) First we see that the p-component of $\pi_{N+2k(p-1)-2}(L_h^N)$ vanishes for $k, h \leqq p-1$, in the complex L_h^N as above. This follows from the exact sequence $\pi_{N+2k(p-1)-2}(L_{h-1}^N) \to \pi_{N+2k(p-1)-2}(L_h^N) \to \pi_{N+2k(p-1)-2}(L_h^N, L_{h-1}^N) \approx G_{2(k-h)(p-1)-2}$ and from Theorem 3.13. Next we remark that $\pi_{N+i}(L_h^N)$ is finite for $i > 2h(p-1)$, because of the finiteness of G_i, $i \neq 0$. L_1^N exists and it has $x\alpha_1$, $x \not\equiv 0$, as the characteristic class. Assume that we have proved the existence of complex $L_{k-1}^N = L_{k-2}^N \cup e^{N+2(k-1)(p-1)}$ satisfying the condition of i). Let $p: L_{k-1}^N \to S^{N+2(k-1)(p-1)}$ be a mapping which shrinks L_{k-2}^N to a single point, then $p_*: \pi_{N+2k(p-1)-1}(L_{k-1}^N, L_{k-2}^N) \approx \pi_{N+2k(p-1)-1}(S^{N+2(k-1)(p-1)}) \approx G_{2p-3}$. $\partial(p_*^{-1}(\alpha_1))$ belongs to the p-component of $\pi_{N+2k(p-1)-2}(L_{k-1}^N)$, which vanishes. Then $\partial(p_*^{-1}(\alpha_1))=0$ and there exists an element α of the p-component of $\pi_{N+2k(p-1)-1}(L_{k-1}^N)$ such that $j_*\alpha = p_*^{-1}(\alpha_1)$. By attaching a cell $e^{N+2k(p-1)}$ to L_{k-1}^N with α, we have a complex $L_k^N = L_{k-1}^N \cup e^{N+2k(p-1)}$. By shrinking L_{k-2}^N of L_k^N we have a complex of a characteristic class α_1. Then $\mathcal{P}^1 \neq 0$ in this complex. Concerning the shrinking map, it follows that $\mathcal{P}^1 H^{N+2(k-1)(p-1)} \neq 0$ in L_k^N. Therefore $\mathcal{P}^k = (1/k)\mathcal{P}^1\mathcal{P}^{k-1} \neq 0$ in L_k^N, and i) is proved by the induction on k.

ii) The existence of A follows from that the class of the attaching map of $e^{N+2p-3+2k(p-1)}$ is carried into the p-component of $\pi_{N+2p-3+2k(p-1)-1}(S^N)$ which vanishes for $1 \leqq k \leqq p-2$. Since B represents an element of the p-component, $A \circ B$ represents an element of $\pi_{N+2p(p-1)-2}(S^N; p) = Z_p$. Now assume that $A \circ B$ represents the zero. Then A can be extended over $\bar{A}: L_{p-1}^{N+2p-3} \to S^N$. Let $Z = S^N \cup L_{p-1}^{N+2p-3} \times (0, 1]$ be a mapping-cylinder of $\bar{A}$ and let $Z^* = S^N \cup e^{N+2p-2} \cup \cdots \cup e^{N+2p(p-1)}$ be a complex obtained from Z by shrinking $L_{p-1}^{N+2p-3} \times (1) \cup e_0 \times [0, 1]$. Since e^{N+2p-2} is attached by α_1, then $\mathcal{P}^1 H^N(Z^*, Z_p) = H^{N+2p-2}(Z^*, Z_p)$. It follows from i) that $\mathcal{P}^{p-1} H^{N+2p-2}(Z^*, Z_p) = H^{N+2p(p-1)}(Z^*, Z_p)$. Thus $\mathcal{P}^{p-1}\mathcal{P}^1 \neq 0$ in Z^*. But this contradicts to Adem's relation $\mathcal{P}^{p-1}\mathcal{P}^1 = 0$. Therefore $A \circ B$ does not represent the zero and it represents a generator of $\pi_{N+2p(p-1)-2}(S^N; p)$. q. e. d.

Now this § is devoted to prove the following theorem. For the simplicity, we denote by

$$\beta_1^t = \overbrace{\beta_1 \circ \cdots \circ \beta_1}^{t} \in G_{2tp(p-1)-2t}$$

the t-fold iterated composition of β_1 ($\beta_1^0 = \iota$). Obviously $p(\beta_1^t \circ \beta_s) = p(\alpha_1 \circ \beta_1^t \circ \beta_s) = 0$.

Theorem 4.11. *If* $1 \leqq s \leqq s+t < p$, *then* $\alpha_1 \circ \beta_1^t \circ \beta_s \neq 0$ *and thus* $\beta_1^t \circ \beta_s \neq 0$. $\alpha_1 \circ \beta_1^{p-1}$ *is not divisible by* p *and* $\beta_1^p \neq 0$. $\alpha_1 \circ \beta_1^p \neq 0$ *if and only if* $\pi_{2p^2(p-1)-3}(\mathfrak{S}; p) \neq 0$.

Before beginning the proof, we remark

(4.10). *Let K be a finite cell complex and let L be its subcomplex. Assume that* $\dim(K-L)\geqq N+k$ *and there are mappings* $f: K\rightarrow S^N$ *and* $g: L\rightarrow X_k$ *such that* $p_k\circ g=f|L$. *Then there exists an extension* $\bar{g}: K\rightarrow X_k$ *of g such that* $p_k\circ\bar{g}=f$.

This follows from the covering homotopy theorem for the fibering $p_k: X_k\rightarrow S^N$ and from the fact that the homotopy groups of the fibre vanish for dimensions greater than $N+k-2$.

First we prove

a). If $1\leqq s\leqq s+t<p$ *and* $(s, t)\neq(p-1, 0)$, *then* $\beta_1^t\circ\beta_s\neq 0$ *implies* $\alpha_1\circ\beta_1^t\circ\beta_s\neq 0$.

Let $k=2((s+t)p+s-1)(p-1)-2(t+1)$ and let $f: S^{N+k}\rightarrow S^N$ be a representative of $\beta_1^t\circ\beta_s$. Assume that $\alpha_1\circ\beta_1^t\circ\beta_s=\beta_1^t\circ\beta_s\circ\alpha_1=0$. Then, by Lemma 4.8, i), there exists a complex $K=S^{N+k}\cup e^{N+k+2p-2}$ having a characteristic class α_1 and there exists an extension $F: K\rightarrow S^N$ of f. $\mathcal{P}^1\neq 0$ in K. By (4.10), there exists a mapping $\bar{F}: K\rightarrow X_k$ such that $p_k\circ\bar{F}=F$. Since $\beta_1^t\circ\beta_s\neq 0$ and since $\pi_k(\mathfrak{S}; p)=Z_p$ (Theorem 3.13), $\bar{F}|S^{N+k}$ represents a generator of the p-component of $\pi_{N+k}(X_k)$. By Hurewicz's isomorphism and by the duality, it follows that $\bar{F}^*: H^{N+k}(X_k, Z_p)\rightarrow H^{N+k}(K, Z_p)$ is an isomorphism, and thus $\bar{F}^*\mathcal{P}^1H^{N+k}(X_k, Z_p)=\mathcal{P}^1\bar{F}^*H^{N+k}(X_k, Z_p)$ $=\mathcal{P}^1H^{N+k}(K, Z_p)=H^{N+k+2p-2}(K, Z_p)\neq 0$. But this contradicts to the relation $\mathcal{P}^1(b^{(s-1)}_{s+t})=0$ of (4.9), i), since $b^{(s-1)}_{s+t}$ generates $H^{N+k}(X_k, Z_p)$. Therefore we conclude that $\alpha_1\circ\beta_1^t\circ\beta_s\neq 0$.

Next we prove

b). If $1\leqq s<s+t<p$, *then* $\alpha_1\circ\beta_1^{t-1}\circ\beta_s\neq 0$ *implies* $\beta_1^t\circ\beta_s\neq 0$.

Let $k=2((s+t-1)p+s)(p-1)-2t-1$, $j=2((s+t)p+s-1)(p-1)$ $-2(t+1)$ and let $f: S^{N+k-2p+3}\rightarrow S^N$ and $a: S^{N+k}\rightarrow S^{N+k-2p+3}$ be representatives of $\beta_1^{t-1}\circ\beta_s$ and α_1 respectively. By Lemma 4.9, ii), $\beta_1^t\circ\beta_s=\beta_1^{t-1}\circ\beta_s\circ\beta_1$ is represented by a composition $f\circ A\circ B: S^{N+j}$ $\rightarrow L_{p-2}^{N+k}\rightarrow S^{N+k-2p+3}\rightarrow S^N$, where $A|S^{N+k}=a$ and B is the attaching map of $L_{p-1}^{N+k}-L_{p-2}^{N+k}$. Assume that $\beta_1^t\circ\beta_s=0$, then $f\circ A\circ B$ is null-homotopic and thus the mapping $f\circ A$ can be extended over $F: L_{p-1}^{N+k}\rightarrow S^N$ such that $F|L_{p-2}^{N+k}=f\circ A$. By (4.10), we lift F up to $\bar{F}: L_{p-1}^{N+k}\rightarrow X_k$ such as $F=p_k\circ\bar{F}$. Since $\alpha_1\circ\beta_1^{t-1}\circ\beta_s\neq 0$ and since $\pi_k(\mathfrak{S}; p)=Z_p$ (Theorem 3.13), $\bar{F}|S^{N+k}$ represents a generator of the p-component of $\pi_{N+k}(X_k)$. Thus $\bar{F}^*: H^{N+k}(X_k, Z_p)$ $\rightarrow H^{N+k}(L_{p-1}^{N+k}, Z_p)$ is an isomorphism. By Lemma 4.9, ii),

$\bar{F}^*\mathcal{P}^{p-1}H^{N+k}(X_k,\ Z_p)=\mathcal{P}^{p-1}\bar{F}^*H^{N+k}(X_k,\ Z_p)=\mathcal{P}^{p-1}H^{N+k}(L_{p-1}^{N+k},\ Z_p)$ $=H^{N+j+1}(L_{p-1}^{N+k},\ Z_p)\neq 0$. But this contradicts to the relation $\mathcal{P}^{p-1}(c_{s+t-1}^{(s-1)})=0$ of (4.9), iii), since $c_{s+t-1}^{(s-1)}$ generates $H^{N+k}(X_k,\ Z_p)$. Therefore we conclude $\beta_1^t\circ\beta_s\neq 0$.

c). If $\alpha_1\circ\beta_1^{p-1}\neq 0$, then $\alpha_1\circ\beta_1^{p-1}$ is not divisible by p and $\beta_1^p\neq 0$.

We use the notation of *b*) for the case $(s, t)=(0, p)$. In particular, $k=2(p-1)p(p-1)-1$. Put $k'=k-2p+3$. $f: S^{N+k'}\to S^N$ and $a: S^{N+k}\to S^{N+k'}$ are representatives of β_1^{p-1} and α_1 respectively. By (4.10), we lift the mappings f and $f\circ a$ up to $\bar{f}: S^{N+k'}\to X_{k'}$ and $\bar{g}: S^{N+k}\to X_k$ such that $p_{k'}\circ\bar{f}=f$ and $p_k\circ\bar{g}=f\circ a$. Let $i: X_k\to X_{k'}$ be the injection, then $\bar{f}\circ a$ and $i\circ\bar{g}$ represent the same element $p_{k'*}^{-1}(\beta_1^{p-1}\circ\alpha_1)$. Thus $\bar{f}\circ a$ and $i\circ\bar{g}$ are homotopic to each other, and the following diagram is commutative.

$$\begin{array}{ccc} H_{N+k}(S^{N+k}) & \xrightarrow{\bar{g}_*} & H_{N+k}(X_k) \\ \downarrow a_* & & \downarrow i_* \\ H_{N+k}(S^{N+k'}) & \xrightarrow{\bar{f}_*} & H_{N+k}(X_{k'})\,. \end{array}$$

Obviously $a_*=0$. Therefore $\bar{g}_*H_{N+k}(S^{N+k})$ is in the kernel of i_*. Since $\bar{g}$ represents $p_{k*}^{-1}(\beta_1^{p-1}\circ\alpha_1)\neq 0$, then $\bar{g}_*H_{N+k}(S^{N+k})\approx Z_p$ by Hurewicz's isomorphism. By (4.9), v), Lemma 4.9, Theorem 3.10 and by the commutativity of (4.8), we know that $H^{N+k}(X_k,\ Z_p)$ $=\{c_{p-1}^{(0)},\ a_{p(p-1)}\}$, $\Delta c_{p-1}^{(0)}\neq 0$, $\Delta_2 a_{p(p-1)}=a'_{p(p-1)}$, and that $a_{p(p-1)}$ and $a'_{p(p-1)}$ are in the image of $i^*: H^*(X_{k'},\ Z_p)\to H^*(X_k,\ Z_p)$ but $c_{p-1}^{(0)}$ and $\Delta c_{p-1}^{(0)}$ are not. Then by the duality, the p-component of $H_{N+k}(X_k)$ is isomorphic to $Z_p+Z_{p^2}$ and i_* maps the p-component onto a subgroup of $H_{N+k}(X_{k'})$ isomorphic to Z_{p^2}. Therefore the p-component of the kernel of i_* is a direct factor isomorphic to Z_p, and thus it coincides with $\bar{g}_*H_{N+k}(S^{N+k})$. This means that $p_{k*}^{-1}(\beta_1^{p-1}\circ\alpha_1)$ is not divisible by p. Therefore $\beta_1^{p-1}\circ\alpha_1$ is not divisible by p, since $p_{k*}: \pi_{N+k}(X_k)\to\pi_{N+k}(S^N)$ is an isomorphism.

The proof of $\beta_1^p\neq 0$ is similar to *b*). The only difficulty is to show that $\bar{F}^*c_{p-1}^{(0)}$ generates $H^{N+k}(L_{p-1}^{N+k},\ Z_p)$. We may identify $\bar{F}|S^{N+k}$ with $\bar{g}$ and $H^{N+k}(L_{p-1}^{N+k},\ Z_p)$ with $H^{N+k}(S^{N+k},\ Z_p)$. Since $\bar{g}_*H_{N+k}(S^{N+k})\approx Z_p$ is not divisible by p, it follows from the duality that $\bar{g}^*: H^{N+k}(X_k,\ Z_p)\to H^{N+k}(S^{N+k},\ Z_p)$ is onto. The above diagram is commutative also for the cohomological case, and $\bar{g}^*\circ i^*$ $=a^*\circ\bar{f}^*=0$. Since $a_{(p-1)p}$ is an i^*-image, then $\bar{g}^*a_{(p-1)p}=0$. Since

$H^{N+k}(X_k, Z_p)$ has two generators $a_{(p-1)p}$ and $c_{p-1}^{(0)}$, then it follows that $\bar{g}^*c_{p-1}^{(0)} \neq 0$ and this generates $H^{N+k}(S^{N+k}, Z_p)$. Thus $\beta_1^p \neq 0$ is proved.

d). $\beta_{p-1} \neq 0$ *and* $\beta_1^p \neq 0$ *imply* $\alpha_1 \circ \beta_{p-1} \neq 0$.

Since $\beta_1^p \neq 0$, $\pi_{2(p^2-1)(p-1)-2}(\mathfrak{S}; p) \neq 0$. By (4.6), this implies $\mathcal{P}^1 b_{p-1}^{(p-2)} = 0$. Now *d*) is proved similarly to the case *a*).

e). *Assume that* $\beta_1^p \neq 0$, *then* $\alpha_1 \circ \beta_1^p \neq 0$ *if and only if* $\pi_{2p^2(p-1)-3}(\mathfrak{S}; p) = Z_p$.

By (3.16), $\pi_{2p^2(p-1)-3}(\mathfrak{S}; p) = Z_p$ implies $\mathcal{P}^1 b_p^{(0)} = 0$. Similarly to *a*), it follows from $\beta_1^p \neq 0$ and $\mathcal{P}^1 b_p^{(0)} = 0$ that $\alpha_1 \circ \beta_1^p \neq 0$. If $\pi_{2p^2(p-1)-3}(\mathfrak{S}; p) \neq Z_p$, then $\pi_{2p^2(p-1)-3}(\mathfrak{S}; p) = 0$ and thus $\alpha_1 \circ \beta_1^p = 0$.

Now the proof of Theorem 4.11 is accomplished, because β_s, $1 \leqq s < p$ does not vanish by the definition.

Corollary 4.12. *For an arbitrary positive integer r, there exists an element* β *of* G_k *for some k such that the iterated r-fold composition* $\beta \circ \cdots \circ \beta$ *does not vanish.*

In fact, take $\beta = \beta_1$ for a prime p not less than r.

§ Generators of stable groups given by toric constructions.

First we prove the following lemma.

Lemma 4.13. *Let X be a simply connected topological space having a finite number of generators for each homology groups. Let* $\{u_\alpha^n\}$ *be a* Z_p*-base of* $H^n(X, Z_p)$. *Then there exist a CW-complex K and a mapping f of K into X satisfying the following conditions. K consists of a vertex and n-cells* e_α^n *corresponding to* u_α^n. *f induces isomorphisms* $f^*: H^n(X, Z_p) \to H^n(K, Z_p)$ *such that* $f^*u_\alpha^n$ *is the cohomology class of* e_α^n. *If the homotopy class of the attaching map of a cell* e_α^n *has an finite order, then the order is a power of p.*

Proof. Assume that the $(n-1)$-section K^{n-1} of K and a mapping $f_{n-1}: K^{n-1} \to X$ are given such that the conditions of the lemma are satisfied in dimensions less than n. Let $Z = X \cup K^{n-1} \times (0, 1]$ be a mapping-cylinder of f_{n-1}. Identify $K^{n-1} \times (1)$ with K^{n-1}. Then it is verified that $j^*: H^n(Z, K^{n-1}, Z_p) \to H^n(Z, Z_p) \approx H^n(X, Z_p)$ is an isomorphism and $H^i(Z, K^{n-1}, Z_p) = 0$ for $i < n$. By the duality, $H_n(Z, K^{n-1}, Z_p) \approx H_n(X, Z_p)$ and $H_i(Z, K^{n-1}, Z_p) = 0$ for $i < n$. By

Serre's theory of classes, it follows that $\pi_n(Z, K^{n-1})\otimes Z_p$ and $H_n(Z, K^{n-1}, Z_p)$ are isomorphic by Hurewicz's homomorphism. We choose elements λ_α of $\pi_n(Z, K^{n-1})$ such that λ_α corresponds to the dual of u_α^n and that if $\partial\lambda_\alpha\in\pi_{n-1}(K^{n-1})$ has a finite order then the order is a power of p. Let $g_\alpha:(E^n, S^{n-1})\to(Z, K^{n-1})$ be representatives of λ_α. Attaching n-cells e_α^n by $g_\alpha|S^{n-1}$ to K^{n-1}, we obtain a complex $K^n=K^{n-1}\cup\cdots\cup e_\alpha^n\cup\cdots$. Let $\bar{g}_\alpha:(E^n, S^{n-1})\to(\bar{e}_\alpha^n, \partial e_\alpha^n)$ be the characteristic maps of e_α^n. Then $f_n: K^n\to X$ is defined by $f_n|K^{n-1}=f_{n-1}$ and $f_n\circ\bar{g}_\alpha=p\circ g_\alpha$ where $p: Z\to X$ is the natural projection. It is verified that K^n and f_n satisfy the conditions of the lemma in dimensions less than $n+1$. Thus the lemma is proved by the induction. q. e. d.

Consider an element $\alpha\in G_h$ such that $p\alpha=\alpha\circ p\iota=0$, then $\{\alpha, p\iota, \alpha_1\}\in G_{h+2(p-1)}/(\alpha_1\circ G_{h+1}+\alpha\circ G_{2p-2})$ is defined. By (4. 4), i) and by Corollary 4. 5,

$$\{\alpha, p\iota, \alpha_1\}\circ p\iota=\alpha\circ\{p\iota, \alpha_1, p\iota\}=\alpha\circ pG_{2p-2}=p\alpha\circ G_{2p-2}=0,$$

and thus any element α' of $\{\alpha, p\iota, \alpha_1\}$ satisfies $p\alpha'=\alpha'\circ p\iota=0$. Then we may choose inductively elements

$$\alpha_t\in\{\alpha_{t-1}, p\iota, \alpha_1\}\subset G_{2t(p-1)-1},\quad t=2, 3, \cdots.$$

Since $p\alpha_t=0$, α_t belongs to the p component. Since the p-component of G_{2p-2} vanishes, $pG_{2p-2}=G_{2p-2}$ and thus $\alpha_{t-1}\circ G_{2p-2}=p\alpha_{t-1}\circ G_{2p-2}=0$. Similarly, $\alpha_1\circ G_{2(t-1)(p-1)}=0$ if $G_{2(t-1)(p-1)}$ has no p-component, in particular, if $0<t-1<p^2$ and $t-1\neq p^2-p-1$ (Theorem 3. 13). By Theorem 3. 13 and Theorem 4. 11, the p-component of $G_{2(p^2-p-1)(p-1)}$ is generated by β_1^{p-1}. Then we have the following remark.

(4. 11). $\{\alpha_{t-1}, p\iota, \alpha_1\}$ *consists of only one element* α_t *if* $1<t\leqq p^2$ *and* $t\neq(p-1)p$. $\{\alpha_{(p-1)p-1}, p\iota, \alpha_1\}=\alpha_{(p-1)p}+\{\alpha_1\circ\beta_1^{p-1}\}$. *If we add a condition* $\alpha_{(p-1)p}\in pG_{2(p-1)p(p-1)-1}$, *then* $\alpha_{(p-1)p}$ *is determined uniquely.*

The last assertion follows from the fact that the p-component of $G_{2(p-1)p(p-1)-1}$ is isomorphic to $Z_p+Z_{p^2}$ and that $\alpha_1\circ\beta_1^{p-1}$ is not divisible by p (Theorem 4. 11).

By Theorem 3. 13, the p-component of $G_{2rp(p-1)-1}$, $1\leqq r<p-1$, is a cyclic group of order p^2. Thus

(4. 12). *For* $1 \leqq r < p$, *there exists an element* α'_{rp} *of* $G_{2rp(p-1)-1}$ *such that* $p\alpha'_{rp} = \alpha_{rp}$.

Theorem 4. 14. i). *If* $1 \leqq t < p^2$, *then* $\alpha_t \neq 0$ *and it is of order* p. *For* $1 \leqq r < p$, α'_{rp} *is of order* p^2.

ii). $\alpha_t \circ \alpha_1 = 0$ *for* $1 \leqq t \leqq p^2 - 1$ *and* $\alpha'_{rp} \circ \alpha_1 = 0$ *for* $1 \leqq r < p$ *and whence we have the relations* $t\{\alpha_t, p\iota, \alpha_1\} \equiv (t+1)\{\alpha_t, \alpha_1, p\iota\}$ *and* $r\{\alpha_{rp}, p\iota, \alpha_1\} \equiv \{\alpha'_{rp}, \alpha_1, p\iota\}$. α'_{rp} *belongs to* $-r\{\alpha_{rp-1}, \alpha_1, p\iota\}$.

Proof. i). We shall prove $\alpha_t \neq 0$ by induction on $t < p^2$. Then i) is proved. Obviously $\alpha_1 \neq 0$. Assume that $\alpha_{t-1} \neq 0$ is already proved ($1 < t < p^2$).

First consider the case $t - 1 \not\equiv 0$ (mod p) and let $k = 2(t-1)(p-1) - 1$. Assume that $\alpha_t = 0$, then $\{\alpha_{t-1}, p\iota, \alpha_1\} \equiv 0$ and, by Lemma 4. 8, ii), there exist a cell complex $K = S^{N+k} \cup e^{N+k+1} \cup e^{N+k+2p-1}$ and a mapping $F: K \to S^N$ such that $S^{N+k} \cup e^{N+k+1}$ and $K/S^{N+k} = S^{N+k+1} \cup e^{N+k+2p-1}$ have the characteristic classes $p\iota$ and α_1 respectively and that $F|S^{N+k}$ represents α_{t-1}. As is easily seen, $\mathcal{P}^1\Delta \neq 0$ in K. By (4. 10), lift the mapping F up to $\bar{F}: K \to X_k$ such that $p_k \circ \bar{F} = F$. Since $\bar{F}|S^{N+k}$ represents $p_{k*}^{-1}(\alpha_{t-1}) \neq 0$ and since $\pi_{N+k}(X_k; p) \approx \pi_{N+k}(S^N; p) \approx Z_p$ (Theorem 3. 13), then $\bar{F}_*: H_{N+k}(K, Z_p) \to H_{N+k}(X_k, Z_p)$ is an isomorphism by Hurewicz's homomorphisms. Thus $\bar{F}^*: H^{N+k}(X_k, Z_p) \to H^{N+k}(K, Z_p)$ is an isomorphism. It follows from (4. 9), iii) that $\bar{F}^*(a_{t-1}) \neq 0$ and thus

$$\bar{F}^*(\mathcal{P}^1\Delta a_{t-1}) = \mathcal{P}^1\Delta\bar{F}^*(a_{t-1}) \neq 0\,.$$

Since $H^{N+k+2p-2}(K, Z_p) = 0$, then

$$\bar{F}^*(\Delta_r\mathcal{P}^1 a_{t-1}) = \Delta_r\mathcal{P}^1\bar{F}^*(a_{t-1}) = 0\,, \quad r = 1, 2\,.$$

But these two relations contradict to the relations $R_{t-1}a_{t-1} = t\mathcal{P}^1\Delta a_{t-1} - (t-1)\Delta\mathcal{P}^1 a_{t-1} = 0$ of (4. 9), iii) or $-r\mathcal{P}^1\Delta a_{rp-1} = \Delta_2\mathcal{P}^1 a_{rp-1}$ of Lemma 4. 9. Consequently the assumption $\alpha_t = 0$ lead us to the contradiction, and thus $\alpha_t \neq 0$.

Next consider the case $t - 1 = rp$, $1 \leqq r < p$ and let $k = 2rp(p-1) - 1$. $\alpha_{rp+1} \in \{\alpha_{rp}, p\iota, \alpha_1\} = \{p\alpha'_{rp}, p\iota, \alpha_1\} \subset \{\alpha'_{rp}, p^2\iota, \alpha_1\}$ by (4. 3), ii). Assume that $\alpha_{rp+1} = 0$ then $\{\alpha'_{rp}, p^2\iota, \alpha_1\} \equiv 0$. By Lemma 4. 8, ii), there exist a complex $K = S^{N+k} \cup e^{N+k+1} \cup e^{N+k+2p-1}$ and a mapping $F: K \to S^N$ such that $F|S^{N+k}$ represents α'_{rp} and $\Delta = 0$ and $\mathcal{P}^1\Delta_2 \neq 0$ in K. Lift F up to $\bar{F}: K \to X_k$ such that $p_k \circ \bar{F} = F$. By (4. 9) and by Lemma 4. 9, we have a relation

$(\Delta\mathcal{P}^1-r\mathcal{P}^1\Delta_2)a_{rp}=0$. Since $H^{N+k+2p-2}(K, Z_p)=0$, then $\bar{F}^*(\Delta\mathcal{P}^1a_{rp})=\Delta\bar{F}^*(\mathcal{P}^1a_{rp})=0$ and thus $\mathcal{P}^1\Delta_2\bar{F}^*(a_{rp})=\bar{F}^*\mathcal{P}^1\Delta_2(a_{rp})=0$. If $\bar{F}^*a_{rp}\neq 0$, then $\mathcal{P}^1\Delta_2\bar{F}^*(a_{rp})\neq 0$ and this contradicts to the last equality, and therefore the assumption $\alpha_{rp+1}=0$ will be a contradiction. So, it is sufficient to prove that $\bar{F}^*a_{rp}\neq 0$. In the case $1\leqq r<p-1$, this follows easily from the fact that $F|S^{N+k}$ represents a generator $p_{k*}^{-1}(\alpha'_{rp})$ of the p-component of $\pi_{N+k}(X_k)$. Consider the case $r=p-1$. Then $\bar{F}_*$ maps the p-component, isomorphic to Z_{p^2}, of $\pi_{N+k}(K)\approx H_{N+k}(K)$ isomorphically into the p-component, isomorphic to $Z_p+Z_{p^2}$, of $\pi_{N+k}(X_k)\approx H_{N+k}(X_k)$. By the duality, $\bar{F}^*: H^i(X_k, Z_p)\to H^i(K, Z_p)$ are onto for $i=N+k$ and $i=N+k+1$. Since $\Delta=0$ in K, $\bar{F}^*(\Delta c_{p-1}^{(0)})=\Delta\bar{F}^*(c_{p-1}^{(0)})=0$. By (4. 9), v) and by Lemma 4. 9, $H^{N+k+1}(X_k, Z_p)$ is generated by $\Delta c_{p-1}^{(0)}$ and $a'_{(p-1)p}=-\Delta_2a_{(p-1)p}$. Since $F^*H^{N+k+1}(X_k, Z_p)=H^{N+k+1}(K, Z_p)\neq 0$, it follows that $\Delta_2\bar{F}^*a_{(p-1)p}\neq 0$ and thus $\bar{F}^*a_{(p-1)p}\neq 0$.

ii). Let $k=2t(p-1)-1$ and $1\leqq t<p^2-1$. Apply Lemma 4. 13 to X_k and let K and $F: K\to X_k$ be a complex and a mapping as in Lemma 4. 13, where generators of (4. 9) are used for bases $\{u_\alpha^n\}$ of Lemma 4. 13.

First consider the case $t\not\equiv 0 \pmod p$ and $t\neq(p-1)p-1$. In this case, the vertex and the cells corresponding to a_t, Δa_t, $\mathcal{P}^1a_t$, $\mathcal{P}^1\Delta a_t$ form a subcomplex $K'=S^{N+k}\cup e^{N+k+1}\cup e^{N+k+2p-2}\cup e^{N+k+2p-1}$ of K by a suitable choice of K, up to homotopy-equivalence. This follows from the fact that each cell $e^n\in K$ of dimension $N+k+1<n<N+k+2p-2$ does not cover $S^{N+k}\cup e^{N+k+1}$, because the p-component of G_i vanishes for $0\leqq i<2p-3$ and the attaching map of e^n represents an element of p-component. Also we remark that $e^{N+k+2p-2}$ does not cover e^{N+k+1} essentially, and so we may take K such that $S^{N+k}\cup e^{N+k+2p-2}$ is a subcomplex L'. Further we may take F such that $F|S^{N+k}$ represents an element of the p-component.

Now apply Lemma 4. 8, ii) and iii) to K' and $F|K'$, $L=S^{N+k}\cup e^{N+k+1}$. Then $\alpha\circ\beta=\alpha\circ\beta'=0$ and $\{\alpha, \beta, \gamma\}+\{\alpha, \beta', \gamma'\}\equiv 0$, where α is represented by $F|S^{N+k}$ and $\beta\in G_0$, $\beta'\in G_{2p-3}$, $\gamma'\in G_{2p-3}$, $\gamma'\in G_0$ are characteristic classes of L, L', K'/L' and K'/L respectively, with respect to some given orientations. Concerning the correspondence of the cohomological operations Δ and $\mathcal{P}^1$ in K', induced by F^* from X, and concerning the relation $(t+1)\mathcal{P}^1\Delta a_t=t\Delta(\mathcal{P}^1a_t)$ of (4. 9), iii) and $\mathcal{P}^1(\Delta a_{rp-1})=-(1/r)\Delta_2(\mathcal{P}^1a_{rp-1})$, we have that $\beta\equiv(-1)^{N+k}p\iota \pmod{p^2\iota}$, $\beta'\equiv(-1)^{N+k}\alpha_1$, $\gamma=(-1)^{N+k+1}\alpha_1$ and

$\gamma' \equiv (-1)^{N+k}((t+1)p/t)\iota \pmod{p^2\iota}$ if $t+1\equiv 0 \pmod p$ and $\gamma' \equiv (-1)^{N+k+1}(1/r)p^2\iota \pmod{p^3\iota}$ if $t+1=rp$. Since $F^*: H^{N+k}(X_k, Z_p) \to H^{N+k}(K, Z_p)$ is an isomorphism, then $\alpha \neq 0$ and thus $\alpha = x\alpha_t$ for some $x \not\equiv 0 \pmod p$. Therefore it follows from $\alpha\circ\beta'=0$ and $\{\alpha, \beta, \gamma\}+\{\alpha, \beta', \gamma'\}\equiv 0$ that $\alpha_t\circ\alpha_1=0$, $\{\alpha_t, p\iota, \alpha_1\}-((t+1)/t)\{\alpha_t, \alpha_1, p\iota\}\equiv 0$ if $t+1\not\equiv 0 \pmod p$ and $\{\alpha_t, p\iota, \alpha_1\}+(p/r)\{\alpha_t, \alpha_1, p\iota\}\equiv 0$ if $t+1=rp$. Consequently we obtain the relation $t\{\alpha_t, p\iota, \alpha_1\}\equiv(t+1)\{\alpha_t, \alpha_1, p\iota\}$ for the case $t\not\equiv 0 \pmod p$ and $t \neq (p-1)p-1$.

For the case $t=(p-1)p-1$ the proof is similar to the above case. The only remark for this case is the fact that K has two $(N+k+1)$-cells, one of which corresponds to $b_{p-1}^{(0)}$ and it is not covered by $e^{N+k+2p-1}$ because $\mathcal{P}^1 b_{p-k}^{(0)}=0$. Then we may neglect this $(N+k+1)$-cell.

Next consider the case $t=rp$ and $1\leqq r<p-1$. By similar discussions to the above, we have a subcomplex K' which consists of a vertex and the cells corresponding to a_{rp}, a'_{rp}, $\mathcal{P}^1 a_{rp}$ and $\mathcal{P}^1 a'_{rp}$. From the relations $\Delta\mathcal{P}^1 a_{rp}=\mathcal{P}^1 a'_{rp}$ of (4.9), iv) and $\Delta_2 a_{rp}=ra'_{rp}$ of Lemma 4.9, we have $\beta\equiv(-1)^{N+k}rp^2\iota$ (mod. $p^3\iota$), $\beta'=(-1)^{N+k+1}\alpha_1$, $\gamma=(-1)^{N+k}\alpha_1$ and $\gamma'\equiv(-1)^{N+k}p\iota$ (mod. $p^2\iota$). $\alpha=x\alpha'_{rp}$ for some $x\not\equiv 0 \pmod p$. Then it follows from the relations $\alpha\circ\beta'=0$ and $\{a, \beta, \gamma\}+\{\alpha, \beta', \gamma'\}\equiv 0$ of Lemma 4.8 that $\alpha'_{rp}\circ\alpha_1=0$, $\alpha_{rp}\circ\alpha_1=p\alpha'_{rp}\circ\alpha_1=0$ and $r\{\alpha'_{rp}, p^2\iota, \alpha_1\}\equiv\{\alpha'_{rp}, \alpha_1, p\iota\}$. By (4.3), ii), $\{\alpha'_{rp}, p^2\iota, \alpha_1\}\equiv\{p\alpha'_{rp}, p\iota, \alpha_1\}=\{\alpha_{rp}, p\iota, \alpha_1\}$. Therefore the relation $r\{\alpha_{rp}, p\iota, \alpha_1\}\equiv\{\alpha'_{rp}, \alpha_1, p\iota\}$ is obtained.

Finally consider the case $t=(p-1)p$ (($k=2(p-1)p(p-1)-1$). Obviously $\alpha_{(p-1)p}\circ\alpha_1=p\alpha'_{(p-1)p}\circ\alpha_1=\alpha'_{(p-1)p}\circ p\alpha_1=0$. By Theorem 3.13, $\pi_{k+2(p-1)-1}(\mathfrak{S}; p)=0$ and this contains $\alpha'_{(p-1)p}\circ\alpha_1$. Thus $\alpha'_{(p-1)p}\circ\alpha_1=0$ and $\{\alpha'_{(p-1)p}, \alpha_1, p\iota\}$ is defined. Since $\{\alpha'_{(p-1)p}, \alpha_1, p\iota\}$ is a coset of $pG_{k+2(p-1)}+\alpha'_{(p-1)p}\circ G_{2p-2}=pG_{k+2(p-1)}$, it contains an element of the p-component. By Theorem 3.13 and by $\alpha_{(p-1)p+1}\neq 0$ of i), $\pi_{k+2(p-1)}(\mathfrak{S}; p)\approx Z_p$ is generated by $\alpha_{(p-1)p+1}=\{\alpha_{(p-1)p}, p\iota, \alpha_1\}\equiv\{\alpha'_{(p-1)p}, p^2\iota, \alpha_1\}$. Therefore $\{\alpha'_{(p-1)p}, \alpha_1, p\iota\}+x\{\alpha'_{(p-1)p}, p^2\iota, \alpha_1\}\equiv 0$ for some coefficient x. Apply Lemma 4.8, iii) to this relation, then there exist a complex $K=S^{N+k}\cup e^{N+k+1}\cup e^{N+k+2p-2}\cup e^{N+k+2p-1}$ and a mapping $F: K\to S^N$ such that $F|S^{N+k}$ represents $\alpha'_{(p-1)p}$ and $L=S^{N+k}\cup e^{N+k+1}$, $L'=S^{N+k}\cup e^{N+k+2p-2}$, K/L' and K/L have the characteristic classes $p\iota$, α_1, $x\alpha_1$ and $p\iota$ respectively. Let u_1, u_2, u_3 and u_4 be the cohomology classes mod p of S^{N+k}, e^{N+k+1}, $e^{N+k+2p-2}$, and

$e^{N+k+2p-1}$, respectively, given by the orientations. Then $\Delta_2 u_1 = (-1)^{N+k} u_2$, $\mathcal{P}^1 u_1 = (-1)^{N+k-1} u_3$, $\mathcal{P}^1 u_2 = (-1)^{N+k} x u_4$ and $\Delta u_3 = (-1)^{N+k} u_4$. Therefore the relation $(x\mathcal{P}^1\Delta_2 + \Delta\mathcal{P}^1)u_1 = 0$ holds. By (4.10), we lift F up to $\bar{F}: K \to X_k$ such that $F = p_k \circ \bar{F}$. Consider $\bar{F}^*: H^*(X_k, Z_p) \to H^*(K, Z_p)$. As in the last part of the proof of i), we see that $\bar{F}^*(a_{(p-1)p}) \neq 0$ and thus $\bar{F}^*(a_{(p-1)p}) = y u_1$ for some $y \not\equiv 0 \pmod{p}$. By (4.9), v) and by Lemma 4.9, a relation $(\mathcal{P}^1\Delta_2 + \Delta\mathcal{P}^1) a_{(p-1)p} = 0$ holds, and it is translated to $(\mathcal{P}^1\Delta_2 + \Delta\mathcal{P}^1) u_1 = 0$ by operating $\bar{F}^*$. Therefore we have that $x \equiv 1$ and $\{\alpha'_{(p-1)p}, \alpha_1, p\iota\} \equiv -\{\alpha'_{(p-1)p}, p^2\iota, \alpha_1\} \equiv -\{\alpha_{(p-1)p}, p\iota, \alpha_1\}$.

The fact $-\alpha'_{rp} \in r\{\alpha_{rp-1}, \alpha_1, p\iota\}$ follows from (4.12) and $p\alpha'_{rp} \in p\{\alpha_{rp-1}, \alpha_1, p\iota\} \equiv (r/(rp-1))\{\alpha_{rp-1}, p\iota, \alpha_1\}$. q. e. d.

Let $k = 2(p^2-1)(p-1)-2$. By Theorem 3.10, $H^{N+k}(X_k, Z_p) = \{b^{(p-2)}_{p-1}\}$, $H^{N+k+1}(X_k, Z_p) = \{\Delta b^{(p-2)}_{p-1}, a_{p^2-1}\}$, $\Delta a_{p^2-1} \neq 0$, $H^{N+k+2p-2}(X_k, Z_p) = \{b^{(0)}_p\}$, $H^{N+k+2p-1}(X_k, Z_p) = \{\mathcal{P}^1\Delta b^{(p-2)}_{p-1}, \mathcal{P}^1 a_{p^2-1}\}$ and $\Delta\mathcal{P}^1\Delta b^{(p-2)}_{p-1} \neq 0$. By (4.6) and by Theorem 4.11, $\mathcal{P}^1 b^{(p-2)}_{p-1} = 0$. According to Lemma 4.13, we can construct a complex K and a mapping $F: K \to X_k$ such that the vertex and the cells corresponding to $b^{(p-2)}_{p-1}$, $\Delta b^{(p-2)}_{p-1}$ and $\mathcal{P}^1\Delta b^{(p-2)}_{p-1}$ form a subcomplex $K' = S^{N+k} \cup e^{N+k+1} \cup e^{N+k+2p-1}$ of K and $F | S^{N+k}$ represents $p_{k*}^{-1}(x\beta_{p-1})$ for some $x \not\equiv 0 \pmod{p}$. Then it follows from ii) of Lemma 4.8

(4.13). $\{\beta_{p-1}, p\iota, \alpha_1\} \equiv 0$.

§ Relations in stable groups.

Summarizing Theorem 3.13, Theorem 4.11 and Theorem 4.14, we obtain

Theorem 4.15.

$\pi_{2rp(p-1)-1}(\mathfrak{S}; p) = Z_{p^2} = \{\alpha'_{rp}\}$ *for* $1 \leqq r < p-1$,

$\qquad = Z_{p^2} + Z_p = \{\alpha'_{(p-1)p}\} + \{\alpha_1 \circ \beta_1^{p-1}\}$ *for* $r = p-1$,

$\pi_{2t(p-1)-1}(\mathfrak{S}; p) = Z_p = \{\alpha_t\}$ *for* $1 \leqq t < p^2$ *and* $t \not\equiv 0 \pmod{p}$,

$\pi_{2(rp+s)(p-1)-2(r-s)}(\mathfrak{S}; p) = Z_p = \{\beta_1^{r-s-1} \circ \beta_{s+1}\}$ *for* $0 \leqq s < r \leqq p-1$,

$\pi_{2(rp+s+1)(p-1)-2(r-s)-1}(\mathfrak{S}; p) = Z_p = \{\alpha_1 \circ \beta_1^{r-s-1} \circ \beta_{s+1}\}$

for $0 \leqq s < r \leqq p-1$ *and* $r-s \neq p-1$,

$\pi_{2p^2(p-1)-2p}(\mathfrak{S}; p) = Z_p = \{\beta_1^p\}$,

$\pi_k(\mathfrak{S}; p) = 0$ *otherwise for* $k < 2p^2(p-1)-3$,

$\pi_{2p^2(p-1)-3}(\mathfrak{S}\,;p) = \{\alpha_1\circ\beta_1^p\} = Z_p$ *or* 0 .

In the following, we shall investigate compositions and toric constructions of these generators.

First consider toric constructions $(s>0,\ t>0)$

$$\{\alpha_s,\ p\iota,\ \alpha_t\}\in G_{2(s+t)(p-1)-1}/(\alpha_s\circ G_{2t(p-1)}+\alpha_t\circ G_{2s(p-1)})\ .$$

If $0<t<p^2$ and $t\neq(p-1)p-1$, then the p-component of $G_{2t(p-1)}$ vanishes, and whence $\alpha_s\circ G_{2t(p-1)}=0$. If $1<s\leqq p+1$, then the p-component of $G_{2(p+s)(p-1)-3}$ vanishes and thus $\alpha_s\circ\beta_1=0$ and $\alpha_s\circ\beta_1^{p-1}=0$. Since the p-component of $G_{2t(p-1)}$ is generated by β_1^{p-1} if $t=(p-1)p-1$, it follows that if $t=(p-1)p-1$ then $\alpha_s\circ G_{2t(p-1)}=0$ for $1<s\leqq p+1$ and $\alpha_1\circ G_{2t(p-1)}=\{\alpha_1\circ\beta_1^{p-1}\}$. Similar results hold for $\alpha_t\circ G_{2s(p-1)}$. It follows then

(4.14). $\{\alpha_s,\ p\iota,\ \alpha_t\}$ *consists of a single element if* $s+t\leqq p^2$ *and* $(s,t)\neq(1,\ (p-1)p-1)\neq(t,s)$. $\{\alpha_1,\ p\iota,\ \alpha_{(p-1)p-1}\}$ *and* $\{\alpha_{(p-1)p-1},\ p\iota,\ \alpha_1\}$ *are cosets of* $\{\alpha_1\circ\beta_1^{p-1}\}$,

Proposition 4.16. $\{\alpha_s,\ p\iota,\ \alpha_t\}=\alpha_{s+t}$ *if* $s+t\leqq p^2$ *and* $(s,t)\neq(1,\ (p-1)p-1)\neq(t,s)$. $\{\alpha_1,\ p\iota,\ \alpha_{(p-1)p-1}\}=\{\alpha_{(p-1)p-1},\ p\iota,\ \alpha_1\}=\alpha_{(p-1)p}+\{\alpha_1\circ\beta_1^{p-1}\}$,

Proof. By i) of Theorem 4.4, $\{\alpha_r,\ p\iota,\ \alpha_1\}=\{\alpha_1,\ p\iota,\ \alpha_r\}$. Let $s>1$ and $s+t\leqq p^2$. By ii) of (4.4), we have a relation $\{\{\alpha_{s-1},\ p\iota,\ \alpha_1\},\ p\iota,\ \alpha_t\}-\{a_{s-1},\ \{p\iota,\ \alpha_1,\ p\iota\},\ \alpha_t\}-\{\alpha_{s-1},\ p\iota,\ \{\alpha_1,\ p\iota,\ \alpha_t\}\}\equiv 0$. This means that for some $\lambda\in\{\alpha_{s-1},\ p\iota,\ \alpha_1\}$, $\mu\in\{p\iota,\ \alpha_1,\ p\iota\}$ and $\nu\in\{\alpha_1,\ p\iota,\ \alpha_t\}$, the following relation holds:

$$\{\lambda,\ p\iota,\ \alpha_t\}\equiv\{\alpha_{s-1},\ \mu,\ \alpha_t\}+\{\alpha_{s-1},\ p\iota,\ \nu\}\ .$$

Since $\{p\iota\ ,\alpha_1,\ p\iota\}\subset G_{2(p-1)}$ and since the p-component of $G_{2(p-1)}$ vanishes, there exists an element μ' of $G_{2(p-1)}$ such that $p\mu'=\mu$. By ii) of (4.3), $\{\alpha_{s-1},\ \mu,\ \alpha_t\}=\{\alpha_{s-1},\ p\mu',\ \alpha_t\}\supset\{\alpha_{s-1},\mu',\ p\alpha_t\}=\{\alpha_{s-1},\ \mu',\ 0\}\ni 0$. Thus $\{\alpha_{s-1},\ \mu,\ \alpha_t\}=\alpha_{s-1}\circ G_{2(t+1)(p-1)}+\alpha_t\circ G_{2s(p-1)}$. Next consider $\{\lambda,\ p\iota,\ \alpha_t\}$. If $s\neq(p-1)p$, then $\lambda=\alpha_s$ by (4.11). Thus $\{\lambda,\ p\iota,\ \alpha_t\}=\{\alpha_s,\ p\iota,\ \alpha_t\}$. If $s=(p-1)p$, then $\lambda=\alpha_s+x\alpha_1\circ\beta_1^{p-1}$ for some coefficient x by (4.11) and $\{\lambda,\ p\iota,\ \alpha_t\}\subset\{\alpha_s,\ p\iota,\ \alpha_t\}+x\{\beta_1^{p-1}\circ\alpha_1,\ p\iota,\ \alpha_t\}$ by (4.3), i). By ii) of (4.3), $\{\beta_1^{p-1}\circ\alpha_1,\ p\iota,\ \alpha_t\}\supset\beta_1^{p-1}\circ\{\alpha_1,\ p\iota,\ \alpha_t\}=\beta_1^{p-1}\circ\alpha_{t+1}=0$ $(1<t+1\leqq p+1)$ and thus $\{\beta_1^{p-1}\circ\alpha_1,\ p\iota,\ \alpha_t\}=\beta_1^{p-1}\circ\alpha_1\circ G_{2t(p-1)}+\alpha_t\circ G_{2(p-1)p(p-1)}=0$. Then we have the

equality $\{\lambda, p\iota, \alpha_t\} = \{\alpha_s, p\iota, \alpha_t\}$ since $\lambda\circ G_{2t(p-1)} = \alpha_s\circ G_{2t(p-1)}$. Similarly $\{\alpha_{s-1}, p\iota, \nu\} = \{\alpha_{s-1}, p\iota, \alpha_{t+1}\}$. Consequently

$$\{\alpha_s, p\iota, \alpha_t\} \equiv \{\alpha_{s-1}, p\iota, \alpha_{t+1}\}\ . \qquad \text{mod. } M_{s,t}\,,$$

where $M_{s,t} = \alpha_s\circ G_{2t(p-1)} + \alpha_t\circ G_{2s(p-1)} + \alpha_{s-1}\circ G_{2(t+1)(p-1)} + \alpha_{t+1}\circ G_{2(s-1)(p-1)}$.

As is seen in (4.14), $M_{s,t}=0$ if $s+t\neq(p-1)p$ or $s-1\neq 1\neq t$ and $M_{2,(p-1)p-2} = M_{(p-1)p-1,1} = \{\alpha_1\circ\beta_1^{p-1}\}$. In the case $s+t\neq(p-1)p$ it follows from $\alpha_{s+t}\in\{\alpha_{s+t-1}, p\iota, \alpha_1\}$ that $\alpha_{s+t}=\{\alpha_s, p\iota, \alpha_t\}$ by the above relation. The case $(s, t)=((p-1)p-1, 1)$ is obvious by (4.11). In the case $s+t=(p-1)p$ and $2<s<(p-1)p-1$, it follows from $\{\alpha_{(p-1)p-1}, p\iota, \alpha_1\} = \alpha_{(p-1)p} + \{\alpha_1\circ\beta_1^{p-1}\}$ that $\{\alpha_s, p\iota, \alpha_t\} = \alpha_{(p-1)p} + y\alpha_1\circ\beta_1^{p-1}$ for some coefficient y which does not depend on s. By Theorem 4.11 and by (4.11), $\alpha_1\circ\beta_1^{p-1}$ is not divisible by p and $\alpha_{(p-1)p}$ is divisible by p. Thus $y=0$ if and only if $\alpha_{(p-1)p}+y\alpha_1\circ\beta_1^{p-1}$ is divisible by p. Since $p\alpha'_p=\alpha_p$ and $p\alpha'_{(p-2)p} = \alpha_{(p-2)p}$, then $\{\alpha_p, p\iota, \alpha_{(p-2)p}\} = \{\alpha'_p\circ p\iota, p\iota, \alpha'_{(p-2)p}\circ p\iota\} \equiv \{\alpha'_p, p^2\iota, \alpha'_{(p-2)p}\}\circ p\iota$ by (4.3), ii). This indicates that $\{\alpha_p, p\iota, \alpha_{(p-2)p}\} = \alpha_{(p-1)p}+y\alpha_1\circ\beta_1^{p-1}$ is divisible by p. Therefore $y=0$ and $\{\alpha_s, p\iota, \alpha_t\} = \alpha_{s+t}$ for $s+t=(p-1)p$ and $1<s<(p-1)p-1$. q. e. d.

As is seen in the last part of the preceding proof, we have $p\{\alpha'_p, p^2\iota, \alpha'_{rp}\} \equiv \{\alpha_p, p\iota, \alpha_{rp}\} = \alpha_{(r+1)p}$ for $1\leqq r\leqq p-1$. Similarly to the proof of (4.11), we have that $\{\alpha'_p, p^2\iota, \alpha'_{rp}\}$ consists of a single element.

(4.15). *Thus, for fixed* α'_p *such that* $p\alpha'_p=\alpha_p$, *the condition* $\alpha'_{rp}\in\{\alpha'_p, p^2\iota, \alpha'_{(r-1)p}\}$, $2\leqq r\leqq p$ *determines an* α'_{rp} *which satisfies the condition of* (4.12).

Next we shall prove

Proposition 4.17. i). $\alpha_t\circ\alpha_s=0$ *for* $s+t\leqq p^2$ *and* $\alpha'_{rp}\circ\alpha_s=0$ *for* $rp+s\leqq p^2$.

ii).

$$\{\alpha_t, \alpha_s, p\iota\} = (t/(s+t))\alpha_{s+t} + pG_{2(s+t)(p-1)-1}$$
$$\text{if } s+t<p^2 \text{ and } s+t\not\equiv 0 \pmod p,$$
$$= (t/r)\alpha'_{rp} + M_{t,s} \qquad \text{if } s+t=rp \text{ and } 1\leqq r<p,$$
$$\{\alpha'_{rp}, \alpha_s, p\iota\} = (r/s)\alpha_{rp+s} + pG_{2(rp+s)(p-1)-1}$$
$$\text{if } rp+s<p^2 \text{ and } s\not\equiv 0 \pmod p,$$

$$= (r/(r+u))\alpha'_{(r+u)p} + pG_{2(r+u)p(p-1)-1}$$

if $s=up$ *and* $r+u<p$,

where $M_{t,s}=pG_{2(s+t)(p-1)-1}$ *if* $(t, s)\neq(1, (p-1)p-1)$ *and* $M_{t,s}=pG_{2(s+t)(p-1)-1}+\{\alpha_1\circ\beta_1^{p-1}\}$ *if* $(t, s)=(1, (p-1)p-1)$.

iii). $\{\alpha_t, \alpha_s, p\iota\}=\{p\iota, \alpha_s, \alpha_t\}$ *and* $\{\alpha'_{rp}, \alpha_s, p\iota\}=\{p\iota, \alpha_s, \alpha'_{rp}\}$.

Proof. i). We use the induction on s. The case $s=1$ is proved by ii) of Theorem 4.14. By (4.4), i) and by Proposition 4.16, $\alpha_t\circ\alpha_s\in\alpha_t\circ\{\alpha_1, p\iota, \alpha_{s-1}\}=\{\alpha_t, \alpha_1, p\iota\}\circ\alpha_{s-1}$ and $\alpha'_{rp}\circ\alpha_s\in\alpha'_{rp}\circ\{\alpha_1, p\iota, \alpha_{s-1}\}=\{\alpha'_{rp}, \alpha_1, p\iota\}\circ\alpha_{s-1}$. Thus $\alpha_t\circ\alpha_s$ and $\alpha'_{rp}\circ\alpha_s$ belong to $G_{2(t+1)(p-1)-1}\circ\alpha_{s-1}$ ($t=rp$ for the latter element). Since the p-component of $G_{2(t+1)(p-1)-1}$ is generated by one or two of α_{t+1}, α'_{t+1} and $\alpha_1\circ\beta_1^{p-1}$, and since $\alpha_{t+1}\circ\alpha_{s-1}=\alpha'_{t+1}\circ\alpha_{s-1}=\beta_1^{p-1}\circ\alpha_1\circ\alpha_{s-1}=0$ by the assumption of induction, we have $G_{2(t+1)(p-1)-1}\circ\alpha_{s-1}=0$ and hence $\alpha_t\circ\alpha_s=\alpha'_{rp}\circ\alpha_s=0$. Thus i) is proved by induction.

ii). Similarly to (4.11), $\{\alpha_t, \alpha_s, p\iota\}$ and $\{\alpha'_{rp}, \alpha_s, p\iota\}$ are cosets of $pG_{2(s+t)(p-1)-1}$ or $M_{t,s}$. Then it follows from Theorem 4.14, ii) that ii) of this proposition is true for the case $s=1$. We use the induction on s.

By ii) of (4.4), for some elements $\lambda\in\{\alpha_t, \alpha_1, p\iota\}$, $\lambda'\in\{\alpha'_{rp}, \alpha_1, p\iota\}$, $\mu\in\{\alpha_1, p\iota, \alpha_{s-1}\}$ and $\nu\in\{p\iota, \alpha_{s-1}, p\iota\}$ the following relations hold:

$$\{\lambda, \alpha_{s-1}, p\iota\}-\{\alpha_t, \mu, p\iota\}+\{\alpha_t, \alpha_1, \nu\}\equiv 0,$$
$$\{\lambda', \alpha_{s-1}, p\iota\}-\{\alpha'_{rp}, \mu, p\iota\}+\{\alpha'_{rp}, \alpha_1, \nu\}\equiv 0.$$

By Corollary 4.5, $\nu\in pG_{2(s-1)(p-1)}$. Since $pG_{2(s-1)(p-1)}$ has vanishing p-component, $\nu=p^2\nu'$ for some ν'. It follows easily that $\{\alpha_t, \alpha_1, \nu\}\equiv 0$ and $\{\alpha'_{rp}, \alpha_1, \nu\}\equiv 0$ and thus these terms may be neglected in the above relations. By Proposition 4.16, $\mu=\alpha_s$ or $\mu=\alpha_{(p-1)p}+x\alpha_1\circ\beta_1^{p-1}$. By (4.3), ii), $\{\alpha_t, \alpha_1\circ\beta_1^{p-1}, p\iota\}\supset\{\alpha_t, \alpha_1, p\beta_1^{p-1}\}=\{\alpha_t, \alpha_1, 0\}\ni 0$. Therefore $\{\alpha_t, \mu, p\iota\}=\{\alpha_t, \alpha_s, p\iota\}$. Similarly, $\{\alpha'_{rp}, \mu, p\iota\}=\{\alpha'_{rp}, \alpha_s, p\iota\}$. Next, from the case $s=1$, $\lambda=(t/(t+1))\alpha_{t+1}+p\gamma$ if $t+1\not\equiv 0 \pmod p$, $\lambda=(-1/r)\alpha'_{rp}+p\gamma$ if $t+1=rp$ and $\lambda'=r\alpha_{rp+1}+p\gamma$, where γ is some element of $G_{2(t+1)(p-1)-1}$. By (4.3), ii) and by Corollary 4.5, $\{p\gamma, \alpha_{s-1}, p\iota\}\supset\gamma\circ\{p\iota, \alpha_{s-1}, p\iota\}=\gamma\circ pG_{2(s-1)(p-1)}=p(\gamma\circ G_{2(s-1)(p-1)})$. Then $\{p\gamma, \alpha_{s-1}, p\iota\}\equiv 0$ and thus $\{\lambda, \alpha_{s-1}, p\iota\}\equiv\{\lambda-p\gamma, \alpha_{s-1}, p\iota\}$. Similarly $\{\lambda', \alpha_{s-1}, p\iota\}\equiv\{\lambda'-p\gamma, \alpha_{s-1}, p\iota\}$. Consequently we obtain the following relations.

$$(t/(t+1))\{\alpha_{t+1},\ \alpha_{s-1},\ p\iota\} \equiv \{\alpha_t,\ \alpha_s,\ p\iota\} \quad \text{if } t+1 \not\equiv 0 \pmod p,$$
$$(-1/r)\{\alpha'_{rp},\ \alpha_{s-1},\ p\iota\} \equiv \{\alpha_{rp-1},\ \alpha_s,\ p\iota\},$$
$$r\{\alpha_{rp+1},\ \alpha_{s-1},\ p\iota\} \equiv \{\alpha'_{rp},\ \alpha_s,\ p\iota\}.$$

Then ii) is verified by induction on s.

iii) is a direct consequence of Theorem 4.4, i). q.e.d.

Consider $\{\alpha_1,\ p\iota,\ \beta_s\}$. By (4.4), i) and by Corollary 4.5, $p\{\alpha_1,\ p\iota,\ \beta_s\} = \{\alpha_1,\ p\iota,\ \beta_s\}\circ p\iota = \alpha_1\circ\{p\iota,\ \beta_s,\ p\iota\} = \alpha_1\circ pG_{2(sp+s-1)(p-1)-1} = 0$. If $1 \leqq s < p-1$, then the p-component of $G_{2(sp+s)(p-1)-2}$ vanishes by Theorem 4.15. Thus $p\{\alpha_1,\ p\iota,\ \beta_s\}=0$ implies $\{\alpha_1,\ p\iota,\ \beta_s\}=0$. By (4.13), $\{\beta_{p-1},\ p\iota,\ \alpha_1\} = \beta_{p-1}\circ G_{2p-2} + \alpha_1\circ G_{2(p^2-1)(p-1)-1}$. Since G_{2p-2} has no p-component, then $\beta_{p-1}\circ G_{2p-2}=0$. Since the p-component of $G_{2(p^2-1)(p-1)-1}$ is generated by α_{p^2-1} and since $\alpha_1\circ\alpha_{p^2-1}=0$ by Proposition 4.17, then $\alpha_1\circ G_{2(p^2-1)(p-1)-1}=0$. Therefore $\{\beta_{p-1},\ p\iota,\ \alpha_1\} = 0$. Concerning i) of Theorem 4.4, we have

(4.16). $\{\beta_s,\ p\iota,\ \alpha_1\} = \{\alpha_1,\ p\iota,\ \beta_s\} = 0$ *for* $1 \leqq s < p$.

We shall prove

Proposition 4.18. *Let* $1 \leqq s < p$.

i). $\alpha_t\circ\beta_s = 0$ *for* $2 \leqq t < p^2$. $\alpha'_{rp}\circ\beta_s = 0$ *for* $1 \leqq r < p$.

ii). $\{\alpha_t,\ p\iota,\ \beta_s\} = \{\beta_s,\ p\iota,\ \alpha_t\} = 0$ *for* $1 \leqq t < p^2$ *and* $t \neq (p-1)p-1$. $\{\alpha_{(p-1)p-1},\ p\iota,\ \beta_s\} = \{\beta_s,\ p\iota,\ \alpha_{(p-1)p-1}\} = \{\beta_1^{p-1}\circ\beta_s\}$.

iii). $\{\alpha_t,\ \beta_s,\ p\iota\} = \{p\iota,\ \beta_s,\ \alpha_t\} = pG_{2(sp+s-1+t)(p-1)-2}$ *for* $2 \leqq t < p^2$. $\{\beta_s,\ \alpha_t,\ p\iota\} = \{p\iota,\ \alpha_t,\ \beta_s\} = M'_{s,t}$ *for* $2 \leqq t < p^2$, *where* $M'_{s,t} = pG_{2(sp+s-1+t)(p-1)-2}$ *if* $t \neq (p-1)p-1$ *and* $M'_{s,t} = \{\beta_1^{p-1}\circ\beta_s\} + pG_{2(sp+s-1+t)(p-1)-2}$ *if* $t=(p-1)p-1$.

Proof. i). By (4.4), i), Theorem 4.14, ii) and by (4.16), $\alpha_t\circ\beta_s \in (t/(t-1))\{\alpha_{t-1},\ \alpha_1,\ p\iota\}\circ\beta_s = (t/(t-1))\alpha_{t-1}\circ\{\alpha_1,\ p\iota,\ \beta_s\} = 0$ for $t \not\equiv 1 \pmod p$, $\alpha_{rp+1}\circ\beta_s \in (1/r)\{\alpha'_{rp},\ \alpha_1,\ p\iota\}\circ\beta_s = (1/r)\alpha'_{rp}\circ\{\alpha_1,\ p\iota,\ \beta_s\} = 0$ and $\alpha'_{rp}\circ\beta_s \in -r\{\alpha_{rp-1},\ \alpha_1,\ p\iota\}\circ\beta_s = -r\alpha_{rp-1}\circ\{\alpha_1,\ p\iota,\ \beta_s\} = 0$. Then i) is proved.

ii). (4.16) is the case $t=1$. We proceed by induction on t.

The proof is quite similar to that of Proposition 4.16. By ii) of (4.4), we have a relation $\{\beta_s,\ p\iota,\ \{\alpha_1,\ p\iota,\ \alpha_{t-1}\}\} + \{\beta_s,\ \{p\iota,\ \alpha_1,\ p\iota\},\ \alpha_{t-1}\} + \{\{\beta_s,\ p\iota,\ \alpha_1\},\ p\iota,\ \alpha_{t-1}\} \equiv 0$. By Corollary 4.5, $\{p\iota,\ \alpha_1,\ p\iota\} \equiv 0$. By (4.16), $\{\beta_s,\ p\iota,\ \alpha_1\} \equiv 0$. By (4.11), $\{\alpha_1,\ p\iota,\ \alpha_{t-1}\} \ni \alpha_t$. Then we obtain a relation $\{\beta_s,\ p\iota,\ \alpha_t\} \equiv 0$, and ii) is proved. For the details, see the proof of Proposition 4.16.

iii). Consider a relation $\{\{\beta_s, p\iota, \alpha_1\}, \alpha_1, p\iota\}+\{\beta_s, \{p\iota, \alpha_1, \alpha_1\}, p\iota\}+\{\beta_s, p\iota, \{\alpha_1, \alpha_1, p\iota\}\}\equiv 0$ from ii) of (4.4). By Proposition 4.17, $\{\alpha_1, \alpha_1, p\iota\}=\{p\iota, \alpha_1, \alpha_1\}\ni(1/2)\alpha_2$. By ii), $\{\beta_s, p\iota, \alpha_1\}\equiv 0$ and $\{\beta_s, p\iota, \{\alpha_1, \alpha_1, p\iota\}\}\equiv 0$. Then it follows $\{\beta_s, \alpha_2, p\iota\}\equiv 0$. By i) and ii) of Theorem 4.4, $\{p\iota, \alpha_2, \beta_s\}\equiv 0$ and $\{\alpha_2, \beta_s, p\iota\}=\{p\iota, \beta_s, \alpha_2\}\equiv-\{\beta_s, \alpha_2, p\iota\}-\{\alpha_2, p\iota, \beta_s\}\equiv 0$. Thus iii) is proved for $t=2$. Next let $t>2$ and consider a relation $\{\{\alpha_{t-2}, p\iota, \alpha_2\}, \beta_s, p\iota\}\equiv\{\alpha_{t-2}, \{p\iota, \alpha_2, \beta_s\}, p\iota\}+\{\alpha_{t-2}, p\iota, \{\alpha_2, \beta_s, p\iota\}\}$ from (4.4), ii). Thus we have $\{\alpha_t, \beta_s, p\iota\}\equiv 0$. By Theorem 4.4, it follows $\{\beta_s, \alpha_t, p\iota\}\equiv 0$, and iii) is proved. The details of the proof are left to the reader. q.e.d.

It follows also from a relation $\{\{\alpha_{rp-1}, \alpha_1, p\iota\}, \beta_s, p\iota\}\equiv\{\alpha_{rp-1}, \{\alpha_1, p\iota, \beta_s\}, p\iota\}-\{\alpha_{rp-1}, \alpha_1, \{p\iota, \beta_s, p\iota\}\}$ that

$$(4.17).\qquad \{\alpha'_{rp}, \beta_s, p\iota\}=\{p\iota, \beta_s, \alpha'_{rp}\}\equiv 0 \quad \textit{for } 1\leqq s<p \textit{ and } 1\leqq r<p.$$

By (4.4), i) and by Corollary 4.5, $p\{\beta_r, p\iota, \beta_s\}=\beta_r\circ\{p\iota, \beta_s, p\iota\}=\beta_r\circ pG_{2(sp+s-1)(p-1)-1}=0$. Since the p-component of $G_{2((r+s)p+r+s-2)(p-1)-3}$ vanishes for $r+s<p$ (Theorem 4.15), then $\{\beta_r, p\iota, \beta_s\}=0$ for $r+s<p$. By i) of Theorem 4.4, $\{\beta_s, p\iota, \beta_s\}=-\{\beta_s, p\iota, \beta_s\}$. Thus we have

$$(4.18).\qquad \{\beta_r, p\iota, \beta_s\}=0 \quad \textit{if } r+s<p \textit{ or } r=s<p.$$

We have no information to compute $\beta_r\circ\beta_s$ for $r, s>2$.

Problem. Is the composition $\beta_r\circ\beta_s$ zero or not $(r, s\geqq 2)$?

§ **Applications for some elementary complexes.**

In this §, N will denote a sufficiently large integer, so that homotopy groups $\pi_{N+i}(S^N\cup\cdots)$ will be stable with respect to suspensions.

i) $S^N\cup e^{N+k+1}$.

Consider a cell complex $K=S^N\cup e^{N+k+1}$ having a characteristic class $\alpha\in G_k$. We may identify $\pi_{N+i}(S^N)$ with G_i and $\pi_{N+i+1}(K, S^N)$ with G_{i-k} (*cf.* [2]). Then we have an exact sequence

$$\cdots \xrightarrow{j_*} G_{i-k} \xrightarrow{\alpha_*} G_i \xrightarrow{i_*} \pi_{N+i}(K) \xrightarrow{j_*} G_{i-k-1} \xrightarrow{\alpha_*} \cdots,$$

where $\alpha_*(\beta)=\alpha\circ\beta$ for $\beta\in G_{i-k}$. Thus it follows the following exact sequence.

$$(4.19).\quad 0 \to G_i/(\alpha\circ G_{i-k}) \xrightarrow{i_*} \pi_{N+i}(K) \xrightarrow{j_*} G_{i-k-1}\cap Ker\,\alpha_* \to 0.$$

Consider an element β of $G_{i-k-1}\cap Ker\ \alpha_*$ and let $\tilde{\beta}$ be an element of $\pi_{N+i}(K)$ such that $j_*(\tilde{\beta})=\beta$. Let r be the order of β, then $j_*(r\tilde{\beta})=\beta\circ r\iota=0$ and thus $r\tilde{\beta}$ is in the image of i_*. By i) of Proposition 4.2,

$$r\tilde{\beta}\in i_*\{\alpha, \beta, r\iota\} \qquad \text{mod. } i_*(rG_i).$$

We have

Lemma 4.19. *For an arbitrary element γ of $\{\alpha, \beta, r\iota\}\subset G_i$, there exists an element $\tilde{\beta}$ of $\pi_{N+i}(K)$ such that $j_*\tilde{\beta}=\beta$ and $r\tilde{\beta}=i_*\gamma$.*

First consider the case $\alpha=p^t\iota$, $t>0$. Then $r=p^s$ for some $s\leqq t$. By ii) of (4.2) and Corollary 4.5, $\{p^t\iota, \beta, p^s\iota\}\supset p^{t-s}\iota\circ\{p^s\iota, \beta, p^s\iota\}=p^{t-s}\iota\circ p^sG_i\ni 0$. It follows from Lemma 4.19, that for any $\beta\in G_{i-k-1}\cap Ker\ (p^t\iota)_*$ of order r there exists an element $\tilde{\beta}\in\pi_{N+i}(K)$ such that $r\tilde{\beta}=0$ and $j_*\tilde{\beta}=\beta$. This means that the sequence (4.19) splits for this case and we have

Proposition 4.20. *If $\alpha=p^t\iota$, $t>0$, then $\pi_{N+i}(S^N\cup_\alpha e^{N+1}) \approx G_i/p^tG_i+\{\beta\,|\,\beta\in G_{i-1},\ p^t\beta=0\}$.*

Next consider the case $k=2p-3$ and $\alpha=\alpha_1\in G_{2p-3}$. Let $i\leqq 2p^2(p-1)-3$. Then, by Theorem 4.15 and Proposition 4.17, $G_i/(\alpha_1\circ G_{i-2p+3})$ is spanned by α_t $(2\leqq t)$, α'_{rp} and $\beta_1^t\circ\beta_s$. Also $G_{i-2p+2}\cap Ker\,\alpha_{1*}$ is spanned by α_t, α'_{rp} and $\alpha_1\circ\beta_1^t\circ\beta_s$. Let $\tilde{\alpha}_t$ and $\tilde{\alpha}'_{rp}$ be elements of $\pi_{N+i}(K)$ such that $j_*\tilde{\alpha}_t=\alpha_t$ and $j_*\tilde{\alpha}'_{rp}=\alpha'_{rp}$.

Proposition 4.21. *For $i\leqq 2p^2(p-1)-3$, the p-component of $\pi_{N+i}(S^N\cup_{\alpha_1} e^{N+2p-2})$ is generated by $\tilde{\alpha}_t$ $(1\leqq t<p^2-1)$, $\tilde{\alpha}'_{rp}$ $(1\leqq r<p)$, $\tilde{\alpha}_1\circ\beta_1^t\circ\beta_s$ $(s+t<p)$ and $i_*(\beta_1^t\circ\beta_s)$ $(s+t<p\,;(s, t)=(1, p-1))$.*

The orders of $\tilde{\alpha}_t$ $(t\not\equiv -1$ (mod p)), $\tilde{\alpha}_{rp-1}$, $\tilde{\alpha}'_{rp}$, $\tilde{\alpha}_1\circ\beta_1^t\circ\beta_s$ and $i_*(\beta_1^t\circ\beta_s)$ are p^2, p^3, p^3, p and p respectively.

The following relations hold for suitable choices of $\tilde{\alpha}_t$ and $\tilde{\alpha}'_{rp}$;

$p\tilde{\alpha}'_{rp}=\alpha_{rp}$, $p\tilde{\alpha}_t=(1/(t+1))\ i_*\alpha_{t+1}$ *for* $t\not\equiv -1 \pmod p$ *and* $p\tilde{\alpha}_{rp-1} =(1/r)\, i_*\alpha'_{rp}$.

Proof. Obviously $j_*(\tilde{\alpha}_1\circ\beta_1^t\circ\beta_s)=\alpha_1\circ\beta_1^t\circ\beta_s$. It follows from (4.19) that $\pi_{N+i}(K)$ is generated by $\tilde{\alpha}_t$, $\tilde{\alpha}'_{rp}$, $\tilde{\alpha}_1\circ\beta_1^t\circ\beta_s$, $i_*\alpha_t$, $i_*\alpha'_{rp}$ and $i_*(\beta_1^t\circ\beta_s)$. The relations follow from Lemma 4.19 and ii) of Proposition 4.17. Thus the proposition is established. q. e. d.

Remark that Proposition 4.20 still holds for $\alpha=x\alpha_1$ with some $x\not\equiv 0 \pmod p$, by multiplying x to the right sides of the last two equations.

Similar discussions imply the following results. The proofs are left to the reader.

(4.20). *Let* $q>1$, $q\not\equiv 0 \pmod p$ *and* $i<2p^2(p-1)-3$. *Then the p-component of* $\pi_{N+i}(S^N\cup_{\alpha_q} e^{N+2q(p-1)})$ *is generated by* $i_*\alpha_t$ $(t<q)$, $i_*\alpha'_{rp}$ $(rp<q)$, $\tilde{\alpha}_t$ $(q+t\not\equiv 0 \pmod p))$, $\tilde{\alpha}_t$ $(q+t\equiv 0 \pmod p))$, $\tilde{\alpha}'_{rp}$, $\tilde{\beta}_s\circ\beta_1^t$, $\tilde{\alpha}_1\circ\beta_1^t\circ\beta_s$, $i_*(\beta_1^t\circ\beta_s)$ *and* $i_*(\alpha_1\circ\beta_1^t\circ\beta_s)$, *the orders of which are* p, p^2, p^2, p^3, p^3, p, p, p *and* p *respectively.*

(4.21). *For* $i<2p^2(p-1)-3$ *and* $1\leqq r<p$, *the p-component of* $\pi_{N+i}(S^N\cup_{\alpha'_{rp}} e^{N+2rp(p-1)})$ *is generated by* $i_*\alpha_t$ $(t<rp)$, $i_*\alpha'_{sp}$ $(s<r)$, $\tilde{\alpha}_t$, $\tilde{\alpha}'_{rp}$, $\tilde{\beta}_s\circ\beta_1^t$, $\tilde{\alpha}_1\circ\beta_1^t\circ\beta_s$, $i_*(\beta_1^t\circ\beta_s)$ *and* $i_*(\alpha_1\circ\beta_1^t\circ\beta_s)$, *the orders of which are* p, p^2, p^2, p^4, p, p, p *and* p *respectively.*

(4.22). *For* $i<2p^2(p-1)-3$, *the p-component of* $\pi_{N+i}(S^N\cup_{\beta_1} e^{N+2p(p-1)-1})$ *is generated by* $i_*\alpha_t$, $i_*\alpha'_{rp}$, $\tilde{\alpha}_t$, $\tilde{\alpha}'_{rp}$, $i_*\beta_s$ *and* $i_*(\alpha_1\circ\beta_s)$, $s>1$, *the orders of which are* p, p^2, p, p^2, p *and* p *respectively.*

ii). $S^N\cup e^{N+k+1}\cup e^{N+k+h+2}$.

Let $K=S^N\cup e^{N+k+1}\cup e^{N+k+h+2}$ be a cell complex such that its subcomplex $L=S^N\cup e^{N+k+1}$ and its quotient complex K/S^N have characteristic classes $\alpha\in G_k$ and $\beta\in G_h$ respectively, with respect to given orientations of the cells of K. α and β have to satisfy

$$\alpha\circ\beta=0\,.$$

The groups $\pi_{N+i}(K)$ will be calculated by means of the following two exact sequences:

$$\cdots \to \pi_{N+i}(L) \longrightarrow \pi_{N+i}(K) \longrightarrow G_{i-k-h-2} \xrightarrow{\partial_1} \pi_{N+i-1}(L) \to \cdots,$$

$$\cdots \longrightarrow G_i \longrightarrow \pi_{N+i}(K) \xrightarrow{j_*} \pi_{N+i}(K/S^N) \xrightarrow{\partial_2} G_{i-1} \longrightarrow \cdots,$$

where we regard that $G_{i-k-h-2}=\pi_{N+i}(K, L)$ and $\pi_{N+i}(K/S^N)=\pi_{N+i}(K, S^N)$. Consider shrinking maps $p_1: L\to L/S^N=S^{N+k+1}$ and $p_2: K/S^N\to K/L=S^{N+k+h+2}$, then as is seen in i), $\pi_{N+i}(L)$ and $\pi_{N+i}(K)$ are calculated by the following two exact sequences:

$$\cdots \longrightarrow G_i \xrightarrow{i_{1*}} \pi_{N+i}(L) \xrightarrow{p_{1*}} G_{i-k-1} \xrightarrow{\alpha_*} G_{i-1} \longrightarrow \cdots,$$

$$\cdots \to G_{i-k-1} \xrightarrow{i_{2*}} \pi_{N+i}(K/S^N) \xrightarrow{p_{2*}} G_{i-k-h-2} \xrightarrow{\beta_*} G_{i-k-2} \to \cdots.$$

To clarify the homomorphisms ∂_1 and ∂_2 the following lemma will be useful.

Lemma 4.22. i). $p_{1*}\partial_1\gamma=\beta\circ\gamma$. *If* $\beta\circ\gamma=0$, *then* $\partial_1\gamma\in i_{1*}\{\alpha, \beta, \gamma\}$.

ii). $\partial_2 i_{2*}\gamma=\alpha\circ\gamma$. *If* $p_{2*}\tilde{\gamma}=\gamma$, *then* $\beta\circ\gamma=0$ *and* $\partial_2\tilde{\gamma}\in -\{\alpha, \beta, \gamma\}$.

This lemma is a direct consequence of Proposition 4.2.

iii). $S^N\cup e^{N+1}\cup e^{N+2p-2}\cup e^{N+2p-1}$.

Let $K=S^N\cup e^{N+1}\cup e^{N+2p-2}\cup e^{N+2p-1}$ be a cell complex such that $L=S^N\cup e^{N+1}$ and $L'=S^N\cup e^{N+2p-2}$ are subcomplexes and that $L, L', K/L=S^{N+2p-2}\cup e^{N+2p-1}$ and $K/L'=S^{N+1}\cup e^{N+2p-1}$ have characteristic classes $p\iota$, $x\alpha_1$, $p\iota$ and $y\alpha_1$ respectively for some integers x and y, with respect to given orientations of the cells of K.

Since N is so large, we identify homotopy groups of a pair with that of their quotient space, for example $\pi_{N+i}(K, L)=\pi_{N+i}(K/L)$. Then we have exact sequences

$$\cdots \to \pi_{N+i}(L) \xrightarrow{i_*} \pi_{N+i}(K) \xrightarrow{j_*} \pi_{N+i}(K/L) \xrightarrow{\partial} \pi_{N+i-1}(L) \to \cdots,$$

$$\cdots \to G_{i-2p+2} \xrightarrow{(p\iota)_*} G_{i-2p+2} \xrightarrow{i_{1*}} \pi_{N+i}(K/L) \xrightarrow{j_{1*}} G_{i-2p+1} \to \cdots,$$

$$\cdots \longrightarrow G_i \xrightarrow{(p\iota)_*} G_i \xrightarrow{i_{2*}} \pi_{N+i}(L) \xrightarrow{j_{2*}} G_{i-1} \longrightarrow \cdots.$$

The last two sequences are clarified in Proposition 4.20.
First we consider the boundary homomorphism ∂.

Lemma 4.23. *There exists an element* $\tilde{\alpha}_t$ *of* $\pi_{N+2(t+1)(p-1)}(K/L)$ *such that* $j_{1*}(\tilde{\alpha}_t)=\alpha_t$ $(1\leqq t<p^2,\ 1\leqq r<p)$ *and*

$$\partial(\tilde{\alpha}_t) = i_{2*}(x+yt/(t+1))\alpha_{t+1} \quad \textit{if}\ t+1\not\equiv 0 \pmod p,$$
$$\partial(\tilde{\alpha}_{rp-1}) = i_{2*}(-y/r)\alpha'_{rp}.$$

Proof. The following diagram is commutative:

$$\begin{array}{ccccc}
 & & \pi_{N+i}(K/L) & \xrightarrow{j_1{}_*} & \pi_{N+i}(S^{N+2p-1})=G_{i-2p+1} \\
 & \nearrow^{i_{1*}} & \downarrow \partial & & \downarrow \partial' \\
G_{i-2p+2}=\pi_{N+i}(S^{N+2p-2}) & \xrightarrow{\partial''} & \pi_{N+i-1}(L) & \xrightarrow{i''_*} & \pi_{N+i-1}(L\cup e^{N+2p-2}) \\
 & & \nwarrow^{i_{2*}} & & \nearrow^{i'_*} \\
 & & & \pi_{N+i-1}(S^N) = G_{i-1}. &
\end{array}$$

The sequence $\xrightarrow{\partial''}\xrightarrow{i''_*}$ is exact. Then the lemma follows from the following results:

$$\partial'(\alpha_t) = i'_*(x+ty/(t+1))\alpha_{t+1}, \quad t+1\not\equiv 0 \pmod p,$$
$$\partial'(\alpha_{rp-1}) = i'_*(-y/r)\alpha'_{rp}.$$

Consider the injection homomorphism $j'_*:\pi_{N+i-1}(L\cup e^{N+2p-2})\to\pi_{N+i-1}(L\cup e^{N+2p-2}/S^N)=\pi_{N+i-1}(S^{N+1}\vee S^{N+2p-2})=G_{i-2}+G_{i-2p+1}$. By the properties of characteristic classes, we have $j'_*(\partial'\iota)=y\alpha_1+p\iota$. It follows then

$$\partial'\iota = y\tilde{\alpha}_1+\widetilde{p\iota},$$

for elements $\tilde{\alpha}_1$ and $\widetilde{p\iota}$ satisfying $j_{1*}\tilde{\alpha}_1=\alpha_1$ and $j_{1*}\widetilde{p\iota}=p\iota$, and they are given as in Proposition 4.2. By Proposition 4.2,

$$\partial'(\alpha_t) = \partial'\iota\circ\alpha_t \in i'_*(\{p\iota,\ y\alpha_1,\ \alpha_t\}+\{x\alpha_1,\ p\iota,\ \alpha_t\}).$$

By Propositions 4.17 and 4.16,

$$\partial'(\alpha_t)-i'_*(x+yt/(t+1))\alpha_{t+1}\in i'_*(pG_{2(t+1)(p-1)-1})$$

or $$\partial'(\alpha_{rp-1})-i'_*(-y/r)\alpha'_{rp}\in i'_*(pG_{2rp(p-1)-1}+x\alpha_1\circ G_{2(rp-1)(p-1)}).$$

Since e^{N+1} is attached to S^N by $p\iota$, $i'_*(pG_{2(t+1)(p-1)-1})=0$. Also, since e^{N+2p-2} is attached to S^N by $x\alpha_1$, $i'_*(x\alpha_1\circ G_{2(rp-1)(p-1)})=0$. Therefore the required relations on ∂' are established. q.e.d.

It follows from the lemma $(1\leqq t<p^2)$

(4.23) $\quad \partial(\tilde{\alpha}_t) = 0$ *if and only if* $(t+1)x+ty\equiv 0 \pmod p$.

In the case $(t+1)x+ty\equiv 0$, $\tilde{\alpha}_t$ and $i_{1*}\alpha_{t+1}(i'_{1*}\alpha'_{rp})$ are not cancelled by ∂', and the groups $\pi_{N+i}(K)$ will be more complicated than the other cases.

Now we consider the case $(t+1)x+ty\equiv 0$ and $x\not\equiv 0$ or $y\not\equiv 0$.

Lemma 4.24. i). *Let $1\leqq t<p^2-1$, $(t+1)x+ty\equiv 0$ and $y\not\equiv 0$. Then, for an element A_t such that $j_*A_t=\tilde{\alpha}_t$, we have $pA_t\neq 0$.*

ii). *Let $1\leqq t<p^2-1$, $(t+1)x+ty\equiv 0$ and $x\not\equiv 0$ or $y\not\equiv 0$. Then for an element A_t' such that $j_*A_t'=i_{1*}\alpha_t$ if $t\not\equiv 0$ and $j_*A_t'=i_{1*}\alpha_t'$ if $t\equiv 0$, we have $pA_t'\neq 0$.*

Proof. i). It is sufficient to prove that $j_{0*}(pA_t)\neq 0$ for $j_{0*}:\pi_{N+i}(K)\to\pi_{N+i}(K/L')$. Applying Proposition 4.21 to our case K/L', i) is proved.

ii). For the case $x\not\equiv 0$, we consider the element A_t' in L' and apply Proposition 4.21, then we see that $pA_t'=i_*((1/(t+1))\alpha_{t+1})$ if $t+1\not\equiv 0$ and $pA_t'=i_*((1/r)\alpha'_{rp})$ if $t+1=rp$. Then ii) is proved for this case. Next let $x\equiv 0$, then $y\not\equiv 0$ and $t\equiv 0$. In this case we may consider that e^{N+2p-2} is attached by a trivial maping, and $L'=S^N\vee S^{N+2p-2}$, $L\cup e^{N+2p-2}=L\vee S^{N+2p-2}$. Consider $\partial':\pi_{N+i}(S^{N-2p-1})\to\pi_{N+i-1}(L\vee S^{N+2p-2})$ and $j'_*:\pi_{N+i-1}(L\vee S^{N+2p-2})\to\pi_{N+i-1}(L/S^N\vee S^{N+2p-2})=G_{i-2}+G_{i-2p+1}$. Then, as is seen in the proof of the previous lemma,

$$\partial'(\alpha'_{rp})=\partial'\iota\circ\alpha'_{rp}\in i'_*(\{p\iota,\ y\alpha_1,\ \alpha'_{rp}\})+i_{0*}(p\alpha'_{rp}),$$

where $i_0:S^{N+2p-2}\to L\vee S^{N+2p-2}$ is the injection. Since $\partial'(\alpha'_{rp})$ gives a relation in K and since $i_{0*}(p\alpha'_{rp})=i_{0*}(\alpha_{rp})$ corresponds to pA'_{rp}, we see that $pA'_{rp}=i_*(-i_{0*}(ry\alpha_{rp+1}))$ by Proposition 4.17, ii). Therefore $pA'_{rp}\neq 0$. q. e. d.

These two lemmas will be applied to investigate the 4-fold iterated suspension $E^4:\pi_i(S^n)\to\pi_{i+4}(S^{n+4})$ in the next section V.

REFERENCES

[1] M. G. Barratt and P. J. Hilton, *On join operations of homotopy groups,* Proc. London Math. Soc. (3), 3 (1953), 430–445.

[2] A. L. Blakers and W. S. Massey, *The Homotopy groups of a triad II,* Ann. of Math., 55 (1952), 192–201.

[3] J-P. Serre, *Groupes d'homotopie et classès de groupes abéliens,* Ann. of Math., 58 (1953), 258–294.

[4] H. Toda, *Generalized Whitehead products and homotopy groups of spheres,* J. of Inst. Poly. Osaka City Univ., 3 (1952), 43–82.

[5] H. Toda, *Reduced join and Whitehead product*, J. of Inst. Poly. Osaka City Univ., 8 (1957), 15–30.

[6] ———, *p-primary components of homotopy groups, I. Exact sequences in Steenrod algebra; II. mod p Hopf invariant; III. Stable groups of the sphere*, these Memoirs, 31 (1958), 129–142: 143–160; 191–210.

[7] G. W. Whitehead, *A generalization of the Hopf invariant*, Ann. of Math., 51 (1950), 192–237.