MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES A
Vol. XXXII, Mathematics No. 2, 1959.

Theory of Abelian integrals and its applications to conformal mappings

By

Yukio KUSUNOKI

(Received Apr. 2, 1959)

Introduction.

The main aim of the present paper is to develop the theory of Abelian integrals on an arbitrary open Riemann surface R. For this purpose we shall introduce in sec. 5 the notion of canonical potentials on R which is a generalization of the normalized potentials. Roughly speaking, a normalized potential takes the constant value zero on the ideal boundary of R, while a canonical one is characterized by the fact that it takes respective real constant value on each ideal boundary component, and canonical differentials are defined as meromorphic differentials derived from canonical potentials. However, the one that attracts our interest particularly is the class $\mathfrak{R}$ of the semi-exact canonical differentials (or integrals of these) which have, by definition, no periods along dividing cycles. Then we are able to establish theorems of Riemann-Roch and Abel on R in terms of elements of $\mathfrak{R}$ which have the analogous formulations as in classical theories. Further finding that the functions of $\mathfrak{R}$ possess an extremal property, we know that our theory have close connections with canonical conformal mappings. Now we show in the following the brief program of this paper.

§ I contains some notes on harmonic measures which are simple fundamental canonical potentials. In § II the definition of canonical differentials is given along with some of their properties. Above all, the uniqueness theorem (Lemma 5) will be powerful for later use. Next, the existence of three kinds of elementary differentials in $\mathfrak{R}$ is proved by using the theory of orthogonal decomposition due to Nevanlinna-Virtanen. Another treatment of this existence theorem will be given in § IV.

§ III concerns with the theorems of Riemann-Roch and Abel on R. The subspace $\mathfrak{K}_0$ of single-valued functions in $\mathfrak{K}$ has the remarkable properties (sec. 11):

(i) If $f \in \mathfrak{K}_0$ is regular everywhere on R, f must be a constant.

(ii) Let the genus of R be finite and q be the number of poles of $f \in \mathfrak{K}_0$, then f is at most q-valent on R.

Thus $\mathfrak{K}_0$ on R seems to play a role corresponding to the totality of single-valued functions on compact surfaces. Further we made clear the relation of these results to my previous ones [3], hence to those in classical theory.

§ IV contains some applications of our theory to conformal mapping. For example, every open Riemann surface of finite genus g can be mapped conformally onto an at most $(g+1)$-time covered plane (Theorem 12). We give a classical parallel slit mapping of a domain of finite or infinite connectivity in explicit form. This is a different point from the extremal method, but it will be immediately shown that the mapping function is identical with the one obtained by the extremal method.

§ I. Preliminaries

Throughout the present paper we shall denote by R an arbitrary open Riemann surface (of finite or infinite genus), unless otherwise stated.

1. Here we shall recall some properties of normalized potentials[1]. Let G be a non-compact domain on R whose relative boundary on R consists of a finite number of analytic Jordan closed curves Γ_0. Let u be a normalized potential on G, which is a single-valued harmonic function on $\bar{G}=G \cup \Gamma_0$ satisfying the "normalization condition"

$$(1.1) \qquad u(P) = \int_{\Gamma_0} u(Q) d\omega(Q, P), \quad P \in G$$

where $d\omega$ stands for the harmonic measure on G with respect to the arc element on Γ_0, then u is also a normalized potential on any subregion G_0 of G with analytic boundaries such that $\bar{G}-G_0$ is compact. Let $df=du+i{}^*du$ be the analytic differential on $\bar{G}$ derived from u and $\varphi=dU+i{}^*dU$ be any analytic differential on G such that

1) Nevanlinna [5] p. 320-333.

$$||\varphi||_G^2 = \frac{i}{2}\iint_G \varphi \wedge \bar{\varphi} < \infty .$$

Then by Green's formula we have

$$(df, \varphi)_G = \frac{i}{2}\iint_G df \wedge \bar{\varphi} = i\int_{\Gamma_0} u\bar{\varphi} \tag{1.2}$$

where the line integral is taken in the positive direction with respect to G. Taking the real parts of (1.2) it follows

$$D_G(du, dU) = \iint_G du \wedge {}^*dU = \int_{\Gamma_0} u^*dU . \tag{1.3}$$

2. For simplicity we shall call an analytic Jordan closed curve (cycle) C on R a *dividing curve* (cycle) if C divides R into two disjoint non-compact domains, and call a compact domain on R an *elementary domain* if the boundary consists of a finite number of dividing curves. Take a dividing curve γ which divides R into R_γ and $\tilde{R}_\gamma$. We consider the harmonic measures ω_γ and $\tilde{\omega}_\gamma$[1] on R with respect to the ideal boundaries of R_γ and $\tilde{R}_\gamma$, which are determined by the homology class of γ. These functions reduce to identically constants (1 or 0) if and only if at least one of the harmonic measures on R_γ, $\tilde{R}_\gamma$ with respect to their ideal boundaries vanishes. Except such cases we have

$$\omega_\gamma + \tilde{\omega}_\gamma = 1 , \quad d\omega_\gamma = -d\tilde{\omega}_\gamma \tag{1.4}$$

The ω_γ (resp. $\tilde{\omega}_\gamma$) is a normalized potential on $\tilde{R}_\gamma$ (resp. R_γ) and have a finite Dirichlet integral over R:

$$D_R(d\omega_\gamma) = \int_\gamma {}^*d\omega_\gamma . \qquad \text{(by (1.3), (1.4))}$$

Let $\varphi = dU + i^*dU$ be any analytic differential on R with finite norm $||\varphi||_R$, then for $\varphi_\gamma = d\omega_\gamma + i^*d\omega_\gamma$ we have by (1.2), (1.3) and (1.4)

$$(\varphi_\gamma, \varphi)_R = i\int_\gamma \bar{\varphi} \tag{1.5}$$

$$D_R(d\omega_\gamma, dU) = \int_\gamma {}^*dU . \tag{1.6}$$

In particular, for any two dividing curves γ and γ'

---

1) Nevanlinna [5] p. 351.

$$D_R(d\omega_\gamma, d\omega_{\gamma'}) = \int_\gamma {}^*d\omega_{\gamma'} = \int_{\gamma'} {}^*d\omega_\gamma . \tag{1.7}$$

3. Let $\gamma_1, \cdots, \gamma_k$ be dividing cycles such that $c_1\gamma_1 + \cdots + c_k\gamma_k$ is homologous to zero. Here we say temporally that $c_1\gamma_1 + \cdots + c_h\gamma_h$ ($h \leq k$) are *weakly homologous to zero* (~ 0) if $d\omega_{\gamma_i} = 0$ ($i = h+1, \cdots, k$). Then we have

$$\varepsilon_1 c_1 d\omega_1 + \cdots + \varepsilon_h c_h d\omega_h \equiv 0 , \tag{1.8}$$

where $\omega_i \equiv \omega_{\gamma_i}$ and $\varepsilon_i = \pm 1$. Especially, if $\gamma_1, \cdots, \gamma_h$ are the boundaries of an elementary domain B and if we take as R_{γ_i} the non-compact domains G_i not containing B, then we have more precisely $\omega_1 + \cdots + \omega_h \equiv 1$. These are seen, for instance, if it is noted that $\omega_i(P)$ can be expressed as $\frac{1}{2\pi}\int_{\gamma_i} \frac{\partial g}{\partial \nu} ds$ $\left(\frac{\partial g}{\partial \nu}\right.$ is the normal derivative of a Green function g on $\left.R\right)$ or $1 - \frac{1}{2\pi}\int_{\gamma_i}\frac{\partial g}{\partial \nu} ds$, according as $P \in \tilde{G}_i$ or $P \in G_i$.

LEMMA 1. *Let dividing curves $\gamma_1, \cdots, \gamma_h$ be homologously independent. Suppose that F is an elementary domain containing the $\gamma_1, \cdots, \gamma_h$ and that $\Gamma_1, \cdots, \Gamma_k$ are all boundaries of F homologously independent each other. Then $h \leq k$ and γ_i can be expressed as*

$$\gamma_i \sim \sum_{j=1}^{k} a_{ij}\Gamma_j, \quad i = 1, \cdots, h \tag{1.9}$$

with non-vanishing determinant $\Delta = |a_{ij}|_{ij=1\cdots h}$. Moreover, then $\Gamma_{h+1}, \cdots, \Gamma_k$ are linearly independent to $\gamma_1, \cdots, \gamma_h$. This lemma remains also true for the above weak homology.

Proof. Suppose $k < h$. Since $\gamma_1, \cdots, \gamma_h \subset F$, the γ_i can be written as (1.9). Now for equations $\sum_{i=1}^{h} a_{ij}c_i = 0$ ($j = 1, \cdots, k$), the space of solutions $(c_1, \cdots, c_h)$ is of dimension $\geq h - k \geq 1$. While, we have then $\sum_{i=1}^{h} c_i\gamma_i \sim 0$ which implies $c_1 = \cdots = c_h = 0$. This is a contradiction. We can analogously prove that at most h boundaries, say $\Gamma_1, \cdots, \Gamma_h$, are the linear combinations of $\gamma_1, \cdots, \gamma_h$ and finally that for $h \leq k$ at least one determinant, say Δ, of rank h is different from zero, q. e. d.

LEMMA 2. *If dividing curves $\gamma_1, \cdots, \gamma_h$ be homologously independent in the weak sense, then $d\omega_1, \cdots, d\omega_h$ are linearly independent.*

Proof. Take an elementary domain F as in Lemma 1. Then it suffices to prove that the Lemma is valid for $\Gamma_1, \cdots, \Gamma_k$. In fact, if it were so, under Lemma 1 and (1.8), (1.9) (where the coefficients $\varepsilon_{ij}=\pm 1$ of $a_{ij}d\omega_{\Gamma_j}$ are determined by Γ_j independently to γ_i, as γ_i are contained in F (cf. the integral representation of ω_{Γ_j}), hence we write $\varepsilon_{ij}=\varepsilon_j'$) the relation

$$\sum_{i=1}^{h} c_i d\omega_i = \sum_{j=1}^{k} \varepsilon_j'(\sum_{i=1}^{h} \varepsilon_i c_i a_{ij}) d\omega_{\Gamma_j} = 0$$

implies that $\sum_{i=1}^{h} \varepsilon_i c_i a_{ij}=0$ $(j=1, \cdots, k\geq h)$, hence $c_1= \cdots =c_h=0$ on account of $\Delta\neq 0$, i.e. $d\omega_1, \cdots, d\omega_h$ would be independent. Hence we suppose from the beginning that $\gamma_1, \cdots, \gamma_h$ are the boundaries of an elementary domain F and $c_1\omega_1+ \cdots +c_h\omega_h\equiv c$, where in G_i $(\partial G_i=\gamma_i, G_i\not\supset F)$ $(i=1, \cdots, h)$ the $\omega_1, \cdots, \omega_{i-1}, \tilde{\omega}_i, \omega_{i+1}, \cdots, \omega_h$ are normalized potentials. Hence for $j\neq i$

$$\omega_j(P) = \int_{\gamma_i} \omega_j(Q) d\Omega_i(Q, P), \qquad P\in G_i \tag{1.10}$$

where $d\Omega_i$ denotes the harmonic measure on G_i with respect to the arc element on γ_i. While for $P\in G_i$

$$\omega_i(P) = 1-\tilde{\omega}_i(P) = 1-\Omega_i(P, \gamma_i)+ \int_{\gamma_i} \omega_i(Q) d\Omega_i(Q, P)\,,$$

therefore

$$c = \sum_{i=1}^{h} c_i\omega_i = c_i+(c-c_i)\Omega_i(P, \gamma_i)\,, \quad P\in G_i\,.$$

Since $\Omega_i(P, \gamma_i)\not\equiv$const., we have $\inf_{P\in G_i} \Omega_i(P, \gamma_i)=0$, hence there exists a sequence of points $\{P_n\}$ on G_i for which $\Omega_i(P_n, \gamma_i)\to 0$ $(n\to\infty)$. We find therefore $c=c_i$ $(i=1, \cdots, h)$. Now since $\gamma_1, \cdots, \gamma_h$ are weakly independent, there is at least another boundary curve, γ_{h+1}, of F for which $\Omega_{h+1}(P, \gamma_{h+1})\not\equiv$ const. Then in G_{h+1} (1.10) is valid for $j=1, \cdots, h$. Thus we have $c=c\Omega_{h+1}(P, \gamma_{h+1})$ from which we conclude $c=0$, i.e. $c_1= \cdots =c_h=0$, q.e.d.

Now we consider dividing curves $\gamma_1, \cdots, \gamma_k$ homologously independent in the weak sense. Then for any real numbers $x_1, \cdots, x_k$ not simultaneously zero, we have by Lemma 2 and (1.7)

$$0< D_R(x_1 d\omega_1+ \cdots +x_k d\omega_k) = \sum_{i,j=1}^{k} x_i x_j D(d\omega_i, d\omega_j)$$
$$= \sum_{i,j} x_i x_j \int_{\gamma_i} {}^*d\omega_j$$

which shows that the last symmetric quadratic form is positive definite. Hence it follows

$$(1.11) \qquad \Delta_k = \Delta(\gamma_1, \cdots, \gamma_k) = \det. \left| \int_{\gamma_i} {}^*d\omega_j \right|_{ij=1\cdots k} > 0 .$$

4. Let H denotes the space consisting of analytic differentials square integrable on R. Then H constitutes a Hilbert space by the inner product

$$(\varphi, \psi) = \frac{i}{2} \iint_R \varphi \wedge \bar{\psi}, \quad \varphi, \psi \in H .$$

Let H_1 be the subspace of H composed of total differentials and H_2 be the subspace of H whose elements are orthogonal to H_1 and have no periods along dividing cycles. The H_1 and H_2 are also Hilbert spaces. Then Nevanlinna-Virtanen's orthogonal decomposition theorem asserts

$$(1.12) \qquad H = H_1 \oplus H_2 \oplus H_3 ,$$

where the H_2 and H_3, the orthogonal complement of $H_1 \oplus H_2$, have respectively the following another interpretation. That is, let $\{A_n, B_n\}$ $(n=1, 2, \cdots)$ be the canonical homology basis and $\{\gamma_n\}$ $(n=1, 2, \cdots)$ be the basis to dividing cycles on R, then the space H_3 is spread by differentials

$$(1.13) \qquad \varphi_n \equiv \varphi_{\gamma_n} = d\omega_n + i^*d\omega_n, \quad \omega_n = \omega_{\gamma_n}, \quad n=1, 2, \cdots ,$$

and the H_2 by the "normals" toward H_3 of elementary differentials of the first kind

$$(1.14) \quad \varphi_{A_n} = du_{A_n} + i^*du_{A_n}, \quad \varphi_{B_n} = du_{B_n} + i^*du_{B_n}, \quad n = 1, 2, \cdots ,$$

where u_{A_n} (resp. u_{B_n}) is a normalized potential outside of A_n (resp. B_n) and has the only non-vanishing period 1 (resp. -1) along B_n (resp. A_n).

Now the consideration in the preceding section shows that H_3 consists of $\varphi_n = \varphi_{\gamma_n}$ associated with dividing curves γ_n homologously independent in the weak sense and those φ_n are linearly independent each other.

Finally we note that in this paper we shall deal with differentials square integrable outside of a compact set E containing the possible singularities for which the sums of residues vanish, and so, as is well known, those have no periods along dividing cycles $\gamma \in R-E$ for which $d\omega_\gamma=0$.

§ II. Canonical differentials.

5. Now we shall slightly generalize the notion of normalized potentials. Let K be an elementary domain on R, then a harmonic function on R is called a *canonical potential associated with* K, if on each complementary domain of K it is a normalized potential except a possible real constant. (The constants may be distinct for each domain). It may have in K a finite number of additive periods and singularities. Let $T=T_0+T_1$ be the real vector space of canonical potentials associated with elementary domains, where the subspace T_0[1] consists of those single-valued, regular on R. While, T_0 is at the same time a subset of the Hilbert space consisting of single-valued harmonic functions u with finite norms $\|u\|=D_R(du)$. Let $\tilde{T}_0$ be the completion of T_0 by this metric and $\tilde{T}=\tilde{T}_0+T_1\supset T$. Now we shall call any element of $\tilde{T}$ a *canonical potential* (on R). We note that for any $u\in\tilde{T}_0$ $D_R(du)<\infty$ and there exists a sequence $\{u_n\}$ $(u_n\in T_0)$ such that $D_R(du_n-du)\to 0$ for $n\to\infty$ and the convergence $u_n\to u$ is uniform on every compact set on R. The Abelian differentials φ such that $Re\int\varphi$ are, except constants, canonical potentials are called *canonical differentials*, provided that the sums of residues vanish.

For our later purposes we prepare some lemmas related to canonical potentials.

LEMMA 3. *Let $u\in T$ be a canonical potential associated with an elementary domain B bounded by dividing curves $\gamma_1, \cdots, \gamma_k$. Let $dU+i{*dU}$ be any differential square integrable over $R-B$ such that $\int_{\gamma_i}{*dU}=0$ $(i=1, \cdots, k)$, then we have*

$$D_{R-B}(du, dU) = \int_{\partial B} u{*dU}.$$

Proof. Let G_i be the non-compact domains adjacent B along γ_i. Since u can be written as u_i+c_i in G_i, where u_i are normalized potentials and c_i constants, we have by (1.3)

1) Two elements of T_0 are identified if the difference is a constant.

$$\int_{\partial B} u^*dU = \sum_{i=1}^{k}\int_{\gamma_i}(u_i+c_i)^*dU = \sum_i \int_{\gamma_i} u_i{}^*dU$$
$$= \sum_i D_{G_i}(du_i, dU) = D_{R-B}(du, dU)\,, \qquad \text{q. e. d.}$$

LEMMA 4. *Let $u\in\tilde{T}$ be a canonical potential on R and U be a harmonic function such that dU is square integrable outside of a compact set B and $\int_\gamma {}^*dU=0$ for every dividing curve $\gamma\subset R-B$, then for any exhaustion $\{R_n\}$ of R we have*

$$\lim_{n\to\infty}\int_{\partial R_n} u^*dU = 0\,.$$

Proof. We prove the case $u\in\tilde{T}_0$. For any element of T_1 it is more easily proved. Let $\{u_m\}$ be a sequence of functions in T_0 such that u_m converge uniformly to u and $D_R(du_m-du)\to 0$ for $m\to\infty$. Let $\{B_m\}$ be a sequence of corresponding elementary domains. We may assume that $\{B_m\}$ is an exhaustion of R. Otherwise, take an exhaustion $\{B'_m\}$ of elementary domains such that $B_m\subset B'_m$, then u_m would be also canonical potentials associated with B'_m (cf. sec. 1). Now for sufficiently large m $B_m\supset R_n$ (for fixed n) and we have by Lemm 3

$$(2.1)\quad D_{R-R_n}(du_m, dU) = D_{Bm-R_n}(du_m, dU)+D_{R-B_m}(du_m, dU)$$
$$= \int_{\partial R_n+\partial B_m} u_m{}^*dU+\int_{\partial B_m} u_m{}^*dU = \int_{\partial R_n} u_m{}^*dU\,.$$

While by Schwarz's inequality

$$(2.2)\qquad |D_{R-R_n}(du_m, dU)|^2\leq D_{R-R_n}(du_m)D_{R-R_n}(dU)$$
$$\leq D_R(du_m)D_{R-R_n}(dU)\leq (D_R(du)+\varepsilon)D_{R-R_n}(dU)\,.$$

Since $D_{R-B}(dU)<\infty$, we have therefore by (2.1) and (2.2) the desired result for $m\to\infty$ successively, $n\to\infty$, q. e. d.

After Ahlfors we say that a (meromorphic) differential on R is *semi-exact* if it has no periods along every dividing cycle on R. Then as corollary to Lemma 4 we get immediately the following

LEMMA 5 (*Uniqueness theorem*) *Let $\varphi=du+i^*du$ be a semi-exact canonical differential such that u is single-valued, regular on R, then φ is identically zero.*

LEMMA 6. *Let $df_j=du_j+idv_j$ $(j=1, 2)$ be any two semi-exact canonical differentials on R such that u_j are single-valued and regular* *outside of a compact domain B, then for every dividing curve $C \subset R-B$ we have*

$$Im \int_C f_1 df_2 = 0 .$$

Proof. Let $\{B_n\}$ be a sequence of elementary domains which exhaust a non-compact domain ($\supset B$) adjacent to C. Then by Riemann's first period relation[1] under our assumptions we have

$$Im \int_C f_1 df_2 = Im \int_{\Gamma_n} f_1 df_2 = \int_{\Gamma_n} v_1 du_2 + u_1 dv_2$$

where $C+\Gamma_n = \partial B_n$. Under our assumptions it is easily seen that $\int_{\Gamma_n} v_1 du_2 = -\int_{\Gamma_n} u_2 dv_1$ and Lemma 4 is also valid for such a component $\bigcup_{n=1}^{\infty} B_n$. Hence we have

$$\lim_{n\to\infty} \int_{\Gamma_n} v_1 du_2 = \lim_{n\to\infty} \int_{\Gamma_n} u_1 dv_2 = 0 , \qquad \text{q. e. d.}$$

6. To get the following fundamental Abelian differentials we consider the space H_3[2], and construct a complete orthonormal system $\{\phi_n\}$ in H_3 by Schmidt's method;

$$(2.3) \qquad \phi_1 = \varphi_1/||\varphi_1|| , \quad \phi_k = g_k/||g_k|| , \quad k = 2, 3, \cdots$$

where $g_k = \varphi_k - \sum_{s=1}^{k-1} (\varphi_k, \phi_s)\phi_s$. Now ϕ_k is a linear combination of $\varphi_1, \cdots, \varphi_k$ with real coefficients. In fact, suppose that in the expressions

$$(2.4) \qquad \phi_s = c_1^s \varphi_1 + \cdots + c_s^s \varphi_s \qquad (s = 1, \cdots, k-1)$$

all coefficients c_i^s are real, then in

$$\phi_k = ||g_k||^{-1}\{\varphi_k - \sum_{s=1}^{k-1} a_s (\sum_{j=1}^{s} c_j^s \varphi_j)\}$$

the constants a_s become real, for by (1.5)

$$a_s = \sum_{j=1}^{s} c_j^s (\varphi_k, \varphi_j) = \sum_{j=1}^{s} c_j^s \int_{\gamma_k} {*d\omega_j} .$$

Therefore by induction we find that ϕ_k ($k=1, 2, \cdots$) are linear

1) Cf. Kusunoki [2].

2) We suppose H_3 is not empty: $H_3 \neq \phi$.

combinations of $\varphi_1, \cdots, \varphi_k$ with real coefficients, and so $Re\int\phi_k$ are canonical potentials associated with an elementary domain containing the cycles $\gamma_1, \cdots, \gamma_k$.

Now we take any one cycle, say A, of the basis $\{A_n, B_n\}$. Let $\tilde{\varphi}_A$ and $\hat{\varphi}_A$ denote respectively the normal and projection of φ_A (cf. (1. 14)) to the space H_3:

$$\varphi_A = \tilde{\varphi}_A + \hat{\varphi}_A ,$$

then $\hat{\varphi}_A \in H_3$ can be written as

$$\hat{\varphi}_A = \sum_{n=1}^{\infty} b_n \phi_n$$

where the coefficients b_n become real, because by (1. 2) and (2. 4) with real c_i^s $(s=1, 2, \cdots)$

$$b_n = (\hat{\varphi}_A, \phi_n) = (\varphi_A, \phi_n) = \sum_{j=1}^{n} c_j^n (\varphi_A, \varphi_j) = \sum_{j=1}^{n} c_j^n \int_A {}^*d\omega_j .$$

It follows therefore that $\tilde{\varphi}_A = \varphi_A - \sum_{n=1}^{\infty} b_n \phi_n$ is a canonical differential, moreover semi-exact on account of $\tilde{\varphi}_A \in H_2$ and $Re\int_C \tilde{\varphi}_A = Re\int_C \varphi_A$ for any cycle c. Thus we have

THEOREM 1. *Let $\{A_n, B_n\}$ be the canonical homology basis on R. Then there exist semi-exact canonical differentials $\varphi^*_{A_n}, \varphi^*_{B_n}$ $(n=1, 2, \cdots)$ of the first kind such that $Re\,\varphi^*_{A_n}$, $Re\,\varphi^*_{B_n}$ have only non-vanishing periods $+1$ and -1 along B_n and A_n respectively.*

Proof. In case of $H_3 \neq \phi$, $\varphi^*_{A_n} = \tilde{\varphi}_{A_n}$, $\varphi^*_{B_n} = \tilde{\varphi}_{B_n}$ satisfy the required conditions. If $H_3 = \phi$ i. e. $d\omega_i \equiv 0$ for the basis $\{\gamma_i\}$ of dividing cycles it is enough to take $\varphi^*_{A_n} = \varphi_{A_n}$, $\varphi^*_{B_n} = \varphi_{B_n}$, for by a remark in sec. 4 those become semi-exact.

COROLLARY. *In the decomposition $H = H_1 \oplus H_2 \oplus H_3$ the space H_2 is composed of semi-exact canonical differentials $\varphi^*_{A_n}, \varphi^*_{B_n}$ $(n=1, 2, \cdots)$.*

Proof. Since $Re\int(\varphi^*_{A_n} - \tilde{\varphi}_{A_n})$, $Re\int(\varphi^*_{B_n} - \tilde{\varphi}_{B_n})$ are single-valued and regular on R, it follows $\varphi^*_{A_n} = \tilde{\varphi}_{A_n}$, $\varphi^*_{B_n} = \tilde{\varphi}_{B_n}$ by the uniqueness theorem (Lemma 5).

7. To obtain other kinds of fundamental differentials we need another complete orthonormal system in H_3 which is constructed by the normalization of periods along dividing cycles. Let ψ'_n be a linear combination of $\varphi_1, \cdots, \varphi_n$ with real coefficients such that

(2. 5) $$\psi'_n = a_1^n\varphi_1 + \cdots + a_n^n\varphi_n .$$

Here we choose these constants so that the conditions

(2. 6) $$\begin{cases} \int_{\gamma_j} \psi'_n = 0 \quad \text{for } j = 1, \cdots, n-1 \\ \int_{\gamma_n} \psi'_n = i \end{cases}$$

that is,

$$\begin{cases} \sum_{i=1}^{n} a_i^n \int_{\gamma_j} *d\omega_i = 0 \quad \text{for } \ j=1, \cdots, n-1 \\ \sum_{i=1}^{n} a_i^n \int_{\gamma_n} *d\omega_i = 1 \end{cases}$$

are fulfilled. This is possible, indeed, by (1. 11) we have

(2. 7) $$a_j^n = \Delta_n^j / \Delta_n \ (j = 1, \cdots, n-1), \ a_n^n = \Delta_{n-1} / \Delta_n$$

where Δ_n^j denote the cofactors of (n, j)-elements in Δ_n. Note that a_n^n is positive. Now it is seen that for $m \neq n$

$$(\psi'_m, \psi'_n) = 0 .$$

Indeed, by (1. 5) and (2. 6) we have for $j<m$ $(\varphi_j, \psi'_m) = i \int_{\gamma_j} \overline{\psi'_m} = 0$. Hence in case of $m>n$

$$(\psi'_n, \psi'_m) = \sum_{j=1}^{n} a_j^n (\varphi_j, \psi'_m) = 0 .$$

In the case $m<n$ we have also $(\psi'_n, \psi'_m) = \overline{(\psi'_m, \psi'_n)} = 0$. Next, for $m=n$ we have

$$||\psi'_n||^2 = \sum_{j=1}^{n} a_j^n (\varphi_j, \psi'_n) = a_n^n$$

which shows again $a_n^n > 0$. Thus, the system $\{\psi_n\}$ where $\psi_n = \psi'_n / \sqrt{a_n^n}$ constitutes a complete orthonormal system in the space H_3.

THEOREM 2. *Let φ be any differential of the space H and*

$$\int_{\gamma_n} \varphi = c_n, \ n = 1, 2, \cdots ,$$

then the series

(2. 8) $$\sum_{n=1}^{\infty} \left| \sum_{j=1}^{n} \frac{a_j^n c_j}{\sqrt{a_n^n}} \right|^2$$

is convergent, where a_j^n are given by (2. 7).

Proof. By the orthogonal decomposition theorem φ can be written as

$$\varphi = \varphi_1^* + \varphi_2^* + \varphi_3^*, \quad \varphi_i^* \in H_i \quad (i = 1, 2, 3)$$

Since φ_1^* and φ_2^* have no period along any γ_n, we have

$$\int_{\gamma_n} \varphi = \int_{\gamma_n} \varphi_3^* = c_n .$$

Using the above constructed orthonormal system

$$\varphi_3^* = \sum_{n=1}^{\infty} \alpha_n \psi_n$$

where $\alpha_n = (\varphi_3^*, \psi_n)$ and the Bessel inequality

$$\sum_{n=1}^{\infty} |\alpha_n|^2 \leq \|\varphi_3^*\|^2 < \infty$$

holds. While, we have

$$\alpha_n = \overline{(\psi_n, \varphi_3^*)} = \sum_{j=1}^{n} \frac{a_j^n}{\sqrt{a_n^n}} \overline{(\varphi_j, \varphi_3^*)} = -i \sum_{j=1}^{n} \frac{a_j^n}{\sqrt{a_n^n}} \int_{\gamma_j} \varphi_3^* = -i \sum_{j=1}^{n} \frac{a_j^n c_j}{\sqrt{a_n^n}},$$

q. e. d.

Conversely, we have the following

THEOREM 3. *Let $\{c_n\}$ be a sequence of given complex numbers such that the series* (2. 8) *is convergent. Then there exists a differential $\varphi \in H_3 (\subset H)$ such that*

$$\int_{\gamma_n} \varphi = c_n, \quad n = 1, 2, \cdots$$

Proof. By Riesz-Fischer's theorem the differential

$$\varphi^* = \sum_{n=1}^{\infty} \left(-i \sum_{j=1}^{n} \frac{a_j^n c_j}{\sqrt{a_n^n}} \right) \psi_n$$

represents an element of H_3 such that

$$-i \sum_{j=1}^{n} \frac{a_j^n c_j}{\sqrt{a_n^n}} = (\varphi^*, \psi_n) .$$

Let $\int_{\gamma_n} \varphi^* = c_n'$, then we have

$$\sum_{j=1}^{n} \frac{a_j^n c_j}{\sqrt{a_n^n}} = \sum_{j=1}^{n} \frac{a_j^n c_j'}{\sqrt{a_n^n}} \qquad n = 1, 2, \cdots$$

Therefore we have successively $c_j = c_j'$ $(j = 1, 2, \cdots)$, q. e. d.

THEOREM .4 *Let for a differential* $\varphi \in H$

$$\int_{\gamma_n} \varphi = a_n + ib_n$$

with real a_n *and* b_n, *then there exist differentials* $\Phi, \Psi \in H_3(\subset H)$ *such that*

$$\int_{\gamma_n} \Phi = a_n, \quad \int_{\gamma_n} \Psi = ib_n, \quad n = 1, 2, \cdots$$

Proof. Under our assumption the series

$$\sum_{n=1}^{\infty} \left| \sum_{j=1}^{n} \frac{a_j^n}{\sqrt{a_n^n}} a_j \right|^2 \quad \text{and} \quad \sum_{n=1}^{\infty} \left| \sum_{j=1}^{n} \frac{a_j^n}{\sqrt{a_n^n}} b_j \right|^2$$

are convergent, q. e. d.

8. Let P be a point given on R. Then there exists an Abelian differential φ_P[1] such that φ_P has a given singularity S_P of the second kind at P and $Re \int \varphi_P$ is single-valued and satisfies the normalization condition on $R - U_P$ (U_P is a neighborhood of P). Let φ'_P denotes the corresponding differential with the singularity $-iS_P$, then $\hat{\varphi}_P = i\varphi'_P$ has the same singularity as φ_P and $Im \int \hat{\varphi}_P$ is single-valued. Then

$$\phi = \varphi_P - \hat{\varphi}_P$$

has no singularity and hence belongs to the space H. Let

$$\int_{\gamma_n} \phi = \alpha_n + i\beta_n, \quad n = 1, 2, \cdots$$

then by Theorem 4 there exists a differential $\Psi \in H_3$ such that

$$\int_{\gamma_n} \Psi = i\beta_n$$

Here we note that since all coefficients in the expression

$$(2.9) \quad \Psi = \sum_{n=1}^{\infty} \left(\sum_{j=1}^{n} \frac{a_j^n \beta_j}{a_n^n} \right) \psi'_n = \sum_{n=1}^{\infty} \left(\sum_{j=1}^{n} \frac{a_j^n \beta_j}{a_n^n} \sum_{i=1}^{n} a_i^n \varphi_i \right), \quad \beta_n = -i \int_{\gamma_n} \varphi_P$$

are real, Ψ is a canonical differential. Finally if we set

$$(2.10) \qquad \Phi_P = \varphi_P - \Psi$$

1) Nevanlinna [5] p. 332.

we find that Φ_P is also canonical and semi-exact, moreover, $Re\int \Phi_P$ is single-valued on R. Therefore we have the following theorem, whose remaining part can be proved analogously.

THEOREM 5. *Let P and Q be given points on R, then there exist the semi-exact canonical differentials $\psi_P^{(\mu)*}$, $\tilde{\psi}_P^{(\mu)*}$[1] of the second kind whose integrals have single-valued real parts, and singularities such as $1/z^\mu$ and i/z^μ ($\mu\geq 1$) at $P(z)$ respectively. Also there exist the semi-exact canonical differentials ϕ_{PQ}^*, $\tilde{\phi}_{PQ}^*$ of the third kind whose integrals have single-valued real parts except an arc PQ, and logarithmic singularities at P and Q with residues -1, $-i$ (at P) and $+1$, $+i$ (at Q) respectively.*

§ III. Riemann-Roch's theorem and Abel's theorem.

9. We shall denote by $\mathfrak{R}$ the class of semi-exact canonical differentials (or integrals of these) on R. Then with differentials and integrals (functions) of $\mathfrak{R}$ we shall establish Riemann-Roch's and Abel's theorems on an arbitrary open Riemann surface R, which have the analogous formulations as in the compact case. The fundamental tool in proof is the following bilinear relation.

THEOREM 6. *Let $d\Omega$ (1st or 2nd kind) and φ be any two differentials of $\mathfrak{R}$ which, outside of an elementary domain B, have only pure imaginary periods and no singularities, then we have*

$$Im\,[2\pi i \sum Res.\, \varphi\Omega] = Im\left[\sum_{i=1}^{p}\int_{A_i}\varphi\int_{B_i}d\Omega - \int_{B_i}\varphi\int_{A_i}d\Omega\right] \tag{3.1}$$

where p denotes the genus of B.

This is immediately derived under the use of Lemma 6 (cf. Kusunoki [2]).

10. Let W be a fixed elementary domain of genus p on R and δ be a given finite divisor on W such that

$$\delta = \frac{\delta_{(P)}}{\delta_{(Q)}} = \frac{P_1^{m_1}P_2^{m_2}\cdots P_r^{m_r}}{Q_1^{n_1}Q_2^{n_2}\cdots Q_s^{n_s}}, \quad G = \sum_{i=1}^{r} m_i - \sum_{j=1}^{s} n_j\,. \tag{3.2}$$

With W and δ we associate the following four vector spaces in the real field;

---

1) If $H_3=\phi$, it is sufficient to take $\psi_P^{(\mu)*}=\psi_P^{(\mu)}$ etc.

E: The vector space consisting of differentials $\varphi = du + idv \in \mathfrak{R}$ which are multiples of $1/\delta_{(Q)}$ and u are single-valued on $R-W$.

D: The vector space of differentials $\in E$ which are multiples of δ.

M: The vector space of functions $\Omega \in \mathfrak{R}$ which are multiples of $1/\delta_{(P)}$ and $Re\,\Omega$ are single-valued on R. In case of $\delta_{(Q)} \neq 1$ we normalize so that $\Omega(Q_1)=0$.

S: The vector space of functions $\in M$ which are multiples of $1/\delta$ and single-valued on W.

Obviously $D \subset E$, $S \subset M$. It is seen by the uniqueness theorem that the E is composed of differentials $\varphi^*_{A_n}$, $\varphi^*_{B_n}$ (Theorem 1) $\in \mathfrak{R}$ ($n=1, \cdots, p$), $\psi^{(\mu-1)*}_{Q_\nu}$, $\tilde{\psi}^{(\mu-1)*}_{Q_\nu} \in \mathfrak{R}$ ($\mu=2, \cdots, n_\nu$, $\nu=1, \cdots, s$) and $\phi^*_{Q_1 Q_2}$, $\tilde{\phi}^*_{Q_1 Q_2}$, $\cdots$, $\phi^*_{Q_1 Q_s}$, $\tilde{\phi}^*_{Q_1 Q_s} \in \mathfrak{R}$ (Theorem 5) and the M consists of constants and integrals $\int \psi^{(\mu)*}_{P_\nu}$, $\int \tilde{\psi}^{(\mu)*}_{P_\nu} \in \mathfrak{R}$ ($\mu=1, \cdots, m_\nu$, $\nu=1, \cdots, r$)[1]. Hence

$$\dim E = \begin{cases} 2(\sum n_j + p - 1) \\ 2p \text{ if } \delta \text{ is integral.} \end{cases} \qquad \dim M = \begin{cases} 2\sum m_i \\ 2(\sum m_i + 1) \text{ if } \delta \text{ is integral.} \end{cases}$$

Thus by the same reasoning as before (cf. [3]) we can prove the following theorems.

THEOREM 7.[2] *Let W be an elementary domain of genus p on R and E, D, M and S be the spaces associated with W and divisor* (3.2) *of total order G, then the orthogonal space of D (resp. S) in the dual space E^* (resp. M^*) is identical with the quotient space M/S (resp. E/D), in other words, M/S and E/D are mutually dual. Thus we have*

$$A = B + 2(G - p + 1) \tag{3.3}$$

where B (resp. A) denotes the dimension of D (resp. S).

THEOREM 8 (*Riemann-Roch's theorem on R*) *Let δ be a divisor $\dfrac{P_1^{m_1} \cdots P_r^{m_r}}{Q_1^{n_1} Q_2^{n_2} \cdots}$ $(m = \sum_{i=1}^{r} m_i < \infty)$ given on R and $\{R_n\}$ be an exhaustion of elementary domains. Let $\delta_n = \delta \cap R_n$ be the restrictions of δ in R_n whose total orders are $m - \kappa_n$, then for sufficiently large $n (> N)$*

$$A = B_n + 2(m - \kappa_n - p_n + 1) \tag{3.4}$$

---

1) If $\delta_{(Q)} \neq 1$, we normalize so that these integrals vanish at Q_1, hence in this case M consists of these integrals only.

2) This has a similar form to Köthe's duality theorem on R which holds, roughly speaking, between single-valued analytic functions on W and differentials on $R-W$.

where p_n denote the genus of R_n and B_n the number of linearly independent (in the real sense) differentials $\varphi \in \mathfrak{K}$ which are multiples of δ_n and $Re\int\varphi$ are single-valued on $R-R_n$, and A the number of linearly independent functions $\in \mathfrak{K}$ which are single-valued on R and multiples of $1/\delta$.

In particular, if the genus p of R is finite, then for a divisor (3. 2) *on R we have the relation* (3. 3) *and, here, B is the number of linearly independent (in the real sense) differentials which are multiples of δ and A the number of linearly independent functions $\in \mathfrak{K}$ which are single-valued on R and multiples of $1/\delta$.*

11. To illustrate the analogies to the classical theory we take up a subcalass $\mathfrak{K}_0$ of $\mathfrak{K}$ consisting of single-valued meromorphic functions on R. The functions of $\mathfrak{K}_0$ have the following remarkable properties:

(i) *If $f \in \mathfrak{K}_0$ is regular everywhere on R, then f must be a constant.*

(ii) *Let the genus of R be finite and q be the number (counted with multiplicities) of poles of $f \in \mathfrak{K}_0$, then f is at most q-valent on R*[1].

(i) is a direct consequence of Lemma 5. (ii) will be proved in § IV. Any single-valued function on a closed Riemann surface takes every complex value (included ∞) the same times and also have the property (i). On general R these are not valid even if the functions belong to an important restricted class $\mathfrak{D}$[2]. While $\mathfrak{K}_0$ considered in Theorem 8 seems to play a role corresponding to the totality of single-valued functions on a compact surface.

Further, the connections to my previous results [3], hence to those in classical theory, are as follows. Let O_{KD} be the class of Riemann surfaces introduced by Sario. That is, on any surface $\in O_{KD}$ there is no single-valued non-constant harmonic function with a finite Dirichlet integral, whose conjugate has no periods along dividing cycles. Obviously $O_{HD} \subsetneq O_{KD}$. It is known (Sario [6]) that $O_{KD} = S_Q$, where S_Q is the class of Riemann surfaces with vanishing Q-span, moreover for planar surface $S_Q = O_{AD} \supsetneqq O_{HD}$. We note here that every single-valued meromorphic function $f \in \mathfrak{D}$

1) I conjecture further that f may map R conformally onto a q-times covered plane with slits parallel to the imaginary axis.

2) Kusunoki [3].

defined on $R \in O_{KD}$ becomes canonical. Indeed, since the sum of residues of df vanishes, we can construct a semi-exact differential $\varphi \in \mathfrak{K}$ with the same singularities as df. Now $Re\left(f-\int\varphi\right)$ is a single-valued function with a finite Dirichlet integral, moreover $df-\varphi$ is semi-exact, hence $df \equiv \varphi$. Thus we have

THEOREM 9. *Let R be an open Riemann surface $\in O_{KD}$ ($\supsetneqq O_{HD}$), then we have the formula* (3.4), *where the $A/2$ denotes the number of single-valued meromorphic functions $\in \mathfrak{D}$ on R which are multiples of $1/\delta$ and linearly independent in the complex sense, and B_n $(n>N)$, i.e. the dimensions of $D(R_n)$ become even. Further if $R \in O_{HD}$, $\mathfrak{K}=\mathfrak{D}$ i.e. we have Theorem* 2.1 *in* [3].

12. Abel's theorem. As an extension of Abel's theorem to an open Riemann surface R it is known (Behnke-Stein) that for any two sequences $\{P_n\}$ and $\{Q_n\}$, clustering nowhere on R, there exists a single-valued meromorphic function on R with just prescribed zeros P_n and poles Q_n. But from our point of view we can state here an analogous extension of the classical Abel's theorem. Let $\hat{\mathfrak{K}}$ be a class of single-valued meromorphic functions on R which can be written as $\exp.\int\varphi$, $\varphi \in \mathfrak{K}$.

THEOREM 10.[1] *The necessary and sufficient condition for the existence of a single-valued meromorphic function $f \in \hat{\mathfrak{K}}$ on R possessing a finite number of zeros P_ν and poles Q_ν ($\nu=1, \cdots, n$) is that the conditions*

$$(3.5) \qquad Re\sum_{\nu=1}^{n}\{\Phi_{\substack{A_\mu\\B_\mu}}(P_\nu)-\Phi_{\substack{A_\mu\\B_\mu}}(Q_\nu)\} = 0 \quad (\text{mod. } 1) \;\; \mu = 1, 2, \cdots$$

hold for the integrals $\Phi_{A_\mu}=\int\varphi^*_{A_\mu}$, $\Phi_{B_\mu}=\int\varphi^*_{B_\mu} \in \mathfrak{K}$ of the first kind.

Proof. If such a function f exists, then $d\log f=\frac{f'}{f}dz \equiv \varphi^* \in \mathfrak{K}$ is an Abelian differential of the third kind such that $\int_{A_\mu}\varphi^* = i\int_{A_\mu} d\arg f=0$, $\int_{B_\mu}\varphi^*=0$ (mod. $2\pi i$) and $\underset{P_\nu}{\text{Res.}}\,\varphi^*=m_\nu$, $\underset{Q_\nu}{\text{Res.}}\,\varphi^*=-n_\nu$, where m_ν and n_ν denote the multiplicities of φ^* at P_ν, Q_ν respectively. Hence (3.5) is obtained from the period relation (3.1) between φ^* and $d\Phi_{A_\mu}$(or $d\Phi_{B_\mu}$) $\in \mathfrak{K}$. Conversely, let $k_\mu = Re\sum_{\nu=1}^{n}(\Phi_{A_\mu}(P_\nu)-$

---

1) Cf. Weyl [9] p. 123.

$\Phi_{A\mu}(Q_\nu))$, $l_\mu = Re \sum_{\nu=1}^{n} (\Phi_{B\mu}(P_\nu) - \Phi_{B\mu}(Q_\nu))$ be integers, then by (3. 1) for $d\Phi_{A\mu}$ (or $d\Phi_{B\mu}$) and $\varphi_0 \equiv \sum_{\nu=1}^{n} \phi^*_{P_\nu Q_\nu} \in \mathfrak{R}$ we have immediately $\int_{A\mu} \varphi_0 = -2\pi i k_\mu$, $\int_{B\mu} \varphi_0 = -2\pi i l_\mu$ $(\mu = 1, 2, \cdots)$. Since φ_0 is semi-exact, $f = \exp. \int \varphi_0 \in \hat{\mathfrak{R}}$ is a single-valued function on R with the prescribed zeros P_ν and poles Q_ν, q. e. d.

§ IV. Applications to conformal mappings.

13. First of all we state an extremal property of functions in $\mathfrak{R}$. Let R be an arbitrary open Riemann surface and $\{Q\}$ be the class of analytic functions $f = u + iv$ on R with the properties

(i) $f \in \{Q\}$ possesses the given singularities at $P_i(z)$ $(i = 1, \cdots, r < \infty)$ such that

$$f = \sum_{j=1}^{m_i} \frac{a_{ij}}{z^j} + \sum_{k=0}^{\infty} b_{ik} z^k \text{ at } P_i \tag{4. 1}$$

where a_{ij} are given.

(ii) u is single-valued on R and $\int_\gamma dv = 0$ for dividing curves γ.

(iii) $\int_{\mathfrak{I}} u dv \leq 0$ where the integral means the limit of increasing boundary integrals, and $\mathfrak{I}$ the ideal boundary of R.

Obviously $D_{R-U_\rho}(du) < \infty$ where U_ρ denotes the union of ρ-neighbourhoods of P_i $(i = 1, \cdots, r)$. Now let f_0 be the function of $\mathfrak{R}$ with the given singularities such that

$$f_0 = u_0 + iv_0 = \sum_{i=1}^{r} \sum_{j=1}^{m_i} \left\{ (Re\, a_{ij}) \int \psi_{P_i}^{(j)*} + (Im\, a_{ij}) \int \tilde{\psi}_{P_i}^{(j)*} \right\}. \tag{4. 2}$$

Since $\int_{\mathfrak{I}} u_0 dv_0 = 0$ (Lemma 4), we find immediately $f_0 \in \{Q\}$. Next we have also by Lemma 4

$$D_{R-U_\rho}(du_0) = \int_{\partial U_\rho} u_0 dv_0, \quad D_{R-U_\rho}(du_0, d(u - u_0)) = \int_{\partial U_\rho} u_0 d(v - v_0) \tag{4. 3}$$

$$D_{R-U_\rho}(du) = \int_{\partial U_\rho} u\, dv + \int_{\mathfrak{I}} u\, dv \leq \int_{\partial U_\rho} u\, dv.$$

By direct calculation we have

$$\int_{\partial U_\rho} u_0 dv_0 = \pi \sum_{i=1}^{r} \sum_{k=1}^{m_i} \frac{k}{\rho^{2k}} |a_{ik}|^2 + O(\rho^2) \tag{4. 4}$$

$$\int_{\partial U_\rho} u_0 d(v-v_0) = -\pi\, Re \sum_{i=1}^{r}\sum_{k=1}^{m_i} ka_{ik}(b_{ik}-b_{ik}^0)+O(\rho^2)$$

$$\int_{\partial U_\rho} u\, dv = \pi \sum_{i=1}^{r}\sum_{k=1}^{m_i} \frac{k}{\rho^{2k}}|a_{ik}|^2+O(\rho^2)\,.$$

where b_{ik}^0 are the coefficients of f_0 corresponding to b_{ik} in (4. 1). By (4. 3) and (4. 4) we have

$$(4.5)\quad 0\leq D_{R-U_\rho}(du-du_0) = D_{R-U_\rho}(du_0)-2D_{R-U_\rho}(du, du_0)+D_{R-U_\rho}(du)$$
$$\leq 2\pi\, Re \sum_{i=1}^{r}\sum_{k=1}^{m_i} ka_{ik}(b_{ik}-b_{ik}^0)+O(\rho^2)\,,$$

hence for $\rho\to 0$ the following theorem.

THEOREM 11. *$f_0\in\mathfrak{R}$ given by* (4. 2) *is a unique minimizing function of the expression Re* $\{\sum_{i=1}^{r}\sum_{k=1}^{m_i} ka_{ik}b_{ik}\}$ *among the class* $\{Q\}$.

The uniqueness follows from (4. 5). This shows that our method is a converse approach to the extremal method due to *R*. de Possel etc. (cf. Sario [6]).

14. *Another characterization of differentials of* $\mathfrak{R}$. Let $\{R_n\}$ be an exhaustion of elementary domains. Since each R_n on R is an open Riemann surface with analytic boundaries, there exist the semi-exact canonical differentials (on R_n) $\varphi_{A_k}^{*n}$, $\varphi_{B_k}^{*n}$, $\psi_P^{(\mu)*n}$ etc., the real parts of whose integrals take actually constants on ∂R_n. In the following we drop the asterisk for simplicity. Now it holds that

$$(4.6)\qquad \varphi_{\substack{A_k\\B_k}}^{n}\to\varphi_{\substack{A_k\\B_k}},\quad \psi_P^{(\mu)n}\to\psi_P^{(\mu)}\quad (n\to\infty)$$

For the proof write $\varphi_{A_k}^n=du_k^n+i\,dv_k^n$. By bilinear relation on R_n we have

$$(4.7)\qquad D_{R_n}(du_k^n) = -\int_{A_k} dv_k^n \qquad (A_k\subset R_n)$$

and also for $m>n$

$$(4.8)\quad 0\leq D_{R_n}(du_k^m-du_k^n) = D_{R_n}(du_k^n)-2D_{R_n}(du_k^n, du_k^m)+D_{R_n}(du_k^m)$$
$$= -\int_{A_k} dv_k^n+2\int_{A_k} dv_k^m-\int_{A_k} dv_k^m+\int_{\partial R_n} u_k^m dv_k^m$$
$$\leq -\int_{A_k} dv_k^n-\left(-\int_{A_k} dv_k^m\right),$$

because of

$$(4.9)\qquad \int_{\partial R_n} u_k^m dv_k^m = -D_{R_m-R_n}(du_k^m)\leq 0.$$

This shows that the quantities in (4. 7) are monoton decreasing for $n\rightarrow\infty$ and uniformly bounded. Hence suitable subsequence, say again $\{u_k^n\}$, converges to a harmonic function u_k' on R, where the convergence is uniform on every compact set. It remains to prove that the semi-exact differential $\varphi'_{A_k}=du_k'+i\,dv_k'$ is identical with $\varphi_{A_k}=du_k+i\,dv_k$. Letting $m\rightarrow\infty$ in (4. 8), (4. 9) we find that

$$0\leq-\int_{\partial R_n}u_k'\,dv_k'\leq-\int_{A_k}dv_k^n+\int_{A_k}dv_k'$$

which tend to zero for $n\rightarrow\infty$. Since

$$\left|\int_{\partial R_n}u_k^m\,dv_k\right|=\left|D_{R_m-R_n}(du_k^m,\,du_k)\right|\leq\sqrt{D_{R_m-R_n}(du_k^m)D_{R_m-R_n}(du_k)}\ ,$$

let $m\rightarrow\infty$, successively $n\rightarrow\infty$, then we find $\int_{\partial R_n}u_k'dv_k\rightarrow 0$. Thus, we have under the use of Lemma 4

$$D_{R_n}(du_k'-du_k)=\int_{\partial R_n}(u_k'-u_k)\,d(v_k'-v_k)\rightarrow 0\quad\text{for }n\rightarrow\infty$$

i. e. $\varphi'_{A_k}\equiv\varphi_{A_k}$. Analogously we have $\varphi_{B_k}^n\rightarrow\varphi_{B_k}$. Next, the brief proof of the latter in (4. 6) is as follows. Let

$$\int\psi_P^{(\mu)n}=\int du_n+i\,dv_n=\frac{1}{z^\mu}+\sum_{k=0}^{\infty}b_{kn}z^k\quad\text{at }P(\subset R_n),\ n=1,2,\cdots$$

By Theorem 11 we have immediately

$$Re\,b_{\mu n}\leq Re\,b_{\mu n+1}\leq\cdots\leq Re\,b_\mu^0$$

where b_μ^0 is the coefficient of z^μ in the expansion of the extremal function $f_0=\int\psi_P^{(\mu)}$ on R. Hence $\lim_{n\rightarrow\infty}Re\,b_{\mu n}$ exists, and by (4. 5) we have for any compact K

$$\begin{aligned}0\leq D_K(du_m-du_n)&\leq D_{R_n}(du_m-du_n)\\&\leq 2\pi\,Re\,\mu(b_{\mu m}-b_{\mu n}),\qquad m>n\ (>N)\end{aligned}$$

from which we see that a suitable subsequence, say again $\{u_n\}$, converges uniformly to u' harmonic on $R-P$ and $D_{R-U_P}(du')<\infty$. Finally we have $du'\equiv du_0$. Indeed, for $m>n$

$$D_{R_n}(du_0-du_m)\leq-\int_{\partial R_n}u_m dv_0+\int_{\partial R_n}u_0 d(v_0-v_m),$$

$$\left|-\int_{\partial R_n}u_m dv_0\right|=\left|D_{R_m-R_n}(du_m,\,du_0)\right|\leq\sqrt{D_{R_m-U_P}(du_m)D_{R-R_n}(du_0)}\ .$$

Since $D_{R_m-U_P}(du_m)\to\int_{\partial U_P}u'dv'$ $(m\to\infty)$, letting $m\to\infty$ successively $n\to\infty$, we get the conclusion.

15. In this section we confine R to an arbitrary open Riemann surface of finite genus p. Let δ be an integral divisor of order m $(p+1\leq m<\infty)$, then by Theorem 8 the number $A\geq 4$. Hence we have the following theorem under the property (ii) in sec. 11.

THEOREM 12. *Any open Riemann surface R of finite genus p can be mapped conformally onto an at most $(p+1)$-times covered plane.*[1]

Let P_1 be a point of R. Since there does not exist a non-constant function $\in\mathfrak{K}_0$ of a multiple of P_1 under the consideration of (ii) in sec. 11, the number $B=B(P_1)$ for the divisor P_1 is equal to $2(p-1)$ by Riemann-Roch's theorem. If $p>1$, there are at least two linearly independent differentials φ_1,φ_2 which vanish at P_1. Then we can take a point P_2 such that

$$\begin{vmatrix} Re\,\check{\varphi}_1(P_2) & Re\,\check{\varphi}_2(P_2) \\ Im\,\check{\varphi}_1(P_2) & Im\,\check{\varphi}_2(P_2) \end{vmatrix} \neq 0$$

where we write in general $\varphi(P)=\check{\varphi}(z)dz$ (z is a local parameter at P). Indeed, otherwise we would have $\check{\varphi}_1/\check{\varphi}_2=\overline{(\check{\varphi}_1/\check{\varphi}_2)}$ in an open set, which implies that analytic function φ_1/φ_2 reduces to a real constant. This is absurd. Thus any differential $\varphi=c_1\varphi_1+c_2\varphi_2$ with real constants c_1, c_2 is a multiple of P_1, but not of P_1P_2. While $B(P_1P_2)\geq 2(p-2)$, hence we have immediately $B(P_1P_2)=2(p-2)$. Repeating this argument p times we find that there exist p distinct points on R such that $B(\delta)=0$ for $\delta=P_1P_2\cdots P_p$ and consequently $A(1/\delta)=2$. That is, we have at once

THEOREM 13. *Let R_0 be an open Riemann surface of genus p whose boundary consists of a finite number of Jordan curves. Then it is possible to find p points $P_1,\cdots,P_p$ on R_0 such that there does not exist any parallel slit mapping of R_0 with possible poles $P_1,\cdots,P_p$.*

Next, consider the divisor $\delta=P_1P_2\cdots P_r$ for simplicity, where P_i are mutually distinct. Then we note that any non-constant function $f\in\mathfrak{K}_0$ of a multiple of $1/\delta$ can be written by uniqueness theorem that

$$f=\sum_{i=1}^{r}\left(a_i\int\psi_{P_i}^{(1)}+b_i\int\tilde{\psi}_{P_i}^{(1)}\right) \tag{4.10}$$

---

1) If the conjecture in footnote 1) p. 250 would be valid, this will give a parallel slit mapping of R.

where a_i and b_i real numbers which satisfy by (3.1) $2p$ linear equations

$$(4.11) \qquad \sum_{i=1}^{r} \{a_i\, Re\, \check{\varphi}_{\substack{A_k\\B_k}}(z_i) - b_i\, Im\, \check{\varphi}_{\substack{A_k\\B_k}}(z_i)\} = 0\,, \quad k = 1, 2, \cdots, p$$

where z_i are local parameters at P_i. Conversely we see that every single-valued function $\in \mathfrak{K}_0$ of a multiple of $1/\delta$ is constructed by this way. Now we shall prove

THEOREM 14.[1] *Let P_1 be a given point on the same surface R_0 as above. Then for suitable choices of p points $P_2, P_3, \cdots, P_{p+1}$ on R_0 there exists a function F_1 (resp. F_2) which maps R_0 conformally onto an at most $(p+1)$-times covered plane with slits parallel to the imaginary (resp. real) axis, and have their poles at P_1 and some of $P_2, \cdots, P_{p+1}$ where* $\operatorname{Res}_{P_1} F_1 = \operatorname{Res}_{P_1} F_2 = 1$.

First we recall that $Re\, \check{\varphi}_{A_k}(P_1)$ and $Re\, \check{\varphi}_{B_k}(P_1)$ (or $Im\, \check{\varphi}_{A_k}(P_1)$ and $Im\, \check{\varphi}_{B_k}(P_1)$) $(k=1, \cdots, p)$ do not vanish simultaneously. Next, we take p distinct points $P_2, P_3, \cdots, P_{p+1}$ on R_0 such that $A(1/P_2P_3 \cdots P_{p+1}) = 2$ and consequently the deteriminant of coefficients to a_i, b_i $(i=2, 3, \cdots, r=p+1)$ in (4.11) is different from zero. Then we see that the function F_1 (resp. iF_2) (4.10) determined by (4.11) $(r=p+1)$ after the choice $a_1=1$, $b_1=0$ (resp. $a_1=0$, $b_1=1$) is the required, q. e. d.

16. *Proof of* (ii) *in sec.* 11. We shall consider for simplicity the case of integral divisor $\delta = P_1P_2 \cdots P_r$. Then any non-constant function $f \in \mathfrak{K}_0$ of a multiple of $1/\delta$ can be written as (4.10) with a_i, b_i satisfying (4.11). Take an elementary domain R_{n_0} so large that $R - R_{n_0}$ becomes planar character and does not contain any pole of f. Then the functions (with suitable constants)

$$f_m = \sum_{i=1}^{r} \left(a_i \int \tilde{\psi}_{P_i}^{(1)m} + b_i \int \tilde{\psi}_{P_i}^{(1)m} \right) \quad m > n_0\,,$$

are single-valued on $R_m - R_{n_0}$ and converge to $f\, (m \to \infty)$ on every compact set on $R - R_{n_0}$. Now by the argument principle we have for any complex number α

$$n(f, \alpha, R_n) - n(f, \infty, R_n) = \frac{1}{2\pi i} \int_{\partial R_n} \frac{df}{f - \alpha} \quad (n > n_0)$$

where $n(f, \infty, R_n) = q \leq r$ and we may assume $f \neq \alpha$ on ∂R_n. If

1) Cf. the corresponding theorem in Nehari's paper [4], sec. 6.

m ($>n>n_0$) are sufficiently large, the integral (integer) on the right hand side is equal to

$$\frac{1}{2\pi i}\int_{\partial R_n}\frac{df_m}{f_m-\alpha}=\frac{1}{2\pi}\int_{\partial R_m} d\arg(f_m-\alpha)-n(f_m,\alpha,R_m-R_n)$$

which is negative, because the integral along ∂R_m vanishes. Hence $n(f,\alpha,R_n)\leq q$ for any $n>n_0$ and we have $n(f,\alpha,R)\leq q$ for $n\to\infty$, which completes the proof.

17. Finally we consider any Riemann surface R of genus zero (planar character). First we note that $\mathfrak{K}=\mathfrak{K}_0$ and every function $f\in\mathfrak{K}$ on R is the uniform limit of above parallel slit mappings f_m of R_m. That the limit function f gives also a parallel slit mapping of R can be proved by the usual method. (e.g. Nevanlinna [5], pp. 295–297). Therefore the integral $\int\Phi_p\in\mathfrak{K}$ of (2.10) shows the explicit form of a canonical slit mapping. For example, any plane domain G can be mapped onto a vertical slit domain by the function

$$F(z,\zeta)=\frac{\partial p(z,\zeta)}{\partial\xi}+i\sum_{n=1}^{\infty}\left\{\sum_{j=1}^{n}\frac{\Delta_n^j}{\Delta_{n-1}}\frac{\partial}{\partial\xi}\int_{\gamma_j}dp(z,\zeta)\sum_{k=1}^{n}\frac{\Delta_n^k}{\Delta_n}\Omega_k(z)\right\},\qquad \zeta=\xi+i\eta \tag{4.12}$$

whose residue at a simple pole $z=\zeta$ is 1 where $p=g+i^*g$, (g is a Green function of G) $\Omega_k=\omega_k+i^*\omega_k$ and Δ's ($\Delta_0=1$, $\Delta_n^n=\Delta_{n-1}$) are given by (1.11) and (2.7).

Department of Mathematics
Kyoto University

REFERENCES

[1] Köthe, G. Dualität in der Funktionentheorie. Journ. für die reine u. angew. Math. 191, 1953 pp. 30–49.
[2] Kusunoki, Y. On Riemann's period relations on open Riemann surfaces. Mem. Coll. of Sci. Univ. of Kyoto Ser. A. Math. 30, 1956 pp. 1–22.
[3] Kusunoki, Y. Contributions to Riemann-Roch's theorem. Ibid. 31, 1958 pp. 161–180.
[4] Nehari, Z. Conformal mapping of open Riemann surfaces. Trans. Amer. Math. Soc. 68, 1950 pp. 258–277.
[5] Nevanlinna, R. Uniformisierung. Berlin, 1953.
[6] Sario, L. An extremal method on arbitrary Riemann surfaces. Trans. Amer. Math. Soc. 73, 1952 pp. 459–470.

[7] Virtanen, K. I. Über Abelsche Integrale auf nullberandeten Riemannschen Flächen von unendlichem Geschlecht. Ann. Acad. Sci. Fenn. Ser A. I. 56, 1949.
[8] Virtanen, K. I. Über eine Integraldarstellung von quadratisch integrierbaren analytischen Differentialen. Ibid. 69, 1950.
[9] Weyl, H. Die Idee der Riemannshen Fläche. Dritte Auflage. 1955.