

A global foundation of Finsler geometry

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Introduction

The purpose of the present paper is to give a global foundation of connections in Finsler spaces by means of the general theory of connections in differentiable fibre bundles.

Since the notion of connections in fibre bundles has been given a modern formulation, many authors have tried to establish the theory of Finsler geometry from this new point of view. Recently, T. Ōtsuki [8; 9; 10] dealt with our subject in detail, and his treatment seems to be rather complicated. Besides, M. Hashiguchi [4] discussed the parallel displacements and showed that the euclidean connection determined by E. Cartan [2] is the shortest and the fittest from a natural standpoint. Further, L. Auslander [1] generalised to Finsler geometry some global theorems concerning positive curvature. His starting point was also the connection for a Finsler manifold as calculated by E. Cartan and used the equations of structure which were given by S. S. Chern [3]. This theory of connections in a Finsler space given by S. S. Chern was formulated rigorously by S. Kashiwabara [5] in the viewpoint of theory of connections in fibre bundles.

Roughly speaking, the euclidean connection considered by E. Cartan is the one in the principal bundle over the line bundle of a given manifold. If we try to define such a connection in a general differentiable fibre bundle, it is quite natural to pay attention to the *centre* Z of the structural Lie group G . Because the centre of the general linear group $GL(n, R)$ is the set of matrices, which are of the form $(a\delta_j^i)$, where $a \in R-0$, and δ_j^i ($i, j=1, \dots, n$) are the Kronecker's deltas. In this point of view, we shall define, in Chapters I and II, a general Finsler bundle \mathfrak{Q} and a Finsler

connection, which is *invariant under central transformation*. That is, given a differentiable fibre bundle \mathfrak{B} over a differentiable manifold M , we consider the associated principal bundle \mathfrak{P} . Then, the Finsler bundle \mathfrak{Q} is defined as the principal bundle over the bundle space B of \mathfrak{B} induced from \mathfrak{P} . The central transformation of the bundle space Q of \mathfrak{Q} by an element of Z will be defined in § 2. In Chapters III and IV, we shall develop the theory of linear Finsler connection, where the original fibre bundle \mathfrak{B} is the tangent bundle of M . In the last section, we shall devote ourself to the study of the euclidean connection determined by E. Cartan.

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Chapter I. Finsler bundles and trivial connections

§ 1. Differentiable fibre bundles \mathfrak{Q} and $\tilde{\mathfrak{B}}$

We consider a differentiable manifold M of dimension n . By differentiability we shall always understand that of class C^∞ . We denote by $\mathfrak{B} = \{B, \tau, M, V, G\}$ a differentiable fibre bundle over the base space M . The differentiable mapping τ of the bundle space B onto M is the projection of \mathfrak{B} . The structural group G is the Lie group which acts differentiably on the standard fibre V to the left.

Let $\mathfrak{P} = \{P, \rho, M, G, G\}$ be the principal bundle associated with \mathfrak{B} [12, p. 35]. The mapping ρ is the projection $P \rightarrow M$. Each point $p \in P$ may be considered as an admissible mapping of V onto the fibre $V(x)$ over the point $x = \rho(p) \in M$. We denote by R_g the right translation of P by an element $g \in G$. Let $\{U_\alpha\}$ be an open covering of M by coordinate neighborhoods and $\chi_\alpha: U_\alpha \times G \rightarrow \rho^{-1}(U_\alpha)$ the coordinate functions corresponding to U_α . Then the right translation R_g is expressed by

$$R_g(p) = p \cdot g = \chi_\alpha(x, \chi_{\alpha,x}^{-1}(p) \cdot g), \quad x = \rho(p) \in U_\alpha.$$

Next, we construct the principal bundle $\tau^{-1}(\mathfrak{B})$ induced from \mathfrak{B} by the projection $\tau: B \rightarrow M$ of \mathfrak{B} , and we denote $\tau^{-1}(\mathfrak{B})$ by $\mathfrak{Q} = \{Q, \sigma, B, G, G\}$, [12, p. 47]. The bundle space Q is the submanifold of the product $B \times P$ and is defined by

$$Q = \{(b, p) \in B \times P: \tau(b) = \rho(p)\}.$$

The projection σ of \mathfrak{Q} is the canonical mapping $Q \rightarrow B$, such that

$\sigma((b, p))=b$. Let R'_g be the right translation of Q by an element $g \in G$. If we take an open set $V_\alpha = \tau^{-1}(U_\alpha)$, then $\{V_\alpha\}$ is an open covering of B considered as the base space of \mathfrak{Q} , and coordinate functions ψ_α corresponding to V_α are given by

$$\psi_\alpha(b, g) = (b, \chi_\alpha(\tau(b), g)), \quad b \in V_\alpha, g \in G,$$

where χ_α are the coordinate functions of \mathfrak{F} . Since a right translation R'_g of Q is expressed by the similar equation to the one of a right translation R_g of P in terms of ψ_α , then we obtain immediately

$$(1) \quad q \cdot g = (\sigma(q), \eta(q) \cdot g), \quad q \in Q, g \in G,$$

where η is the canonical mapping $Q \rightarrow P$, such that $\eta((b, p))=p$.

We consider next the fibre bundle $\tilde{\mathfrak{B}} = \{\tilde{B}, \tilde{\sigma}, B, V, G\}$ over the manifold B , which is associated with \mathfrak{Q} and has V as the standard fibre. In order to construct the bundle space \tilde{B} , we introduce an equivalence relation in the product $Q \times V$ as follows: $(q, v) \in Q \times V$ is equivalent to $(q', v') \in Q \times V$ if and only if there exists an element $g \in G$ such that $q' = q \cdot g$ and $v' = g^{-1} \cdot v$. We define now the space \tilde{B} as the quotient space of $Q \times V$ by the above equivalence relation, and write simply by qv the equivalence class of (q, v) . The projection $\tilde{\sigma}$ of $\tilde{\mathfrak{B}}$ is given by $\tilde{\sigma}(qv) = \sigma(q)$. Each point $q \in Q$ is considered as an admissible mapping of V onto the fibre $V(b)$ of $\tilde{\mathfrak{B}}$ over the point $b = \sigma(q) \in B$, such that $q(v) = qv, v \in V$.

Now, we shall show that *the bundle space \tilde{B} may be identified with $\bigcup_{x \in M} V(x) \times V(x)$* , where $V(x)$ is a fibre over a point $x \in M$ of the original bundle \mathfrak{B} . In order to prove this, we shall introduce a mapping $i: \tilde{B} \rightarrow B \times B$, such that

$$(2) \quad i(qv) = (\sigma(q), (\eta(q))(v)), \quad qv \in \tilde{B}.$$

If we put $q = (b, p)$, then the mapping i is expressed simply in the form $i((b, p)v) = (b, p(v))$. Since $\tau(b) = \rho(p)$, the mapping i transforms \tilde{B} into $\bigcup_{x \in M} V(x) \times V(x)$. It is easily seen that i as thus defined is one-to-one, and hence the above statement is established. On using the identification i , the projection $\tilde{\sigma}$ is rewritten in the form $\tilde{\sigma}(qv) = b$, where $qv = (b, b')$, $b, b' \in V(x)$, $x = \tau(b) = \tau(b') \in M$.

§2. Finsler fibre bundles

We shall define, in this section, central transformations C_z and

C'_z of the manifolds B and Q respectively, which will be important in the following. Let Z be the *centre* of the structural Lie group G of the original bundle \mathfrak{B} . A *central transformation* C_z of the bundle space B of \mathfrak{B} by an element $z \in Z$ is given by

$$C_z(b) = z \cdot b = p(z \cdot p^{-1}(b)), \quad p \in \rho^{-1} \circ \tau(b),$$

where $p \in P$ is to be considered as an admissible mapping of \mathfrak{B} . If we take account of the fact that $z \in Z$ commutes with any $g \in G$, then we see that the definition of C_z is independent of the choice of $p \in \rho^{-1} \circ \tau(b)$. It is clear that C_z acts on each fibre and gives an equivalence relation in B . Therefore, if we denote by B^* the quotient space of B by central transformations, then we have naturally a fibre bundle $\mathfrak{B}^* = \{B^*, \tau^*, M, V^*, G/Z\}$, where the projection τ^* is the mapping induced from τ , and the standard fibre V^* is the quotient space of V by Z .

The central transformation C_z induces naturally a transformation C'_z of the bundle space Q of \mathfrak{Q} over B , which is given by

$$C'_z(q) = z \cdot q = (z \cdot \sigma(q), \eta(q)), \quad q \in Q.$$

This is called also *the central transformation of Q by an element $z \in Z$* . From the relation $\sigma \circ C'_z = C_z \circ \sigma$, it follows that C'_z is a mapping of $G(b)$ onto $G(z \cdot b)$, where $G(b)$ is a fibre over $b \in B$ of \mathfrak{Q} . Furthermore, we see easily that C'_z commutes with a right translation R'_g of Q .

In the first section, we defined the principal bundle \mathfrak{Q} as the induced bundle $\tau^{-1}(\mathfrak{A})$. Similarly, we can introduce an induced bundle $\tau^{*-1}(\mathfrak{A})$ from \mathfrak{A} by the projection $\tau^*: B^* \rightarrow M$, and denote it by $\mathfrak{Q}^* = \{Q^*, \sigma^*, B^*, G, G\}$. We write by b^* the equivalence class of $b \in B$ given by central transformations, and define the mapping

$$r: B \rightarrow B^*, \quad r(b) = b^*, \quad b \in B.$$

Let $\bar{r}: Q \rightarrow Q^*$ be the induced mapping from r , such that $\bar{r}(q) = (r \circ \sigma(q), \eta(q))$, $q \in Q$. We can easily prove that *the mapping \bar{r} gives the bundle mapping $\mathfrak{Q} \rightarrow \mathfrak{Q}^*$* , such that all of the mapping transformations are equal to the identity [12, p. 9]. The induced bundle \mathfrak{Q}^* will be called *the Finsler fibre bundle of the manifold M constructed from the fibre bundle \mathfrak{B} over M* .

We constructed, in §1, the associated bundle $\tilde{\mathfrak{B}}$ with \mathfrak{Q} . We have now similarly the fibre bundle $\tilde{\mathfrak{B}}^* = \{\tilde{B}^*, \tilde{\sigma}^*, B^*, V, G\}$ associated with the Finsler bundle \mathfrak{Q}^* . In this case also, a point

of the bundle space \tilde{B}^* is identified with a pair (b^*, b') , $b^* \in B^*$, $b' \in B$, such that $\tau^*(b^*) = \tau(b')$. Then the projection σ^* is given by $\sigma^*((b^*, b')) = b^*$, and b^* is called *the element of support of b'* [2, p. 4].

§ 3. Trivial connections in the principal bundle \mathfrak{Q}

Let $\mathfrak{P} = \{P, \rho, M, G, G\}$ be the principal bundle as mentioned in § 1. A *connection* γ in \mathfrak{P} is a differentiable distribution $p \in P \rightarrow \gamma_p$ (=subspace of the tangent space P_p at p of P) which satisfies the following two conditions [7, p. 25]:

- 1) The tangent space P_p at $p \in P$ is the direct sum of the subspace γ_p and $G(x)_p$, where $G(x)_p$ is the tangent space at the point p of the fibre $G(x)$ over the point $x \in \rho(p)$;
- 2) The distribution γ is right invariant: $R_g(\gamma) = \gamma^1$, where R_g is a right translation of P by $g \in G$.

The subspace $G(x)_p$ of P_p is called the vertical subspace and γ_p the horizontal subspace. According to 1), if we take a tangent vector $X \in P_p$, then we have the decomposition

$$X = v(X) + h(X), \quad v(X) \in G(x)_p, \quad h(X) \in \gamma_p.$$

The vector $v(X)$ (resp. $h(X)$) is called the vertical (resp. horizontal) component of X .

We shall define next so-called connection forms of connections. We denote by \hat{G} the Lie algebra of G , and identify \hat{G} with the tangent space G_e at the identity e of G . Using the right translation R_g , we have the mapping

$$L_p: G \rightarrow G(x), \quad L_p(g) = p \cdot g, \quad p \in P, \quad x = \rho(p) \in M.$$

The fundamental vector field A on P corresponding to an element $\hat{A} \in \hat{G}$ is the vertical vector field, such that the value A_p of A at a point $p \in P$ is given by $L_p(\hat{A})$. Now, the connection form π of the above connection γ is a \hat{G} -valued differentiable 1-form, which satisfies the following two conditions:

- 1*) If A is a fundamental vector field on P corresponding to an element $\hat{A} \in \hat{G}$, then $\pi(A) = \hat{A}$.
- 2*) For any horizontal vector $X \in \gamma_p$, we have $\pi(X) = 0$.

It is well-known that $R_g(A)$ is the fundamental vector field

1) Later on, for the differential $\delta\varphi$ of a differentiable mapping φ , we shall use the same letter φ in case there is no danger of confusion. The letter φ^* means the dual of the differential $\delta\varphi$.

corresponding to $ad(g^{-1}) \cdot \hat{A} \in \hat{G}$, from which it follows the further property of π as follows:

3*) The connection form π is of $ad(G)$ -type: $R_g^*(\pi) = ad(g^{-1}) \cdot \pi$.

Conversely, given a differentiable 1-form π on P with value in \hat{G} satisfying 1*) and 3*), we can define a connection γ whose connection form is this π . In this case, the horizontal subspace γ_p at $p \in P$ is defined as the set of tangent vectors at p which are mapped by π into zero. This correspondence of connections and connection forms is one-to-one.

We shall treat mainly a connection Γ in the principal bundle \mathfrak{Q} , which is defined in the similar manner to the case of γ .

Theorem 1.1. *Let η be the canonical mapping $Q \rightarrow P$, and π a connection form on P . Then the induced 1-form $\omega = \eta^*(\pi)$ is a connection form on Q .*

Proof. It is sufficient to show that the similar conditions to 1*) and 3*) are also satisfied by ω . Let A' be the fundamental vector field on Q corresponding to $\hat{A} \in \hat{G}$. The value A'_q of A' at $q \in Q$ is $A'_q = L'_q(\hat{A})$, where the mapping L'_q is defined by means of R'_g in the similar manner to the case of L_p . In virtue of (1), we get $\eta \circ L'_q = L_p$, $p = \eta(q)$, and hence we obtain

$$\omega(A'_q) = \pi(\eta \circ L'_q(\hat{A})) = \pi(A_p) = \hat{A}, \quad p = \eta(q).$$

Next, using $\eta \circ R'_g = R_g \circ \eta$, we get

$$R'_g(\omega) = (\eta \circ R'_g)^*(\pi) = \eta^* \circ R_g^*(\pi) = ad(g^{-1}) \cdot \omega.$$

Thus it concludes the proof.

If there is a connection Γ in \mathfrak{Q} and its connection form ω is the induced one by η from the connection form π of the connection γ in \mathfrak{P} , then the connection Γ and its connection form ω are called *to be trivial*.

We consider next a connection Γ in \mathfrak{Q} , which is not necessarily to be trivial. The horizontal subspace Γ'_q of the tangent space Q_q at $q \in Q$ is isomorphic to the tangent space B_b at $b = \sigma(q)$ of the base B . By this isomorphism, for a tangent vector $X \in B_b$, we obtain the horizontal vector $l_q(X)$ at a point $q \in \sigma^{-1}(b)$, such that X and $l_q(X)$ are σ -related: $\sigma(l_q(X)) = X$. The vector $l_q(X)$ is called the lift at q of X [7, p. 26]. Let X be a vertical vector at $b \in B$, that is, $\tau(X) = 0$, and B_b^v the set of vertical vectors at b . The set of lifts at $q \in \sigma^{-1}(b)$ of vertical vectors is clearly a subspace

of the horizontal subspace Γ_q , which is called *the zero-horizontal subspace of Q_q* and denoted by Γ_q^0 . It is easily seen that $\tau \circ \sigma(\bar{X}) = 0$ for $\bar{X} \in \Gamma_q^0$.

Lemma 1.1. *Let η be the canonical mapping $Q \rightarrow P$.*

1) *The vertical subspace at $q \in Q$ is isomorphic to the vertical subspace at $\eta(q) = p \in P$ by the differential of η .*

2) *We suppose that a connection Γ is given in \mathfrak{Q} . Then, for any zero-horizontal vector $\bar{X} \in \Gamma_q^0$ at $q \in Q$, $\eta(\bar{X})$ is vertical at $\eta(q) = p \in P$.*

Proof. 1) is a direct result from the definition of \mathfrak{Q} . From $\tau \circ \sigma = \rho \circ \eta$ it follows that $\tau \circ \sigma(\bar{X}) = \rho \circ \eta(\bar{X}) = 0$, $\bar{X} \in \Gamma_q^0$. Hence $\eta(\bar{X})$ is vertical and this proves 2).

The trivial connections in \mathfrak{Q} have particular properties among general connections in \mathfrak{Q} . We shall give some of them in the following theorem.

Theorem 1.2. *Let Γ be a trivial connection in \mathfrak{Q} induced from a connection γ in \mathfrak{P} , and ω and π the connection forms of Γ and γ respectively.*

1) $\eta(\Gamma_q) = \gamma_p$, $p = \eta(q) \in P$;

2) *The connection form ω is central-invariant: $C'_z{}^*(\omega) = \omega$, where C'_z is any central transformation of Q .*

3) *The kernel of the differential $\delta\eta$ is the zero-horizontal subspace Γ^0 .*

4) *If we take $q = (b, p) \in Q$ and $v = p^{-1}(b) \in V$, and define a mapping*

$$\mu_v: P \rightarrow Q, \quad \mu_v(p') = (p'(v), p'), \quad p' \in P,$$

then the horizontal subspace Γ_q is the direct sum of $\mu_v(\gamma_p)$ and Γ_q^0 .

Proof. 1) If we take $\bar{X} \in \Gamma_q$, then we have $\pi(\eta(\bar{X})) = \omega(\bar{X}) = 0$, and hence $\eta(\bar{X})$ is horizontal and thus we see $\eta(\Gamma_q) \subset \gamma_p$. Conversely, if we take $X \in \gamma_p$, then there exists $\bar{X} \in Q_q$ such that $\eta(\bar{X}) = X$. Then we have $\eta(\bar{X}) = \eta(v(\bar{X})) + \eta(h(\bar{X})) = X$. In virtue of Lemma 1.1, 1), we see that $\eta(v(\bar{X}))$ is vertical. On the other hand, $\eta(h(\bar{X}))$ is horizontal as above shown. Since X is horizontal, we have $\eta(v(\bar{X})) = 0$, and hence $v(\bar{X}) = 0$. Therefore \bar{X} is horizontal and consequently, we have $\eta(\Gamma_q) \supset \gamma_p$.

2) By means of $\eta \circ C'_z = \eta$, we obtain

$$C'_z{}^*(\omega) = (\eta \circ C'_z)^*(\pi) = \eta^*(\pi) = \omega.$$

This prove 2).

3) We take $\bar{X} \in Q_q$, such that $\eta(\bar{X})=0$. It is easily seen that $\sigma(\bar{X})$ is vertical at $b=\sigma(q) \in B$. If we take the lift $l_q(\sigma(\bar{X}))$ at q , then $\bar{X}-l_q(\sigma(\bar{X}))$ is vertical at q . On the other hand, $\eta(\bar{X})=\eta(v(\bar{X}))+\eta(h(\bar{X}))=0$, and hence $\eta(v(\bar{X}))=0$, from which it follows that $v(\bar{X})=0$ and \bar{X} is horizontal. Therefore $\bar{X}-l_q(\sigma(\bar{X}))$ is horizontal at q . Then we have $\bar{X}=l_q(\sigma(\bar{X}))$. Consequently the kernel of η at $q \in Q$ is contained in Γ_q^0 . Conversely, $\eta(\bar{X}), \bar{X} \in \Gamma_q^0$, is vertical by means of Lemma 1.1, 2), and horizontal by 1). Thus we have $\eta(\bar{X})=0$.

4) We shall first show that $\mu_v(\gamma_p) \subset \Gamma_q$. In fact, we take $X^* \in \gamma_p$ and then, according to $\eta \circ \mu_v = \text{identity}$, we have

$$\omega(\mu_v(X^*)) = \pi(\eta \circ \mu_v(X^*)) = \pi(X^*) = 0.$$

Hence $\mu_v(X^*)$ is horizontal at q . Next, let \bar{X} be any horizontal vector at q . In virtue of 1), $\eta(\bar{X})$ is horizontal at $p=\eta(q)$ and hence $\mu_v \circ \eta(\bar{X})$ is horizontal at q . Since $\eta(\bar{X}-\mu_v \circ \eta(\bar{X}))=0$, we have $\bar{X}-\mu_v \circ \eta(\bar{X})=\bar{Y} \in \Gamma_q^0$ by means of 3). Thus we obtain the decomposition $\Gamma_q = \mu_v(\gamma_p) + \Gamma_q^0$. With the aid of consideration of dimensions, we see easily that this decomposition is direct sum, and then we prove 4).

Theorem 1.3. *The necessary and sufficient condition that a connection Γ in the principal bundle \mathfrak{Q} be trivial is that*

- 1) *for any $q, q' \in Q$, such that $\eta(q)=\eta(q')$, we obtain $\eta(\Gamma_q)=\eta(\Gamma_{q'})$.*
- 2) *$\eta(\Gamma_q^0)=0, q \in Q$.*

Proof. The necessity is clear from 1) and 2) of the above theorem, and hence we show the sufficiency. Given a connection Γ in \mathfrak{Q} satisfying the above conditions, we take a point $q \in Q$ for a given point $p \in P$, such that $\eta(q)=p$, and define $\gamma_p = \eta(\Gamma_q)$. The subspace γ_p of the tangent space P_p is independent of the choice of a point $q \in Q, \eta(q)=p$, in consequence of 1). We shall show that the distribution $\gamma: p \rightarrow \gamma_p$ of P is a connection in \mathfrak{F} . For a tangent vector $X \in P_p$, we take a vector $\bar{X} \in Q_q$, such that $\eta(\bar{X})=X$. The vector \bar{X} is written in the direct sum $\bar{X}=v(\bar{X})+h(\bar{X})$ with respect to the connection Γ . Then we obtain $X=\eta(v(\bar{X}))+\eta(h(\bar{X}))$, and $\eta(v(\bar{X}))$ is vertical in virtue of Lemma 1.1, 1), while we see $\eta(h(\bar{X})) \in \gamma_p$ in consequence of the definition of γ . Thus we get the decomposition $P_p = G(x)_p + \gamma_p$. If we put $\dim. M=n$ and $\dim. V=v$, then we have $\dim. \Gamma_q = n+v$ and $\dim. \Gamma_q^0 = v$. By applying 2) we see $\dim. \gamma_p \leq n$. Because of the possibility of the above decomposition,

it follows that the decomposition is direct sum. Furthermore we obtain

$$R_g(\gamma_p) = R_g \circ \eta(\Gamma_q) = \eta \circ R'_g(\Gamma_q) = \eta(\Gamma_{q',g}) = \gamma_{p',g},$$

and hence γ is right invariant. Therefore γ is a connection in \mathfrak{B} . It is easily seen that the original connection form ω on Q is induced from the connection form π of the connection γ as above defined.

Chapter II. Finsler connections

§4. Central invariant connections in the principal bundle Ω

In §2, we have defined the Finsler bundle Ω^* of the manifold M constructed from the fibre bundle \mathfrak{B} over M , and we have shown that $\bar{r}: \Omega \rightarrow \Omega^*$ is the bundle mapping. The Finsler connection of M is defined to be a connection in the Finsler bundle Ω^* .

On the other hand, a connection Γ in Ω is called to be *central invariant*, if $C'_z(\Gamma) = \Gamma$ for any central transformation C'_z of Q by an element $z \in Z$. If we use its connection form ω , then the central invariance of the connection is clearly expressed by $C'_z{}^*(\omega) = \omega$.

Theorem 2.1. *There is a natural one-to-one correspondence between the set of central invariant connections in Ω and the set of Finsler connections in Ω^* by the bundle mapping $\bar{r}: \Omega \rightarrow \Omega^*$.*

Before proving the theorem, we shall define the central-horizontal subspace Γ_q^z , $q \in Q$, of the zero-horizontal subspace Γ_q^0 . We consider a point $b \in B$ and the equivalence class $r(b) = b^* \in B^*$. The point b^* may be thought of as a submanifold of the fibre $V(x)$, $x = \tau(b)$, through the point b , and hence the set of the lifts at $q \in \sigma^{-1}(b)$ of tangent vectors at b of the submanifold b^* is a subspace Γ_q^z of Γ_q^0 , which is called *the central-horizontal subspace at $q \in Q$* . The mapping $r: B \rightarrow B^*$ is constant on b^* , and hence, in consequence of the definition of \bar{r} , we have immediately

Lemma 2.1. *The differential of the bundle mapping $\bar{r}: Q \rightarrow Q^*$ carries a central-horizontal subspace to zero.*

We shall prove now the theorem. Given a central invariant connection Γ in Ω and take a point $q \in Q$ for a given point $q^* \in Q^*$, such that $\bar{r}(q) = q^*$. We put $\Gamma_{q^*} = \bar{r}(\Gamma_q)$. The subspace Γ_{q^*} is independent of the choice of a point $q \in Q$, $\bar{r}(q) = q^*$, in virtue of the central invariance of Γ . We shall show that the distribution $\Gamma^*: q^* \rightarrow \Gamma_{q^*}$ of Q^* is a connection. In fact, given a tangent vector

X^* at q^* of Q^* and take $X \in Q_q$, such that $\bar{r}(X) = X^*$. Let $X = v(X) + h(X)$ be the decomposition of X with respect to Γ . Then we get $X^* = \bar{r}(v(X)) + \bar{r}(h(X))$. The vector $\bar{r}(v(X))$ is clearly vertical, while $\bar{r}(h(X)) \in \Gamma_{q^*}^*$ by the definition of $\Gamma_{q^*}^*$. That the above decomposition of X^* is direct sum is immediately seen in consequence of Lemma 2.1. Next, let R'_g be the right translation of Q^* by $g \in G$, and then it follows that $R''_g \circ \bar{r} = \bar{r} \circ R'_g$, where R'_g is the right translation of Q . Hence we obtain

$$R''_g(\Gamma^*) = R'_g \circ \bar{r}(\Gamma) = \bar{r} \circ R'_g(\Gamma) = \bar{r}(\Gamma) = \Gamma^*.$$

Therefore we prove that Γ^* is a Finsler connection.

Conversely, if we have a Finsler connection Γ^* , then we define $\omega = \bar{r}^*(\omega^*)$, where ω^* is the connection form of Γ^* . We shall show that ω is a connection form on Q . Let A (resp. A^*) be the fundamental vector field on Q (resp. Q^*) corresponding to $\hat{A} \in \hat{G}$. We see easily $\bar{r}(A) = A^*$, and then we obtain

$$\omega(A) = \omega^*(\bar{r}(A)) = \omega^*(A^*) = \hat{A}.$$

Also we get, for a right translation R'_g

$$R'_g{}^*(\omega) = (\bar{r} \circ R'_g)^*(\omega^*) = \bar{r}^* \circ R''_g{}^*(\omega^*) = ad(g^{-1}) \cdot \omega.$$

Hence the form ω is a connection form on Q . Next, for any central transformation C'_z of Q , we have

$$C'_z{}^*(\omega) = (\bar{r} \circ C'_z)^*(\omega^*) = \bar{r}^*(\omega^*) = \omega,$$

and thus ω is central invariant. Hence we have a central invariant connection Γ , whose connection form is ω as above defined. It is easily verified that the connection in Ω^* constructed from Γ by the process as shown in the first step coincides with the original Finsler connection Γ^* . This concludes the proof of our theorem.

The structure of the principal bundle Ω seems to be simpler than that of the Finsler bundle Ω^* , especially on local coordinates. And the above theorem shows that the study of the Finsler connection is equivalent to that of the central invariant connection of Ω . Because of these points of view, our attention will be confined, in subsequent sections, to the principal bundle Ω with a central invariant connection. It is understood hereafter that Ω will be called the Finsler bundle of the manifold M constructed from \mathfrak{B} and a central invariant connection in Ω a Finsler connection. From this standpoint, for a point $\tilde{b} = (b, b') \in B$, we shall say that the point $\sigma(\tilde{b}) = b \in B$ is the element of support of b' .

As a result of Theorem 1.2, we see that a trivial connection of Ω is considered as one of the Finsler connection, and hence the existence of a Finsler connection is assured [7, p. 41]. By means of the well-known theorem [6, p. 68], we have

Theorem 2.2. *Suppose that we have a trivial connection form ω_0 in the Finsler bundle Ω . Then there is an one-to-one correspondence between the set $\{\Gamma\}$ of Finsler connections and the set $\{\mu\}$ of \hat{G} -valued, $ad(G)$ -type, central invariant, tensorial 1-forms on the bundle space Q , and the correspondence is given by*

$$\omega = \omega_0 + \mu,$$

where ω is the connection form of the connection $\Gamma \in \{\Gamma\}$ corresponding to the form $\mu \in \{\mu\}$.

We consider the local expressions of the connection form ω . Let U_α and $V_\alpha = \tau^{-1}(U_\alpha)$ be coordinate neighborhoods of M and B respectively, which were considered in §1. We suppose that we have a local section $\rho_\alpha: U_\alpha \rightarrow \rho^{-1}(U_\alpha)$ of \mathfrak{F} , and then we obtain the induced local section $\sigma_\alpha: V_\alpha \rightarrow \sigma^{-1}(V_\alpha)$ of Ω , such that

$$\sigma_\alpha(b) = (b, \rho_\alpha \circ \tau(b)), \quad b \in V_\alpha.$$

We denote by $\omega_\alpha, \omega_{0\alpha}$ and μ_α the local expressions on V_α of ω, ω_0 and μ respectively in the above theorem [6, pp. 60, 67], which are defined by

$$\omega_\alpha = \sigma_\alpha^*(\omega), \quad \omega_{0\alpha} = \sigma_\alpha^*(\omega_0), \quad \mu_\alpha = \sigma_\alpha^*(\mu).$$

Let π be the connection form on P , from which ω_0 is induced, and π_α the local expressions $\rho_\alpha^*(\pi)$. Making use of $\eta \circ \sigma_\alpha = \rho_\alpha \circ \tau$, we get $\omega_{0\alpha} = \tau^*(\pi_\alpha)$. Thus we have the equations

$$(3) \quad \omega_\alpha = \tau^*(\pi_\alpha) + \mu_\alpha.$$

It is easily verified that the forms $\omega_\alpha, \pi^*(\pi_\alpha)$ and μ_α on V_α are all central invariant with respect to C_z .

§5. Various parallel displacements

Throughout the remainder of this chapter it is understood that we have a Finsler connection Γ in the Finsler bundle Ω . Given a (piece-wise differentiable) curve $C = \{b_t, 0 \leq t \leq 1\}$ in the base space B , we have a lift of C to the bundle space Q issuing from a point $q_0 = (b_0, p_0) \in \sigma^{-1}(b_0)$, which is a horizontal curve $\bar{C} = \{q_t, 0 \leq t \leq 1\}$, such that $\sigma(q_t) = b_t$ [7, p. 27]. The existence and uniqueness

are proved, and it is well-known that the curve $\bar{C} \cdot g$, $g \in G$, is also a lift of C to Q issuing from the point $q_0 \cdot g$. We say that *the end point q_1 of \bar{C} is obtained from q_0 by parallel displacement along the curve C* . Furthermore we have the definition of a *holonomy group* Φ_q with reference point at q , which is the set $\{g\}$ of elements of the structural group G , such that the point $q \cdot g$, $g \in \{g\}$, is obtained from q by parallel displacement along a closed curve at $b = \sigma(q)$. Since the base space B of Ω is the bundle space of the original fibre bundle \mathfrak{B} , we have a special subgroup Φ_q^v of the holonomy group Φ_q . That is, if a curve C in B is contained in a fibre $V(x)$ over $x \in M$, then C is said to be vertical. Then, the subgroup Φ_q^v is defined as the set of elements of Φ_q corresponding to all of closed vertical curve at $b = \sigma(q)$, and we shall call Φ_q^v *the vertical subgroup of the holonomy group Φ_q* .

We shall define next the parallel displacement of points of P along a curve $C = \{b_t, 0 \leq t \leq 1\}$ in B , where P is the bundle space of \mathfrak{B} . Take a point $p_0 \in \rho^{-1} \cdot \tau(b_0)$ of P , we have a point $q_0 = (b_0, p_0) \in Q$. Then we obtain the lift $\bar{C} = \{q_t = (b_t, p_t), 0 \leq t \leq 1\}$ of C to Q issuing from q_0 . Then we obtain the curve $\eta(\bar{C}) = C^* = \{p_t, 0 \leq t \leq 1\}$ issuing from the given point p_0 . The curve C^* is determined by C and the starting point p_0 , and we call C^* *the lift of C to P issuing from p_0* . It is clear that the lift of C to P issuing from $p = p_0 \cdot g$, $g \in G$, is then given by $C^* \cdot g$. Now we say that *the end point p_1 of the lift C^* is obtained from p_0 by parallel displacement along the curve C* .

We consider two curves $C = \{b_t, 0 \leq t \leq 1\}$ and $C' = \{b'_t, 0 \leq t \leq 1\}$ in B , such that there exists an element $z \in Z$ and $b'_t = z \cdot b_t$ for any $t \in [0, 1]$. Then C and C' are called to be *z -related*. Since $\tau(b_t) = \tau(b'_t) = x_t \in M$, we can define the parallel displacements of a point $p_0 \in \rho^{-1}(x_0)$ along both of C and C' . We take the lifts $\bar{C} = \{(b_t, p_t)\}$ and $\bar{C}' = \{(b'_t, p'_t)\}$ to Q of C and C' respectively. Since the Finsler connection is central invariant, the curve $z \cdot \bar{C} = \{(b'_t, p_t)\}$ is also horizontal and covers C' . From the uniqueness of a lift, it follows that $z \cdot \bar{C}$ is to coincide with \bar{C}' , and hence we obtain $p_t = p'_t$. Therefore we have

Theorem 2.3. *The parallel displacements of a point of P along curves C and C' coincide, if C and C' are z -related.*

We consider next the associated fibre bundle $\tilde{\mathfrak{B}}$ with the Finsler bundle Ω as defined in §1. The bundle space \tilde{B} of $\tilde{\mathfrak{B}}$ has

been identified with $\bigcup_{x \in M} V(x) \times V(x)$. It is well-known [7, p. 43] that there is a natural one-to-one correspondence between the set of connections in \mathfrak{Q} and the set of connections in \mathfrak{B} . Hence we have a lift of a curve $C = \{b_t, 0 \leq t \leq 1\}$ in B to \tilde{B} , and we can define a parallel displacement of a point of \tilde{B} along C .

Let $\bar{C} = \{q_t = (b_t, p_t), 0 \leq t \leq 1\}$ and $\tilde{C} = \{\tilde{b}_t = (b_t, b'_t), 0 \leq t \leq 1\}$ be lifts of C to Q and \tilde{B} respectively. The relation between \bar{C} and \tilde{C} is given by

$$(4) \quad q_t \circ q_0^{-1}(\tilde{b}_0) = \tilde{b}_t, \quad 0 \leq t \leq 1.$$

The end point \tilde{b}_1 of \tilde{C} is said to be obtained from \tilde{b}_0 by parallel displacement along C . If we put $v_0 = q_0^{-1}(\tilde{b}_0) \in V$, then we have $(b_0, b'_0) = q_0 v_0 = (b_0, p_0(v_0))$, and hence $v_0 = p_0^{-1}(b'_0)$. Thus we obtain $q_t \circ q_0^{-1}(\tilde{b}_0) = q_t v_0 = (b_t, p_t(v_0))$. It follows from (4) that

$$(5) \quad p_t \circ p_0^{-1}(b'_0) = b'_t, \quad 0 \leq t \leq 1,$$

where $C^* = \{p_t, 0 \leq t \leq 1\}$ is the lift of C to P .

In virtue of the equation (5), we can define the parallel displacement of points of B . Let C be a curve in B as above considered, and b'_0 a point of the fibre through b_0 , which is the starting point of C . Then we have a curve $C' = \{b'_t, 0 \leq t \leq 1\}$ in B issuing from b'_0 , which is given by (5). We say that C' is parallel to C , and the end point b'_1 of C' is obtained from b'_0 by parallel displacement along the curve C of its element of support b_0 .

Theorem 2.4. *If a curve C' in B be parallel to a curve C in B , then C' is also parallel to any z -related curve with C . That is, let $b'_1 \in B$ be obtained from $b'_0 \in B$ by the parallel displacement along C of its element of support b_0 . Then b'_1 is obtained from b'_0 by the parallel displacement along $z \cdot C$, $z \in Z$, of its element of support $z \cdot b_0$.*

This theorem is a direct result from Theorem 2.3 and the equation (5).

Theorem 2.5. *If a curve C' in B be parallel to a curve C in B , then a z -related curve with C' is also parallel to C . That is, let $b'_1 \in B$ be obtained from $b'_0 \in B$ by the parallel displacement along C of its element of support b_0 . Then a point $z \cdot b'_1$, $z \in Z$, is obtained from $z \cdot b'_0$ by the same parallel displacement along C .*

Proof. From the definition of the central transformation C_z of B , it follows that

$$z \cdot p^{-1}(b) = p^{-1}(z, b), \quad p(z \cdot v) = z \cdot p(v), \quad p \in P, \quad b \in B, \quad v \in V, \\ \rho(p) = \tau(b).$$

Making use of (5), we see that, for any t ,

$$p_t \circ p_0^{-1}(z \cdot b_0) = p_t(z \cdot p_0^{-1}(b_0)) = z \cdot (p_t \circ p_0^{-1}(b_0)) = z \cdot b'_t.$$

This proves the theorem.

§ 6. Property D of a Finsler connection

Let $C = \{b_t, 0 \leq t \leq 1\}$ be a *vertical* curve contained in the fiber $V(x_0)$ in B , and $\bar{C} = \{q_t = (b_t, p_t), 0 \leq t \leq 1\}$ the lift of C to Q . Since we have

$$\rho(p_t) = \rho \circ \eta(q_t) = \tau \circ \sigma(q_t) = \tau(b_t) = x_0,$$

the lift $C^* = \eta(\bar{C})$ of C to P is also vertical and contained in the fibre $G(x_0)$. Therefore there exists a curve $C_G = \{g_t, 0 \leq t \leq 1\}$ in the structural group G , such that $p_t = p_0 \cdot g_t$ for any t . The curve C_G is called *the development of the vertical curve C with reference point at $p_0 \in P$* . It is clear that any development of C in G is given by $g^{-1} \cdot C_G \cdot g$, $g \in G$. On the other hand, if we take a curve C' in B , which is parallel to C , then C' is also vertical and contained in the above fibre $V(x_0)$.

Now, in order to determine so-called euclidean connection in a Finsler manifold by means of the fundamental function, E. Cartan introduced five postulates A, B, C, D and E [2, p. 10]. The fourth of them is expressible as follows [11, p. 68]:

“If the direction of a vector with fixed components coincides with that of its element of support, then its covariant differential corresponding to an infinitesimal rotation of its element of support about its own centre vanishes identically”.

In our case of general Finsler bundles, the centre of the element of support $b \in B$ of a point $b' \in B$ is to be defined as the point $x = \tau(b) \ni M$, and hence the rotation of b means that b moves along a vertical curve in B . Therefore we shall give the property D of a Finsler connection as follows:

Definition. *The Finsler connection is said to have the property D , when any curve issuing from a point b_0 and being parallel to a vertical curve C is a single point b_0 , if the starting point of C coincides with b_0 .*

That is, a point obtained from b_0 by parallel displacement

along any vertical curve C of the element of support b_0 coincides with the original point b_0 at any time t . Hence, by virtue of (5), the analytic expression of the property D is that

$$(6) \quad p_t \circ p_0^{-1}(b_0) = b_0,$$

where $C^* = \{p_t, 0 \leq t \leq 1\}$ is a lift to P of a vertical curve $C = \{b_t, 0 \leq t \leq 1\}$ issuing from b_0 . In this case C^* is also vertical and expressed as $p_t = p_0 \cdot g_t$, where $C_G = \{g_t, 0 \leq t \leq 1\}$ is the development of C in G with the reference point $p_0 \in P$. If we put $p_0^{-1}(b_0) = v_0 \in V$, then (6) is written in the form $p_0(g_t \cdot v_0) = p_0(v_0)$, and hence we get $g_t \cdot v_0 = v_0$. Consequently the development C_G is a curve in an isotropy subgroup $G(v_0)$ of G , the elements of which map v_0 into itself. It is clear that any development $g^{-1} \cdot C_G \cdot g$ is contained in an isotropy subgroup $G(g^{-1} \cdot v_0)$.

Conversely, if a development $C_G = \{g_t, 0 \leq t \leq 1\}$ of any vertical curve $C = \{b_t, 0 \leq t \leq 1\}$ with reference point p_0 be contained in an isotropy subgroup $G(v_0)$, $v_0 = p_0^{-1}(b_0)$, of G , then we take a lift $C^* = \{p_t, 0 \leq t \leq 1\}$ to P issuing from p_0 , and we have $p_t = p_0 \cdot g_t$, and hence

$$p_t \circ p_0^{-1}(b_0) = p_0(g_t \cdot v_0) = p_0(v_0) = b_0,$$

from which it follows that (6) is satisfied. Thus we establish

Theorem 2.6. *A Finsler connection has the property D, if and only if a development of any vertical curve in B is a curve contained in an isotropy subgroup $G(v)$, $v \in V$, of the structural group G .*

In the last section we have defined the vertical subgroup Φ_q^v of the holonomy group Φ_q . From the above theorem we have

Corollary. *If a Finsler connection has the property D, then the vertical subgroup of the holonomy group with reference point $q = (b, p)$ is contained in an isotropy subgroup $G(v)$ of G , where $v = p^{-1}(b)$.*

Chapter III. Linear Finsler connections

§7. Canonical coordinates

Throughout the remainder of this paper we shall confine ourselves to the case where the original fibre bundle \mathfrak{B} is the tangent bundle of the n -dimensional differentiable manifold M , so that the structural Lie group is the general linear group $GL(n, R)$ and the standard fibre V is the n -dimensional vector space over the real field R . Then the associated principal bundle \mathfrak{P} is the bundle of

frames on M , and a point $q=(b, p)$ of the bundle space Q of the Finsler bundle \mathfrak{Q} is a pair of a tangent vector $b \in B$ and a frame $p=(p_1, \dots, p_n) \in P$, such that b, p_1, \dots, p_n are tangent vectors at the same point of M . Then \mathfrak{Q} is called *the linear Finsler bundle of M* . A point of the bundle space \tilde{B} of the associated bundle $\tilde{\mathfrak{B}}$ with \mathfrak{Q} can be considered as a pair of tangent vectors at a point of M . In the following it is understood that the element of support b of b' is not a zero vector.

The centre of $GL(n, R)$ is the set of n -matrices, which are expressed in the forms $((z) \cdot \delta_j^i)$, where $(z) \in R-0$ and δ_j^i , $(i, j=1, \dots, n)$ are the Kronecker's deltas. Thus there is an one-to-one correspondence between elements of Z and real numbers $\neq 0$. We denote by (z) the real number corresponding to an element $z \in Z$.

Let (x^i) , $(i=1, \dots, n)$ be the local coordinates of a point in the coordinate neighborhood U_α of M , and (e_i) , $(i=1, \dots, n)$ a fixed base of V . A tangent vector $b \in B$ at $\tau(b)=x \in U_\alpha$ is expressed by the canonical frame $(\partial/\partial x^i)$ in the form $b=b^i(\partial/\partial x^i)_x$ ²⁾ and hence (x^i, b^i) are considered as coordinates of a point in $V_\alpha=\tau^{-1}(U_\alpha)$ of the base space B of \mathfrak{Q} . We shall call (x^i, b^i) *the canonical coordinates in V_α* . On the other hand, a vector p_i , $(i=1, \dots, n)$, of a frame $p=(p_1, \dots, p_n) \in P$, $\rho(p)=x \in V_\alpha$, is expressible in the form $p_i=p_i^j(\partial/\partial x^j)_x$, and hence a point $q=(b, p) \in Q$, $b \in V_\alpha$, is expressed by the set of real numbers (x^i, b^i, p_i^j) , which are called also *the canonical coordinates in $\sigma^{-1}(V_\alpha)$* . If we take $q=(b, p)=(x^i, b^i, p_i^j) \in Q$, then $p^{-1}(b)=v \in V$ is of the form $v=p_j^{-1}b^j \cdot e_i$, where the matrix (p_j^{-1}) is the inverse of (p_i^j) . Thus the central transformation C_z of B by $z \in Z$ is given by

$$C_z(b) = (x^i, (z)b^i), \quad \text{where } b = (x^i, b^i).$$

Now we consider a connection form π on P . It is well known [6, p. 85] that the local expressions π_α of π are given by

$$\pi_\alpha = \gamma_{jk}^i(x^1, \dots, x^n) dx^k \cdot \hat{g}_i^j, \quad i, j, k = 1, \dots, n,$$

where (\hat{g}_i^j) is a fixed base of the Lie algebra of $GL(n, R)$. If we write the 1-form μ_α in the equations (3) by

$$\mu_\alpha = [\mu_{jk}^i(x^1, \dots, x^n, b^1, \dots, b^n) dx^k + C_{jk}^i(x^1, \dots, x^n, b^1, \dots, b^n) db^k] \cdot \hat{g}_i^j, \\ i, j, k = 1, \dots, n,$$

2) We make free use of the summation convention.

then, in virtue of (3), the local expressions ω_ω of a general linear Finsler connection form ω are expressed in terms of canonical coordinates as follows :

$$(7) \quad \omega_\omega = (\Gamma_{jk}^i dx^k + C_{jk}^i db^k) \cdot \hat{g}_i^j, \quad i, j, k = 1, \dots, n,$$

where we put

$$\Gamma_{jk}^i = \gamma_{jk}^i(x^1, \dots, x^n) + \mu_{jk}^i(x^1, \dots, x^n, b^1, \dots, b^n).$$

Since μ_ω is central invariant, it is immediately verified that, with respect to the variables b^1, \dots, b^n , the coefficients μ_{jk}^i and C_{jk}^i are homogeneous of degree 0 and -1 respectively. Therefore ω_ω as above are the local expressions of Finsler connection form, if and only if, with respect to the variables b^1, \dots, b^n , the coefficients Γ_{jk}^i and C_{jk}^i are homogeneous of degree 0 and -1 respectively, and further the well-known formula of transformation [6, p. 60] is satisfied.

It can be easily shown that the lift $l_q(X)$ of a tangent vector $X = X^i(\partial/\partial x^i)_b + X^{(i)}(\partial/\partial b^i)_b$ with respect to the Finsler connection (7) is given by

$$(8) \quad l_q(X) = X^i \left(\frac{\partial}{\partial x^i} \right)_q + X^{(i)} \left(\frac{\partial}{\partial b^i} \right)_q - p_j^k (\Gamma_{ki}^j(b)) X^i + C_{ki}^j(b) X^{(i)} \left(\frac{\partial}{\partial p_j^i} \right)_q, \\ i, j, k, l = 1, \dots, n.$$

Now, we shall treat parallel displacements. Let $C: b_t = (x^i(t), b^i(t))$ be a curve in B . Making use of (8), we have the condition for $\bar{C}: q_t = (x^i(t), b^i(t), p_j^i(t))$ to be a lift of C as follows :

$$(9) \quad \frac{dp_j^i}{dt} + p_j^k \left(\Gamma_{ki}^j(b) \frac{dx^l}{dt} + C_{ki}^j(b) \frac{db^l}{dt} \right) = 0.$$

In particular, when C is vertical, then $x^i(t), i=1, \dots, n$, are identical to $x^i(o)$, and $p_j^i(t), i, j=1, \dots, n$, are written in the forms $p_j^i(t) = p_k^i(o) \cdot g_j^k(t)$, where $C_G: g_t = (g_j^i(t))$ is a development of C in $GL(n, R)$. Hence the lift \bar{C} is given by

$$(10) \quad p_k^i(o) \frac{dg_j^k}{dt} + p_h^k(o) g_j^h C_{ki}^j(b_t) \frac{db^l}{dt} = 0.$$

We consider the property D of a Finsler connection. In consequence of Theorem 2.6, given a vertical curve $C: b_t = (x_o^i, b^i(t))$, we take its development $C_G: g_t = (g_j^i(t))$ with reference point (x_o^i, p_{jo}^i) and

a vector $v = v^i \cdot e_i$, where $v^i = p_{j_0}^{-1i} b^j(o)$. Then we obtain $g_j^k(t)v^j = v^k$, and hence contraction of (10) by v^j gives easily

$$(11) \quad b^j C_{j,k}^i(b) = 0, \quad i, j, k = 1, \dots, n,$$

which is the well-known equation [4, (10)], [11, p. 69]. Thus we get

Theorem 3.1. *In terms of canonical coordinates, the property D of the Finsler connection (7) is written in the form (11).*

Finally we shall find the geometrical meaning of the well-known equation [11, p. 15]

$$(12) \quad C_{j,k}^i(b)b^k = 0, \quad i, j, k = 1, \dots, n.$$

For this purpose, we consider curves $C = \{b_t\}$ and $C' = \{b'_t\}$ in B . If there exists a curve $C_Z = \{z_t\}$ in Z , such that $b'_t = z_t \cdot b_t$ for any t , then we say that C and C' are central-related. In particular, when C_Z is a single point z , then C and C' are simply z -related, as defined in §5. We shall show that

Theorem 3.2. *Let C' be a curve in B , which is parallel to a curve C in B . The curve C' is also parallel to any curve, which is central-related with C , if and only if (12) be satisfied.*

Proof. First we shall find the equation which shows that C' is parallel to C . If we put $C : b_t = (x^i(t), b^i(t))$ and $C' : b'_t = (x^i(t), b'^i(t))$, then we have from (5) that

$$p_j^i(t)p_k^{-1j}(o)b'^k(o) = b'^i(t),$$

where $p_j^i(t)$ satisfy (9). Therefore we have

$$(13) \quad \frac{db'^i}{dt} + b'^k \left(\Gamma_{ki}^i(b) \frac{dx^k}{dt} + C_{ki}^i(b) \frac{db^k}{dt} \right) = 0.$$

This is the condition that C' be parallel to C [4, (11)]. Now we take a central-related curve $\bar{C} : b_t = (x^i(t), (z(t))b^i(t))$ with C and then, in virtue of (13), the condition that C' be also parallel to \bar{C} is given by $b'^k C_{ki}^i(b)b'^i((z)^{-1})(d(z)/dt) = 0$, and hence the theorem is immediately proved.

§8. Vertical vector bundle and characteristic vector field

We have denoted, in §3, by B_b^v the set of vertical vectors at a point b of B . We put now $B^v = \bigcup_{b \in B} B_b^v$ and then define a mapping $\lambda : \tilde{B} = \bigcup_{x \in M} V(x) \times V(x) \rightarrow B^v$ as follows :

$$\lambda((b, b')) = \delta p((p^{-1}(b'))_v) \in B_b^v, \quad p \in \sigma^{-1} \circ \tau(b), \quad v = p^{-1}(b).$$

The letters in the right hand side will be explained in the following. Given a point $\bar{b} = (b, b') \in \bar{B}$, we take an arbitrary point $p \in \rho^{-1} \circ \tau(b)$, which is considered as an admissible mapping of V onto $V(x)$. Hence $p^{-1}(b')$ is a point of V , namely, a vector, so that we can identify $p^{-1}(b')$ with a tangent vector at $v = p^{-1}(b) \in V$ in the ordinary manner, and denote it by $(p^{-1}(b'))_v$. Then the differential δp of the admissible mapping p maps this to the vertical vector at b of B . It is easily verified that the vertical vector as thus obtained is independent of the choice of p .

Conversely, given a vertical vector $X \in B_b^v$, we have a tangent vector $\delta p^{-1}(X)$ at $v = p^{-1}(b)$, and then can identify with a point of V . Hence $p(\delta p^{-1}(X))$ is a point of the fibre through b , and thus we obtain the point $(b, p(\delta p^{-1}(X))) \in \bar{B}$. It is easily seen that $\lambda((b, p(\delta p^{-1}(X)))) = X$. Therefore \bar{B} can be identified with B^v by the mapping λ , and we shall call $\bar{\mathfrak{B}}$ the vertical tangent bundle of B .

Next, we consider a tangent vector $X \in B_b$ and put $X^v = \lambda((b, \tau(X)))$, where $\tau(X)$ is a tangent vector at $x = \tau(b)$ of M , which is considered as a point of the fibre over x . Thus, for a vector $X \in B_b$, we have an unique vertical vector X^v , which will be called the induced vertical vector from X . In terms of canonical coordinates, if $X = X^i(\partial/\partial x^i)_b + X^{(i)}(\partial/\partial b^i)_b$, then we have $X^v = X^i(\partial/\partial b^i)_b$. The next lemma is a direct result from the definition $X \rightarrow X^v$.

Lemma 3.1. *Let X and $Y \in B_b$, and f be a real function on B .*

1) *The mapping $X \rightarrow X^v$ is linear :*

$$(X + Y)^v = X^v + Y^v, \quad (f \cdot X)^v = f \cdot X^v.$$

2) *$X^v = 0$, if and only if X be vertical.*

The relation between the mapping $X \rightarrow X^v$ and central transformations of B is given by the following lemma.

Lemma 3.2. *Let C_z be a central transformation of B by $z \in Z$. Then, for $X \in B_b$, we obtain $(C_z(X))^v = (z)^{-1} \cdot C_z(X^v)$.*

Proof. If we identify a point $u \in V$ with a tangent vector u_v at $v \in V$, then it is immediately obtained that

$$(14) \quad g \cdot u_v = (g \cdot u)_{g \cdot v}.$$

It follows from (14) and the definition of $X \rightarrow X^v$, that, for $X \in B_b$, $p \in \rho^{-1} \circ \tau(b)$, and $v = p^{-1}(b)$,

$$\begin{aligned} (C_z(X))^v &= \delta p((p^{-1} \circ \tau(C_z(X)))_{z \cdot v}) = \delta p((p^{-1} \circ \tau(X))_{z \cdot v}) \\ &= \delta p(z \cdot (z^{-1} \cdot p^{-1}(\tau(X)))_v) = C_z \circ \delta p((z^{-1} \cdot p^{-1}(\tau(X)))_v) = C_z((z)^{-1} \cdot X^v). \end{aligned}$$

Thus we have proved the lemma.

We shall now define a special vertical vector field on B , which will play a rôle in the following. Let $\vec{b} = (b, b)$ be a diagonal point of \vec{B} , and then we have a vertical vector $\lambda((b, b))$ at b of B , which will be denoted by \vec{b} . The vector field $b \in B \rightarrow \vec{b}$ is called *the characteristic vector field on B* . In terms of canonical coordinates, the vector \vec{b} at b is given by $b^i(\partial/\partial b^i)_b$, where $b = (x^i, b^i)$.

Lemma 3.3. *The characteristic vector field \vec{b} is invariant by central transformations of B .*

Proof. In virtue of the definition of \vec{b} and (14), we get

$$C_z(\vec{b}) = C_z \circ \delta p(v_v) = \delta p(z \cdot v_v) = \delta p((z \cdot v)_{z \cdot v}) = \overrightarrow{(z \cdot b)},$$

where $v = p^{-1}(b)$. This prove the lemma.

The following two lemmas can be easily verified, using canonical coordinates.

Lemma 3.4. *Let X and Y be vector fields on B . Then the bracket of X^v and Y^v is written as follows:*

$$[X^v, Y^v] = [X, Y^v]^v + [X^v, Y]^v.$$

Lemma 3.5. *Let X be a vector field on B and \vec{b} the characteristic vector field on B . Then we obtain*

$$[\vec{b}, X]^v = [\vec{b}, X^v] + X^v.$$

The inverse mapping of $X \rightarrow X^v$ is not uniquely determined. But we shall denote by X^h a vector, from which the vertical vector X is induced. By means of Lemma 3.1, 2), we see that such a vector X^h is determined within vertical vector. X^h is called a *vector inducing the vector X* .

Lemma 3.6. *Let X be a vector field on B , and then the vector $X + [\vec{b}^h, X^v]$ is vertical.*

Proof. In consequence of Lemma 3.1, it is sufficient to show that $X^v + [\vec{b}^h, X^v]^v = 0$ identically. It follows from Lemma 3.4 that

$$[\vec{b}, X]^v = [\vec{b}^h, X^v]^v + [\vec{b}, X]^v.$$

Then the lemma is assured by Lemma 3.5.

§ 9. Horizontal forms and tensors

Let α be a $s(\geq 1)$ -form B and $X_1, \dots, X_s \in B_b$, such that one at least of X_1, \dots, X_s is vertical. If $\alpha(X_1, \dots, X_s) = 0$ for any point b of B , then α will be called to be *horizontal*. For such a form α , we put

$$\alpha_0(X_2, \dots, X_s) = \alpha(\vec{b}^h, X_2, \dots, X_s), \quad X_2, \dots, X_s \in B_b.$$

Since α is horizontal, the $(s-1)$ -form α_0 as thus defined is independent of the choice of \vec{b}^h . The form α_0 will be called *the reduced form from α* for $s > 1$, which is clearly horizontal. If $s = 1$, then α_0 will be called *the reduced function from α* [2, p. 18]. For $s > 1$, we get

$$\alpha(X_1, \dots, \overset{\wedge}{\underset{(i)}{b}}^h, \dots, X_s) = (-1)^{i-1} \cdot \alpha_0(X_1, \dots, \hat{X}_i, \dots, X_s)^3,$$

where $X_1, \dots, \hat{X}_i, \dots, X_s \in B_b$.

Next, let $\underline{X}_1, \dots, \underline{X}_s \in M_x$ be a set of tangent vectors at x . Then we have tangent vectors $X_1, \dots, X_s \in B_b$, $b \in V(x)$, such that $\tau(X_i) = \underline{X}_i$, $i = 1, \dots, s$. Such X_i are determined within vertical vectors, and hence we can define $\alpha(b)$ uniquely as follows:

$$\alpha(b)(\underline{X}_1, \dots, \underline{X}_s) = \alpha(X_1, \dots, X_s).$$

The mapping $\alpha(b)$ as thus defined will be called *the projection of the horizontal form α* .

Next, we consider a $s(\geq 1)$ -form β on B , which is not necessarily to be horizontal. If we put

$$\beta^h(X_1, \dots, X_s) = \beta(X_1^v, \dots, X_s^v), \quad X_1, \dots, X_s \in B_b,$$

then we obtain a s -form β^h on B , which is horizontal by virtue of Lemma 3.1, 2). We shall call β^h *the horizontal form induced from β* . It is clear that

Lemma 3.7. *If α is a horizontal $s(\geq 1)$ -form on B , then the induced horizontal form α^h vanishes identically.*

The above notions and processes for forms on B can be applied equally well to covariant $s(\geq 1)$ -tensors on B . We can define notions of a horizontal tensor and induced horizontal tensor, and further we obtain the projection of a tensor. On the other hand,

3) The sign (i) under a letter indicates that we replace X_i to this letter, and the sign (\wedge) over a letter does that this letter is to be omitted.

in the case of horizontal covariant s -tensor T , we have to remark that T is not necessarily skew-symmetric, so that we obtain from T reduced tensors of different type, and hence we shall write

$$T_{0(i)}(X_1, \dots, \hat{X}_i, \dots, X_s) = T(X_1, \dots, \vec{b}_{(i)}^h, \dots, X_s), \\ X, \dots, \hat{X}_i, \dots, X_s \in B_b.$$

§ 10. Covariant vertical derivatives

E. Cartan introduced a simple process of a covariant differentiation [2, p. 12]. We shall define such a process in our case. In the first place, we consider a real function f on B , and put

$$\Delta^v f(X) = X^v(f), \quad X \in B_b.$$

This covariant vector $\Delta^v f$ is clearly horizontal in virtue of Lemma 3.1, which will be called *the covariant vertical derivative*, or, for brevity, *covariant v -derivative of f* .

Generally, let T be a horizontal covariant $s(\geq 1)$ -tensor on B and put, for $X_1, \dots, X_s, Y \in B_b$,

$$\Delta^v T(X_1, \dots, X_s, Y) \\ = Y^v(T(X_1, \dots, X_s)) + \sum_{i=1}^s T(X_1, \dots, [X_i, Y^v], \dots, X_s)$$

We shall verify that $\Delta^v T$ as thus defined is a covariant $(s+1)$ -tensor. In fact, we can easily show that

$$\Delta^v T(X+X', X_2, \dots, X_s, Y) = \Delta^v T(X, \dots, X_s, Y) + \Delta^v T(X', \dots, X_s, Y), \\ \Delta^v T(X_1, \dots, X_s, Y+Y') = \Delta^v T(X_1, \dots, X_s, Y) + \Delta^v T(X_1, \dots, X_s, Y'),$$

where $X, X', X_2, \dots, X_s, Y, Y' \in B_b$. Next, let φ and ψ be functions on B , and X and Y vector fields on B . Then it is well-known [7, p. 4] that

$$[\varphi X, \psi Y] = \varphi \cdot \psi \cdot [X, Y] + \varphi \cdot (X(\psi)) \cdot Y - \psi \cdot (Y(\varphi)) \cdot X.$$

Making use of this we obtain immediately, for $X_1, \dots, X_s, Y \in B_b$,

$$\Delta^v T(\varphi X_1, X_2, \dots, X_s, \psi Y) = \varphi \cdot \psi \cdot \Delta^v T(X_1, \dots, X_s, Y) \\ + \varphi \cdot (X_1(\psi)) \cdot T(Y^v, X_2, \dots, X_s) + \varphi \cdot \sum_{i=2}^s X_i(\psi) \cdot T(X_1, \dots, \vec{Y}_{(i)}^v, \dots, X_s).$$

Since T is assumed to be horizontal, then the last two terms of the above equation vanish, and hence we conclude that $\Delta^v T$ is a covariant tensor. Since $[X_i, Y^v]$ is vertical for any vertical

vector X_i , then it is easily seen that $\Delta^v T$ is horizontal. Then we have the horizontal covariant $(s+1)$ -tensor $\Delta^v T$, which will be called *the covariant v -derivative of T* .

Lemma 3.8. 1) *If T is a horizontal covariant symmetric tensor on B , then its covariant v -derivative $\Delta^v T$ has the property that $\Delta^v T(X_1, \dots, X_s, Y), X_1, \dots, X_s, Y \in B_b$, is symmetric with respect to X_1, \dots, X_s .*

2) *Let f be a real function on B , and then we have the horizontal covariant s -tensor $\underbrace{\Delta^v \dots \Delta^v f}_s$. This tensor is symmetric for $s \geq 2$.*

Proof. From the definition of the operator Δ^v , 1) is clearly satisfied, and hence we shall prove 2) by the process of mathematical induction. We observe first that, for $X, Y \in B_b$,

$$\Delta^v \Delta^v f(X, Y) = Y^v(X^v(f)) + [X, Y^v]^v(f).$$

In consequence of this and Lemma 3.4, it is easily seen that $\Delta^v \Delta^v f$ is symmetric. Next, we assume that $\underbrace{\Delta^v \dots \Delta^v f}_r$ be symmetric for $r=2, \dots, s$. If we write $\underbrace{\Delta^v \dots \Delta^v f}_{s-1} = T$, then, by direct calculation, we obtain

$$\begin{aligned} & \underbrace{\Delta^v \dots \Delta^v f}_{s+1}(Y, X_2, \dots, X_s, X) - \underbrace{\Delta^v \dots \Delta^v f}_{s+1}(X, X_2, \dots, X_s, Y) \\ &= ([X^v, Y^v] + [Y, X^v]^v - [X, Y^v]^v)(T(X_2, \dots, X_s)) \\ &+ \sum_{i=2}^s T(X_2, \dots, \underset{(i)}{Z_i}, \dots, X_s), \end{aligned}$$

where $X, Y, X_2, \dots, X_s \in B_b$, and we put

$$\begin{aligned} Z_i &= [X_i, [Y, X^v]^v] + [[X_i, X^v], Y^v] - [X_i, [X, Y^v]^v] \\ &\quad - [[X_i, Y^v], X^v]. \end{aligned}$$

As a result of Lemma 3.4, we see that the first term of the right hand members vanishes and that Z_i are equal to zero by means of the Jacobi's identity for the brackets. Thus we establish the lemma.

Lemma 3.9. *Let T and T' be horizontal covariant tensors or functions on B and $T \cdot T'$ their product. Then we have*

$$\Delta^v(T \cdot T') = \Delta^v T \cdot T' + T \cdot \Delta^v T'.$$

The proof of this lemma is quite easy. Now, the next lemma

gives the relation between the covariant v -differentiation and the dual of central transformation of B .

Lemma 3.10. *Let T be a horizontal covariant tensor or a function on B . Then we obtain*

$$\Delta^v(C_z^*(T)) = (z) \cdot C_z^*(\Delta^v T),$$

where (z) is a real number corresponding to the element $z \in Z$.

Proof. We remark first that $C_z^*(T)$ is horizontal, because C_z maps a vertical vector into a vector of the same kind. We shall prove the lemma for a horizontal covariant $s(\geq 1)$ -tensor T . For $X_1, \dots, X_s, Y \in B_b$, we have

$$\begin{aligned} (\Delta^v(C_z^*(T)))(X_1, \dots, X_s, Y) &= Y^v(T(C_z(X_1), \dots, C_z(X_s)) \circ C_z) \\ &+ \sum_{i=1}^s T(C_z(X_1), \dots, [C_z(X_i), C_z(Y^v)], \dots, C_z(X_s)). \end{aligned}$$

According to Lemma 3.2, this is rewritten in the form

$$\begin{aligned} &= (z) \cdot (C_z(Y))^v(T(C_z(X_1), \dots, C_z(X_s))) \\ &+ (z) \cdot \sum_{i=1}^s T(C_z(X_1), \dots, [C_z(X_i), (C_z(Y))^v], \dots, C_z(X_s)) \\ &= (z) \cdot \Delta^v T(C_z(X_1), \dots, C_z(X_s), C_z(Y)). \end{aligned}$$

This concludes the proof of the lemma.

Lemma 3.11. 1) *For a real function f on B , we have the reduced function $(\Delta^v f)_0$ from the covariant v -derivative $\Delta^v f$ of f . Then we obtain $(\Delta^v f)_0(b) = \vec{b}(f)$, $b \in B$.*

2) *Let T be a horizontal covariant $s(\geq 1)$ -tensor on B , and then we have, for $X_1, \dots, \hat{X}_i, \dots, X_s, Y \in B_b$, and $i=1, \dots, s$,*

$$\begin{aligned} &(\Delta^v T)_{0(i)}(X_1, \dots, \hat{X}_i, \dots, X_s, Y) \\ &= (\Delta^v(T_{0(i)}))(X_1, \dots, \hat{X}_i, \dots, X_s, Y) - T(X_1, \dots, Y, \dots, X_s). \end{aligned}$$

Proof. Since $(\vec{b}^h)^v = \vec{b}$, we have 1) easily. We shall show 2).

$$\begin{aligned} &(\Delta^v T)_{0(i)}(X_1, \dots, \hat{X}_i, \dots, X_s, Y) = \Delta^v T(X, \dots, \vec{b}_{(i)}^h, \dots, X_s, Y) \\ &= Y^v(T(X_1, \dots, \vec{b}_{(i)}^h, \dots, X_s)) + \sum_{a=1}^{i-1} T(X_1, \dots, [X_a, Y^v], \dots, \vec{b}_{(i)}^h, \dots, X_s) \\ &+ T(X_1, \dots, [\vec{b}_{(i)}^h, Y^v], \dots, X_s) \\ &+ \sum_{a=i+1}^s T(X_1, \dots, \vec{b}_{(i)}^h, \dots, [X_a, Y^v], \dots, X_s). \end{aligned}$$

In the third term of the right hand members, we can replace $[\vec{b}_{(i)}^h, Y^v]$ by $-Y$ in virtue of Lemma 3.6 and hence 2) is proved.

Chapter IV. Euclidean connections in linear Finsler bundles

§11. The fundamental function and metric tensor

We consider a covariant $s(\geq 0)$ -tensor T on B , and suppose that T satisfy the equation

$$C_z^*(T) = (z)^r \cdot T,$$

where C_z is any central transformation of B . Then T will be said to be r -homogeneous. In particular, that T is 0-homogeneous means that T is central invariant. We write by R^+ the set of positive numbers, and if the above equation holds for any z , $(z) \in R^+$ only, then T will be said to be *positively r -homogeneous*. We shall define the homogeneity for differentiable forms similarly. On making use of the covariant v -derivative, the Euler's theorem for homogeneous functions leads us to the fact that a covariant s -tensor T is r -homogeneous, if and only if the equation

$$(\Delta^v T)_{0(s+1)} = r \cdot T$$

be satisfied (cf. Lemma 3.11, 2)).

Lemma 4.1. *Let T be a r -homogeneous covariant $s(\geq 1)$ -tensor on B . Then the induced horizontal tensor T^h is $(r-s)$ -homogeneous. The same is true for a form.*

Proof. From the definition of the horizontal tensor T^h , we have

$$\begin{aligned} (C_z^*(T^h))(X_1, \dots, X_s) &= T^h(C_z(X_1), \dots, C_z(X_s)) \\ &= T((C_z(X_1))^v, \dots, (C_z(X_s))^v), \quad X_1, \dots, X_s \in B_b. \end{aligned}$$

From Lemma 3.2, we can rewrite this in the form

$$\begin{aligned} &= (z)^{-s} \cdot T(C_z(X_1^v), \dots, C_z(X_s^v)) = (z)^{-s} \cdot (C_z^*(T))(X_1^v, \dots, X_s^v) \\ &= (z)^{r-s} \cdot T^h(X_1, \dots, X_s). \end{aligned}$$

This proves the lemma.

Now we consider the local expressions ω_a on V_a of the Finsler connection form ω . Since we have shown that ω_a is central invariant, we obtain, as a result of Lemma 4.1, the following theorem.

Theorem 4.1. *Let ω_a be the local expressions on V_a of the Finsler connection form ω . The local horizontal forms $(\omega_a)^h$ are (-1) -homogeneous.*

The forms $(\omega_a)^h$ are called *the local horizontal connection forms*. Let ω_a and ω_b be local expressions of ω on V_a and V_b respectively. Then it is well known that we obtain the equations

$$\omega_b = ad(g_{a\beta}^{-1}) \cdot \omega_a + g_{a\beta}^{-1} \cdot \delta g_{a\beta},$$

where $g_{a\beta}: V_a \cap V_b \rightarrow G$ satisfy the equations $\sigma_b = \sigma_a \cdot g_{a\beta}$ for the local sections σ_a and σ_b . Since we obtain

$$\eta \circ \sigma_b(b) = \rho_b(x) = \rho_a(x) \cdot g_{a\beta}(b), \quad b \in V_a \cap V_b, \quad x = \tau(b),$$

the mappings $g_{a\beta}$ are constant on each fibre $V(x)$, and hence the differentials $\delta g_{a\beta}$ carry a vertical vector to zero. Then we see

$$(\omega_b)^h(X) = ad(g_{a\beta}^{-1}) \cdot (\omega_a)^h(X), \quad X \in B_b.$$

This means that *the local horizontal connection forms are of $ad(G)$ -type*. Making use of (7), we have easily that

$$(\omega_a)^h = C_{jk}^i(x^1, \dots, x^n, b^1, \dots, b^n) dx^k \cdot \hat{g}_i^j.$$

That $(\omega_a)^h$ is of $ad(G)$ -type means that the set of C_{jk}^i obeys the transformation of the well-known tensor-type by the transformation of canonical coordinates. And the above theorem shows that C_{jk}^i are (-1) -homogeneous functions with respect to the variables b^1, \dots, b^n (cf. § 7).

We shall return to the consideration of general homogeneous tensors and prove that

Lemma 4.2. *Let T be a r -homogeneous covariant $s(\geq 1)$ -tensor on B . Then the reduced tensors $T_{\alpha(i)}$, $i=1, \dots, s$, are $(r+1)$ -homogeneous. The same is true of a form.*

Proof. From Lemmas 3.2 and 3.3, it follows that

$$((z) \cdot C_z(\vec{b}^h))^v = C_z(\vec{b}) = (\vec{z} \cdot \vec{b}), \quad b \in B,$$

and hence we can choose $(z) \cdot C_z(\vec{b}^h)$ as a vector inducing the characteristic vector $(\vec{z} \cdot \vec{b})$ at the point $z \cdot b$. Therefore we obtain, for $X_2, \dots, X_s \in B_b$,

$$\begin{aligned} (C_z^*(T_{\alpha(i)}))(X_2, \dots, X_s) &= T((\vec{z} \cdot \vec{b})^h, C_z(X_2), \dots, C_z(X_s)) \\ &= (z) \cdot T(C_z(\vec{b}^h), C_z(X_2), \dots, C_z(X_s)) = (z)^{r+1} \cdot T(\vec{b}^h, X_2, \dots, X_s) \\ &= (z)^{r+1} \cdot T_{\alpha(i)}(X_2, \dots, X_s). \end{aligned}$$

This proves the lemma.

Lemma 4.3. *Let T be a r -homogeneous covariant $s(\geq 0)$ -tensor on B . Then $\underbrace{\Delta^v \cdots \Delta^v}_u T$ is $(r-u)$ -homogeneous.*

Proof. It follows from Lemma 3.9 that

$$\Delta^v(C_z^*(T)) = \Delta^v((z)^r \cdot T) = (z)^r \cdot \Delta^v T.$$

On the other hand, it follows from Lemma 3.10 that

$$\Delta^v(C_z^*(T)) = (z) \cdot C_z^*(\Delta^v T).$$

Hence $\Delta^v T$ is $(r-1)$ -homogeneous. The lemma will be established if we repeat the above process.

We are now in a position to define the fundamental function of a Finsler manifold. We suppose that we are given a positive valued function L on B , which satisfies

- 1) L is positively 1-homogeneous.

If we put $F = \frac{1}{2} L^2$, then we see that $\Delta^v \Delta^v F$ is positively 0-homogeneous horizontal symmetric covariant 2-tensor on B , in virtue of Lemmas 3.8, 2) and 4.3. Hence we can define the projection $(\Delta^v \Delta^v F)(b)$ on the base manifold M . We impose upon L the further condition :

- 2) $(\Delta^v \Delta^v F)(b)$ is positive-definite, that is,

$$(\Delta^v \Delta^v F)(b)(\underline{X}, \underline{X}) \geq 0, \quad \underline{X} \in M_x, \quad b \in \tau^{-1}(x),$$

and the equality holds if and only if $\underline{X} = 0$.

The function L (or F) satisfying the above two conditions will be called *the fundamental function of the Finsler bundle* Ω , and then M *the Finsler manifold*. The value $L(b)$ of L at a point $b \in B$ is called *the Finslerian length of the tangent vector* $b \in M_x, x = \tau(b)$. Since L is positively 1 homogeneous, then we obtain $L(z \cdot b) = (z) \cdot L(b), (z) \in R^+$.

Next, we shall introduce the metric tensor. We suppose that a field of covariant 2-tensor m is given on B , which satisfies the following conditions :

- 1) m is horizontal and symmetric,
- 2) m is 0-homogeneous.
- 3) the projection $m(b)$ on M is positive-definite.

Then we shall call m *the metric tensor on B* . For a tangent vector $\underline{X} \neq 0$ at x of M , we obtain a positive number $\sqrt{m(b)(\underline{X}, \underline{X})}, b \in \tau^{-1}(x)$, which is called *the relative euclidean length of \underline{X} with*

respect to the element of support b . From the definition of the projection $m(b)$ and the condition 2) of m , it follows immediately that $m(z \cdot b)(\underline{X}, \underline{X}) = m(b)(\underline{X}, \underline{X})$, $z \in Z$. We construct the reduced covariant vector m_0 from m , which is 1-homogeneous in virtue of Lemma 4.2. m_0 is called the characteristic form and denoted by \vec{b} . Thus we have $\vec{b}(X) = m(\vec{b}^h, X)$, $X \in B_b$. The reduced function \vec{b}_0 from \vec{b} is given by $\vec{b}_0(b) = m(\vec{b}^h, \vec{b}^h) = m(b)(b, b)$, $b \in B$, from which it follows that the value of \vec{b}_0 at $b \in B$ is the square of the relative euclidean length of $b \in M_x$, $x = \tau(b)$, with respect to the element of support b itself. The quantity $\sqrt{\vec{b}_0(b)}$ is called the absolute length of the vector b .

§ 12. Pure-horizontal subspaces

We consider the associated bundle $\tilde{\mathfrak{B}}$ with the Finsler bundle \mathfrak{Q} , which has been looked upon as the vertical tangent bundle of B as shown in § 8. Hence we can define the covariant derivative $D_Y X$ of a vertical vector field X on B with respect to a vector field Y on B [7, p. 52]. That is, let $C = \{b_t, 0 \leq t \leq 1\}$ be an integral curve of Y issuing from $b_0 \in B$ and $\tilde{C} = \{q_t, 0 \leq t \leq 1\}$ a lift to Q of C issuing from $q_0 \in \sigma^{-1}(b_0)$. Then $D_Y X$ is given by

$$(D_Y X)_{b_0} = \lim_{t \rightarrow 0} \frac{1}{t} (q_0 \circ q_t^{-1}(X_{b_t}) - X_{b_0}).$$

The vertical vector X_{b_t} is expressed by a pair (b_t, b'_t) and we put $q_t = (b_t, p_t)$, and then we obtain

$$q_0 \circ q_t^{-1}(X_{b_t}) = q_0(p_t^{-1}(b'_t)) = (b_0, p_0 \circ p_t^{-1}(b'_t)).$$

Therefore we have the expression of $D_Y X$ as follows:

$$(15) \quad (D_Y X)_{b_0} = \left(b_0, \lim_{t \rightarrow 0} \frac{1}{t} (p_0 \circ p_t^{-1}(b'_t) - b'_0) \right).$$

We consider a vector field X on B , which is not necessarily to be vertical. Then we have the vertical vector field X^v , and hence we can define the covariant derivative $D_Y X^v$, which will be written in the form $D_Y^v X$.

Next, we consider a horizontal covariant s -tensor T on B , and define the covariant derivative $D_Y^h T$ of T with respect to Y as follows [7, p. 55]:

$$D_Y^v T(X_1, \dots, X_s) = Y(T(X_1, \dots, X_s)) - \sum_{i=1}^s T(X_1, \dots, (D_Y^v X_i)^h, \dots, X_s),$$

where $X_1, \dots, X_s \in B_b$. The covariant differential $D^v T$ of T is the $(s+1)$ -tensor, which is given by $D^v T(X_1, \dots, X_s, Y) = D_Y^v T(X_1, \dots, X_s)$, $X_1, \dots, X_s, Y \in B_b$.

Next, let \underline{X} be a vector field on M and $b_t, 0 \leq t \leq 1$, a vector field defined along a curve $\underline{C} = \{x_t, 0 \leq t \leq 1\}$ on M . The curve $C = \{b_t\}$ is in B and covers \underline{C} . Let $C^* = \{p_t, 0 \leq t \leq 1\}$ be a life to P of C issuing from a point $p_0 \in \rho^{-1}(x_0)$, and we define

$$(D_b \underline{X})_{x_0} = \lim_{t \rightarrow 0} \frac{1}{t} (p_0 \circ p_t^{-1}(\underline{X}_{x_t}) - \underline{X}_{x_0}),$$

which is clearly a tangent vector at x_0 . $D_b \underline{X}$ as thus defined will be called *the covariant derivative of \underline{X} with respect to the element of support b_t* . We take a tangent vector Y of the curve C and a vertical vector field $X = (b_t, \underline{X})$. Then we have from (15) that $D_Y X = (b, D_b \underline{X})$. Making use of (9), we obtain easily

$$(16) \quad D_b \underline{X} = \left[\frac{dX^i}{dt} + X^j \left(\Gamma_{jk}^i(b) \frac{dx^k}{dt} + C_{jk}^i(b) \frac{db^k}{dt} \right) \right] \left(\frac{\partial}{\partial x^i} \right)_x,$$

where $\underline{X} = X^i(\partial/\partial x^i)_x$, $b = (x^i, b^i)$.

In particular, we can consider the case where the curve \underline{C} is a single point x_0 and C a vertical curve in the fibre $V(x_0)$. Then we obtain

$$(D_b \underline{X})_{x_0} = \lim_{t \rightarrow 0} \frac{1}{t} (p_0 \circ p_t^{-1}(\underline{X}) - \underline{X}), \quad \underline{X} \in M_{x_0},$$

and such a covariant derivative of a fixed vector \underline{X} for a rotation of the element of support b_t will be denoted by $D_b^0 \underline{X}$. From (16) it follows that $D_b^0 \underline{X} = X^j C_{jk}^i(b) (db^k/dt) (\partial/\partial x^i)_x$, where $(b^k(t))$ is the vertical curve C . In terms of the local horizontal connection form $(\omega_a)^h$, this equation is written in the form

$$(17) \quad D_b^0 \underline{X} = (\omega_a)^h(Y^h) \cdot \underline{X},$$

where Y is the vertical vector tangent to (b_t) , and the dot denotes the product of matrices $(\omega_a)^h(Y^h)$ and \underline{X} .

We suppose now that the Finsler connection under consideration has the property D . We take a fixed point $x_0 \in M$ and a curve $\underline{C} = \{x_t, 0 \leq t \leq 1\}$ on M issuing from x_0 . Then we shall show that there exists an unique curve $C = \{b_t, 0 \leq t \leq 1\}$ on B ,

covering \underline{C} and issuing from a given point $b_0 \in \tau^{-1}(x_0)$, such that the covariant derivative $D_b b$ of b with respect to the element of support b itself vanishes identically. In fact, from (11) and (16), we obtain the differential equations of such a curve C as follows:

$$(18) \quad \frac{db^i}{dt} + b^j \Gamma_{jk}^i(b) \frac{dx^k}{dt} = 0,$$

with the initial condition $b^i(0) = b_0^i$, where $(x^k(t))$ expresses the curve \underline{C} on M . Therefore the curve C is uniquely determined by the curve \underline{C} on M and the starting point of C . The curve C will be called *the horizontal curve*, covering \underline{C} , or *the lift of \underline{C} to B* . Let H_b be a set of tangent vectors at b of B , which are tangent to lifts of curves on M issuing from $x = \tau(b)$. We shall call H_b *the horizontal subspace of B_b* . It is clear that B_b is the direct sum of H_b and B_b^v . For any $\underline{X} \in M_x$, there is an unique horizontal vector $l_b(\underline{X})$ at $b \in \tau^{-1}(x)$, such that $\tau(l_b(\underline{X})) = \underline{X}$. The vector $l_b(\underline{X})$ is called *the lift at b of \underline{X}* . In consequence of (18), the expression of the lift $l_b(\underline{X})$ at $b = (x^i, b^i)$ of a vector $\underline{X} = X^i(\partial/\partial x^i)_x$ is given by the equation

$$(19) \quad l_b(\underline{X}) = X^i \left(\frac{\partial}{\partial x^i} \right)_b - b^j \Gamma_{jk}^i(b) X^k \left(\frac{\partial}{\partial b^i} \right)_b.$$

For $X \in B_b$, we can write uniquely $X = v(X) + h(X)$, where $v(X)$ is vertical and $h(X)$ horizontal. If we put $X = X^i(\partial/\partial x^i)_b + X^{(i)}(\partial/\partial b^i)_b$, then $h(X)$ is given by the same equation (19), and hence $v(X)$ is given by

$$(19') \quad v(X) = (X^{(i)} + b^j \Gamma_{jk}^i(b) X^k) \left(\frac{\partial}{\partial b^i} \right)_b.$$

It is to be noted here that the vertical component $v(X)$ is different from the induced vertical vector X^v .

We consider a s -form α on B and put

$$\begin{aligned} \alpha^0(X_1, \dots, X_s) &= \alpha(v(X_1), \dots, v(X_s)), \\ \alpha^1(X_1, \dots, X_s) &= \alpha(h(X_1), \dots, h(X_s)), \quad X_1, \dots, X_s \in B_b. \end{aligned}$$

Then we obtain two forms α^0 and α^1 , the latter being horizontal. The form α^0 (resp. α^1) is called *the vertical* (resp. *horizontal*) *component of α* . Further we have the form $\alpha - \alpha^0 - \alpha^1$, which is called *the mixed component of α* . It is clear that the mixed component of a 1-form is equal to zero.

Let ω_a be local expressions of the Finsler connection form ω . The vertical and horizontal components of ω_a are given by

$$(\omega_a)^0 = (\Gamma_{jk}^{0i} dx^k + C_{jk}^i db^k) \hat{g}_i^j, \quad (\omega_a)^1 = \Gamma_{jk}^{*i} dx^k \hat{g}_i^j,$$

where we put

$$\Gamma_{jk}^{*i} = \Gamma_{jk}^i - C_{jh}^i \Gamma_{lk}^h b^l, \quad \Gamma_{jk}^{0i} = \Gamma_{jk}^i - \Gamma_{jk}^{*i}.$$

The coefficients Γ_{jk}^{*i} have been introduced by E. Cartan [2, p. 14] and will play a rôle in the following.

We consider next the bundle space Q of the Finsler bundle Ω . we have defined, in § 3, a zero-horizontal subspace Γ_q^0 of the horizontal subspace Γ_q at $q \in Q$, which is the lift of vertical subspace B_b^0 of B_b , $b = \sigma(q)$. We have now the horizontal subspace H_b , and hence can define the subspace Γ_q^1 , which is the lift of H_b . We shall call Γ_q^1 the *pure-horizontal subspace* of Q_q . It is clear that Γ_q is the direct sum of the zero-horizontal subspace Γ_q^0 and pure-horizontal subspace Γ_q^1 . Thus we obtain the unique decomposition of a vector $\bar{X} \in Q_q$ as follows :

$$\bar{X} = v(\bar{X}) + h^0(\bar{X}) + h^1(\bar{X}), \quad h^0(\bar{X}) \in \Gamma_q^0, \quad h^1(\bar{X}) \in \Gamma_q^1.$$

If we put $\bar{X} = X^i(\partial/\partial x^i)_q + X^{(i)}(\partial/\partial b^i)_q + X_j^i(\partial/\partial p_j^i)_q$, then the pure-horizontal component $h(\bar{X})$ of \bar{X} is given, in virtue of (8) by the equation

$$(20) \quad h(\bar{X}) = X^i \left(\frac{\partial}{\partial x^i} \right)_q - b^j \Gamma_{jk}^i(\sigma(q)) X^k \left(\frac{\partial}{\partial b^i} \right)_q - p_j^i \Gamma_{ki}^{*j}(\sigma(q)) X^l \left(\frac{\partial}{\partial p_j^i} \right)_q.$$

We consider a horizontal s -form α on Q , and then we have two forms α^0 and α^1 , such that, for $X_1, \dots, X_s \in Q_q$,

$$\begin{aligned} \alpha^0(X_1, \dots, X_s) &= \alpha(h^0(X_1), \dots, h^0(X_s)), \\ \alpha^1(X_1, \dots, X_s) &= \alpha(h^1(X_1), \dots, h^1(X_s)), \end{aligned}$$

which will be called *the zero and pure components* of α respectively. The form $\alpha - \alpha^0 - \alpha^1$ will be called *the mixed component* of α . It is clear that the mixed component of a 1-form vanishes.

We consider finally the principal bundle \mathfrak{P} . Given a point $p \in P$, we take an arbitrary point $b \in \tau^{-1} \circ \rho(p)$, and then we obtain a point $q = (b, p) \in Q$. We now define the subspace $H^*(b)_p$ of P_p as $\eta(\Gamma_q^1)$, where η is the canonical projection $Q \rightarrow P$. $H^*(b)_p$ is

called the horizontal subspace of P_p with respect to the point $b \in B$. Let X^* be a tangent vector at p and put $h_b(X^*) = \eta \circ l_q \circ l_b \circ \rho(X^*)$. It is clear that $\rho(h_b(X^*)) = \rho(X^*)$, so that $X^* - h_b(X^*)$ is vertical. Hence P_p is the direct sum of $H^*(b)_p$ and the vertical subspace $G(x)_p$, $x = \rho(p)$. It is to be remarked that this decomposition of P_p will depend upon the choice of a point $b \in \tau^{-1} \circ \rho(p)$ generally. It is easily shown that, if we put $X^* = X^i(\partial/\partial x^i)_p + X^j(\partial/\partial p^j)_p$, then the horizontal component $h_b(X^*)$ is given by

$$(21) \quad h_b(X^*) = X^i \left(\frac{\partial}{\partial x^i} \right)_p - p^j \Gamma_{ki}^{*j}(b) X^k \left(\frac{\partial}{\partial p^j} \right)_p.$$

§ 13. Torsion and curvature forms

In this section, we suppose also that the Finsler connection under consideration has the property D , so that we have the notion of pure-horizontal subspaces.

Let θ^* be a V -valued 1-form on P , such that $\theta_p^*(X^*) = p^{-1} \circ \rho(X^*)$, [7, p. 49]. Then we have the induced form $\theta^1 = \eta^*(\theta^*)$ on Q , where η is the canonical mapping $Q \rightarrow P$. We see easily that

$$\theta_q^1(X) = p^{-1} \circ \tau \circ \sigma(X), \quad q = (b, p), \quad X \in Q_q,$$

from which it follows that θ^1 vanishes on the zero-horizontal subspace. We shall call θ^1 the pure-basic form on Q .

Next, we define a V -valued 1-form θ^0 on Q , such that

$$\theta_q^0(X) = \delta p^{-1}(v(\sigma(X))), \quad q = (b, p), \quad X \in Q_q.$$

where $v(\sigma(X))$ is the vertical component of $\sigma(X) \in B_b$, and $\delta p^{-1}(v(\sigma(X)))$ is the tangent vector at $p^{-1}(b)$ of V , which is identified with a point of V . We shall call θ^0 the zero-basic form on Q . It is clear that θ^0 vanishes on the pure-horizontal subspace.

The following equations can be easily verified.

$$\begin{aligned} R_\sigma^i(\theta^i) &= g^{-1} \cdot \theta^i, & i &= 0, 1, \\ C_z^*(\theta^0) &= z \cdot \theta^0, & C_z^*(\theta^1) &= \theta^1. \end{aligned}$$

Thus the zero-basic form is not central invariant.

We shall now introduce the basic vector fields on Q [7, p. 49]. The pure-basic vector field $B^1(v)$ corresponding to an element $v \in V$ is that the value of $B^1(v)$ at $q \in Q$ is given by

$$B^1(v)_q = l_q \circ l_b \circ p(v), \quad q = (b, p),$$

where l_b (res. l_q) is the operation of taking a life to B (resp. Q) of a vector on M (resp. B). The zero-basic vector field $B^0(v)$ corresponding to $v \in V$ is given by

$$B^0(v)_q = l_q \circ \delta p(v), \quad q = (b, p),$$

where $v \in V$ is considered as the tangent vector at $p^{-1}(b)$. The process which was used in [7, p. 49] can be applied to our $B^i(v)$, $i=0, 1$, and then we see the zero- (resp. pure-) basic vector field X is a horizontal vector field on Q such that $\theta^0(X)$ (resp. $\theta^1(X)$) is constant. Making use of this fact, we can easily prove the following equations.

$$\begin{aligned} R'_q(B^i(v)) &= B^i(g^{-1} \cdot v), \quad i = 0, 1, \\ C'_z(B^0(v)) &= B^0(z \cdot v), \quad C'_z(B^1(v)) = B^1(v). \end{aligned}$$

Thus the zero-basic vector field is not central invariant. By means of the above equations, we obtain [7, p. 50]

$$[A, B^i(v)] = B^i(\hat{A} \cdot v), \quad i = 0, 1,$$

where A is the fundamental vector field corresponding to $\hat{A} \in \hat{G}$.

We consider the covariant differential $D\theta^i = \Theta^i$, $i = 0, 1$, of θ^i . Θ^0 (resp. Θ^1) will be called *the zero* (resp. *pure*) *torsion form*. Then we have the pure, zero and mixed components of Θ^i , which are written by $\Theta^{i(1)}$, $\Theta^{i(0)}$ and $\Theta^{i(01)}$ respectively. It is easily seen that the zero-component of Θ^1 is equal to zero, and hence we have

$$\begin{aligned} \Theta^0 &= \Theta^{0(0)} + \Theta^{0(01)} + \Theta^{0(1)}, \\ \Theta^1 &= \Theta^{1(1)} + \Theta^{1(01)}. \end{aligned}$$

The pure component $\Theta^{1(1)}$ of the pure torsion form Θ^1 will be important in the final section. In terms of canonical coordinates, the local expressions $\Theta^{1(1)}_\alpha$ are given in the form

$$(22) \quad \Theta^{1(1)}_\alpha = \frac{1}{2} (\Gamma_{jk}^{*i} - \Gamma_{kj}^{*i}) dx^k \wedge dx^j \cdot e_i,$$

where (e_i) is a fixed base of V . Since E. Cartan assumed that the pure component $\Theta^{1(1)}$ vanish, then he treated only the mixed component $\Theta^{1(01)}$ of Θ^1 and called it the torsion from [2, p. 33]. On the other hand *the zero-torsion form Θ^0 is not central invariant*.

The additional structure equations [7, p. 51] are also obtained for Θ^0 and Θ^1 , which are immediately given in the following.

$$d\theta^i(X, Y) = \frac{1}{2}(\omega(Y) \cdot \theta^i(X) - \omega(X) \cdot \theta^i(Y)) + \Theta^i(X, Y),$$

where $i=0, 1$ and $X, Y \in Q_q$.

We consider finally the curvature form $\Omega = D\omega$, and then we obtain the pure, zero and mixed components of Ω . E. Cartan introduced the curvature forms of three kinds [2, p. 33]. In our treatment, we can obtain these forms by means of the notion of pure- and zero-horizontal subspace of Q_q . The curvature form Ω is, of course, central invariant.

§ 14. The euclidean connection defined by E. Cartan

E. Cartan introduced the elegant process in order to determine an euclidean connection by the fundamental function. In the final section of this paper, we shall discuss his five postulates.

We have defined, in § 11, the fundamental function L and the metric tensor m . We give now a relation between L and m as follows:

Postulate I. *The Finsler length $L(b)$ and the absolute euclidean length $\sqrt{m(b)(\vec{b}, \vec{b})}$ of any tangent vector $b \in M_x$, $x = \tau(b)$, coincide.*

In terms of the function $F = \frac{1}{2}L^2$ and the reduced function \vec{b}_0 of the characteristic form \vec{b} , the above postulate is expressed by $2F = \vec{b}_0$ at each point of B . Taking the covariant v -derivative of the both sides of this equation, we have

$$2\Delta^v F(X) = X^v(m(\vec{b}^h, \vec{b}^h)), \quad X \in B_b.$$

Taking account of the symmetry of m , we have

$$\Delta^v m(\vec{b}^h, \vec{b}^h, X) = X^v(m(\vec{b}^h, \vec{b}^h)) + 2m([\vec{b}^h, X^v], \vec{b}^h),$$

and, according to Lemma 3.6, we obtain

$$= X^v(m(\vec{b}^h, \vec{b}^h)) - 2m(X, \vec{b}^h).$$

Hence, under the above postulate, we have

$$(23) \quad 2\Delta^v F = (\Delta^v m)_{0(c_1)c_2} + 2\vec{b}_0.$$

Since m is 0-homogeneous, we obtain $(\Delta^v m)_{0(c_3)} = 0$ in virtue of the Euler's theorem, while $(\Delta^v m)_{0(c_1)}$ is equal to $(\Delta^v m)_{0(c_2)}$. These are

written by $(\Delta^v m)_0$ simply. We give now the second postulate as follows :

Postulate II. *The metric tensor m satisfies the condition of normality: $(\Delta^v m)_0 = 0$ [4, p. 372].*

When this postulate is satisfied, the equation (23) is reduced to the simple form

$$(24) \quad \Delta^v F = \underset{\rightarrow}{b}.$$

It follows from Lemma 3.11, 2) that, for $X, Y \in B_b$,

$$\Delta^v \underset{\rightarrow}{b}(X, Y) = (\Delta^v(m_0))(X, Y) = (\Delta^v m)_0(X, Y) + m(X, Y),$$

and hence Postulate II and (24) give

$$(25) \quad \Delta^v \Delta^v F = m.$$

Therefore, *the metric tensor m is determined by the fundamental function F under Postulate I and II.* It is easily seen that the properties 1), 2) and 3) imposed upon m are satisfied by means of the properties of L_* and covariant v -differentiation.

The following postulate is the same as Postulate C of E. Cartan.

Postulate III. *The covariant derivative $D_0^v b'$ of a fixed vector $b' \in M_x$ for a rotation of the element of support b satisfies the equation $m(b)(D_0^v b', b'') = m(b)(b', D_0^v b'')$.*

In the following we shall use letters of matrices. It follows from (17) that the left hand side of the above equation is written in the form ${}^t b'' \cdot (m(\omega_a)^h(Y^h)) \cdot b'$,⁴⁾ Since m is symmetric, Postulate III is expressible in the form

$$(26) \quad {}^t(\omega_a)^h m = m(\omega_a)^h.$$

In terms of canonical coordinates, (26) is written in the well-known form

$$C_{jik} = C_{ijk}, \quad i, j, k = 1, \dots, n, \quad \text{where } C_{ijk} = g_{jt} C_{ik}^t.$$

In order to give the following postulate, we consider the covariant derivative $D^v m$ of m :

$$D^v m(X, Y, Z) = Z(m(X, Y)) - m((D_Z^v X)^h, Y) - m(X, (D_Z^v Y)^h),$$

where $X, Y, Z \in B_b$. In making use of canonical coordinates and taking account of (26), we obtain, from the above equation

4) The sign t on the left shoulder of a matrix indicates its transposed matrix.

$$(27) \quad (D^v m)^h = (dm)^h - 2m(\omega_a)^h.$$

On the other hand, we shall write $\Delta_z^v m(X, Y) = \Delta^v m(X, Y, Z)$, for $X, Y, Z \in B_b$. Then we obtain

$$\begin{aligned} \Delta_z^v m(X, Y) &= Z^v(m(X, Y)) + m(XZ^v - Z^v X, Y) + m(X, YZ^v - Z^v Y) \\ &= Z^v({}^t X m Y) - m(Z^v X, Y) - m(X, Z^v Y) \\ &= {}^t X Z^v m Y = Z^v(m)(X, Y) = (dm)^h(Z)(X, Y). \end{aligned}$$

Hence we have

$$(28) \quad \Delta_z^v m = (dm)^h(Z).$$

Now we require the following relation between the Finsler connection and the metric tensor m .

Postulate IV. *The linear Finsler connection is metrical:*

$$D^v m = 0.$$

In a Riemann manifold, the linear connection is determined by the similar postulate. In our case, from (27) it follows first that $(dm)^h = 2m(\omega_a)^h$. Further, in virtue of (28), we obtain $\Delta^v m = 2m(\omega_a)^h$. We have had the expression (25) of m , from which we obtain by covariant v -differentiation that

$$(29) \quad \Delta^v \Delta^v \Delta^v F = 2m(\omega_a)^h.$$

In terms of canonical coordinates, (29) is written in the well-known form

$$\frac{1}{2} \frac{\partial^3 F}{\partial b^i \partial b^j \partial b^k} = C_{ijk}, \quad i, j, k = 1, \dots, n.$$

Consequently, *the local horizontal connection forms $(\omega_a)^h$ are determined by the fundamental function F under the above four postulates.* The coefficients C_{jk}^i as thus obtained satisfy the equations (11) and (12). Hence we have

Theorem 4.2. *Let Γ be the Finsler connection satisfying the above four postulates. Then we obtain*

- 1) Γ has the property D .
- 2) The reduced functions $((\omega_a)^h)_0$ of the local horizontal connection forms vanish identically.

We had the definition of the Finslerian length of a tangent vector on M , which coincides with the absolute length, in consequence of Postulate I. Therefore we can define the length of a

vector. We consider a curve $\underline{C} = \{x_t, 0 \leq t \leq 1\}$ on M and a curve $C = \{b_t, 0 \leq t \leq 1\}$ which is a field of tangent vectors of \underline{C} . Let $C' = \{b'_t, 0 \leq t \leq 1\}$ be parallel to C . The vector b'_t depends generally upon the choice of C . Under the above postulates, this parallel displacement is independent of the choice of C , provided that C is a field of tangent vectors of \underline{C} , because the condition of Theorem 3.2 is satisfied. Hence we have the notion of *the parallel displacement of a vector b'_0 along \underline{C}* , when C is a field of tangent vectors of \underline{C} . Furthermore, we may define a geodesic on M in the ordinary manner [4].

Finally, we consider the postulate E of E. Cartan, which is expressed as follows [11, p. 68]:

“The coefficients Γ_{jk}^{*i} which appear in the covariant differential when the displacement is such that the element of support is transported parallel to itself from x to $x + dx$ are to be symmetric in their lower indices.”

The coefficients Γ_{jk}^{*i} as above mentioned coincide with the one as defined in § 12, and we can treat the pure component of the pure torsion form. Then, in our case, the above postulate is expressible as follows:

Postulate V. *The pure component $\Theta^{(1)}$ of the pure torsion form Θ^1 vanishes identically.*

Therefore Θ^1 is equal to its mixed component $\Theta^{1(01)}$. From the definition of θ^1 , we have, for $X, Y \in Q_q$,

$$\Theta^{(1)}(X, Y) = d\theta^*(\eta(h^1(X)), \eta(h^1(Y))) .$$

Hence Postulate V means in \mathfrak{P} that $d\theta^*$ vanishes on the horizontal subspace $H^*(b)_p$ of P_p with respect to a point $b \in B$.

Under the above five postulates, we may determine uniquely the Finsler connection, and the calculation in order to show this is well-known [2], [11].

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