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On the structure tensor of G-structure

By

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Introduction

The theory of G-structure on differentiable manifolds has been recently studied by several authors. In the present paper, we shall give the definiton of the structure tensor of G-structure and investigate some properties concerning it. After some preliminaries, we define the structure tensor of G-structure. In § 3 we introduce the concept of G-connexion and establish the relation between the structure tensor and the torsion tensor of G-connexion. Finally in § 4 we obtain, concerning the automorphisms of G-structure, some results which contain a generalization of Riemannian case.

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§ 1. Preliminaries and notations[1]

1. Let X be a differentiable manifold. The differentiability class of all manifolds, mappings, vector fields, differential forms, etc. will be understood to be C^∞. We denote by $T_x(X)$ the tangent vector space of X at $x \in X$. Any differentiable mapping f of X into another manifold X' induces a linear map $f_*: T_x(X) \to T_{f(x)}(X')$. We denote by f^* the dual map of f_*.

Let $P(X, G)$ be a differentiable principal fibre bundle over a base space X with projection p and structural Lie group G. For each $g \in G$, we denote by R_g the right translation of $P(X, G)$ corresponding to g. A tangent vector $t \in T_b(P)$ at $b \in P(X, G)$ is said to be *vertical* if it is tangent to the fibre through b.

Let (r, F) be any differentiable representation of G on a vector

1) For many of the notions introduced in this section, see [5].

space F and $(\bar{r}, F)$ be its induced representation of the Lie algebra $\mathcal{G}$ of G. An F-valued l-form φ on $P(X, G)$ is called a *tensorial l-form of type* (r, F), if it satisfies the following conditions:

(i) $R_g^* \varphi = r(g^{-1})\varphi$ for any $g \in G$.

(ii) $\varphi(t_1, \cdots, t_l) = 0$ if t_1 is vertical.

In the case $l=0$, a differentiable mapping $\varphi: P(X, G) \rightarrow F$ is called a *tensor of type* (r, F), if it satisfies the condition:

$$\varphi \cdot R_g = r(g^{-1})\varphi \qquad \text{for any } g \in G.$$

Let Λ be a $\mathcal{G}$-valued 1-form on $P(X, G)$ and φ be an F-valued l-form on $P(X, G)$. The F-valued $(l+1)$-form $\bar{r}(\Lambda)\varphi$ is defined as follows: for $t_1, \cdots, t_{l+1} \in T_b(P)$, $b \in P$,

$$(1.1) \quad \bar{r}(\Lambda)\varphi(t_1, \cdots, t_{l+1}) = \sum_{a=1}^{l+1} \frac{(-1)^{a-1}}{l+1} \bar{r}(\Lambda(t_a))\varphi(t_1, \cdots, \hat{t}_a, \cdots, t_{l+1}).$$

2. Let G be a closed subgroup of the general linear group $GL(n, R)$ in n variables, and $\mathcal{G}$ be its Lie algebra. We assume $\dim G = s$. We shall denote by $(ad, \mathcal{G})$ the adjoint representation of G. Let E be an n-dimensional vector space over the field of real numbers. We consider the representation (ρ, E) of G defined by

$$(1.2) \qquad \rho(g)e_j = \sum_{i=1}^{n} g_j^i e_i \qquad \text{for any} \quad g = (g_j^i) \in G,$$

where $(e_1, \cdots, e_n)$ is a base of E.

We put $V = E^* \otimes \mathcal{G}$ and $W = (E^* \wedge E^*) \otimes E$, E^* being the dual space of E. From the representations (ρ, E) and $(ad, \mathcal{G})$, we obtain the representations $(\rho^* \otimes ad, V)$ and $((\rho^* \wedge \rho^*) \otimes \rho, W)$, where ρ^* is the dual representation of ρ. For the sake of simplicity, we shall denote these representations by (α_1, V) and (α_2, W) respectively.

Taking a base $(\tilde{e}_1, \cdots, \tilde{e}_s)$ of $\mathcal{G}$, we can express $ad(g)$ by a matrix $||\alpha_\sigma^\tau(g)||$[2]:

$$(1.3) \qquad ad(g)\tilde{e}_\sigma = \sum_\tau \alpha_\sigma^\tau(g)\tilde{e}_\tau.$$

Since $\bar{\rho}(\tilde{e}_\sigma)$ is an endomorphism of E, $\bar{\rho}(\tilde{e}_\sigma)$ can be represented by a matrix $||a_{\sigma j}^i||$:

$$(1.4) \qquad \bar{\rho}(\tilde{e}_\sigma)e_j = \sum_i a_{\sigma j}^i e_i.$$

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2) Unless otherwise stated, we use the following ranges of indices:

$$i, j, k = 1, 2, \cdots, n, \qquad \sigma, \tau = 1, 2, \cdots, s.$$

From the well-known relation

$$\rho(g)\bar{\rho}(X)\rho(g^{-1}) = \bar{\rho}(ad(g)X) \tag{1.5}$$

for $g \in G$ and $X \in \mathcal{G}$, it follows that

$$\sum_k g^i_k a^k_{\sigma j} = \sum_{\tau,k} a^i_{\tau k}\alpha^\tau_\sigma(g) g^k_j \qquad \text{for } g = (g^i_j) \in G. \tag{1.6}$$

DEFINITION 1.1. Let $(\tilde{e}_1, \cdots, \tilde{e}_s)$ be a base of $\mathcal{G}$, and let $(e_1, \cdots, e_n)$ be a base in E and $(e^1, \cdots, e^n)$ its dual base. We define the linear map $\mathfrak{A}$: $V \to W$ as follows:

$$\mathfrak{A}(\sum_{\sigma,k} \xi^\sigma_k e^k \otimes \tilde{e}_\sigma) = \sum_{\sigma,i,j,k} (a^i_{\sigma j}\xi^\sigma_k - a^i_{\sigma k}\xi^\sigma_j) e^j \wedge e^k \otimes e_i . \tag{1.7}$$

It is straightfoward to verify that this definition does not depend on the choice of the base. By making use of (1.5), we have

$$\mathfrak{A}\circ\alpha_1(g) = \alpha_2\circ(g)\mathfrak{A} \qquad \text{for any } g \in G. \tag{1.8}$$

We put $N = \mathfrak{A}(V)$ and $Q = W/N$. Let q denote the natural projection: $W \to Q$. Then, it follows from (1.8) that the subspace N of W is invariant under $\alpha_2(g)$. Therefore $\alpha_2(g)$ induces an automorphism $\alpha_3(g)$ of Q. Thus we obtain the representation (α_3, Q) which satisfies the relation

$$\alpha_3(g)\circ q = q\circ\alpha_2(g) \qquad \text{for any } g \in G. \tag{1.9}$$

DEFINITION 1.2. We say that the *group G has the property (I)* if there exists a linear map h: $N \to V$, satisfying the following conditions

(i) $\mathfrak{A}\circ h = I_N$.

(ii) $h\circ\alpha_2(g) = \alpha_1(g)\circ h$ for any $g \in G$.

DEFINITION 1.3. We say that the *group G has the property (II)* if there exists a linear map k: $Q \to W$, satisfying the following conditions

(i) $q\circ k = I_Q$.

(ii) $k\circ\alpha_3(g) = \alpha_2(g)\circ k$ for any $g \in G$.

The following two propositions are easily proved, and so we omit the proofs.

PROPOSITION 1.1. *If the kernel of $\mathfrak{A}$ is zero, then the group G has the property (I). In the case $Ker\,\mathfrak{A} \neq 0$, the group G has property (I) if and only if there exists a subspace B of V such that*

(i) $V = Ker\,\mathfrak{A} + B$ (*direct sum*).

(ii) $\alpha_1(g)B \subset B$ *for any* $g \in G$.

PROPOSITION 1.2. *The group G has the property (II) if and only if there exists a subspace Z of W such that*

(i) $W = N + Z$ (*direct sum*).

(ii) $\alpha_2(g)Z \subset Z$ *for any* $g \in G$.

§ 2. *G*-structure and its structure tensor

3. In this and following sections we denote by G a closed subgroup of $GL(n, R)$ and assume $\dim G = s$.

DEFINITION 2.1. We say that an n-dimensional differentiable manifold M possesses a *G-structure* when the structural group of the frame bundle of M is reducible to G. We shall denote the reduced bundle by $H(M, G)$.

Hereafter we shall use the following notations:
For the principal fibre bundle $H(M, G)$,

U_α, U_β, U_γ denote the coordinate neighborhoods for $H(M, G)$;
p denotes the projection of $H(M, G)$;
φ_α denotes the coordinate function of $H(M, G)$;
p_α denotes the cross-projection of $H(M, G)$;
$g_{\alpha\beta}$ denotes the coordinate transformation of $H(M, G)$;
R_g denotes the right translation corresponding to $g \in G$;
$\chi(b)$ denotes the admissible map corresponding to $b \in H(M, G)$.

For the Lie group G,

$\mathfrak{G}$ denotes the Lie algebra of G;
γ denotes the Maurer-Cartan form of G.

As is well-known[3], it holds that

$$p_\alpha(R_g b) = p_\alpha(b)\cdot g \quad \text{for } g \in G \text{ and } b \in p^{-1}(U_\alpha). \tag{2.1}$$

$$p_\beta(b) = g_{\beta\alpha}(p(b))p_\alpha(b),\ p(b) \in U_\alpha \cap U_\beta. \tag{2.2}$$

An element $b \in H(M, G)$, such that $p(b) = x$, is called a *distinguished frame at x*.

Define the mapping τ_α: $U_\alpha \to U_\alpha \times G$ by

$$\tau_\alpha(x) = (x, e)$$

for $x \in U_\alpha$, e being the neutral element of G. If we put

3) Cf. [4].

(2.3) $$Y_\alpha = \varphi_\alpha \tau_\alpha$$

then Y_α is a local cross-section on U_α. Hence $Y_\alpha(x)$ can be expressed in the form

(2.4) $$Y_\alpha(x) = (x, z_{(\alpha)1}(x), \cdots, z_{(\alpha)n}(x))$$

for $x \in U_\alpha$, where $z_{(\alpha)1}, \cdots, z_{(\alpha)n}$ are linearly independent vector fields on U_α. Let $\theta^1_\alpha, \cdots, \theta^n_\alpha$ be the 1-forms on U_α such that

(2.5) $$\theta^i_\alpha(z_{(\alpha)j}) = \delta^i_j .$$

Define the E-valued 1-form on U_α by

$$\theta_\alpha = \sum_i \theta^i_\alpha \otimes e_i .$$

If $U_\alpha \cap U_\beta \neq 0$, then it holds that[4]

(2.6) $$\theta_{\alpha,x} = \rho(g_{\alpha\beta}(x))\theta_{\beta,x} \qquad \text{for } x \in U_\alpha \cap U_\beta .$$

Now we define the E-valued 1-form ω_0 on $H(M, G)$ by

(2.7) $$\omega_{0,b} = \rho(p_\alpha^{-1}(b))p^*\theta_{\alpha,x}, \qquad x = p(b) \in U_\alpha .$$

From (2.2) and (2.6) it follows that this definition is independent of the choice of coordinate neighborhood.

By the definition of ω_0, the following proposition is obvious.

PROPOSITION 2.1. *The form ω_0 is a tensorial 1-form on $H(M, G)$ of type (ρ, E) and satisfies the following condition*:

(2.8) *If $\omega_0(t) = 0$, then t is vertical.*

4. Let Λ be a tensorial 1-form on $H(M, G)$ of type $(ad, \mathcal{G})$. The E-valued 1-form ω_0 and the $\mathcal{G}$-valued 1-form Λ can be expressed by

$$\omega_0 = \sum_i \omega^i \otimes e_i \quad \text{and} \quad \Lambda = \sum_\sigma \Lambda^\sigma \otimes \tilde{e}_\sigma ,$$

where ω^i and Λ^σ are real valued 1-forms on $H(M, G)$. By making use of (2.8), it is easily seen that the forms $\omega^1, \cdots, \omega^n$ are linearly independent. Hence we can set

$$\Lambda^\sigma = \sum_i \lambda^\sigma_i \omega^i ,$$

where λ^σ_i are functions on $H(M, G)$. If we put

$$\lambda = \sum_{\sigma,i} \lambda^\sigma_i e^i \otimes \tilde{e}_\sigma ,$$

4) Cf. [1], [3].

then λ becomes a tensor on $H(M, G)$ of type (α_1, V). The tensor λ thus obtained is called the *tensor corresponding to* Λ. It is easily seen that this correspondence is one-to-one.

A tensorial 2-form Ξ on $H(M, G)$ of type (ρ, E) can be written as

$$\Xi = \sum_{i,j,k} \Xi^i_{jk} \omega^j \wedge \omega^k \otimes e_i ,$$

where Ξ^i_{jk} are functions on $H(M, G)$ satisfying the relation

$$\Xi^i_{jk} + \Xi^i_{kj} = 0 .$$

Putting

$$\xi = \sum_{i,j,k} \Xi^i_{jk} e^j \wedge e^k \otimes e_i ,$$

we obtain the tensor ξ on $H(M, G)$ of type (α_2, W). We call this tensor ξ the *tensor corresponding to a tensorial 2-form* Ξ *of type* (ρ, E). In this case, we define the correspondence ψ by $\psi\Xi = \xi$. Then, as is easily seen, ψ is a one-to-one correspondence between the set of all tensorial 2-forms of type (ρ, E) and the set of all tensors of type (α_2, W).

Since $p^{-1}(U_\alpha)$ is regarded as a principal fibre bundle over base space U_α, we can consider tensorial forms on $p^{-1}(U_\alpha)$. We call them local tensorial forms over U_α. We obtain, in the same way as above, a one-to-one correspondence between the set of all local tensorial 1-forms over U_α of type $(ad, \mathfrak{G})$ and the set of all local tensors over U_α of type (α_1, V), and a one-to-one correspondence ψ_α between the set of all local tensorial 2-forms over U_α of type (ρ, E) and the set of all local tensors over U_α of type (α_2, W). When we restrict ourselves to local tensorial forms over $U_\alpha \cap U_\beta$, then we denote the latter correspondence by $\psi_{\alpha\beta}$.

LEMMA 2. 1. *Let* λ *be the tensor of type* (α_1, V) *corresponding to a tensorial 1-form* Λ *of type* $(ad, \mathfrak{G})$. *Then the 2-form* $\bar{\rho}(\Lambda)\cdot\omega_0$ *is the tensorial 2-form of type* (ρ, E) *corresponding to the tensor* $-\frac{1}{2}\mathfrak{A}(\lambda)$ *of type* (α_2, W). *The same result holds for local tensorial forms.*

Proof. From the definition (1. 1), it follows that the form $\bar{\rho}(\Lambda)\omega_0$ is a tensorial 2-form of type (ρ, E). The forms ω_0 and Λ can be written as

$$\omega_0 = \sum_i \omega^i \otimes e_i , \qquad \Lambda = \sum_{\sigma,k} \lambda^\sigma_k \omega^k \otimes \tilde{e}_\sigma .$$

Then, by virtue of (1. 4), we have

$$\bar{\rho}(\Lambda)\omega_0 = -\frac{1}{2}\sum_{\sigma,i,j,k}(a^i_{\sigma j}\lambda^\sigma_k - a^i_{\sigma k}\lambda^\sigma_j)\omega^j \wedge \omega^k \otimes e_i .$$

This implies that the tensorial 2-form $\bar{\rho}(\Lambda)\omega_0$ goes under the correspondence ψ into the tensor $-\frac{1}{2}\mathfrak{a}(\lambda)$.

5. We define the E-valued 2-form Ω_α on $p^{-1}(U_\alpha)$ by

$$\Omega_{\alpha,b} = \rho(p_\alpha^{-1}(b))p^*d\theta_{\alpha,p(b)} \qquad \text{for } p(b) \in U_\alpha . \tag{2.9}$$

From (2.1) we see that the E-valued 2-form Ω_α is a local tensorial form over U_α of type (ρ, E). We denote by S_α the local tensor over U_α of type (α_1, V) corresponding to Ω_α. Applying exterior differentiation to the equation (2.6) and taking account of (2.2), we have

$$\Omega_\beta = \Omega_\alpha + \bar{\rho}(ad(p_\alpha^{-1})p^*g^*_{\beta\alpha}\gamma)\omega_0 \qquad \text{on } p^{-1}(U_\alpha \cap U_\beta) . \tag{2.10}$$

It is easily verified that $ad(p_\alpha^{-1})p^*g^*_{\beta\alpha}\gamma$ is a local tensorial 1-form of type $(ad, \mathfrak{G})$ over $U_\alpha \cap U_\beta$, and hence we denote by $\mu_{\beta\alpha}$ the local tensor over $U_\alpha \cap U_\beta$ of type (α_1, V) corresponding to $ad(p_\alpha^{-1})p^*g^*_{\beta\alpha}\gamma$. By Lemma 2.1, the equation (2.10) goes under the correspondence $\psi_{\beta\alpha}$ into the equation

$$S_\beta = S_\alpha - \frac{1}{2}\mathfrak{A}(\mu_{\beta\alpha}) . \tag{2.11}$$

We define the differentiable mapping S: $H(M, G) \to Q$ by

$$S(b) = q\cdot S_\alpha(b) \qquad \text{for } b \in p^{-1}(U_\alpha) . \tag{2.12}$$

Then, from (2.11) we see that S is globally defined. Moreover from the fact that S_α is a local tensor of type (α_2, W) and from (1.9), we get for $b \in p^{-1}(U_\alpha)$ and $g \in G$

$$\begin{aligned} S(R_g b) &= q\cdot S_\alpha(R_g b) = q\cdot\alpha_2(g^{-1})S_\alpha(b) = \alpha_3(g^{-1})q\cdot S_\alpha(b) \\ &= \alpha_3(g^{-1})S(b) . \end{aligned}$$

This shows that S is a tensor on $H(M, G)$ of type (α_3, Q). Following Bernard[5], we shall call the tensor S the *structure tensor of G-structure.* Thus we have the following:

PROPOSITION 2.2.[5] *The structure tensor of a G-structure is of type* (α_3, Q).

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5) Cf. [1].

§ 3. *G*-connexion and its torsion tensor

6. From now on we shall suppose a G-structure to be given in an n-dimensional differentiable manifold M.

DEFINITION 3. 1. By a *G-connexion* on M we mean a connexion on the reduced bundle $H(M, G)$.

A G-connexion on M is given by a differential 1-form ω_1 satisfying the following conditions[6]

(i) ω_1 is a 1-form on $H(M, G)$ with values in the Lie algebra $\mathcal{G}$ of G.

(ii) If a vector $t \in T_b(H)$ is vertical, then $\omega_1(t) = \chi(b)_*^{-1}t$.

(iii) For any $g \in G$, $R_g^*\omega_1 = ad(g^{-1})\omega_1$.

We shall call the form ω_1 the *connexion form* of G-connexion or merely the G-connexion.

A G-connexion on M is also given by a system $\pi = \{\pi_\alpha\}$ of $\mathcal{G}$-valued 1-forms in M satisfying the following conditions[7]

(i) Each component π_α is defined in the coordinate neighborhood U_α.

(ii) If $U_\alpha \cap U_\beta \neq 0$, then π_α and π_β are related by the equation

$$\pi_\alpha = ad(g_{\beta\alpha}^{-1})\pi_\beta + g_{\beta\alpha}^*\gamma\,. \tag{3. 1}$$

The relation between above two definitions of G-connexion is given by[7]

$$\omega_1 = ad(p_\alpha^{-1})p^*\pi_\alpha + p_\alpha^*\gamma\,. \tag{3. 2}$$

The *torsion form* Ω_0 of a G-connexion ω_1 is given by

$$\Omega_0 = d\omega_0 + \bar{\rho}(\omega_1)\omega_0\,. \tag{3. 3}$$

As is well-known, the torsion form Ω_0 is a tensorial 2-form of type (ρ, E). The tensor of type (α_2, W) which corresponds to Ω_0 is called the *torsion tensor* of the G-connexion.

PROPOSITION 3. 1.[8] *Let T and T' denote the torsion tensors of two G-connexions ω_1 and ω_1' respectively. Then it holds that*

$$q \circ T = q \circ T'\,.$$

Proof. Let Ω_0 and Ω_0' be the torsion forms of ω_1 and ω_1' respectively. Then by (3. 3) we have

6) Cf. [2], [5]
7) Cf. [2].
8) Cf. [5].

(3. 4) $$\Omega_0-\Omega_0' = \bar{\rho}(\omega_1-\omega_1')\omega_0 .$$

From the first definition of G-connexion, we see that the form $\omega_1-\omega_1'$ is a tensorial 1-form of type $(ad, \mathcal{G})$. Therefore we denote by Γ the tensor of type (α_1, V) corresponding to $\omega_1-\omega_1'$. Then, by Lemma 2.1, (3. 4) goes under the correspondence ψ into the equation

$$T-T' = -\frac{1}{2}\mathfrak{A}(\Gamma) .$$

Hence proposition is proved.

PROPOSITION 3. 2. *Let S be the structure tensor of the G-structure, and T be the torsion tensor of any G-connexion. Then it holds that*

$$q\circ T = S .$$

Proof. Using the relation (3. 2), we obtain

(3. 5) $$d\rho(p_\alpha^{-1}) = -\bar{\rho}(\omega_1)\rho(p_\alpha^{-1})+\rho(p_\alpha^{-1})\bar{\rho}(p^*\pi_\alpha) .$$

Taking the exterior derivative of (2. 7) and using (3. 5), we get

$$d\omega_0 = -\bar{\rho}(\omega_1)\omega_0+\bar{\rho}(ad(p_\alpha^{-1})p^*\pi_\alpha)\omega_0+\Omega_\alpha \qquad \text{on } p^{-1}(U_\alpha) .$$

Thus we get

(3. 6) $$\Omega_0 = \bar{\rho}(ad(p_\alpha^{-1})p^*\pi_\alpha)\omega_0+\Omega_\alpha \qquad \text{on } p^{-1}(U_\alpha) .$$

We denote by ξ_α the local tensor over U_α of type (α_1, V) corresponding to the local tensorial 1-form $ad(p_\alpha^{-1})p^*\pi_\alpha$ of type $(ad, \mathcal{G})$, and by S_α the local tensor of type (α_2, W) corresponding to Ω_α. Then (3. 6) goes under the correspondence ψ_α into the equation

$$T = -\frac{1}{2}\mathfrak{A}(\xi_\alpha)+S_\alpha \qquad \text{on } p^{-1}(U_\alpha) .$$

Hence we have

$$q\circ T = S .$$

LEMMA 3. 1. *Assume that the group G has the property (I). If a tensor u of type (α_2, W) satisfies*

$$q\circ u = 0 ,$$

then there exists a tensor Γ of type (α_1, V) such that

$$u = \mathfrak{A}(\Gamma) .$$

Prof. Take the linear map h described in Definition 1. 2 and define Γ by

$$\Gamma(b) = h(u(b)) \qquad \text{for } b \in H(M, G) .$$

Then Γ is a tensor with the desired properties. In fact, we have

$$\mathfrak{A}\cdot\Gamma(b) = \mathfrak{A}\cdot h(u(b)) = u(b) ,$$

and

$$\Gamma(R_g b) = h(u(R_g b)) = h\cdot\alpha_1(g^{-1})u(b) = \alpha_2(g^{-1})h(u(b)) = \alpha_2(g^{-1})\Gamma(b) .$$

PROPOSITION 3. 3. *Assume that the group G has the property (I). Let R be a tensor of type (α_2, W). In order that R be a torsion tensor of a G-connexion it is necessary and sufficient that R satisfies*

$$q\circ R = S ,$$

where S is the structure tensor.

Proof. We need only to prove the sufficiency. Take an arbitrary but fixed G-connexion ω_1, and let T denote its torsion tensor. According to Proposition 3.2, we have

$$q\circ T = S .$$

Therefore we have

$$q\circ(R-T) = 0 .$$

Hence, by Lemma 3. 1, there exists a tensor Γ of type (α_1, V) such that

$$R = T+\mathfrak{A}(\Gamma) .$$

Denoting by β the tensorial 1-form of type $(ad, \mathfrak{G})$ which corresponds to 2Γ, we see that $\omega_1-\beta$ is a G-connexion with torsion tensor R. q.e.d.

By Lemma 3. 1 and Proposition 3. 3, we have the following

PROPOSITION 3. 4. *Assume that the group G has the properties (I) and (II). Let k be the linear map $Q\rightarrow W$ satisfying the conditions in Definition* 1. 3. *Let S denote the structure tensor. Then the torsion tensor of any G-connexion is written in the form*

$$T = \mathfrak{A}(\Gamma)+k\circ S$$

with some tensor Γ of type (α_1, V).

Conversely, for an arbitrary tensor Γ of type (α_1, V) there exists a G-connexion whose torsion tensor is $\mathfrak{A}\circ\Gamma+k\circ S$.

In particular, there exists a G-connexion whose torsion tensor is $k\circ S$. We call this G-connexion the *canonical G-connexion.*

COROLLARY 1. *Under the same assumption of Proposition* 3.4, *the structure tensor vanishes if and only if it is possible to introduce a G-connexion without torsion.*

§4. Automorphism

7. Let M be an n-dimensional differentiable manifold which possesses a G-structure and let f be a differentiable transformation of M onto itself. f induces a natural differentiable transformation of the frame bundle of M. Thus any distinguished frame $Y=(x, t_1, \cdots, t_n)$ at $x \in M$ is mapped into the frame $(f(x), f_*t_1, \cdots, f_*t_n)$ at $f(x)$. But, in general, the frame $(f(x), f_*t_1, \cdots, f_*t_n)$ is not distinguished.

DEFINITION 4.1. Given a G-structure on a differentiable manifold M, a differentiable transformation f of M is called an *automorphism of the G-structure* if, for any distinguished frame $Y=(x, t_1, \cdots, t_n)$ at x the frame $(f(x), f_*t_1, \cdots, f_*t_n)$ is distinguished. In this case, we shall denote the distinguished frame $(f(x), f_*t_1, \cdots, f_*t_n)$ by $\tilde{f}Y$ and call the mapping $\tilde{f}$ the *prolongation* of f.

The prolongation $\tilde{f}$ of an automorphism f of the G-structure is clearly an automorphism of the reduced bundle $H(M, G)$ that is, $\tilde{f}$ satisfies the conditions: $p\cdot\tilde{f}=\tilde{f}\cdot p$ and $\tilde{f}\cdot R_g=R_g\tilde{f}$ for any $g \in G$.

PROPOSITION 4.1. *If a differentiable transformation f of M is an automorphism of the G-structure, then the prolongation $\tilde{f}$ of f leaves the form ω_0 invariant*:

$$\tilde{f}^*\omega_0 = \omega_0 .$$

Proof. As in §2, we define a local cross-section Y_α on U_α by

$$Y_\alpha(x) = \varphi_\alpha(x, e) .$$

If $f(x) \in U_\beta$ for $x \in U_\alpha$, then $\tilde{f}Y_\alpha(x)$ can be written in the form

$$\tilde{f}Y_\alpha(x) = \varphi_\beta(f(x), a_{\beta\alpha}(f(x))) . \tag{4.1}$$

where $a_{\beta\alpha}$ is a differentiable mapping $f(U_\alpha)\cap U_\beta \to G$ such that

$$a_{\beta\alpha}(x') = g_{\beta\gamma}(x')a_{\gamma\alpha}(x') \qquad \text{for } x' \in f(U_\alpha)\cap U_\beta \cap U_\gamma . \tag{4.2}$$

Hence, we have

$$f^*\theta_\beta = \rho(a_{\beta\alpha}\circ f)\theta_\alpha . \tag{4.3}$$

For $b=\varphi_\alpha(x, g) \in p^{-1}(U_\alpha)$, we have $b=R_gY_\alpha(x)$. Since $\tilde{f}$ is an automorphism of $H(M, G)$, we have

$$\tilde{f}(b) = R_g\tilde{f}(Y_\alpha(x)) = \varphi_\beta(f(x), a_{\beta\alpha}(f(x))g).$$

By virtue of (4.2), this expression is independent of the choice of coordinate neighborhood. Thus we have

$$p_\beta(\tilde{f}(b)) = a_{\beta\alpha}(f\circ p(b))p_\alpha(b) \qquad \text{for } b\in p^{-1}(U_\alpha). \tag{4.4}$$

From (4.3) and (4.4) we see that $\tilde{f}$ leaves ω_0 invariant.

PROPOSITION 4.2. *If a differentiable transformation f of M is an automorphism of the G-structure, then the prolongation $\tilde{f}$ of f leaves the structure tensor S invariant*:

$$S\circ\tilde{f} = S.$$

Proof. Suppose $x\in U_\alpha$ and $f(x)\in U_\beta$. Using (4.3) and (4.4), we get

$$\tilde{f}^*\Omega_{\beta,\tilde{f}(b)} = \Omega_{\alpha,b}+\bar{\rho}(ad(p_\alpha^{-1}(b))p^*(a_{\beta\alpha}\cdot f)^*\gamma)\omega_{0,b},\ p(b)\in U_\alpha\cap f^{-1}(U_\beta).$$

Since $\tilde{f}$ leaves ω_0 invariant, this equation goes under the correspondence ψ_α into the equation

$$S\circ\tilde{f} = S. \qquad \text{q.e.d.}$$

LEMMA 4.1. *Let ω_1 be a G-connexion with torsion form Ω_0 and let $\tilde{f}$ be the prolongation of an automorphism of the G-structure. Then $\tilde{f}^*\omega_1$ is a G-connexion with torsion form $\tilde{f}^*\Omega_0$.*

Proof. Since $\tilde{f}$ commutes with R_g, we have

$$R_g^*\tilde{f}^*\omega_1 = \tilde{f}R_g^*\omega_1 = ad(g^{-1})\tilde{f}^*\omega_1.$$

On the other hand, it holds that $\chi(\tilde{f}(b))=\tilde{f}\circ\chi(b)$ for $b\in H(M, G)$. Since $\tilde{f}_*$ maps any vertical vector into a vertical vector, it follows that for a vertical vector $t\in T_b(H)$

$$\tilde{f}^*\omega_1(t) = \omega_1(\tilde{f}_*t) = \chi(\tilde{f}(b))_*^{-1}\tilde{f}_*t = \chi(b)_*^{-1}t.$$

Consequently $\tilde{f}^*\omega_1$ is a G-connexion. Since $\tilde{f}$ leaves ω_0 invariant, the torsion form of $\tilde{f}^*\omega_1$ is given by

$$d\omega_0+\bar{\rho}(\tilde{f}^*\omega_1)\cdot\omega_0 = \tilde{f}^*[d\omega_0+\bar{\rho}(\omega_1)\omega_0] = \tilde{f}^*\Omega_0. \qquad \text{q.e.d.}$$

Given a G-connexion ω_1, an automorphism f of the G-structure is called an *automorphism of the G-connexion* ω_1 if its prolongation $\tilde{f}$ preserves the G-connexion ω_1: $\tilde{f}^*\omega_1=\omega_1$. If this is the case, $\tilde{f}$ leaves the torsion form Ω_0 of ω_1 invariant. Conversely we have the following

PROPOSITION 4.3. *Assume that the group G satisfies the condition*

Ker $\mathfrak{A}=0$. *If the prolongation* $\tilde{f}$ *of an automorphism* f *of the G-structure leaves the torsion form of a G-connexion invariant, then* f *is an automorphism of the G-connexion.*

To prove this we need the following lemma.

LEMMA 4.2. *Assume that the group G satisfies the condition Ker* $\mathfrak{A}=0$. *Then two G-connexions* ω_1 *and* ω_1' *are identical if and only if their respective torsion form* Ω_0 *and* Ω_0' *coincide.*

Proof. The condition $\Omega_0=\Omega_0'$ implies $\bar{\rho}(\omega_1-\omega_1')\omega_0=0$. Denoting by Γ the tensor of type (α_1, V) corresponding to $\omega_1-\omega_1'$, we have $\mathfrak{A}(\Gamma)=0$. By assumption we have $\Gamma=0$ and hence $\omega_1=\omega_1'$. q.e.d.

Proof of Proposition 4.3. Suppose that $\tilde{f}$ leaves the torsion form Ω_0 of a G-connexion ω_1 invariant: $\tilde{f}^*\Omega_0=\Omega_0$. Then by Lemma 4.1 the G-conexion $\tilde{f}^*\omega_1$ must have the torsion form Ω_0, so by Lemma 4.2 we have $\tilde{f}^*\omega_1=\omega_1$ and consequently f is an automorphism of the G-connexion. q.e.d.

According to Propositions 4.2 and 4.3, we have the following

PROPOSITION 4.4. *Assume that the group G satisfies the condition Ker* $\mathfrak{A}=0$ *and has the property (II). Then an automorphism of the G-structure is an automorphism of the canonical G-connexion.*

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