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On the theory of Martin boundaries induced by countable Markov processes

By

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INTRODUCTION

J. L. Doob [5] has established the Martin boundary theory for countable Markov processes with a discrete time parameter. Independently, the author [15] has outlined the almost same results to apply them to Hausdorff moment problem. In this paper, we shall show that the boundary theory holds even in the continuous parameter case by an approach somewhat different from Doob's one. We shall not discuss the dual boundary theory as well as the potential theory of set functions. But we shall discuss some problems which were not treated by Doob.

A countable Markov process with a discrete or continuous time parameter defines the family of functions called x_t-superharmonic or x_t-harmonic over the countable space; those functions are, respectively, the exact counterpart of the ordinary superharmonic or harmonic functions. The purpose of this paper is to obtain some representation theorems for the above functions, by modifying R. S. Martin's approach [12] to the ordinary harmonic functions from a probabilistic point of view. This work was motivated by W. Feller's paper [6], in which he has shown that a substochastic matrix induces a boundary for the countable space. It seems that the Martin boundary which we shall introduce is more advantageous in concrete construction than Feller's. In general, the relation of the both boundaries is still unknown. But many results of Feller's would be also able to be derived from our standpoint, though it is not discussed in this paper.

Chapter 1 consists of three sections and contains some fundamental facts on countable Markov processes which will be abbreviated as CMP's in the sequel. In Section 1, after the precise descriptions of a CMP and its related terminologies, we shall introduce some important quantities: For example, the distribution H_A of the hitting time for the set A, the transition probabilities $\{H^t; t \in T\}$, the Green measures $\{G_\alpha: \alpha \geqq 0\}$, the mean q and distribution Π of the first jumping time, the generator $\mathfrak{G}$ and so on. Section 2 contains some properties of the quantities introduced in Section 1 and the decomposition of the state space X to the indecomposable recurrent sets $\bigcup_i R_i$ and the nonrecurrent part N. In Section 3, it is shown that the system $\{H^t; t \in T\}$ or $\{q, \Pi\}$ which satisfies some conditions determines a CMP uniquely.

Chapter 2 consists of three sections devoted to the potential theory of x_t-superharmonic functions. Section 4 contains the definition and elementary properties of x_t-superharmonic or x_t-harmonic functions. In Section 5, we shall study the function $H_A u$ (in which u is nonnegative and x_t-superharmonic) and the potential of a function and, among all, we shall prove two theorems, i.e. Theorem 2.2 and 2.7 which play basic roles in the following chapters as well as in that chapter. The other theorems in this section are easily derived from the two theorems cited above. In Section 6, we shall show that the class of nonnegative x_t-superharmonic functions coincides with the class of Hunt's excessive functions and moreover we shall discuss the relation of x_t-harmonic functions with H^t-invariant functions.

In Chapters 3 and 4, we establish the Martin boundary theory. Its procedure is essentially the same as in Martin's original paper [**12**]. But some modifications are necessary to prove the main representation theorem (Section 11), for our boundary is not necessarily compact, differently from Martin's case. Our method will be based on Choquet's capacity theorem.

Chapter 3 consists of two sections and introduces the Martin space M, the Martin boundary $\partial\hat{X}$ and the réduite $u_D(x)$ (in which u is a nonnegative x_t-superharmonic function and D is a compact subset of M). In the beginning of Section 7, we shall introduce the center condition (CMP. 4), the K-function and the function families X_c, M_c. Next it is shown that M_c is compact with a suitable metric ρ. M and $\partial\hat{X}$ are defined as the set homeomorphic to M_c and $M_c - X_c$, respectively. Finally we can see that, under some additional conditions, M and $\partial\hat{X}$ have the same properties as in Martin's case. Section 8 is concerned with the réduite. Using the fact that $H_A u$ is alternating of order 2 in A, it follows that $u_D(x)$ is an alternating capacity of order 2 over the class of all compact sets in M and therefore it can be extended to any Borel set in M.

Chapter 4, consisting of three sections, is devoted to the representation theory for x_t-superharmonic functions. Section 9 contains some auxiliary representation theorems and the fact that, if u is nonnegative and x_t-harmonic, then $u = u_{\partial\hat{X}+\cup\hat{R}_i}$. In Section 10, we shall introduce the concept of minimal x_t-superharmonic functions, the minimal part M_1 and nonminimal part M_0 of M. Theorem 4.4 which gives the classification of M_1 and M_0 by means of the $\hat{K}$-function is useful both theoretically and practically. Moreover it is shown that both M_1 and M_0 are Borel in M, and M_0 is a subset of $\partial\hat{X}$. In the beginning of Section 11, the canonical representation is defined as the $\hat{K}$-representation with the measure whose total mass is carried on M_1. Our main theorem is stated as follows: *Any nonnegative x_t-superharmonic function u admits of exactly one canonical representation.* The explicite determination of the corresponding measure is given by the réduite of u and the potential of $-\mathfrak{G}u$. Finally we shall list some results derived from the main theorem.

Chapter 5, consisting of three sections, contains several examples and brief comments on the extention of the boundary theory to general Markov processes and the dual boundary theory.

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CHAPTER 1. PRELIMINARIES

1. Definition of countable Markov Processes. Notations and terminologies. We shall start with the definition of a countable Markov process with a discrete or continuous time parameter which has right continuous paths and cannot survive after infinitely many jumps. Our definition is essentially the same as that of Doob [3] except some modification for convenience of probabilistic treatment.

Let X be a countable (*state*) space with the discrete topology and ∞ an extra point to be added to X as an isolated point, and let $\tilde{X}$ denote $X \cup \{\infty\}$. Each state, i.e. each point of $\tilde{X}$ will be denoted by x, y and so on, and the set of all subsets of $\tilde{X}$ by $\mathfrak{B}_{\tilde{X}}$. Let the time parameter space T be a compact set $\{0, 1, 2, \cdots, +\infty\}$ or $[0, +\infty]$ with the ordinary topology and $\mathfrak{B}_T$ the topological Borel field over T. Any function (*path*) of $t \in T$ over $\tilde{X}$ will be denoted by w and its value at time t by w_t or $x_t(w)$. The *hitting time* σ_A *for a subset* A *of* $\tilde{X}$ is defined by

$$
(1.1) \qquad \begin{aligned} \sigma_A(w) &= \inf \{t\,;\, x_t(w) \in A\} && \text{if } x_t(w) \in A \text{ for some } t \geqq 0, \\ &= +\infty && \text{otherwise.} \end{aligned}
$$

Now consider the set W of all the paths which satisfy the following conditions:

(W. 1) $\qquad x_{+\infty}(w) = \infty$

(W. 2) $\qquad x_t(w) = \infty \quad$ for every $\; t \geqq \sigma_\infty(w)$.

(W. 3)[1] $\quad x_t(w)$ is right continuous for every t and has at most discontinuities of the first kind for $t < \sigma_\infty(w)$.

We shall denote by $\mathfrak{B}_W$ the Borel field generated by the sets $\{w\,;\, x_t(w) \in E\}$, where E runs over $\mathfrak{B}_{\tilde{X}}$ and t over T. Given any path w and any *random time* $\sigma(w)$, i.e. a function from $(W, \mathfrak{B}_W)$ into $(T, \mathfrak{B}_T)$, the *stopped path* w_σ^- and *shifted path* w_σ^+ is defined by

$$
(1.2) \qquad \begin{aligned} (w_\sigma^-)_t &= w_{\min(t,\sigma)} \; (t \neq +\infty), \text{ and } = \infty \; (t = \infty), \\ (w_\sigma^+)_t &= w_{\sigma+t}\,. \end{aligned}
$$

We shall prove that $\varphi_\sigma(w) \equiv w_\sigma^-$ and $\psi_\sigma(w) \equiv w_\sigma^+$ are measurable mappings from $(W, \mathfrak{B}_W)$ into itself. First we shall show that $w_\sigma^- \in W$. Noting that

$$
(1.3) \qquad \begin{aligned} \sigma_\infty(w_\sigma^-) &= \inf \{t\,;\, x_t(w_\sigma^-) = \infty\} \\ &= \inf \{t\,;\, x_{\min(t,\sigma)}(w) = \infty\} \\ &= \inf \{t\,;\, \min(t, \sigma) \geqq \sigma_\infty(w)\}\,, \end{aligned}
$$

we have from (W. 2) that $x_t(w_\sigma^-) = x_{\min(t,\sigma)}(w) = \infty$ for $t \geqq \sigma_\infty(w_\sigma^-)$. This means that w_σ^- satisfies (W. 2). Further if $t < \sigma_\infty(w_\sigma^-)$, $\min(t, \sigma) < \sigma_\infty(w)$. Hence we have

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1) This condition is trivially true for any path in the discrete parameter case.

$$t<\sigma_\infty(w) \quad \text{and} \quad x_{\min(t,\sigma)}(w) = x_t(w) \qquad \text{if} \quad t<\sigma,$$
$$x_{\min(t,\sigma)}(w) = x_\sigma(w) \qquad \text{if} \quad t\geqq\sigma,$$

which shows that (W. 3) is true for w_σ^-. Since (W. 1) is contained in the definition of w_σ^-, it has been proved that w_σ^- is an element of W. Next we shall prove that $\{w\,;\varphi_\sigma(w)\in B\}\in\mathfrak{B}_W$ for any set B of $\mathfrak{B}_W$. From the definition of $\mathfrak{B}_W$, it is enough to show that $\{w\,;(\varphi_\sigma(w))_t=w_{\min(\sigma,t)}\in E\}\in\mathfrak{B}_W$ for any set E of $\mathfrak{B}_X$ and any $t\in T$. This is easily derived from the fact that $x_t(w)$ is measurable as a function of (t, w). The argument for $\psi_\sigma(w)$ is quite similar.

We shall now denote by $\mathfrak{B}_\sigma$, the Borel subfield $(\varphi_\sigma)^{-1}\mathfrak{B}_W$ of $\mathfrak{B}_W$. Note that the $\mathfrak{B}_t$ for the constant random time t coincides with the Borel field generated by all the sets $\{w\,;x_s(w)\in E\}$ for any $s\leqq t$.

DEFINITION 1. 1. A random time σ is a *Markov time* if

$$(1.4) \qquad \{w\,;\sigma(w)\leqq t\}\in\mathfrak{B}_t \qquad \text{for any} \quad t\in T.$$ [2]

Given any hitting time σ_A and any $t\in T$, using (W. 2) and (W. 3), we can see that

$$\{w\,;\sigma_A(w)>t\}=\{w\,;x_r(w)\notin A \quad \text{for any rational } r<t \text{ and } x_t(w)\notin A\}$$
$$=[\bigcap_{r<t}\{w\,;\,x_r(w)\notin A\}]\cap\{w\,;\,x_t(w)\notin A\}\in\mathfrak{B}_t\,.$$

Hence we have

LEMMA 1. 1. *Any hitting time is a Markov time.*

Further we shall list some properties of Markov times which are used later and will be proved in Itô and McKean [**10**].

LEMMA 1. 2. (i) *If* $\sigma(w)$ *is a Markov time, we have* $\sigma(w)=\sigma(w_t^-)$ *for every* w *and every* $t\geqq\sigma$. *This means that* σ *is a* $\mathfrak{B}_\sigma$*-measurable function and that, if* σ_1 *and* σ_2 *are Markovian and* $\sigma_1\geqq\sigma_2$, *then* $\mathfrak{B}_{\sigma_1}\supseteqq\mathfrak{B}_{\sigma_2}$.

(ii) *If* $\sigma_1(w)$ *and* $\sigma_2(w)$ *are Markovian,* $\sigma(w)=\sigma_1(w)+\sigma_2(w_{\sigma_1}^+)$ *is also Markovian.*

(iii) *If* $\sigma_1(w)$ *and* $\sigma_2(w)$ *are Markovian, the set* $\{w\,;\sigma_1(w)<\sigma_2(w)\}$ *belongs to both* $\mathfrak{B}_{\sigma_1}$ *and* $\mathfrak{B}_{\sigma_2}$.[3]

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2) In the continuous time parameter case, we can use a weaker condition (*) $\{w\,;\,\sigma(w)<t\}\in B_t$ instead of (1.4). In fact, if X is a more general state space, the condition (*) is more desirable for the continuous parameter case than (1.4), though it is not suitable for the discrete one. See also Itô and McKean [**10**] and our paper [**14**].

3) This assertion implies that the sets $\{w\,;\,\sigma_1(w)\leqq\sigma_2(w)\}$ and $\{w\,;\,\sigma_1(w)=\sigma_2(w)\}$ belong also to both B_{σ_1} and B_{σ_2}.

A countable Markov process (CMP) is a system $(P_x, x \in \tilde{X})$ of measures over $(W, \mathfrak{B}_W)$, satisfying the following conditions:

(CMP. 1) For any fixed x, $P_x(\cdot)$ is a probability measure over $(W, \mathfrak{B}_W)$.

(CMP. 2) Any state x is not *fictitious*, that is,

$$P_x\{w\,;\, x_0(w) = x\} = 1 \qquad \text{for any} \quad x \in \tilde{X}.$$

(CMP. 3) (Markov Property) For any $x \in \tilde{X}$, $t \in T$ and $B \in \mathfrak{B}_W$, we have

$$(1.5) \qquad P_x(w\,;\, w_t^+ \in B \,|\, \mathfrak{B}_t) = P_{x_t}(B) \qquad \text{with } P_x\text{-probability } 1,$$

where the left side denotes the conditional probability of the set $\{w\,;\, w \in B\}$ relative to $\mathfrak{B}_t$ under P_x.

Remark 1. 1. Using the same argument as in Doob [**3**], p. 81, (1. 5) is reduced to

$$(1.6) \qquad P_x(w\,;\, w_{t_n+s} \in E \,|\, x_{t_1}, x_{t_2}, \cdots, x_{t_n}) = P_{x_{t_n}}(w\,;\, w_s \in E)\,,$$

in which $E \in \mathfrak{B}_{\tilde{X}}$ and $t_n > t_{n-1} > \cdots t_1 \geqq 0$. This remark will be used in Section 3.

Now given a real valued measurable function $f(w)$ and a set $B \in \mathfrak{B}_W$, we shall define

$$(1.7) \qquad E_x(f(w)\,;\, B) = \int_B f(w) P_x(dw)\,.$$

In particular, if B coincides with W up to P_x-probability 0, $E_x(f(w)\,;\, B)$ will be denoted by $E_x(f(w))$.

Theorem 1. 1. (Strong Markov Property) *For any Markov time σ, $x \in \tilde{X}$ and $B \in \mathfrak{B}_W$, we have*

$$(1.8) \qquad P_x(w\,;\, w_\sigma^+ \in B \,|\, \mathfrak{B}_\sigma) = P_x(B) \qquad \textit{with } P_x\textit{-probability } 1.$$

This fact was established by many authors even for more general Markov processes. We shall here sketch a proof which is due to Itô [**9**].

Proof. By the definition of $\mathfrak{B}_W$, it is enough to show that

$$(1.9) \qquad P_x\{(w\,;\, w_\sigma^+ \in B) \cap \Lambda\} = E_x(P_{x_\sigma}(B)\,;\, \Lambda)$$

holds for any $\Lambda \in \mathfrak{B}_\sigma$ and any cylinder set B, i.e. a set of the form $\{w\,;\, x_{s_i}(w) \in E_i,\ i = 1, 2, \cdots, k\}$.

First suppose that σ is a discrete valued Markov time whose range is denoted by $\{t_i\,;\ i=1,2,\cdots\}$. From the fact that there exists a set $\Gamma\in\mathfrak{B}_W$ such that $\Lambda=\{w\,;\ w_\sigma^-\in\Gamma\}$, we have

$$\begin{aligned}\Lambda_i &\equiv \Lambda\cap\{w\,;\ \sigma(w)=t_i\}\\ &=\{w\,;\ w_{t_i}^-\in\Gamma,\ \sigma(w)=t_i\}\in\mathfrak{B}_{t_i}.\end{aligned}$$

Consequently
$$\begin{aligned}P_x\{(w\,;\ w_\sigma^+\in B)\cap\Lambda\} &= \sum_i P_x\{(w\,;\ w_\sigma^+\in B)\cap\Lambda_i\}\\ &= \sum_i P_x\{(w\,;\ w_{t_i}^+\in B)\cap\Lambda_i\}\,,\end{aligned}$$

using (CMP. 3)

$$= \sum_i E_x(P_{x_{t_i}}(B)\,;\ \Lambda_i) = E_x(P_{x_\sigma}(B)\,;\ \Lambda)\,,$$

which completes the proof for the discrete parameter case.

If T is continuous and σ is a general Markov time, consider a sequence $\{\sigma_n\}$ of discrete valued Markov times approximating σ from above, for example $\sigma_n=\frac{[n\sigma]+1}{n}$. Since $\mathfrak{B}_\sigma\subseteqq\mathfrak{B}_{\sigma_n}$ from Lemma 1.2. (i), (1.9) holds for any $\Lambda\in\mathfrak{B}_\sigma$ and any σ_n. Noting that B is a cylinder set and using (W. 3), we have $(w\,;\ w_{\sigma_n}^+\in B)\to(w\,;\ w_\sigma^+\in B)$ and $P_{x_{\sigma_n}}(B)\to P_{x_\sigma}(B)$, and hence (1.9) is also true for σ.

LEMMA 1.3. *Given any Markov time σ and any measurable functions $f(w)$, $g(w)$, we have*

$$E_x[f(w_\sigma^-)g(w_\sigma^+)] = E_x[f(w_\sigma^-)E_{x_\sigma}(g(w))]\,, \tag{1.10}$$

which we understand in the sense that if the one side of (1.10) *is well defined* (*admitting* $\pm\infty$), *the other is so too and the both sides are equal to each other.*

PROOF. We may assume, with no loss of generality, that f and g are nonnegative. By approximating f and g from below by a sequence of step functions, we can derive this case from the case f and g are step functions, for which (1.10) follows immediately from Theorem 1.1.

We shall now introduce several definitions, notations and terminologies.

DEFINITION 1.2. A state x is a *trap* if it satisfies

$$P_x\{w\,;\ x_t(w)=x \quad \text{for every} \quad t\neq+\infty\} = 1\,, \tag{1.11}$$

or equivalently

$$P_x\{w\,;\ \sigma_{\{x\}^c}(w)<+\infty\} = 0\,. \tag{1.11$'$}$$

DEFINITION 1.3. A state x is *recurrent* if it satisfies

$$P_x\{w\,;\,\sigma_2(w^+_{\sigma_1}) < +\infty \,|\, \sigma_1(w) < +\infty\} = 1\,, \tag{1.12}$$

where $\sigma_1(w) = \sigma_{\{x\}^c}(w)$ and $\sigma_2(w) = \sigma_x(w)$.

To see that our definition is natual, consider the recurrence time at x, $\sigma(w) = \sigma_1(w) + \sigma_2(w^+_{\sigma_1})$, which is a Markov time by Lemma 1.2. (ii). If x is not a trap, $P_x\{w\,;\,\sigma_1(w) < +\infty\} = 1$ as is shown in Seciion 2 and therefore (1.12) is equuivalent to

$$P_x\{w\,;\,\sigma(w) < +\infty\} = 1\,, \tag{1.12$'$}$$

while a trap is trivially recurrent.

To continue, we shall define

$$p_\alpha(x, y) = E_x(e^{-\alpha\sigma_y}) \qquad \text{for} \quad 0 \leqq \alpha < +\infty\,. \tag{1.13}$$

Note that $p_\alpha(x, y)$ is a monotone nonincreasing continuous function of α, that $p_0(x, y)$ (or simply $p(x, y)$) is equal to the accessible probability from x to y, i.e. $P_x\{w\,;\,\sigma_y(w) < +\infty\}$ and that $p_\alpha(x, y)$ is either strictly positive or identically zero.

DEFINITION 1.4. If $p(x, y) > 0$, y is *accessible* from x, in symbols $x \to y$.

REMARK 1.2. Using the strong Markov property and the formula $\sigma_y(w) \leqq \sigma_z(w) + \sigma_y(w^+_{\sigma_z})$ for any path starting at x, we have

$$p_\alpha(x, y) \geqq p_\alpha(x, z)p_\alpha(z, y)\,, \tag{1.14}$$

which shows that the accessible relation is transitive, namely, that if $x \to z$ and $z \to y$. then $x \to y$.

DEFINITION 1.5. A state x is *conservative* over X if ∞ is inaccessible from x, that is, if $p(x, \infty) = 0$.

If general, if a real valued function $H(x, E)$ defined over $\tilde{X} \times \mathfrak{B}_{\tilde{X}}$ is a measurable function of x for each E and a measure over $(\tilde{X}, \mathfrak{B}_{\tilde{X}})$ for each x, it is called a *kernel*, following Hunt [8]. For any kernel $H(x, E)$, a transformation H of functions of x is defined by the formula

$$Hf.(x) = \int_{\tilde{X}} f(y)H(x, dy)\,,$$

if the integral on the right side is well defined (admitting $\pm\infty$). In particular, to the kernal defined by

$$\delta(x, E) = 1 \quad \text{if} \quad E \ni x, \text{ and } = 0 \text{ otherwise,}$$

corresponds the unit transformation I.

We shall introduce several important kernels induced by a CMP. Given a Markov time σ, the kernel H^σ is given by

$$(1.15) \qquad H^\sigma(x, E) = P_x\{w\,;\, x_\sigma(w) \in E\}\,.$$

Consider two Markov times $\sigma_1(w)$, $\sigma_2(w)$ and put $\sigma(w) = \sigma_1(w) + \sigma_2(w^+_{\sigma_1})$. Then the strong Markov property proves that

$$(1.16) \qquad H^\sigma f.(x) = H^{\sigma_1} H^{\sigma_2} f.(x)$$

holds for every nonnegative function. If σ is the hitting time for a set A, H^σ is denoted by H_A and, for each fixed x, $H_A(x, \cdot)$ is called the *hitting measure for the set A of the process starting at* x. Further we introduce a new notation $\Pi(x, E) = H_{\{x\}^c}(x, E)$.

Putting $\sigma \equiv t$, we shall get a system of usual *transition probabilities* $\{H^t(x, E)\,;\, t \in T\}$, in which case (1.16) is nothing but the well known semigroup property of H^t,

$$(1.17) \qquad H^{t+s} f.(x) = H^t H^s f.(x)\,.$$

Further the Markov property shows that for any integer $n \geqq 1$, if $t_1 < \cdots < t_n$ are paramerter values and $E_1, \cdots, E_n$ are subsets of X,

$$(1.18) \qquad P_x\{w\,;\, x_{t_i}(w) \in E_i,\ i = 1, 2, \cdots, n\} = H^{t_1}\chi_{E_1} H^{t_2-t_1}\chi_{E_2} \cdots H^{t_n-t_{n-1}}\chi_{E_n}.(x)\,,$$

where χ_{E_i} is the indicator function of E_i and, for $i = 1, 2, \cdots, n-1$, it is considered as a transformation of functions in the sense of $\chi_{E_i} f.(x) = \chi_{E_i}(x) f(x)$. Consequently a CMP is uniquely determined by its system of transition probabilities. In particular, a CMP with the discrete time parameter is determined by $H^1(x, E)$, because H^t is determined by (1.17) for $t < +\infty$, while $H^{+\infty}(x, E) = 1$ or 0 according as E contains ∞ or not. Noting that $\{w\,;\, \sigma_\infty(w) > t\} = \{w\,;\, x_t(w) \in X\}$ and (CMP. 2), we can see that a state x is conservative if and only if $H^t(x, X) = 1$ for every $t < +\infty$. Similary, if T is discrete, every state x in X is conservative if and only if $H^1(x, X) = 1$ for every x in X.

The *Green kernel of order* $\alpha \geqq 0$ is defined by

$$(1.19) \qquad G_\alpha(x, E) \equiv E_x\left\{\int_0^{+\infty} e^{-\alpha t}\chi_E(x_t)dt\right\} = \int_0^{+\infty} e^{-\alpha t} H^t(x, E)dt\,,$$

where the integral notation is understood as the summation $\sum_{t=0}^{+\infty}$ in case of the discrete time parameter. $G_\alpha(x, \cdot)$ is a finite measure for every $x\in\tilde{X}$ and $\alpha>0$ (see (1.21) and (1.22)), but $G_0(x,\cdot)$ (or simply $G(x,\cdot)$) is generally an infinite measure which is not even a σ-finite measure. The following *Dynkin formula* which is a direct consequence of the strong Markov property is useful: For any Markov time σ,

$$G_\alpha f.(x)=E_x\left(\int_0^\sigma e^{-\alpha t}f(x_t)dt\right)+E_x\{e^{-\alpha\sigma}G_\alpha f.(x_\sigma)\}\,. \tag{1.20}$$

As is well known, the system $\{G_\alpha\,;\,\alpha>0\}$ satisfies

$$G_\alpha(x,E)\geqq 0\,,\quad G_\alpha(x,\tilde{X})=\frac{1}{\alpha}\,, \tag{1.21}$$

$$(\alpha-\beta)G_\alpha G_\beta f.(x)+G_\beta f.(x)-G_\alpha f.(x)=0 \text{ (RESOLVENT EQUATION)}$$

in the continuous time parameter case, or

$$G_\alpha(x,E)\geqq 0\,,\quad G_\alpha(x,\tilde{X})=\frac{1}{1-e^{-\alpha}}\,,\quad G_\alpha(x,x)\geqq 1\,, \tag{1.22}$$

$$(e^{-\alpha}-e^{-\beta})G_\alpha G_\beta f.(x)+e^{-\beta}G_\beta f.(x)-e^{-\alpha}G_\alpha f.(x)=0$$

(RESOLVENT EQUATION)

in the discrete one. Conversely,[4] to any system $\{G_\alpha\,;\,\alpha>0\}$ satisfying (1.21) or (1.22) corresponds uniquely a discrete or continuous system $\{H^t\,;\,t\in T\}$ which satisfies (1.17), (1.19) and

$$H^t(x,\tilde{X})=1\,. \tag{1.23}$$

Consequently a CMP is also uniquely determined by the system of Green kernels for $\alpha>0$.

Next we shall introduce another new quantity

$$q(x)=E_x(\sigma_{\{x\}^c})\,, \tag{1.24}$$

which is strictly positive by the definition of CMP and is finite if x is not a trap, as is shown in Section 2. Then the *Dynkin generator* $\mathfrak{G}$ is defined by

4) This fact has been proved in Feller [7] for the continuous time parameter. On the other hand, in case of the dicrete one, replace $e^{-\alpha}$ by s. Then (1.22) proves that $G_s\equiv G_\alpha$ is an analytic function of complex s in $|s|<1$ and its n-th derivative is given by

$$G_s^{[n]}=n!\;G_s\left(\frac{G_s-I}{s}\right)^n, \tag{$*$}$$

which shows that $H^n=\frac{G_s^{[n]}}{n!}$ is a kernel according to $G_s(x,E)\geqq\delta(x,E)$. The semi-group property and (1.23) of $\{H^n\,;\,n=0,1,2,\cdots\}$ are easily derived from (1.22).

$$\mathfrak{G}(x, E) = \frac{\Pi(x, E) - \delta(x, E)}{q(x)}. \tag{1.25}$$

2. Some properties of a CMP and the decomposition of the state space. We start with

LEMMA 1.4. *The distribution of $\sigma(w) = \sigma_{\{x\}^c}(w)$ relative to P_x is of geometric or exponential type according as T is discrete or continuous. The state is a trap if and only if $q(x) = +\infty$. In particular, if T is discrete, we have*

$$q(x) = \frac{1}{H^1(x, \{x\}^c)}. \tag{1.26}$$

PROOF. It is shown by the strong Markov property that $P_x(\sigma > t) = f(t)$ for $t \in T$ satisfies

$$f(t+s) = f(t)f(s).$$

Since we have $P_x(\sigma > t) = E_x\{\chi_{(t,+\infty]}(\sigma(w))\}$ and $\chi_{(t,+\infty]}(t')$ is measurable in (t, t'), $f(t)$ is also measurable in t. Therefore we get

$$f(t) = e^{-\lambda t} \qquad \text{for some} \quad \lambda \geqq 0,$$

which proves the first statement. Moreover it is clear that $\lambda = 1/q(x)$. Suppose now that x is a trap. Then since $f(t) = 1$ for every t, we have $\lambda = 0$ and hence $q(x) = +\infty$. The inverse statement is evident. The last statement immediately follows from $f(1) = H(x, x)$ and $q(x) = \dfrac{1}{1-f(1)}$.

We shall here define the *n-th jumping time* σ_n as follows:

$$\sigma_0(w) = 0,\ \sigma_1(w) = \sigma_{\{x_0(w)\}^c}(w),\ \cdots,\ \sigma_n(w) = \sigma_{n-1}(w) + \sigma_1(w^+_{\sigma_{n-1}}). \tag{1.27}$$

Since $\{\sigma_1(w) > t\} = [\bigcap_{r<t} \{w\,;\, x_r(w) = x_0(w)\}] \cap \{w\,;\, x_t(w) = x_0(w)\} \in \mathfrak{B}_t$, every $\sigma_n(w)$ is a Markov time. According to (CMP. 2) and strong Markov property, we have $q(x) = E_x(\sigma_1)$ and $H^{\sigma_n} = \Pi^n$, in which Π^n is the n product of Π with the convention $\Pi^0 = I$. Next putting $\sigma_{+\infty}(w) = \lim_{n\to\infty} \sigma_n(w)$, it results from (W. 2) and (W. 3) that $\sigma_{+\infty}(w) \geqq \sigma_\infty(w)$ for every w. In fact, suppose that $\sigma_{+\infty}(w) < \sigma_\infty(w)$. Then $\lim_{n\to\infty} x_{\sigma_n}(w)$ exists and belongs to X. This means that $x_{\sigma_{n_0}}(w) = x_{\sigma_{n_0+1}}(w) = \cdots = \lim_{n\to\infty} x_{\sigma_n}(w)$ for some n_0, which contradicts the definition of σ_n. Consequently

$$(1.28)\quad H^{\sigma}(x, E) = \sum_{n=0}^{\infty} P_x(x_\sigma \in E, \sigma_n \leqq \sigma < \sigma_{n+1}) + \delta(\infty, E) P_x(\sigma_{+\infty} \leqq \sigma)$$
$$= \sum_{n=0}^{\infty} P_x(x_{\sigma_n} \in E, \sigma_n \leqq \sigma < \sigma_{n+1}) + \delta(\infty, E) P_x(\sigma_{+\infty} \leqq \sigma)$$

holds for any subset E of $\tilde{X}$.

Suppose now that T is continuous. Then, putting $H^{t}_{(n)}(x, E) = P_x(x_t \in E, \sigma_n \leqq t < \sigma_{n+1})$, $P_x(\sigma_{\{x\}^c} > t) = e^{-\lambda(x)t}$, we get

$$H^{t}_{(n)}(x, E) = \Big[\int_{\substack{t = t_1 + \cdots + t_{n+1} \\ t_i \geqq 0}} e^{-\lambda t_1} \lambda \Pi \cdots e^{-\lambda t_{n+1}} dt_1 \cdots dt_n \Big] \chi_E . (x) ,$$

where $e^{-\lambda t}$ and λ are taken, respectively, as a transformation of functions in the sense of $e^{-\lambda t} f . (x) = e^{-\lambda(x)t} f(x)$, $\lambda f . (x) = \lambda(x) f(x)$. Therefore

$$(1.29)\quad G^{(n)}_{\alpha}(x, E) \equiv \int_0^{+\infty} e^{-\alpha t} H^{t}_{(n)}(x, E) dt$$
$$= [(\alpha + \lambda)^{-1} \lambda \Pi]^n (\alpha + \lambda)^{-1} \chi_E . (x) .$$

This can be also derived from

$$C^{(n)}_{\alpha}(x, E) = E_x \Big(\int_{\sigma_n}^{\sigma_{n+1}} e^{-\alpha t} \chi_E(x_t) dt \Big) = \frac{1}{\alpha} E_x [\chi_E(x_{\sigma_n})(e^{-\alpha\sigma_n} - e^{\alpha\sigma_{n+1}})] ,$$

using the strong Markov property. Consequently, if $E \subset X$,

$$(1.30)\quad G_{\alpha}(x, E) = \sum_{n=0}^{+\infty} G^{(n)}_{\alpha}(x, E)$$
$$= \sum_{n=0}^{+\infty} [(\alpha + \lambda)^{-1} \lambda \Pi]^n (\alpha + \lambda)^{-1} \chi_E . (x) .$$

Since, according to $\lambda(x) = q(x)^{-1}$, we have

$$(1.31)\quad \alpha - \mathfrak{G} = \alpha - \lambda(\Pi - I) = (\alpha + \lambda) [I - (\alpha + \lambda)^{-1} \lambda \Pi] ,$$

the relation

$$(1.32)\quad (\alpha - \mathfrak{G}) G_{\alpha}(x, E) = \delta(x, E)$$

holds for every x and $E \subset X$. Further, noting that $G_{\alpha}(x, \infty) = \frac{1}{\alpha} - G_{\alpha}(x, X)$, (1.32) holds for $E = \infty$ and therefore for any $E \subset \tilde{X}$. Hence G_{α} is an inverse kernel of $\alpha - \mathfrak{G}$.

In the same way, if T is discrete, we can obtain

$$H^{t}_{(n)}(x, E) = [\sum_{\substack{t = t_1 + \cdots + t_{n+1} \\ t_1, \cdots, t_n = 1, 2, \cdots \\ t_{n+1} = 0, 1, 2, \cdots}} e^{-\lambda(t_1 - 1)} (1 - e^{-\lambda}) \Pi \cdots e^{-\lambda t_{n+1}}] \chi_E . (x) ,$$

(1.33) $G_\alpha^{(n)}(x, E) = [(1-e^{-\alpha-\lambda})e^{-\alpha}(1-e^{-\lambda})\Pi)]^n(1-e^{-\alpha-\lambda})^{-1}\chi_E.(x)$,

(1.34) $G_\alpha(x, E) = \sum_{n=0}^{\infty}[(1-e^{-\alpha-\lambda})e^{-\alpha}(1-e^{-\lambda})\Pi]^n(1-e^{-\alpha-\lambda})^{-1}\chi_E.(x)$,

$$q(x) = (1-e^{-\lambda(x)})^{-1},$$

(1.35) $(1-e^{-\alpha})-e^{-\alpha}\mathfrak{G} = (1-e^{-\alpha-\lambda})[I-(1-e^{-\alpha-\lambda})^{-1}e^{-\alpha}(1-e^{-\lambda})\Pi]$,

(1.36) $[(1-e^{-\alpha})-e^{-\alpha}\mathfrak{G}]G_\alpha(x, E) = \delta(x, E)$ for every x and $E \subset \tilde{X}$.

In our case, as will be shown in Chapter 2, $G_\alpha(x, E)$ is the unique nonnegative bounded solution of (1.36).

Using the expression of H^t by means of q and Π, we can prove

THEOREM 1.2. (i) *If T is continuous and f is a bounded function, $H^t f.(x)$ is differentiable in t and the formula*

(1.37) $$\lim_{t\to 0}\frac{H^t f.(x)-f(x)}{t} = \mathfrak{G}f.(x)$$

holds, so that putting $f=\chi_{\{x\}^c}$, we get

$$\lim_{t\to 0}\frac{H^t(x, \{x\}^c)}{t} = q^{-1}(x).$$

(ii) *If T is discrete, the formula*

(1.38) $$H^1 f.(x)-f(x) = \mathfrak{G}f.(x)$$

holds for any function f.

PROOF. (i) Suppose that x is not a trap, for (1.37) is evident for a trap x. Moreover we can assume that f is a nonnegative function with no loss of generality. By a simple calculation we obtain

(1.39) $$H_{(0)}^t f.(x) = e^{-\lambda(x)t}f(x),$$

$$H_{(1)}^t f.(x) = \lambda(x)\int_{\tilde{X}} g(x, y, t)f(y)\Pi(x, dy),$$

where

$$g(x, y, t) = \frac{1-e^{-(\lambda(x)-\lambda(y))t}}{\lambda(x)-\lambda(y)}e^{-\lambda(y)t} \quad \text{if } \lambda(x) \neq \lambda(y)$$

$$= te^{-\lambda(y)t} \quad \text{if } \lambda(x) = \lambda(y).$$

If we now put $E_K = \{y\,;\,\lambda(y)<K\}$, then $E_K \uparrow \tilde{X}$ with $K\to+\infty$. Therefore, given any $\varepsilon>0$, there exists a number $K(\geqq\lambda(x))$ such that

$$\int_{E_K} f(y)\Pi(x, dy) \geqq \Pi f.(x)-\varepsilon$$

Hence using $(1-e^{-\lambda t})/\lambda \geqq t-\frac{|\lambda|}{2}t^2$, we have

$$H_{(1)}^t f.(x) \geqq \lambda(x)e^{-Kt}(t-Kt^2)\{\Pi f.(x)-\varepsilon\},$$

so that

$$H_{(1)}^t f.(x) \geqq \lambda(x)\Pi f.(x)\cdot t-\varepsilon t$$

holds for any $t \leqq$ some $t_0(>0)$. In the same way, we have

$$H_{(1)}^t f.(x) \leqq \lambda(x)\ (t+Kt^2e^{Kt})\Pi f.(x).$$

Consequently

$$H_{(1)}^t f.(x) = \lambda(x)\Pi f.(x)\cdot t+o(t). \tag{1.40}$$

Putting $k=\sup_{y\in\tilde{X}} f(y)$ and applying (1.39) and (1.40) to the unit function $\chi_{\tilde{X}}$, we get

$$\begin{aligned} E_x(f(x_t)\,;\,\sigma_2 \leqq t) &\leqq kE_x(\chi_{\tilde{X}}(x_t)\,;\,\sigma_2 \leqq t) \\ &= k[1-H_{(0)}^t\chi_{\tilde{X}}.(x)-H_{(1)}^t\chi_{\tilde{X}}.(x)] \\ &= k[1-\{1-\lambda(x)t+o(t)\}+\lambda(x)t+o(t)] \\ &= o(t). \end{aligned} \tag{1.41}$$

Summing up (1.39), (1.40) and (1.41), our statement is evident.

(ii) (1.38) is a direct consequence of

$$\begin{aligned} H_{(0)}^1 f.(x) &= e^{-\lambda(x)}f(x) = \{1-q^{-1}(x)\}f(x), \\ H_{(1)}^1 f.(x) &= \{1-e^{-\lambda(x)}\}\Pi f.(x) = q^{-1}(x)\Pi f.(x), \\ H_{(n)}^1 f.(x) &= 0 \qquad \text{for} \quad n=2,3,\cdots. \end{aligned}$$

To proceed to the decomposition of the state space, we shall list the basic results on the recurrence in

LEMMA 1.5. (i) *A state x is recurrent if and only*

$$G(x, x) = \infty. \tag{1.42}$$

(ii) *If x is recurrent and $x\rightarrow y$, y is also recurrent. Further $p(x, y)=p(y, x)=1$.*

(iii) *For any two states x and y,*

$$G_\alpha(x, y) = p_\alpha(x, y)G_\alpha(y, y) \qquad \alpha>0. \tag{1.43}$$

In particular, if y is nonrecurrent, (1.43) *holds also for $\alpha=0$, that is,*

$$G(x, y) = y(x, y)G(y, y) \leqq G(y, y) < +\infty. \tag{1.44}$$

PROOF. (i) and (ii) have been proved by the author [14] for more general Markov processes. (1.43) is a direct consequence of the Dynkin formula (1.20).

DEFINITION 1.6. A subset R of X is an *indecomposable recurrent set* if R contains a recurrent state x which satisfies

$$(1.45) \qquad p(x, y) > 0 \quad \text{for any} \quad y \in R, \\ P_x\{w\,;\,\sigma_{R^c}(w) < +\infty\} = 0\,.$$

THEOREM 1.3. (i) *Any state x of an indecomposable recurrent set R is recurrent and satisfies* (1.45).

(ii) *If a state x is recurrent, there exists a unique indecomposable recurrent set containing x.*

(iii) *The state space X is decomposed uniquely into the direct sum of at most countably many indecomposable recurrent sets and the set consisting of all nonrecurrent states.*

PROOF. (i) is clear by Lemma 1.5 and Remark 1.2. For the statement (ii), put $R(x)=\{y\,;\,p(x, y)>0\}$. It is easily shown that $R(x)$ is what we need. To prove the third statement, first consider any fixed recurrent state x_1 and $R(x_1)$. Next, take any recurrent state x_2 which is not contained in $R(x_1)$ if such state exists. $R(x_1)$ and $R(x_2)$ are disjoint, for otherwise we would have $p(x_1, x_2)>0$, which is a contradiction. In the same way we define $R(x_n)$ if there exists a recurrent state x_n such that $x_n \notin \bigcup_{i=1}^{n-1} R(x_i)$. These indecomposable recurrent sets are mutually disjoint and any state $x \notin \bigcup_{i=1} R(x_i)$ is nonrecurrent. This completes the proof of (iii).

In the sequel, the decomposition of X is denoted by

$$(1.46) \qquad X = \bigcup_i R_i + N\,,$$

where each R_i is an indecomposable recurrent set and N is the nonrecurrent part of X.

Finally we shall prove

LEMMA 1.6. *Let σ_n be the n-th jumping time, $\sigma_{+\infty}=\lim_{n\to\infty}\sigma_n$ and $L_N=\{w\,;\,x_t(w)$ has limit points in N as $t\to\sigma_{+\infty}(w)\}$. Then*

$$(1.47) \qquad P_x(L_N) = 0$$

holds for any state x.

PROOF. Put $L_y = \{w\,;\, x_t(w)$ has y as a limit point when $t \to \sigma_{+\infty}(w)\}$. Then since $L_N = \bigcup_{y \in N} L_y$, it is enough to show that

(1.48) $$P_x(L_y) = 0\,.$$

To prove this we shall define the *n-th hitting time* τ_n *for* y as follows:

$$\begin{aligned} \tau_1(w) &= \sigma_y(w) & \tau_1'(w) &= \tau_1(w) + \sigma_{\{y\}^c}(w^+_{\tau_1}) \\ &\vdots & &\vdots \\ \tau_n(w) &= \tau_{n-1}(w) + \sigma_y(w^+_{\tau'_{n-1}}) & \tau_n'(w) &= \tau_n(w) + \sigma_{\{y\}^c}(w^+_{\tau_n})\,. \end{aligned}$$

It is clear that

$$L_y = [\bigcup_n \{\tau_n(w) < +\infty,\ \tau_n'(w) = +\infty\}] \cup \{\tau_n(w) < +\infty \text{ for any } n\}\,.$$

We now calculate the probability of each set on the right side.

$$\begin{aligned} P_x\{\tau_n(w) < +\infty,\ \tau_n'(w) = +\infty\} &= P_x\{\tau_n(w) < +\infty,\ \sigma_{\{y\}^c}(w^+_{\tau_n}) = +\infty\} \\ &= E_x[P_{x_{\tau_n}}(\sigma_{\{y\}^c}(w) = +\infty)\,;\, \tau_n(w) < +\infty] \\ &= 0\,, \end{aligned}$$

by virtue of $P_y\{\sigma_{\{y\}^c}(w) < +\infty\} = 1$ (Lemma 1.4). Recalling the fact that y is nonrecurrent, we get

$$\begin{aligned} P_x\{\tau_n(w) < +\infty \text{ for any } n\} &= \lim_{n \to \infty} P_x\{\tau_n(w) < +\infty\} \\ &= \lim_{n \to \infty} p(x, y)[P_y\{\tau_2(w) < +\infty\}]^{n-1} \\ &= 0\,. \end{aligned}$$

Thus we have proved (1.48).

COROLLARY. *If F is a finite subset of N,*

$$P_x\{w\,;\, x_t(w) \notin F \quad \textit{for any} \quad t \geqq \textit{some } t_0\} = 1$$

holds for every state x.

3. Construction of a CMP. It is clear that the system $\{H^t\,;\, t \in T\}$ of transition probabilities of a CMP satisfies the following

(1.49) $$H^0(x, E) = \delta(x, E),\quad H^t(\infty, E) = H^{+\infty}(x, E) = \delta(\infty, E)\,,$$

besides the semigroup property (1.17) and the stochastic condition (1.23). Now we shall study the problem whether, for any given system $\{H^t\,;\, t \in T\}$ satisfying (1.17), (1.23) and (1.49), there exists a CMP whose system of transition probabilities is $\{H^t\}$.

We shall start with a preliminary

LEMMA 1.7. Let $(\tilde{W}, \mathfrak{B}_{\tilde{W}}, \tilde{P})$ be an abstract probability field and $y(\tilde{w})=(y_t(\tilde{w})\,;\, t\in T)$ a stochastic process on $\tilde{X}$ which satisfies (i) $y^{-1}W=\{\tilde{w}\,;\, y(\tilde{w})\in W\}\in\mathfrak{B}_{\tilde{W}}$ and (ii) $\tilde{P}(y^{-1}W)=1$. Then the process $y_t(\tilde{w})$ induces a probability measure over $(W, \mathfrak{B}_W)$ by the formula

$$P(B) = \tilde{P}(y^{-1}B)\,.$$

In particular, if $B=\{w\,;\, x_{t_i}(w)\in E_i,\ i=1,2,\cdots,n\}$,

$$P(B) = \tilde{P}\{\tilde{w}\,;\, y_{t_i}(\tilde{w})\in E_i,\quad i = 1,2,\cdots,n\}\,. \tag{1.50}$$

PROOF. If $B=\{w\,;\, x_t(w)\in E\}$, then $y^{-1}B=y^{-1}W\cap\{\tilde{w}\,;\, y_t(\tilde{w})\in E\}\in\mathfrak{B}_{\tilde{W}}$. Therefore $y^{-1}B\in\mathfrak{B}_{\tilde{W}}$ holds for any $B\in\mathfrak{B}_W$ by the definition of $\mathfrak{B}_W$, which completes the proof.

THEOREM 1.4. *Let $H^1(x, E)$ be any kernal satisfying*

$$H^1(x, E)\geqq 0,\quad H^1(x, \tilde{X}) = 1 \quad \textit{and} \quad H^1(\infty, E) = \delta(\infty, E)\,.$$

Then there exists uniquely a CMP with the discrete parameter which satisfies

$$P_x\{x_1(w)\in E\} = H^1(x, E)\,. \tag{1.51}$$

REMARK 1.3. Our kernel H^1 is uniquely determined by its restriction to X. Consequently to any given kernel over X satisfying $H(x, X)\leqq 1$ corresponds one and only one CMP.

PROOF. The uniqueness of the process has been already shown in Section 1. Hence it remains to construct our process.

Consider the system $\{H^t\,;\, t\in T\}$ induced by H^1, using (1.17) and (1.49). As is well known (see Doob [3]), we can construct an abstract probability field $(\tilde{W}_x, \mathfrak{B}_{\tilde{W}_x}, \tilde{P}_x)$ and a stochastic process $y_t^{(x)}(\tilde{w})$ on $\tilde{X}$ such that

$$\tilde{P}_x\{y_{t_i}^{(x)}(\tilde{w})\in E_i,\ i = 1,2,\cdots,n\} = H^{t_1}\chi_{E_1}\cdots H^{t_n-t_{n-1}}\chi_{E_n}.(x) \tag{1.52}$$

holds for any $t_i\in T$, $0\leqq t_1<t_2<\cdots<t_n\leqq +\infty$ and $E_i\in\mathfrak{B}_{\tilde{X}}$. Further we have

$$\begin{aligned}\tilde{W}_x-y^{-1}W &= \{y^{(x)}(\tilde{w})\notin W\}\\ &= [\bigcup_{t=0}\{y_t^{(x)}(\tilde{w}) = \infty,\ y_{t+1}^{(x)}(\tilde{w})\neq\infty\}]\cup\{y_{+\infty}^{(x)}(\tilde{w})\neq\infty\}\in\mathfrak{B}_{\tilde{W}_x},\\ \tilde{P}_x\{y_t^{(x)}(\tilde{w}) &= \infty,\ y_{t+1}^{(x)}(\tilde{w})\neq\infty\} = H^t\chi_\infty H^1\chi_X.(x) = 0\,,\\ \tilde{P}_x\{y_{+\infty}^{(x)}(\tilde{w})\neq\infty\} &= H^{+\infty}\chi_X.(x) = 0\,,\end{aligned}$$

which proves that $y_t^{(x)}(\tilde{w})$ satisfies the conditions (i), (ii) in Lemma

1.7. We denote by P_x the probability measure over $(W, \mathfrak{B}_W)$ induced by $y_t^{(x)}(\tilde{w})$.

We shall show that the system $\{P_x ; x \in \tilde{X}\}$ obtained above is the CMP which we wanted. First the condition (CMP. 1) in the paragraph 1 is evident. Further (CMP. 2) and (1.51) come from (1.50) and (1.52). Finally, by Remark 1.1, (CMP. 3) is reduced to (1.6), i.e.

$$P_x\{x_{t_i}(w) \in E_i,\ i = 1, 2, \cdots, n\} = E_x[P_{x_{t_{n-1}}}(x_{t_n - t_{n-1}}(w) \in E_n) ;\ x_{t_i}(w) \in E_i,\ i = 1, 2, \cdots, n-1]$$

for $0 \leqq t_1 < t_2 < \cdots < t_n \leqq +\infty$, which is also derived from (1.50) and (1.52).

THEOREM 1.5. *Let $\{H^t ; t \in T\}$ be a continuous system satisfying* (1.17), (1.23), (1.49) *and the following conditions:*[5] *For any fixed t, any finite set E and any $\varepsilon > 0$, there exists some finite set F such that*

$$H^t(x, E) < \varepsilon \quad \text{for any} \quad x \notin F. \tag{1.53}$$

Then we have one and only one CMP with the continuous parameter whose system of transition probabilities is $\{H^t ; t \in T\}$.

PROOF. We now understand $\tilde{X}$ as the one-point compactification of X. Then the condition (1.53) implies that H^t makes invariant the family of bounded continuous functions on $\tilde{X}$. In fact, if f is a bounded continuous functions on $\tilde{X}$, $H^t f$ is continuous on X evidently. Moreover, since $E = \{y ; |f(y) - f(\infty)| > \varepsilon\}$ is a finite set for any given $\varepsilon > 0$, we have

$$\begin{aligned} |H^t f.(x) - H^t f.(\infty)| &= |H^t f.(x) - f(\infty)| \\ &\leqq \int_E |f(y) - f(\infty)| H^t(x, dy) + \int_{E^c} |f(y) - f(\infty)| H^t(x, dy) \\ &\leqq K H^t(x, E) + \varepsilon \\ &\leqq 2\varepsilon \quad \text{for any } x \notin \text{ some finite set } F. \end{aligned}$$

Therefore, according to Itô [**9**], we can introduce a probability field $(\tilde{W}_x, \mathfrak{B}_{\tilde{W}_x}, \tilde{P}_x)$ and a stochastic process $y_t^{(x)}(\tilde{w})$ on $\tilde{X}$ such that

---

5) This is not a necessary condition. A simple example which does not satisfy (1.53) is this: $X = \{0, 1, 2, \cdots\}$, $\sum_{x \geqq 1} q(x) < +\infty$, $\Pi(x, x-1) = 1$ for $x \geqq 1$ and $\Pi(0, 0) = 1$. Such process is known as the pure death process and its existence is guaranteed by Theorem 1.6.

(a) $y_t^{(x)}(\tilde{w})$ is right continuous with $\tilde{P}_x$-probability 1 and (b) (1.52) holds. The proof of constructing the requied CMP by means of $y_t^{(x)}(\tilde{w})$ is the same as in Theorem 1.4, so it will be omitted.

Next we shall discuss another construction of a CMP. The system $\{q, \Pi\}$ corresponding to a CMP satisfies the following conditions;

(1.54) $\quad 1 \leqq q(x) \leqq +\infty,\ q(\infty) = +\infty, \quad$ if T is discrete,

(1.55) $\quad 0 < q(x) \leqq +\infty,\ q(\infty) = +\infty, \quad$ if T is continuous,

(1.56) $\quad \Pi(x, \tilde{X}) = 1,$

(1.57) $\quad \Pi(x, x) = 0 \quad$ if $\ q(x) < +\infty,$

$\quad \Pi(x, E) = \delta(\infty, E) \quad$ if $\ q(x) = +\infty.$

Conversely we can prove

THEOREM 1.6. *Suppose that a pair* $\{q, \Pi\}$ *of a function and a kernel satisfies* (1.54) (*or* (1.55)), (1.56) *and* (1.57). *Then we have one and only one* CMP *with the discrete* (*or continuous*) *parameter satisfying*

(1.58) $$E_x(\sigma_{\{x\}^c}) = q(x)\,, \quad P_x\{x_{\sigma_{\{x\}^c}} \in E\} = \Pi(x, E)\,.$$

PROOF. The uniqueness of our process is clear by (1.30) and (1.34). Since our proof of existence is the same as in Doob [3], we shall give only its outline.

In case T is discrete, consider a probability field $(\tilde{W}, \mathfrak{B}_{\tilde{W}}, \tilde{P})$ and a family of random variables satisfying the following conditions: (a) $\tau_k^{(x)}(\tilde{w})$, $a_k^{(x)}(\tilde{w})$ ($k=1, 2, \cdots$, and $x \in \tilde{X}$) are mutually independent. (b) Each $\tau_k^{(x)}(\tilde{w})$ is subject to the geometric distribution with the mean $q(x)$. (c) Each $a_k^{(x)}(\tilde{w})$ is a random variable on $\tilde{X}$ whose distribution is given by $\Pi(x, \cdot)$. Next we define a new family of random variables as follows:

$$\begin{aligned}
\sigma_1^{(x)} &= \tau_1^{(x)} & b_1^{(x)} &= a_1^{(x)} \\
\sigma_2^{(x)} &= a_2^{(b_1^{(x)})} + \sigma_1^{(x)} & b_2^{(x)} &= a_2^{(b_1^{(x)})} \\
&\ \vdots & &\ \vdots \\
\sigma_n^{(x)} &= \tau_n^{(b_{n-1}^{(x)})} + \sigma_{n-1}^{(x)} & b_n^{(x)} &= a_n^{(b_{n-1}^{(x)})}\,.
\end{aligned}$$

Moreover we consider a stochastic process $y^{(x)}(\tilde{w}) = (y_t^{(x)}(\tilde{w})\,;\, t \in T)$ defined by

$$
\begin{aligned}
y_t^{(x)} &= x && \text{if } 0 \leqq t < \sigma_1^{(x)} \\
&\vdots && \qquad \vdots \\
&= b_n^{(x)} && \text{if } \sigma_n^{(x)} \leqq t < \sigma_{n+1}^{(x)} \\
&\vdots && \qquad \vdots \\
&= \infty && \text{if } \lim_{n\to\infty} \sigma_n^{(x)} \leqq t \leqq +\infty .
\end{aligned}
$$

Then the system $(W, \mathfrak{B}_W, P_x; x \in \tilde{X})$ induced by $y_t^{(x)}(\tilde{w})$ (using Lemma 1.7) is the CMP required. In fact, the same argument as in Theorem 1.4 proves that the Markov property of $\{P_x\}$ is reduced to that of $y_t^{(x)}(\tilde{w})$, which has been shown by Doob [3]. The other properties are easily verified by the definition of $y_t^{(x)}(\tilde{w})$.

Our proof is applicable to the continuous parameter case, in which it is assumed that $\tau_k^{(x)}(\tilde{w})$ is of exponential type with the mean $q(x)$.

REMARK 1.4. Another proof for the discrete parameter is as follows: For given $\{q, \Pi\}$, we define $\mathfrak{G}$ and H^1, respectively, by (1.25) and (1.38), where $H^1(x, E) = \delta(x, E)$ for $q(x) = \infty$ by definition. Then it is easily shown that H^1 satisfies the conditions in Theorem 1.4. The CMP corresponding to H^1 is what we wanted.

CHAPTER 2. x_t-SUPERHARMONIC FUNCTIONS

4. Definition and its direct consequences. In the following discussions, we shall denote a CMP by x_t. Moreover consider the new kernel $\hat{\Pi}$ defined by

$$
(2.1) \qquad \hat{\Pi}(x, E) = \Pi(x, E) \quad \text{if } x \text{ is not a trap,} \qquad = \delta(x, E) \quad \text{if } x \text{ is a trap.}
$$

DEFINITION 2.1. (i) Given a fixed state a, a real valued function u over $\tilde{X}$ is *x_t-superharmonic at a* if it satisfies

$$
(2.2) \qquad -\infty < u(a) \leqq +\infty , \quad u(\infty) = 0 \quad \text{and} \quad \hat{\Pi}u.(a) \leqq u(a) .
$$

In particular, if $\hat{\Pi}u.(a) < u(a)$ or $u(a) = +\infty$, u is *strictly x_t-superharmonic at a.*

(ii) A function u is *(strictly) x_t-subharmonic at a* if $(-u)$ is (strictly) x_t-superharmonic at the state.

(iii) A function u is *x_t-harmonic at a* if it is both x_t-superharmonic and x_t-subharmonic at the state.

(iv) A function u which is x_t-superharmonic (x_t-subharmonic or x_t-harmonic) at any state is *x_t-superharmonic* (*x_t-subharmonic* or *x_t-harmonic*).

REMARK 2.1. Since the definition of an x_t-superharmonic function depends only on Π (or $\hat{\Pi}$), to the family of CMP's with the same Π corresponds the same class of x_t-superharmonic functions. Therefore our concept may be understood as an analytic one with repect to a kernel Π.

REMARK 2.2. In our case, the function $u\equiv+\infty$ except $x=\infty$ is x_t-superharmonic. But an x_t-harmonic function is finite valued over X by the definition.

We shall now define several families of functions over $\tilde{X}$. We shall here use the notation $\mathfrak{F}^+(\mathfrak{F}^-)$ to denote the class of all nonnegative (nonpositive) functions in a given function family $\mathfrak{F}$.

$\mathfrak{R}(X)=\{f\,;f$ is a real valued (admitting $\pm\infty$) function over $\tilde{X}$ taking the value 0 at $\infty\}$.

$\mathfrak{F}(X)=\mathfrak{R}(X)\frown\{f\,;-\infty<f(x)<+\infty)$,

$\mathfrak{H}_1(X)=\{u\,;u$ is x_t-superharmonic$\}$,

$\mathfrak{H}_2(X)=\{u\,;u$ is x_t-subharmonic$\}$,

$\mathfrak{H}(X)=\{u\,;u$ is x_t-harmonic$\}=\mathfrak{H}_1(X)\frown\mathfrak{H}_2(X)\subset\mathfrak{F}(X)$.

In the following two lemmas we shall list some elementary properties of x_t-superharmonic functions to be used later.

LEMMA 2.1. *Suppose that u, v, $u_n\in\mathfrak{H}_1(X)$. Then*

(i) *If $k\geqq 0$, $ku\in\mathfrak{H}_1(X)$.*

(ii) *If $[\hat{\Pi}u+\hat{\Pi}v]$ is well defined (admitting $\pm\infty$), $u+v\in\mathfrak{H}_1(X)$.*

(iii) *If $-\infty\leqq\hat{\Pi}u.(x)<+\infty$ for any x, $u\wedge v\in\mathfrak{H}_1(X)$, where $u\wedge v(x)=\min(u(x),v(x))$.*

(iv) *If $-\infty<\hat{\Pi}u_n.(x)\leqq+\infty$ for any x and $u_n\uparrow u_{+\infty}$, then $u_{+\infty}\in\mathfrak{H}_1(X)$.*

(v) *If $-\infty\leqq\hat{\Pi}u.(x)<+\infty$, then $\hat{\Pi}^n u\downarrow u_{-\infty}$ and $\hat{\Pi}u_{-\infty}=u_{-\infty}$. Therefore $u_{-\infty}\in\mathfrak{H}_2(X)$ and in particular, if $u_{-\infty}(a)>-\infty$, it is x_t-harmonic at the state a.*

PROOF. (i) and (ii) are evident. (iii) Since $u-u\wedge v\geqq 0$ and $\hat{\Pi}(u\wedge v)=\hat{\Pi}u-\hat{\Pi}(u-u\wedge v)$, $\hat{\Pi}(u\wedge v)$ is well defined by our assumption. Now suppose that $u\wedge v(a)=u(a)$ for a state a. Then we have

$$\hat{\Pi}(u\wedge v).(a)\leqq\hat{\Pi}u.(a)\leqq u(a)=u\wedge v(a),$$

which completes the proof. (iv) is clear from the well known theorem of Lebesgue. (v) Notice that $\hat{\Pi}^n u$ is well defined for every n in the same way as in (iii). Again using the theorem of Lebesgue, we get

$$\hat{\Pi} u_{-\infty}.(x) = \hat{\Pi}(\lim_{n\to\infty} \hat{\Pi}^n u).(x) = \lim_{n\to\infty} \hat{\Pi}^{n+1} u.(x) = u_{-\infty}(x).$$

Thus Lemma 2. 1 was completely proved.

LEMMA 2. 2. (i) $\chi_X \in \mathfrak{H}_1^+(X)$.

(ii) *If* u *and* $v \in \mathfrak{H}_1^+(X)$, *then also* $u+v$ *and* $u \wedge v \in \mathfrak{H}_1^+(X)$.

(iii) *If* $u_n \in \mathfrak{H}_1^+(X)$ *and* $u_n \to u_{+\infty}$, $u_{+\infty} \in \mathfrak{H}_1^+(X)$.

(iv) *Any* $u \in \mathfrak{H}_1^+(X)$ *can be approximated from below by a sequence of bounded functions* $u_n \in \mathfrak{H}_1^+(X)$.

(v) *If* $u \in \mathfrak{H}_1^+(X)$, *then* $\hat{\Pi}^n u \downarrow u_{-\infty}$ *and* $u_{-\infty} \in \mathfrak{H}_1^+(X)$. *In particular, if* $u \in \mathfrak{H}_1^+(X) \frown \mathfrak{F}(X)$, $u_{-\infty} \in \mathfrak{H}^+(X)$.

PROOF. (i) is clear from the fact that $\hat{\Pi}\chi_X.(x) = \hat{\Pi}(x, X) \leqq 1$ for every $x \in X$. (ii) In our case, $\hat{\Pi}(u+v)$ and $\hat{\Pi}(u \wedge v)$ are always well defined, while the other arguments are the same as in Lemma 2. 1. (iii) Using Fatou's lemma, we get

$$\hat{\Pi} u_{+\infty}.(x) = \hat{\Pi}(\lim_{n\to\infty} u_n).(x) \leqq \liminf_{n\to\infty} \hat{\Pi} u_n.(x) \leqq \lim_{n\to\infty} u_n(x) = u_{+\infty}(x),$$

which is what we wanted. (iv) It is enough to approximate u by $u_n = u \wedge (n\chi_X)$, for u_n is a bounded function of $\mathfrak{H}_1^+(X)$ according to Lemma 2. 1. and this lemma (i), (ii). (v) $u_n = \hat{\Pi}^n u$ is well defined and $0 \leqq \hat{\Pi} u_n = \hat{\Pi}^{n+1} u \leqq \hat{\Pi}^n u = u_n$, so that each u_n belongs to $\mathfrak{H}_1^+(X)$. Therefore, according to the statement (iii), $u_{-\infty} \in \mathfrak{H}_1^+(X)$. The latter part is a special case of Lemma 2. 1. (v).

Now we have

THEOREM 2. 1. *If* $u \in \mathfrak{H}(X)$, *then* $|u| \in \mathfrak{H}_2^+(X)$ *and* $\hat{\Pi}^n |u| \uparrow u_{+\infty} \in \mathfrak{H}_1^+(X)$. *Especially, if* $u_{+\infty} \in \mathfrak{F}(X)$, $u_{+\infty} \in \mathfrak{H}^+(X)$ *and further* u *can be represented in the form*

$$u = u_1 - u_2 \tag{2. 3}$$

by means of some $u_1, u_2 \in \mathfrak{H}^+(X)$. *Conversely, if* u *is expressible in the form* (2. 3), $u_{+\infty} \in \mathfrak{F}^+(X)$.

REMARK 2. 3. Our theorem implies that any bounded function of $\mathfrak{H}(X)$ is always expressible as the difference of two bounded functions in $\mathfrak{H}^+(X)$.

PROOF. Consider the function $u^+ = u \vee 0$. Then $-u^+ = (-u) \wedge 0$ is a function of $\mathfrak{H}_1^-(X)$ according to Lemma 2. 1. (iii). In the same way, putting $u^- = (-u) \vee 0$, we have $-u^- \in \mathfrak{H}_1^-(X)$. Therefore $-|u| = (-u^+) + (-u^-) \in \mathfrak{H}_1^-(X)$, i.e. $|u| \in \mathfrak{H}_2^+(X)$. Applying Lemma 2.1. (v), $-u_{+\infty} \in \mathfrak{H}_2^-(X)$ (i.e. $u_{+\infty} \in \mathfrak{H}_1^+(X)$) and, if $u_{+\infty} \in \mathfrak{F}(X)$, $-u_{+\infty} \in \mathfrak{H}^-(X)$ (i.e. $u_{+\infty} \in \mathfrak{H}^+(X)$). Further noting that $-|u| \leqq -u^+$, we get

$$-\infty < -u_{+\infty}(x) = \lim_{n\to\infty} \hat{\Pi}^n(-|u|).(x) \leqq \lim_{n\to\infty} \hat{\Pi}^n(-u^+).(x) \equiv -u_1(x) \leqq 0,$$

which means that $-u_1 \in \mathfrak{H}^-(X)$ (use again Lemma 2.1. (v)). Similary $\lim_{n\to\infty} \hat{\Pi}^n(-u^-) \equiv -u_2 \in \mathfrak{H}^-(X)$. Thus we have

$$u = u^+ - u^-,$$

$$u = \lim_{n\to\infty} \hat{\Pi}^n u = \lim_{n\to\infty} \hat{\Pi}^n u^+ - \lim_{n\to\infty} \hat{\Pi}^n u^- = u_1 - u_2,$$

which proves (2.3). Conversely, suppose that $u = u_1 - u_2$, where $u_1, u_2 \in \mathfrak{H}^+(X)$. Then since $-|u| \geqq -u_1 - u_2$, we obtain

$$0 \geqq -u_{+\infty}(x) = \lim_{n\to\infty} \hat{\Pi}^n(-|u|).(x) \geqq -\lim_{n\to\infty} \hat{\Pi}^n u_1.(x) - \lim_{n\to\infty} \hat{\Pi}^n u_2.(x)$$
$$= -u_1(x) - u_2(x) > -\infty,$$

which completes the proof of the theorem.

REMARK 2.4. All the results in this section hold for any substochastic kernel H, i.e. a kernel such that $H(x, X) \leqq 1$.

5. The properties of the function $H_A u$. Potentials of functions. Suppose that u is a function of $\mathfrak{H}_1^+(x)$. Then the function $H_A u$ is also a function of $\mathfrak{H}_1^+(X)$ which does not exceed u. Moreover, if A is a finite subset of the nonrecurrent part N of X and u is finite over A, $H_A u$ is a potential of a nonnegative function whose carrier is contained in A. These facts play fundamental roles in the study of nonnegative x_t-superharmonic functions.[6]

First we shall prove a general

LEMMA 2.3. *Let all the functions under considerations belong to* $\mathfrak{R}^+(X)$. *Then we have*

(i) $H_A f = f$ *over* A.

(ii) *If* $f \geqq g$ *over* A, *then* $H_A f \geqq H_A g$ *over* X.

(iii) $H_A(k_1 f + k_2 g) = k_1 H_A f + k_2 H_A g$ *for* $k_1, k_2 \geqq 0$.

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6) Refer to the theorems in Section 1 of Martin [12], noting that our function $H_A u$ is the exact counterpart of the function u_σ^* in [12]. Also see Hunt's paper [8], in which he has discussed another approach to the function $H_A u$ for general Markov processes.

(iv) *If* $f_n \uparrow f$ *over* A, *or if* $f_n \to f$ *and each* f_n *is dominated over* A *by an* $H_A(x, \cdot)$*-integrable function, we have*

$$\lim_{n\to\infty} H_A f_n = H_A f .$$

In general, if $f_n \to f$ *over* A, *then*

$$\liminf_{n\to\infty} H_A f_n \geqq H_A f .$$

(v) *If* $A \subset B$, *then*

$$H_A f = H_A H_B f = H_B H_A f . \tag{2.4}$$

(vi) $$H_{A\cup B} f \leqq H_A f + H_B f .$$

PROOF. (i) is clear according to $H_A(x, \cdot) = \delta(x, \cdot)$ for $x \in A$. (ii)-(iv) are also clear from the fact that $H_A(x, \cdot)$ is a measure over A. (v) Noting that $H_B f = f$ over $A (\subset B)$, the first equality is evident from (ii). On the other hand, since $\sigma_A(w) = \sigma_B(w) + \sigma_A(w^+_{\sigma_B})$, the relation $H_A f = H_B H_A f$ is nothing but a special case of the formula (1.16).

(vi) $H_{A\cup B} f.(x)$

$$\begin{aligned}
&= E_x(f(x_{\sigma_A})\,;\,\sigma_{A\cup B} = \sigma_A) + E_x(f(x_{\sigma_B})\,;\,\sigma_{A\cup B} = \sigma_B,\ \sigma_{A\cup B} \neq \sigma_A) \\
&\leqq E_x[f(x_{\sigma_A})] + E_x[f(x_{\sigma_B})] \\
&= H_A f.(x) + H_B f.(x) .
\end{aligned}$$

Next we have

LEMMA 2.4. *Suppose that* $f \in \Re(X)$, $A \subset X$ *and* $a \notin A$. *Then if* $H_A f$ *is well defined, we have the formula*

$$\hat{\Pi} H_A f.(a) = H_A f.(a) . \tag{2.5}$$

Therefore, if $-\infty < H_A f.(a) < +\infty$, $H_A f$ *is* x_t*-harmonic at* a.

PROOF. It is enough to show that $\Pi H_A u.(a) = H_A u.(a)$ for $a \neq$ a trap. Let σ_1 be the first jumping time. Then, by virtue of $a \notin A$, it holds that $\sigma_A(w) = \sigma_1(w) + \sigma_A(w^+_{\sigma_1})$ with P_a-probability 1. Hence, according to Lemma 1.3, we get

$$\begin{aligned}
H_A f.(a) &= E_a[f(x_{\sigma_A})] = E_a[f(x_{\sigma_A(w^+_{\sigma_1})}(w^+_{\sigma_1}))] \\
&= E_a[E_{x_{\sigma_1}}(f(x_{\sigma_A}))] = E_a[H_A f.(x_{\sigma_1})] \\
&= \Pi H_A f.(a) ,
\end{aligned}$$

which is the formula required.

THEOREM 2.2. *For any* $u \in \mathfrak{H}_1^+(X)$, *we have*

(2.6) $\quad 0 \leqq H_A u \leqq u \qquad$ *for any* A,

(2.7) $\quad H_A u \in \mathfrak{H}_1^+(X)$,

(2.8) $\quad H_A u.(x) = u(x) \qquad$ *if* $x \in A$,

$\qquad = x_t$-*harmonic if* $x \notin A$ *and* $H_A u.(x) < +\infty$.

PROOF. (2.8) is evident by the last lemma. Moreover in case $x \in A$, (2.6) is comprised in (2.8). Now we prove (2.6) for $x \notin A$. For this purpose we can assume that u is a bounded function of $\mathfrak{H}_1^+(X)$. In fact, if (2.6) is true for any bounded function of $\mathfrak{H}_1^+(X)$, it is also true for any function of $\mathfrak{H}_1^+(X)$ according to Lemma 2.2. (iv) and the theorem of Lebesgue.

Now consider the n-th jumping time σ_n of (1.27). Then recalling the formula (1.28) and noting that $\sigma_A = \sigma_n$ if $\sigma_n \leqq \sigma_A < \sigma_{n+1}$, we get

$$\begin{aligned} 0 \leqq H_A u.(x) &= \sum_{n=0}^{\infty} E_x(u(x_{\sigma_n}) ; \sigma_n \leqq \sigma_A < \sigma_{n+1}) \\ &= \sum_{n=0}^{\infty} E_x(u(x_{\sigma_n}) ; \sigma_A = \sigma_n) < +\infty . \end{aligned}$$

It results from the definition of u, Lemma 1.3 and $x \notin A$ that

$$E_x(u(x_{\sigma_0}) ; \sigma_A = \sigma_0) = 0 ,$$

and if $n \geqq 1$,

$$\begin{aligned} &E_x(u(x_{\sigma_n}) ; \sigma_A = \sigma_n) \\ &\quad = E_x[u(x_{\sigma_1(w^+_{\sigma_{n-1}})}(w^+_{\sigma_{n-1}})) ; \sigma_A(w^+_{\sigma_{n-1}}) = \sigma_1(w^+_{\sigma_{n-1}}), \ \sigma_A > \sigma_{n-1}] \\ &\quad = E_x[E_{x_{\sigma_{n-1}}}(u(x_{\sigma_1}) ; \sigma_A = \sigma_1) ; \sigma_A > \sigma_{n-1}] \\ &\quad = E_x[\{\Pi u.(x_{\sigma_{n-1}}) - E_{x_{\sigma_{n-1}}}(u(x_{\sigma_1}) ; \sigma_A > \sigma_1)\} ; \sigma_A > \sigma_{n-1}] \\ &\quad \leqq E_x[u(x_{\sigma_{n-1}}) - E_{x_{\sigma_{n-1}}}(u(x_{\sigma_1}) ; \sigma_A > \sigma_1) ; \sigma_A > \sigma_{n-1}] \\ &\quad = E_x(u(x_{\sigma_{n-1}}) ; \sigma_A > \sigma_{n-1}) - E_x(u(x_{\sigma_n}) ; \sigma_A > \sigma_n) . \end{aligned}$$

Hence we have

$$\begin{aligned} H_A u \cdot (x) &= \lim_{m \to \infty} \sum_{n=0}^{m} E_x(u(x_{\sigma_n}) ; \sigma_A = \sigma_n) \\ &= \lim_{m \to \infty} [E_x(u(x_{\sigma_0}) ; \sigma_A > \sigma_0) - E_x(u(x_{\sigma_m}) ; \sigma_A > \sigma_m)] \\ &\leqq E_x(u(x_{\sigma_0}) ; \sigma_A > \sigma_0) = u(x) . \end{aligned}$$

Finally we shall prove (2.7). In case $x \notin A$, our statement is clear from the formula (2.5). On the other hand, if $x \in A$, (2.6) and (2.8) implies that

$$\hat{\Pi} H_A u.(x) \leqq \hat{\Pi} u.(x) \leqq u(x) = H_A u.(x).$$

Thus our theorem was completely proved.

COROLLARY 1. *If* $u \in \mathfrak{H}_1^+(X)$ *and* $u(x) < +\infty$ *and* $x \to y$, *then* $u(y) < +\infty$. *In particular*, $u(x) = 0$ *implies* $u(y) = 0$.

PROOF. Applying (2.6) for the set $\{y\}$, we have

$$u(x) \geqq H_y u.(x) = p(x, y) u(y) \geqq 0,$$

which proves our statement by virtue of $p(x, y) > 0$.

COROLLARY 2. *If* $u \in \mathfrak{H}_1^+(X)$, $H_A u$ *is the smallest among all the functions in* $\mathfrak{H}_1^+(X)$ *which are* $\geqq u$ *over* A.

PROOF. If $v \geqq u$ over A, it results from Lemma 2.3. (ii) and the formula (2.6) that

$$v \geqq H_A v \geqq H_A u,$$

which completes the proof.

THEOREM 2.3. *For any function* $u \in \mathfrak{H}_1^+(X)$, *we have*

$$H_A u \leqq H_B u \quad \text{if} \quad A \subset B, \tag{2.9}$$

$$H_{A_n} u \uparrow H_A u \quad \text{if} \quad A_n \uparrow A. \tag{2.10}$$

PROOF. Applying (2.6) to $H_B u \in \mathfrak{H}_1^+(X)$ and using (2.4), we get

$$H_B u \geqq H_A H_B u = H_A u \qquad \text{for} \quad A \subset B.$$

Next noting that $H_{A_n} u = u$ over A_n and $A_n \uparrow A$, it follows that $H_{A_n} u \uparrow u$ over A. Theorefore, by virtue of Lemma 2.3. (iv) and (v), we have

$$\lim_{n \to \infty} H_{A_n} u = \lim_{n \to \infty} H_A H_{A_n} u = H_A u.$$

THEOREM 2.4. *The accessible probability* $p(x, y)$, *taken as a function of* x, *is a function of* $\mathfrak{H}_1^+(X)$ *which is* x_t*-harmonic at* $x \neq y$. *Moreover* $p(x, y)$ *is* x_t*-harmonic or strictly* x_t*-superharmonic at* $x = y$ *according as* y *is a recurrent state or not.*

PROOF. Since $p(x, y) = H_y \chi_X.(x)$ and $\chi_X \in \mathfrak{H}_1^+(X)$, the first statement is clear.

For the proof of the latter part, we can assume that y is not a trap. Then recalling the formula (1.16), we get

$$\begin{aligned} \Pi H_y \chi_X.(y) &= \Pi H_y \chi_X.(y) = H_{\{y\}^c} H_y \chi_X.(y) \\ &= H^\sigma \chi_X.(y) = P_y(\sigma < +\infty), \end{aligned}$$

where $\sigma(w)=\sigma_{\{y\}^c}(w)+\sigma_y(w^+_{\sigma_{\{y\}^c}})$, that is, the recurrence time at y. Therefore, from the definition of recurrence, we have

$$P_y(\sigma < +\infty) = 1 = p(y, y) \quad \text{if } y \text{ is recurrent},$$
$$< 1 = p(y, y) \quad \text{otherwise},$$

which completes the proof.

Now recalling that the decomposition of the state space is given by the formula (1. 46), we have

THEOREM 2. 5. *Any function u of $\mathfrak{H}_1^+(X)$ is a constant, say k_i, over each indecomposable recurrent set R_i. Moreover the function $H_{R_i}u$ has the following properties*:

(i) $H_{R_i}u.(x) = 0$ *if* $x \in R_j$, $i \neq j$.

(ii) *For any finite or infinite sum of* R_i,

$$H_{\bigcup R_i}u = \sum H_{R_i}u. \tag{2. 11}$$

(iii) *For any state* $r_i \in R_i$,

$$H_{R_i}u.(x) = H_{r_i}u.(x) = k_i p(x, r_i). \tag{2. 12}$$

Therefore, if $k_i < +\infty$, $H_{R_i}u$ is a function of $\mathfrak{H}^+(X)$.

REMARK 2. 5. Putting $u=\chi_X$, it follows from (2. 12) that

$$P_x(\sigma_{R_i} < +\infty) = p(x, r_i).$$

Note that the above formula can be also derived from the strong Markov property.

REMARK 2. 6. If $u \in \mathfrak{H}_1^+(X) \cap \mathfrak{F}(X)$, we can see that $H_{\bigcup R_i}u$ is a function of $\mathfrak{H}^+(X)$ which does not exceed u. Hence u can be decomposed into

$$u = v + H_{\bigcup R_i}u, \tag{2. 13}$$

where v is a function of $\mathfrak{H}_1^+(X) \cap \mathfrak{F}(X)$ which vanishes over the recurrent part $\bigcup R_i$ of X.

PROOF. Consider any two states $x, y \in R_i$. Then, since $p(x, y)=p(y, x)=1$ (see Lemma 1. 5. (ii) and Theorem 1. 3. (i)), we get

$$u(x) \geqq H_y u.(x) = p(x, y) u(y)$$
$$= u(y) \geqq H_x u.(y) = p(y, x) u(x) = u(x).$$

Hence, for any fixed $r_i \in R_i$, we have

$$u = H_{r_i}u \quad \text{over} \quad R_i.$$

which proves (2.12) by virtue of Lemma 2.3. (ii) and (v). Moreover (i) is clear from the fact that $p(x, r_i)=0$ for $x\in R_j$ $(i\neq j)$.

Finally we shall prove (ii). For this purpose we shall first show that (2.11) holds for any finite sum, i.e.

$$H_{\bigcup_1^n R_i}u=\sum_{i=1}^n H_{R_i}u. \tag{2.14}$$

In fact, Lemma 2.3. (vi) implies that $H_{\bigcup_1^n R_i}u\leqq\sum_{i=1}^n H_{R_i}u$. On the other hand, using (2.9) and this theorem (i), it follows that

$$H_{\bigcup_1^n R_i}u\geqq\sum_{i=1}^n H_{R_i}u \qquad \text{over } \bigvee_1^n R_i.$$

Therefore we get

$$H_{\bigcup_1^n R_i}u=H_{\bigcup_1^n R_i}H_{\bigcup_1^n R_i}u\geqq\sum_{i=1}^n H_{\bigcup_1^n R_i}H_{R_i}u=\sum_{i=1}^n H_{R_i}u,$$

which proves (2.14). Next, if $\bigvee R_i$ is an infinite sum, applying (2.10) to $\bigvee_1^n R_i\uparrow\bigvee R_i$, we can obtain (2.11) immediately.

To continue, we shall now define a potential of a function in

DEFINITION 2.2. If a function $u\in\mathfrak{N}^+(X)$ can be written in the form

$$u=Gf \tag{2.15}$$

by means of some function $f\in\mathfrak{N}^+(X)$, u is the *potential of* f. The family of all the potentials is denoted by $\mathfrak{P}(X)$.

We shall state the main properties of potentials in the following

THEOREM 2.6. (i) *Any $u\in\mathfrak{P}(X)$ is a function of $\mathfrak{H}_1^+(X)$ and, in case $u\neq 0$, it is not x_t-harmonic.*

(ii) *If $u\in\mathfrak{P}(X)\frown\mathfrak{F}(X)$, the function f of the formula* (2.15) *is uniquely determined and given by $f=-\mathfrak{G}u$. Moreover the carrier of f, that is, the set $\{x\,;f(x)\neq 0\}$ is contained in N. Therefore u vanishes over $\bigvee R_i$.*

(iii) *If $u\in\mathfrak{P}(X)\frown\mathfrak{F}(X)$, we have*

$$\Pi^n u\downarrow 0 \qquad (n\to\infty). \tag{2.16}$$

PROOF. Let σ_n be the n-th jumping time and u, the potential of $f\in\mathfrak{N}^+(X)$;

(2. 16) $$u(x)=Gf.(x)=E_x\left[\int_0^{\sigma_\infty} f(x_t)\,dt\right],$$

by virture of $f(\infty)=0$. Then, using Lemma 1. 3, we get

(2. 17) $$\hat{\Pi}Gf.(x)=Gf.(x) \qquad \text{for } x=\text{a trap},$$
$$=E_x\left[\int_{\sigma_1}^{\sigma_\infty} f(x_t)\,dt\right]\leqq Gf.(x) \qquad \text{for } x\neq\text{a trap},$$

so that u belongs to $\mathfrak{H}_1^+(X)$.

Now suppose that $f(a)\neq 0$ for some state $a\in\bigcup R_i$. Then according to (1. 42), we have

$$Gf.(a)\geqq f(a)G(a,a)=+\infty.$$

Therefore if $u=Gf\in\mathfrak{F}(X)$, f vanishes over $\bigcup R_i$ and hence it follows from (1. 44) that

$$u(x)=\int_N f(y)G(x,dy)=\int_N f(y)p(x,y)G(y,dy).$$

This implies that u vanishes over $\bigcup R_i$, because we have $p(x,y)=0$ for any pair of $x\in\bigcup R_i$ and $y\in N$.

Next, if $u=Gf\in\mathfrak{F}(X)$ and $x\in N$, according to (2. 17) we have

(2. 18) $$\hat{\Pi}Gf.(x)=\Pi Gf.(x)=Gf.(x)-E_x\left[\int_0^{\sigma_1} f(x_t)\,dt\right]$$
$$=Gf.(x)-f(x)q(x),$$

which shows that $f=-\mathfrak{G}u$. Moreover if $u\not\equiv 0$, then $f(a)\neq 0$ for some $a\in N$, so that (2. 18) proves that u is strictly x_t-superharmonic at a.

Finally noting that $u=Gf\in\mathfrak{F}(X)$ vanishes over $\bigcup R_i$ which contains all the traps, we get

$$0\leqq\hat{\Pi}^n u.(x)=\Pi^n Gf.(x)=E_x\left[\int_{\sigma_n}^{\sigma_\infty} f(x_t)\,dt\right]\underset{(n\to\infty)}{\downarrow}0.$$

Thus our theorem was proved.

Next we shall establish the most important relation of the function $H_A u$ with the potential.

THEOREM 2.7. *If A is a finite subset of N and u is a function of $\mathfrak{H}_1^+(X)$ which is finite over A, then $H_A u$ is a potential of the function whose carrier is included in A. In general, if u is a function of $\mathfrak{F}(X)$, $H_A u$ can be represented as the difference of two potentials of the functions whose carriers are included in A.*

PROOF. Now suppose that it has been shown that

$$(2.19)\qquad H_A u.(x) = \int_A [-\mathfrak{G} H_A u.(y)] G(x, dy) \quad \text{for any } u \in \mathfrak{F}(X).$$

Then our second statement is easily verified by decomposing the function $-\mathfrak{G} H_A u$ into the positive and negative parts. Moreover if $u \in \mathfrak{H}_1^+(X)$, also $H_A u \in \mathfrak{H}_1^+(X)$ according to (2.7) and hence $-\mathfrak{G} H_A u = q^{-1}[H_A u - \Pi H_A u] \geqq 0$, which proves the first statement. Thus it is enough to show (2.19).

First notice that it follows from (1.44) that

$$(2.20)\qquad \int_A |f(y)| G(x, dy) \leqq \max_{y \in A} |f(y)| \sum_{y \in A} G(x, y) \leqq \max_{y \in A} |f(y)| \sum_{y \in A} G(y, y) < +\infty,$$

for any $f \in \mathfrak{F}(X)$. Second define the *n-th hitting time* τ_n *for* A as follows;

$$\begin{aligned} \tau_1(w) &= \sigma_A(w) & \tau_1'(w) &= \tau_1(w) + \sigma_1(w_{\tau_1}^+), \\ &\vdots & &\vdots \\ \tau_n(w) &= \tau_{n-1}'(w) + \tau_1(w_{\tau'_{n-1}}^+) & \tau_n'(w) &= \tau_n(w) + \sigma_1(w_{\tau_n}^+), \end{aligned}$$

where σ_1 is the first jumping time. Then recalling $q(x) = E_x(\sigma_1)$, we get

$$\begin{aligned} (2.21)\qquad \int_A f(y) G(x, dy) &= E_x\left[\int_0^{\sigma_\infty} \chi_A(x_t) f(x_t)\, dt\right] \\ &= \sum_{n=1}^{\infty} E_x\left[\int_{\tau_n}^{\tau'_n} f(x_t)\, dt\right] \\ &= \sum_{n=1}^{\infty} E_x[f(x_{\tau_n}) \sigma_1(w_{\tau_n}^+)] \\ &= \sum_{n=1}^{\infty} E_x[f(x_{\tau_n}) E_{x_{\tau_n}}(\sigma_1)] \\ &= \sum_{n=1}^{\infty} H^{\tau_n} q f.(x). \end{aligned}$$

Therefore noting that $H^{\tau_n} H_A = H^{\tau_n} H^{\tau_1} = H^{\tau_n}$ and $H^{\tau_n} \Pi H_A = H^{\tau_n} H^{\sigma_1} H^{\tau_1} = H^{\tau_{n+1}}$, we get the formula (2.19) as follows:

$$\begin{aligned} \int_A [-\mathfrak{G} H_A u.(y)] G(x, dy) &= \int_A \left[\frac{H_A u.(y)}{q(y)} - \frac{\Pi H_A u.(y)}{q(y)}\right] G(x, dy) \\ &= \sum_{n=1}^{\infty} [H^{\tau_n} H_A u.(x) - H^{\tau_n} \Pi H_A u.(x)] \\ &= \sum_{n=1}^{\infty} [H^{\tau_n} u.(x) - H^{\tau_{n+1}} u.(x)] \\ &= H^{\tau_1} u.(x) = H_A u.(x), \end{aligned}$$

because $H^{\tau_n}u.(x)\to 0$ according to (2.20) and (2.21).

The following theorem is the exact counterpart of the Riesz decomposition theorem on ordinary superharmonic functions (see Radó [13], p. 45).

THEOREM 2.8. *A function u of $\mathfrak{H}_1(X)\frown\mathfrak{F}(X)$ is decomposed by means of some $v\in\mathfrak{P}(X)$ and some $w\in\mathfrak{H}(X)$ into the form*

$$u = v + w\,, \tag{2.22}$$

if an only if there exists a function of $\mathfrak{H}(X)$ which does not exceed u. In such case, the decomposition (2.22) is unique and we have

$$v(x) = \int_N [-\mathfrak{G}u.(y)]G(x, dy)\,, \tag{2.23}$$

$$w(x) = \lim_{n\to\infty} \hat{\Pi}^n u.(x)\,. \tag{2.24}$$

Moreover $w(x)$ is the greatest among all the functions of $\mathfrak{H}(X)$ which does not exceed u.

PROOF. It is convenient to separate our proof into several steps.

1° The necessity of the first statement is evident, for, if $u=v+w$, then $u\geqq w\in\mathfrak{H}(X)$.

2° Suppose that $u\in\mathfrak{H}_1^+(X)\frown\mathfrak{F}(X)$ and let A be a finite subset of N. Then we have

$$H_A u.(x) = \int_A [-\mathfrak{G}H_A u.(y)]G(x, dy)\,.$$

Letting $A\uparrow N$ and using Fatou's Lemma, we get

$$u(x) \geqq H_N u.(x) \geqq \int_N [-\mathfrak{G}H_N u.(y)]G(x, dy)\,.$$

On the other hand we can see that

$$-\mathfrak{G}H_N u.(y) \geqq -\mathfrak{G}u.(y) \geqq 0 \qquad \text{for any } y\in N,$$

by virtue of $H_N u=u$ and $\Pi H_N u\leqq \Pi u$ over N. So

$$0\leqq v(x) \equiv \int_N [-\mathfrak{G}u.(y)]\,G(x, dy) \leqq u(x)\,,$$

which proves that $\mathfrak{G}v=\mathfrak{G}u$ (see Theorem 2.6. (ii)) and hence $w=u-v\in\mathfrak{H}^+(X)$.

3° Suppose that $u\in\mathfrak{H}_1(X)\frown\mathfrak{F}(X)$ and $u\geqq h\in\mathfrak{H}(X)$. Then, from 2°, we have

$$0 \leqq (u-h)(x) = \int_N [-\mathfrak{G}(u-h).(y)] G(x, dy) + w'(x)$$
$$= \int_N [-\mathfrak{G}u.(y)] G(x, dy) + w'(x) ,$$

whence

$$u(x) = \int_N [-\mathfrak{G}u.(y)] G(x, dy) + (h+w')(x) \tag{2.25}$$

which shows the sufficiency of the first statement.

4° Consider $u \in \mathfrak{H}_1(X)$, $u \geqq h \in \mathfrak{H}(X)$ and any decomposition (2.22) of u. Then, according to Theorem 2.6. (iii), we get

$$h = \hat{\Pi}^n h \leqq \hat{\Pi}^n u = \hat{\Pi}^n v + \hat{\Pi}^n w = \hat{\Pi}^n v + w \underset{(n\to\infty)}{\downarrow} w ,$$

which proves (2.24) and the last statement immediately. Further it follows that our decomposition is unique and hence the potential v of (2.22) has to coincide with the first term of (2.25). Thus we have proved our theorem.

Corollary (Criterion of Potentials) (i) *A function $u \in \mathfrak{H}_1^+(X)$ is a potential if and only if there exists no nontrivial function of $\mathfrak{H}^+(X)$ which does not exceed u.*

(ii) *Let f be a function of $\mathfrak{F}^+(X)$. Then the equation $-\mathfrak{G}u=f$ has at least one solution in $\mathfrak{F}^+(X)$ if and only if the potential Gf is a function of $\mathfrak{F}^+(X)$. In this case, Gf is the smallest one among any solution in $\mathfrak{F}^+(X)$.*

Now using the results obtained hitherto, we shall discuss the solutions of the equations ($\alpha > 0$)

$$(\alpha - \mathfrak{G})u = f \qquad \text{if } T \text{ is continuous,} \tag{2.26}$$

$$[(1-e^{-\alpha}) - e^{-\alpha}\mathfrak{G}]u = f \qquad \text{if } T \text{ is discrete.} \tag{2.27}$$

Theorem 2.9. (i) *Let f be a function of $\mathfrak{F}^+(X)$. Then* (2.26) (*or* (2.27)) *has at least one solution in $\mathfrak{F}^+(X)$ if and only if $G_\alpha f$ belongs to $\mathfrak{F}^+(X)$. In such case, $G_\alpha f$ is the smallest solution in $\mathfrak{F}^+(X)$.*

(ii) *If f is a bounded function of $\mathfrak{F}(X)$, $G_\alpha f$ is always a bounded solution of* (2.26) (*or* (2.27)) *belonging to $\mathfrak{F}(X)$,*

(iii) *In order that, for any function of $\mathfrak{F}(X)$,* (2.26) (*or* (2.27)) *has at most one bounded solution in $\mathfrak{F}(X)$, it is necessary and sufficient that the following condition is satisfied:*

$$P_x(\sigma_{+\infty} = +\infty) = 1 \qquad \textit{for any } x \in X ,^{7)} \tag{2.28}$$

7) Note that this condition is always satisfied for the discrete parameter case.

where $\sigma_{+\infty}$ is the limit of the n-th jumping time σ_n.

PROOF. In case of the continuous parameter, define the process $x_t^{(\alpha)}$ of order $\alpha>0$ attached to x_t as the CMP corresponding to the following system $\{q^{(\alpha)}, \Pi^{(\alpha)}\}$;

$$q^{(\alpha)}(x) = (\alpha+\lambda)^{-1}(x) \quad \text{for } x\in X, \qquad = +\infty \quad \text{for } x=\infty\,,$$
$$\begin{aligned} \Pi^{(\alpha)}(x, E) &= (\alpha+\lambda)^{-1}\lambda\Pi(x, E) && \text{for } E\subset X\,, \\ &= 1-(\alpha+\lambda)^{-1}\lambda\Pi(x, X) && \text{for } E=\infty\,, \end{aligned} \Bigg\} \text{ if } x\in X\,.$$
$$= \delta(\infty, E) \qquad\qquad \text{if } x=\infty\,.$$

where $\lambda(x)=q^{-1}(x)$. Then it follows that the Green kernels and generator of $x_t^{(\alpha)}$ are given by

$$G_\beta^{(\alpha)} = G_{\alpha+\beta} \qquad \text{for} \quad \beta\geqq 0\,, \quad \mathfrak{G}^{(\alpha)} = \mathfrak{G}-\alpha\,.$$

Now applying the corollary of the last theorem to the above process $x_t^{(\alpha)}$, our statement (i) is clear. Moreover the second statement comes from $G_\alpha(x, X)\leqq\dfrac{1}{\alpha}$ immediately.

Finally we proceed to the proof of (iii). First noting that

$$\begin{aligned} E_x(e^{-\alpha\sigma_{+\infty}}) &= \lim_{n\to\infty} E_x(e^{-\alpha\sigma_n}\,;\, \sigma_{n+1}<+\infty) \\ &= \lim_{n\to\infty}[(\alpha+\lambda)^{-1}\lambda\Pi]^n\chi_X.(x) \\ &= \lim_{n\to\infty}[\Pi^{(\alpha)}]^n\chi_X.(x)\,, \end{aligned}$$

it follows that the condition (2.28) is equivalent to

$$\lim_{n\to\infty}[\Pi^{(\alpha)}]^n\chi_X.(x) = 0\,. \tag{2.29}$$

Further applying Theorem 2.8, Lemma 2.2. (v) and Theorem 2.1 to $x_t^{(\alpha)}$, we see that (2.29) holds if and only if the equation

$$(\alpha-\mathfrak{G})u = -\mathfrak{G}^{(\alpha)}u = 0$$

has no nontrivial bounded solutions in $\mathfrak{F}(X)$, which completes the proof for the continuous case.

The similar proof is applicable to the discrete case.

6. Hunt's excessive functions and H^t-invariant functions. An *excessive function* u was defined by Hunt [8] as a function of $\mathfrak{N}^+(X)$ satisfying

$$H^t u\leqq u \qquad \text{for any} \quad t\in T\,. \tag{2.30}$$

In the following arguments we shall denote the family of all the excessive functions by $\mathfrak{E}(X)$.

First we have

LEMMA 2.4. (i) $\chi_X \in \mathfrak{E}(X)$.

(ii) *If* u *and* $v \in \mathfrak{E}(X)$, *then also* $u+v$ *and* $u \wedge v \in \mathfrak{E}(X)$.

(iii) *If* $u_n \in \mathfrak{E}(X)$ *and* $u_n \rightarrow u_{+\infty}$, $u_{+\infty} \in \mathfrak{E}(X)$.

(iv) *Any* $u \in \mathfrak{E}(X)$ *can be approximated from below by a sequence of bounded functions in* $\mathfrak{E}(X)$.

(v) *If* $u \in \mathfrak{E}(X)$, *then* $H_t u$ *increases with the decrease of* t *and* $H_t u \uparrow u (t \downarrow 0)$.

PROOF. (i)—(iv) are proved in the same way as in Lemma 2.2. (v) For any $t > s$, $H_{t-s}u \leqq u$ implies that $H_t u \leqq H_s u$, which is the first statement. The second statement is clear for a bounded function of $\mathfrak{E}(X)$. For a general function u of $\mathfrak{E}(X)$, considering a sequence of bounded functions u_n in $\mathfrak{E}(X)$ such that $u_n \uparrow u$, we have

$$\begin{aligned} 0 \leqq u - H^t u &= (u-u_n)+(u_n-H^t u_n)+H^t(u_n-u) \\ &\leqq (u-u_n)+(u_n-H^t u_n), \end{aligned}$$

which proves our lemma.

LEMMA 2.5. *Let* $\mathfrak{F}_1(X)$ *and* $\mathfrak{F}_2(X)$ *be the subfamilies of* $\mathfrak{R}(X)$ *each of which possesses the following properties*: (*a*) *If* $u_n \in \mathfrak{F}_i(X)$ *and* $u_n \rightarrow u$, *then* $u \in \mathfrak{F}_i(X)$. (*b*) *Any function* u *of* $\mathfrak{F}_i(X)$ *can be approximated by a sequence of bounded functions in* $\mathfrak{F}_i(X)$. *Then in order to prove* $\mathfrak{F}_1(X)=\mathfrak{F}_2(X)$ *it is enough to show that any bounded function of* $\mathfrak{F}_1(X)$ *belongs to* $\mathfrak{F}_2(X)$ *and vice versa.*

The proof is clear.

LEMMA 2.6. *A function* u *of* $\mathfrak{R}^+(X)$ *is excessive if and only if it satisfies for any* $\alpha > 0$

(2.31) $\qquad (1-e^{-\alpha})G_\alpha u \leqq u \qquad$ if T is discrete,

(2.32) $\qquad \alpha G_\alpha u \leqq u \qquad$ if T is continuous.

PROOF. Noting that the kernel $(1-e^{-\alpha})G_\alpha$ (or αG_α) is substochastic over X, it follows that the family of all the functions of $\mathfrak{R}^+(X)$ satisfying (2.31) (or (2.32)) possesses the properties (a) and (b) in the last lemma (see Remark 2.4). Consequently, it suffices to show that the condition (2.31) (or (2.32)) is equivalent to (2.30) for a bounded function of $\mathfrak{R}^+(X)$.

In case T is continuous, considering the Laplace transform of the both sides of (2.30), we have

$$G_\alpha u \leqq \frac{1}{\alpha} u \qquad \text{for any} \quad \alpha > 0, \tag{2.33}$$

which proves the necessity. Conversely, in order to derive the formula (2.30) from (2.32), it is enough to show that

$$(-1)^n G_\alpha^{[n]} u \leqq (-1)^n \left(\frac{1}{\alpha}\right)^{[n]} u, \text{[8]} \tag{2.34}$$

according to the well known theorem on Laplace transform. But it is clear from (2.33) that

$$G_\alpha^n u \leqq \frac{1}{\alpha^n} u,$$

so that, using the familiar formula

$$(-1)^n G_\alpha^{[n]} = n!\, G_\alpha^{n+1},$$

we get

$$(-1)^n G_\alpha^{[n]} u = n!\, G_\alpha^{n+1} u \leqq n!\, \frac{1}{\alpha^{n+1}} u = (-1)^n \left(\frac{1}{\alpha}\right)^{[n]} u,$$

which completes the proof for the continuous case.

The proof for the discrete parameter case is similar to that for the continuous one, so it will be omitted.

THEOREM 2.10. $\mathfrak{E}(X) = \mathfrak{H}_1^+(X)$.

PROOF. By Lemma 2.5, it suffices to prove that any bounded function of $\mathfrak{E}(X)$ belongs to $\mathfrak{H}_1^+(X)$ and vice versa. Now suppose that u is a bounded function of $\mathfrak{E}(X)$. Then it follows from the formulas (1.37) and (1.38) that $\mathfrak{G}u \leqq 0$, i.e. $u \in \mathfrak{H}_1^+(X)$.

Conversely suppose that u is a bounded function of $\mathfrak{H}_1^+(X)$. Then since $-\mathfrak{G}u \geqq 0$, we have

$$g \equiv (\alpha - \mathfrak{G}) u \geqq \alpha u \tag{2.35}$$

$$g' \equiv [(1-e^{-\alpha}) - e^{-\alpha}\mathfrak{G}] u \geqq (1-e^{-\alpha}) u. \tag{2.36}$$

Therefore if T is continuous, recalling Theorem 2.9. (i) and using (2.35), we get

$$u \geqq G_\alpha g \geqq \alpha G_\alpha u,$$

8) The upper suffix $[n]$ means the n-th derivative with respect to α.

which proves that $u \in \mathfrak{H}_1^+(X)$. In the same way, if T is descrete, (2.36) implies that (2.31) holds and so $u \in \mathfrak{H}_1^+(X)$.

COROLLARY. *If u is a function of $\mathfrak{H}_1^+(X)$, $H^t u$ is a function of $\mathfrak{H}_1^+(X)$ which does not exceed u.*

Finally we shall discuss H^t-invariant functions in connection with x_t-harmonic functions. Denoting the family of H^t-invariant functions in $\mathfrak{H}(X)$ by $\mathfrak{J}(X)$, we have

THEOREM 2.11. (i) $\mathfrak{J}^+(X) \subset \mathfrak{H}^+(X)$.

(ii) *In order that any bounded function of $\mathfrak{H}(X)$ belonge to $\mathfrak{J}(X)$, it is necessary and sufficient that the condition* (2.28) *holds.*

(iii) *A sufficient condition that any function of $\mathfrak{H}^+(X)$ belongs to $\mathfrak{J}^+(X)$, namely, that $\mathfrak{H}^+(X) = \mathfrak{J}^+(X)$, is that the following condition holds:*

$$q(x) \geqq k > 0 \qquad \text{for any} \quad x \in X. \tag{2.37}$$

In particular, the above condition is always satisfied for the discrete parameter case.

PROOF. We shall discuss only the continuous parameter case. The similar proof holds for the discrete case.

(i) Any function u of $\mathfrak{J}^+(X)$ satisfies

$$u = \alpha G_\alpha u. \tag{2.38}$$

Therefore according to Theorem 2.9. (i), we have

$$(\alpha - \mathfrak{G})u = \alpha u, \tag{2.39}$$

which shows $u \in \mathfrak{H}^+(X)$.

(ii) Suppose that (2.28) holds and u is a bounded function of $\mathfrak{H}(X)$. Then since u satisfies (2.39), it follows from Theorem 2.9. (ii) and (iii) that (2.38) holds, namely, that $u \in \mathfrak{J}(X)$. Conversely suppose that (2.28) does not hold. Then we have

$$P_a(\sigma_{+\infty} < +\infty) > 0 \qquad \text{for some state} \quad a \in X.$$

We shall now prove that the function $P_x(\sigma_{+\infty} < +\infty)$ is x_t-harmonic but not H^t-invariant. In fact the former assertion is shown by

$$\begin{aligned} E_x[P_{x_{\sigma_1}}(\sigma_{+\infty} < +\infty)] &= P_x\{w\,;\, \sigma_{+\infty}(w_{\sigma_1}^+) < +\infty\} \\ &= P_x(\sigma_{+\infty} < +\infty). \end{aligned}$$

On the other hand, noting that $P_a(\sigma_{+\infty} < t) > 0$ for some $t < +\infty$, we get

$$E_a[P_{x_t}(\sigma_{+\infty} < +\infty)] = P_a\{w\,;\,\sigma_{+\infty}(w_t^+) < +\infty\}$$
$$= P_a\{w\,;\,\sigma_{+\infty}(w_t^+) < +\infty,\ \sigma_{+\infty}(w) \geqq t\}$$
$$= P_a(t \leqq \sigma_{+\infty} < +\infty) < P_a(\sigma_{+\infty} < +\infty)\,,$$

which proves that $P_x(\sigma_{+\infty} < +\infty)$ is not H^+-invariant at a.

(iii) First note that the condition (2.37) implies (2.28) or equivalenty (2.29). Second for a given function u of $\mathfrak{H}^+(X)$, consider a sequence of bounded functions u_n in $\mathfrak{H}_1^+(X)$ such that $u_n \uparrow u$. Putting

$$(2.40) \qquad -\mathfrak{G}u_n = v_n \geqq 0\,,$$

we get

$$0 \leqq v_n \to 0 \quad (n \to \infty)\,, \qquad v_n \leqq q^{-1}u_n \leqq k^{-1}u_n \leqq k^{-1}u\,.$$

Adding αu_n to the both sides of (2.40) and recalling Theorem 2.9. (ii) and (iii), we have

$$u_n = \alpha G_\alpha u_n + G_\alpha v_n\,.$$

But according to Theorem 2.10, u is excessive and hence it follows from (2.32) that

$$0 \leqq G_\alpha u.(x) < +\infty \qquad \text{for any } x \text{ in } X.$$

Consequently using the theorem of Lebesgue, we get

$$u = \lim_{n\to\infty} u_n = \lim_{n\to\infty}[\alpha G_\alpha u_n + G_\alpha v_n] = \alpha G_\alpha u + G_\alpha[\lim_{n\to\infty} v_n] = \alpha G_\alpha u\,,$$

which means that $u \in \mathfrak{J}^+(X)$.

THEOREM 2.12. *Suppose that the condition* (2.37) *holds. Then a function u can be expressed as the defference of two functions in* $\mathfrak{H}^+(X)$ *if and only if it satisfies the conditions*: (α) $u \in \mathfrak{H}(X)$, (β) $u \in \mathfrak{J}(X)$ *and* (γ) $H^t|u|.(x)$ *is bounded in t for each x.*

PROOF. For definiteness we consider the continuous parameter case. Now if $u = u_1 - u_2$ and $u_i \in \mathfrak{H}^+(X) \equiv \mathfrak{J}^+(X)$ then (α) is evident. Moreover $H^t|u| \leqq u_1 + u_2$ and hence (γ) is satisfied. Recalling the formula (1.29) and noting that $(\alpha+\lambda)^{-1}\lambda\Pi = (\alpha+\lambda)^{-1}\lambda\hat{\Pi}$, we have from ($\alpha$)

$$G_\alpha^{(n)}u = [(\alpha+\lambda)^{-1}\lambda\Pi]^n(\alpha+\lambda)^{-1}u$$
$$= \frac{1}{\alpha}[(\alpha+\lambda)^{-1}\lambda\Pi]^{n-1}[(\alpha+\lambda)^{-1}\lambda\Pi]\{u - (\alpha+\lambda)^{-1}\lambda u\}$$
$$= \frac{1}{\alpha}[(\alpha+\lambda)^{-1}\lambda\Pi]^{n-1}(\alpha+\lambda)^{-1}\lambda u - \frac{1}{\alpha}[(\alpha+\lambda)^{-1}\lambda\Pi]^n(\alpha+\lambda)^{-1}\lambda u\,.$$

Consequently it follows that

$$\alpha \sum_{k=0}^{n} G_{\alpha}^{(k)} u = u - [(\alpha+\lambda)^{-1}\lambda\Pi]^{n}(\alpha+\lambda)^{-1}\lambda u . \tag{2.41}$$

But since $\hat{\Pi}^{n}|u|$ is bounded in n from Theorem 2.1 and $(\alpha+\lambda)^{-1}\lambda \leqq \left(\alpha+\frac{1}{k}\right)^{-1}\cdot\frac{1}{k}$ from (2.37), we get

$$|[(\alpha+\lambda)^{-1}\lambda\Pi]^{n}(\alpha+\lambda)^{-1}\lambda u| \leqq \left[\left(\alpha+\frac{1}{k}\right)^{-1}\cdot\frac{1}{k}\right]^{n+1}\hat{\Pi}^{n}|u| \to 0 , \tag{2.42}$$

which shows that $\alpha G_{\alpha} u = u$. Thus u satisfies (β). Conversely assume that u satisfies (α), (β) and (γ). By Remark 2.4, $H^{t}|u|$ increases with t and $v \equiv \lim H^{t}|u| \in \mathfrak{I}^{+}(X)$. Therefore $|u| \leqq v \in \mathfrak{H}^{+}(X)$, which implies that $\hat{\Pi}^{n}|u|$ is bounded in n, namely that $u = u_1 - u_2$, $u_i \in \mathfrak{H}^{+}(X)$ (see Theorem 2.1).

REMARK 2.7. It is hoped that the condition (β) in the above theorem is derived from (α) and (γ). If T is discrete, our statement is true, for the formula (1.38) proves that $\mathfrak{I}(X) = \mathfrak{H}(X)$, i.e. that (β) is equivalent to (α). On the other hand it is not sure whether our statement always holds in the continuous parameter case. A sufficient condition is that $q(x)$ is a constant (say k). In fact, with this condition, the left side of (2.42) is equal to $\left[\left(\alpha+\frac{1}{k}\right)^{-1}\cdot\frac{1}{k}\right]^{n+1}|\Pi^{n}u|$, which converges to 0 for any $u \in \mathfrak{H}(X)$. Moreover the condition (γ) guarantees that $\alpha \sum_{k=0}^{n} G_{\alpha}^{(k)} u \to \alpha G_{\alpha} u$. Therefore it follows from (2.41) that $u = \alpha G_{\alpha} u$, which proves (β).

CHAPTER 3. THE MARTIN SPACE AND BOUNDARY

7. The function family M_c. The definition of the Martin space and boundary. In the sequal we shall only consider the CMP satisfying the following condition:

(CMP. 4) There exists at least one state c such that $p(c, y) > 0$ for any state y in X. Such state is called a *center* of the CMP.

Now define $K(c, x, y)$ by

$$K(c, x, y) = \lim_{\alpha\to 0} \frac{G_{\alpha}(x, y)}{G_{\alpha}(c, y)} , \tag{3.1}$$

where the right side is understood as the density of the measure $G_{\alpha}(x, \cdot)$ relative to $G_{\alpha}(c, \cdot)$ and therefore $K(c, x, y)$, as a point function of y. Since $K(c, x, y) = p(x, y)/p(c, y)$ from the formula

(1. 43), it is well defined and finite. Moreover (1. 44) shows that $K(c, x, y)=G(x, y)/G(c, y)$ for $y \in N$. We shall list some properties of $K(c, x, y)$ in

LEMMA 3. 1. (i) *$K(c, x, y)$ is x_t-superharmonic as a function of x.*

(ii) *It is bounded in y for each fixed x. In fact,*

$$K(c, x, y) \leqq \frac{1}{p(c, x)}. \tag{3. 2}$$

(iii) *If $K(c, x, y)=K(c, x, y')$ for any $x \in X$, then $y=y'$ or both y and y' are in a same indecomposable recurrent set.*

PROOF. (i), (ii) are evident from Theorem 2. 4 and the formula (1. 14). For (iii), assume that $K(c, x, y)=K(c, x, y')$ and $y \neq y'$. Then we have $p(x, y)=kp(x, y')$ for some positive constant k, which implies that $p(x, y)$ is x_t-harmonic and $p(y, y')>0$. Therefore, by virtue of Theorem 2. 4 and Theorem 1. 3, y and y' belong to a common indecomposable recurrent set.

To continue, consider the family of functions of x, $X_c=\{K(c, \cdot, y) ; y \in X\}$. From the above lemma it follows that X_c is a normal family and that to a function of X_c corresponds a non-recurrent state or else an indecomposable recurrent set. Denote the family of all the limit functions of X_c, by M_c and a function of M_c, by $\xi(\cdot)$. Of course, X_c is a subset of M_c. Lemma 2. 2. (iii) proves that any function of M_c belongs to $\mathfrak{H}_1^+(X) \cap \mathfrak{F}(X)$. We now topologize M_c by the metric

$$\rho(\xi, \eta) = \int_X \frac{|\xi(x)-\eta(x)|}{1+|\xi(x)-\eta(x)|} m(dx), \tag{3. 3}$$

where m is a totally finite measure which is positive on any state over X. It follows that ρ-convergence is equivalent to pointwise convergence and therefore the topology of M_c is independent of the choice of m. It is also clear that the natural mapping $\xi(\cdot)=K(c, \cdot, y)$ from $y \in X$ into $\xi \in M_c$ is continuous. Moreover we have

THEOREM 3. 1. (i) *M_c is a compact metric space and is the completion of X_c with respect to ρ-metric.*

(ii) *M_c is homeomorphic to $M_{c'}$ which is derived from another center c'.*

PROOF. (i) Noting that ρ-convergence is replaced by pointwise convergence and that X_c is a normal family, the proof is straightforward and will be omitted. (ii) First note that $\xi(c')\geqq p(c', c)>0$ for any $\xi\in X_c$ and hence for any $\xi\in M_c$. We prove that the mapping $\xi'(\cdot)=\xi(\cdot)/\xi(c')$ is the homeomorphism from M_c to $M_{c'}$. In fact, if $\xi(\cdot)=\lim_{n\to\infty} K(c, \cdot, y_n)$, then $\lim_{n\to\infty} K(c', \cdot, y_n)$ exists and equals to $\xi'(\cdot)$, which shows $\xi'(\cdot)\in M_{c'}$. Moreover it follows from $\xi(c)=1$ for any $\xi\in M_c$ that $\xi(\cdot)/\xi(c')=\eta(\cdot)/\eta(c')$ implies $\xi(\cdot)=\eta(\cdot)$, so that the mapping is one-to-one. Finally the continuity of both the above mapping and the inverse one, $\xi(\cdot)=\xi'(\cdot)/\xi'(c)$, is evident.

DEFINITION 3.1. Any space M which is homeomorphic to M_c is the *Martin space induced by the CMP*. We denote, by θ, the natural continuous mapping from X into $M(X\to M_c\to M)$ and by $\hat{A}$, the image $\theta(A)$ of a subset A of X. $\partial\hat{X}\equiv M-\hat{X}$ is the *Martin boundary.*

An element of M is denoted by ξ and the function of M_c corresponding to ξ, by $\hat{K}(c, \cdot, \xi)$. The topology of M is equivalent to that by the metric

$$\rho(\xi, \eta) = \int_X \frac{|\hat{K}(c, x, \xi)-\hat{K}(c, x, \eta)|}{1+|\hat{K}(c, x, \xi)-\hat{K}(c, x, \eta)|} m(dx), \tag{3.4}$$

so that $\hat{K}(c, x, \xi)$ is continuous in ξ for each x.

In the construction of the Martin space, it is convenient to introduce the following terminology.

DEFINITION 3.2. If $K(c, \cdot, y_n)$ converges to a function of M_c-X_c, the sequence $\{y_n\}$ is a *fundamental sequence.* Two fundamental sequences $\{y_n\}$, $\{z_n\}$ are *equivalent* if they determine the same limit function.

Now choose a state r_i from each R_i. It is the most natural to choose M as the union of N, every r_i's and every equivalent classes of fundamental sequences. Hereafter such M will be referred to as the *canonical Martin space.*

If the process x_t satisfies

(CMP. 5) $\quad X=N$,

θ is the one-to-one mapping but not necessarily the homeomorphism onto $\hat{X}$. For this we require a new condition:

(CMP. 6) For each fixed x, there exists a finite subset F_x such that $\Pi(x, X-F_x)=0$.

THEOREM 3.2. *Suppose that the conditions* (CMP. 5) *and* (CMP.

6) *are satisfied. Then we have*

(i) *X and $\hat{X}$ are homeomorphic with θ. Therefore the canonical Martin space is a compactification of X.*

(ii) *$\hat{X}$ is open in M.*

(iii) *For any $\xi \in \partial\hat{X}$, $\hat{K}(c, \cdot, \xi)$ is x_t-harmonic.*

PROOF. With no loss of generality, we can assume that M is the canonical Martin space. In general, any fundamental sequence has no limit point in X. Therefore if $\xi \in \partial\hat{X}$ and $\{y_n\}$ is a fundamental sequence determining ξ, there exists an integer $n(x)$ such that

$$(3.5) \qquad K(c, x, y_n) = [\hat{\Pi} K(c, \cdot, y_n)].(x) = \sum_{z \in F_x} K(c, z, y_n) \hat{\Pi}(x, z)$$

for any $n \geqq n(x)$. Noting that the right side of the above formula is a finite sum and letting $n \to \infty$, it follows that $\hat{K}(c, x, \xi)$ is x_t-harmonic, which proves (iii). Moreover (CMP. 6) and (iii) imply that if $\xi_n \in \partial\hat{X}$ and $\rho(\xi_n, \xi) \to 0$, $\hat{K}(c, \cdot, \xi)$ is x_t-harmonic and hence, according to (CMP. 5), $\xi \in \partial\hat{X}$. Consequently $\partial\hat{X}$ is closed, so that (ii) was proved.

For (i), we shall first show that any sequence having no limit points in X, say $\{y_n\}$, contains at least one funamental sequence. Since X_c is a normal family, some infinite subsequence of $\{K(c, \cdot, y_n)\}$ converges to a function of M_c, Using the same argument as above, it follows that such limit function is x_t-harmonic and therefore is a function of $M_c - X_c$. Now suppose that ρ-topology in X is not discrete, i.e. that X contains some state y for which there exists an infinite sequence $\{y_n\}$ of different states such that $\rho(y_n, y) \to 0$. From the above result, we can assume that $\{y_n\}$ is a fundamental sequence. This is a contradiction. Thus our theorem was proved.

8. The (generalized) réduite. Usually the réduite of a non-negative superharmonic function is defined only for a boundary subset (see Martin [**12**], Doob [**4**]). But we shall here adopt a little wider definition.

Let D be a closed subset of the Martin space M and $\mathfrak{U}(D)$, the family of open sets in M containing D. Then according to Theorem 3.1. (i), any element G of $\mathfrak{U}(D)$ intersects with $\hat{X}$. Therefore $[G] \equiv \theta^{-1}(G \cap \hat{X})$ is a nonnull subset of X.

DEFINITION 3. 3. For a nonnegative x_t-superharmonic function u and a closed set D in M, the *réduite of u to the set D at x*, denoted by $u_D(x)$, is defined by

(3. 6) $\qquad \inf H_{[G]}u.(x) \qquad$ for every G in $\mathfrak{U}(D)$.

First we shall study some properties of a réduite as a point function of x.

LEMMA 3. 2. *Let $\{G_n\}$ be any sequence in $\mathfrak{U}(D)$ such that $G_n \downarrow$ and $\bigcap_n \bar{G}_n = D$. Then we have*

(3. 7) $$H_{[G_n]}u \downarrow u_D .$$

PROOF. By virtue of (2.9), $H_{[G_n]}u$ decreases with n and hence converges to a function, say v. From the definition, $v \geqq u_D$. To prove $v \leqq u_D$, take any state x and an arbitrary small $\varepsilon > 0$. Then again from the definition, there exists a set G of $\mathfrak{U}(D)$ such that $u_D(x) \geqq H_{[G]}u.(x) - \varepsilon$, where the choice of G depends on u and x. But since $G_{n_0} \subset G$ for some n_0, we get

$$u_D(x) \geqq H_{[G]}u.(x) - \varepsilon \geqq H_{[G_{n_0}]}u.(x) - \varepsilon \geqq v(x) - \varepsilon ,$$

which proves $v \leqq u_D$.

THEOREM 3. 3. (i) *u_D is an x_t-superharmonic function which does not exceed u. Especially, $u_M = u$.*

(ii) *For any G of $\mathfrak{U}(D)$.*

(3. 8) $$u_D = H_{[G]}u_D = [H_{[G]}u]_D .$$

(iii) *If $D \subset \partial\hat{X} + \cup \hat{R}_i$ and $u \in \mathfrak{H}_1^+(X) \cap \mathfrak{F}(X)$, then $u_D \in \mathfrak{H}^+(X)$.*

PROOF. The first two statements are clear from the above lemma and the results in the preceding chapter. For the last statement, take the sequence $\{G_n\}$ of the above lemma. Then the set $\bigcap_n [G_n]$ is disjoint with N. Therefore if $x \in N$, $H_{[G_n]}u$ is x_t-harmonic at x for some $n_0 \leqq$ any n, so that u_D is also x_t-harmonic at x according to $H_{[G_n]}u \downarrow$ and Lebesgue's theorem. Moreover Theorem 2. 5 shows that u_D is x_t-harmonic over $\bigcup R_i$. Thus we have proved our theorem.

THEOREM 3. 4. (i) *If $u \geqq v$, then $u_D \geqq v_D$,*

(ii) $(k_1 u + k_2 v)_D = k_1 u_D + k_2 v_D \qquad$ *for* $k_1, k_2 \geqq 0$.

(iii) *If $u_n \uparrow u$, then it holds that $\lim_{n\to\infty} (u_n)_D = u_D$.*

(iv) *If $D_1 \supset D_2$, then $(u_{D_1})_{D_2} = (u_{D_2})_{D_1} = u_{D_2}$ and hence $u_{D_1} \geqq u_{D_2}$.*

(v) $u_{D_1\cup D_2}+u_{D_1\cap D_2}\leqq u_{D_1}+u_{D_2}$.

(vi) *If* $D_n\downarrow D$, *then* $u_{D_n}\downarrow u_D$.

REMARK. If both D_1 and D_2 are the subsets of $\partial\hat{X}+\bigvee\hat{R}_i$, the assertions (iv) and (v) are strengthened as follows:

(iv)′ $(u_{D_1})_{D_2}=u_{D_1\cap D_2}$,

(v)′ $u_{D_1\cap D_2}+u_{D_1\cup D_2}=u_{D_1}+u_{D_2}$.

These formulas are derived, in a more general form, from the main theorem in the next chapter (see Section 11), though it may be possible to prove them directly.

PROOF. The first four statements are immediate, summing up Lemma 2.3, Theorem 2.2 and Lemma 3.2. To prove (vi), choose the sequence $\{G_n\}$ such that $D_n\subset G_n$ and $\bigwedge_n \bar{G}_n=D$. Then from (iv), we get

$$u_D\leqq u_{D_n}\leqq H_{[G_n]}u\rightarrow u_D\,.$$

Finally we shall prove (v). For this, according to Lemma 3.2, it is enough to show that $H_Au.(x)$ is alternating of order 2[9] as a set function defined over the class of all the subsets of X in the terminology of Choquet [1]. By his paper (Section 14.3), this is equivalent to the statement below: $H_Au.(x)$ increases with A and satisfies

$$H_{A_1\cup A_2}u.(x)+H_{A_1\cap A_2}u.(x)\leqq H_{A_1}u.(x)+H_{A_2}u.(x)\,. \tag{3.9}$$

The first part is nothing but (2.9). Moreover a simple calculation shows that

$$\begin{aligned}&H_{A_1}u+H_{A_2}u-H_{A_1\cup A_2}u-H_{A_1\cap A_2}u\\&\quad=H_{A_1-A_2}(H_{A_2}u-H_{A_1\cap A_2}u)+H_{A_2-A_1}(H_{A_1}u-H_{A_1\cap A_2}u)\,.\end{aligned}$$

By virtue of $H_{A_i}u\geqq H_{A_1\cap A_2}u$, the right side is nonnegative, so that (3.9) was proved.

The last three statements of the above theorem show that, for any fixed x in X, $u_D(x)$ is a capacity which is alternating of order 2 as a set function defined over the class of all the compact subsets of M. Now denote the capacity of any capacitable set C by $u_C(x)$. Then, from Choquet's capacity theorem, we have the statements below:

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9) It is well known that $H_A\chi_X.(x)$ is alternating of order ∞ (see Hunt [8], Doob [4]). In the same way, by induction, we shall be able to prove that $H_Au.(x)$, for any $u\in\mathfrak{H}_1^+(X)$, is alternating of order ∞.

(R. 1)[10] Any Borel subset C of M is capacitable and for any $\varepsilon>0$, there exist an open G and a compact D such that $G\supset C\supset D$ and

$$u_G(x)-\varepsilon \leqq u_C(x) \leqq u_D(x)+\varepsilon . \quad (3.10)$$ [11]

(R. 2) For any Borel subsets C_1, C_2 of M,

$$u_{C_1\cup C_2}+u_{C_1\cap C_2} \leqq u_{C_1}+u_{C_2}, \quad (3.11)$$

which implies the subadditivity of u_C.

Moreover it follows from (2.10) that

(R. 3) $u_G=H_{[G]}u$ for any open set G of M.

THEOREM 3.5. *Let C be any Borel subset of $\partial\hat{X}+\bigcup\hat{R}_i$ and u, and function of $\mathfrak{H}_1^+(X)\cap\mathfrak{F}(X)$. Then there exist a decreasing sequence $\{G_n\}$ of open sets and an increasing sequence $\{D_n\}$ of compact sets such that $G_n\supset C\supset D_n$ and*

$$u_{G_n}(x)\downarrow u_C(x), \quad u_{D_n}(x)\uparrow u_C(x) \qquad \textit{for every } x \textit{ in } X. \quad (3.12)$$

Therefore u_C is an x_t-harmonic function. Moreover if $C\supset C'$,

$$(u_C)_{C'} = (u_{C'})_C = u_{C'} . \quad (3.13)$$

PROOF. It is clear from (R. 1) that there exist the sequences $\{G_n\}$, $\{D_n\}$ which satisfy (3.12) at the center c. Now consider the function $v_n=u_{G_n}-u_{D_n}$. Obviously, v_n is nonnegative and decreasing. Recalling Theorem 3.3. (iii), we have $v_n\in\mathfrak{H}_1^+(X)\cap\mathfrak{F}(X)$, so that the limit function v of v_n is a function of $\mathfrak{H}_1^+(X)$ which takes the value 0 at c. Therefore $v\equiv 0$ by Corollary 1 of Theorem 2.2. On the other hand, we have

$$v_n(x) = |u_{G_n}(x)-u_C(x)| + |u_C(x)-u_{D_n}(x)| \qquad \text{for every } x,$$

which proves (3.12). The formula (3.12) implies that u_C is x_t-harmonic and hence the iteration '$u_C\to(u_C)_{C'}$' is possible as well as '$u_{C'}\to(u_{C'})_C$'. For the last statement, let $\{G_n\}$ be a sequence of open sets in (3.12) corresponding to both u_C and $(u_{C'})_C$, and $\{D_n\}$, a sequence of compact sets to both $u_{C'}$ and $(u_C)_{C'}$. Then it follows from (3.8) that $(u_{G_n})_{D_k}=(u_{D_k})_{G_n}=u_{D_k}$ for any k, n. First letting $n\to\infty$ for a fixed k and next $k\to\infty$, we get $(u_C)_{C'}=u_{C'}$. In the same way, letting $k\to\infty$ for a fixed n and next $n\to\infty$, we have $(u_{C'})_C=u_{C'}$.

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10) In the sequel, all the arguments on u_C hold also for any analytic set of M, though it is not necessary for our purpose.

11) Of course, the choice of G and D depends on u and x.

CHAPTER 4. THE REPRESENTATION THEOREMS FOR x_t-SUPERHARMONIC FUNCTIONS

9. Representation of a réduite and its consequences. In this chapter we shall establish the representation theory for a function of $\mathfrak{H}_1^+(X)\frown\mathfrak{F}(X)$ or equivalently, a nonnegative x_t-superharmonic function being finite at the center c (from Corollary 1 of Theorem 2.2). Our main theorem is stated in Section 11. In this section we shall derive some auxiliary representation theorems.

LEMMA 4.1. *If u is a function of $\mathfrak{H}_1^+(X)\frown\mathfrak{F}(X)$ and A is a finite subset of X, then there exists some positive measure μ such that*

$$(4.1)\qquad H_A u.(x) = \int_A K(c, x, y)\mu(dy) \qquad \textit{for every } x \textit{ in } X,$$

so that

$$(4.2)\qquad \mu(A) = H_A u.(c) \leqq u(c)\,.$$

PROOF. Consider the function $v = H_{A\frown(\smile R_i)}u$. For simplicity, assume that $R_1, R_2, \cdots, R_n$ intersect with A. Take out any r_i from each $A\frown R_i$, $i=1, 2, \cdots, n$. Then, in the same way as in Theorem 2.5, we get

$$v \leqq H_{\overset{n}{\underset{1}{\smile}} R_i} u = \sum_1^n H_{R_i} u = \sum_1^n H_{r_i} u = H_{\overset{n}{\underset{1}{\smile}} r_i} u \leqq v.$$

Therefore it follows from (2.12) that

$$v(x) = \sum_1^n k_i p(x, r_i) = \sum_1^n K(c, x, r_i)\cdot k_i p(c, r_i)\,,$$

so that v has the representation of the form (4.1). Next consider the function $w = H_A u - v$. Then we have

$$H_A w = H_A H_A u - H_A H_{A\frown(\smile R_i)} u = H_A u - H_{A\frown(\smile R_i)} u = w\,,$$

$$H_{A\frown(\smile R_i)} w = H_{A\frown(\smile R_i)} H_A u - H_{A\frown(\smile R_i)} H_{A\frown(\smile R_i)} u = 0\,,$$

so that

$$H_{A\frown N} w \leqq w = H_A w \leqq H_{A\frown(\smile R_i)} w + H_{A\frown N} w = H_{A\frown N} w\,.$$

Consequently, according to Theorem 2.7, we get

$$w(x) = H_{A\frown N} w(x) = \int_{A\frown N} f(y) G(x, dy) \qquad (f \geqq 0)$$

$$= \int_{A\frown N} K(c, x, y) f(y) G(c, dy)\,,$$

in which we have used that $K(c, x, y)=G(x, dy)/G(c, dy)$ for $y\in N$. Thus w is also expressible in the from (4.1). This proves our Lemma.

THEOREM 4.1. *Let G and D be, respectively, an open subset and a closed one of M. Then for any $u\in\mathfrak{H}_1^+(X)\frown\mathfrak{F}(X)$, we have*

$$H_{[G]}u = u_G = \int_{\bar{G}} \hat{K}(c, \cdot, \xi)\mu(d\xi), \tag{4.3}$$

$$u_D = \int_D \hat{K}(c, \cdot, \xi)\mu(d\xi), \tag{4.4}$$

where μ is a positive (Radon) measure over $\bar{G}$ or D.

PROOF. Consider a sequence $\{A_n\}$ of finite subsets in X which approximates $[G]$ from below. From the above lemma and the continuity of the mapping θ, we obtain

$$\begin{aligned} H_{A_n}u &= \int_{A_n} K(c, \cdot, y)\nu_n(dy) \\ &= \int_{\hat{A}_n} \hat{K}(c, \cdot, \xi)\mu_n(d\xi), \end{aligned}$$

where $\hat{A}_n=\theta(A_n)$, $\mu_n(d\xi)=\nu_n[\theta^{-1}(d\xi)]$. Putting $\mu_n(\bar{G}-\hat{A}_n)=0$, $\{\mu_n\}$ is a sequence of positive measures over $\bar{G}$ satisfying $\mu_n(\bar{G})\leqq u(c)$. Therefore some subsequence $\{\mu_{n(k)}\}$ converges weakly to some measure μ over $\bar{G}$. Noting that $\hat{K}(c, x, \cdot)$ is ρ-continuous for any x, we have

$$H_{A_{n(k)}}u.(x) = \int_{\bar{G}} \hat{K}(c, x, \xi)\mu_{n(k)}(d\xi) \to \int_{\bar{G}} \hat{K}(c, x, \xi)\mu(d\xi)$$

for every x in X. On the other hand, $H_{A_n}u\to H_{[G]}u$ (see Theorem 2.3). Thus (4.3) was proved.

Next consider a sequence $\{G_n\}$ of open sets in M which satisfies the conditions in Lemma 3.2. Then any weak limit μ of the sequence $\{\mu_n\}$ each of which corresponds to u_{G_n} in (4.3) is a measure over D. Therefore (4.4) can be easily derived, analogously to the arguments for (4.3).

THEOREM 4.2. *If C is a Borel subset of $\partial\hat{X}+\bigcup\hat{R}_i$, u_C can be expressed in the form*

$$u_C = \int_C \hat{K}(c, \cdot, \xi)\mu(d\xi) \tag{4.5}$$

by means of some positive measure μ over C.

PROOF. Choose an increasing sequence $\{D_n\}$ satisfying (3.12). Then putting $v_n=u_{D_1}$ for $n=1$ and $=u_{D_n}-u_{D_{n-1}}$ for $n\geqq 2$, it follows that $u_C=\sum_{n=1}^{\infty} v_n$. For v_1, we have

$$v_1=\int_{D_1}\hat{K}(c,\cdot,\xi)\mu_1(d\xi).$$

For $n\geqq 2$, $v_n\in\mathfrak{H}^+(X)$ and $(v_n)_{D_n}=(u_{D_n})_{D_n}-(u_{D_{n-1}})_{D_n}=u_{D_n}-u_{D_{n-1}}=v_n$. Therefore v_n is expressible in the form

$$v_n=\int_{D_n}\hat{K}(c,\cdot,\xi)\mu_n(d\xi).$$

Extending each μ_n to the set C by $\mu_n(C-D_n)=0$ and putting $\mu=\sum_{n=1}^{\infty}\mu_n$, we get

$$\mu(C)=\sum_{n=1}^{\infty}\mu_n(C)=\sum_{n=1}^{\infty}v_n(c)=u_C(c),$$

$$u_C=\sum_{n=1}^{\infty}\int_C\hat{K}(c,\cdot,\xi)\mu_n(d\xi)=\int_C\hat{K}(c,\cdot,\xi)\mu(d\xi),$$

so that (4.5) was proved.

To continue, we shall prove that if u is nonnegative and x_t-harmonic, then $u=u_{\partial\hat{X}+\cup\hat{R}_i}$. This fact will play an essential role for the proof of the main theorem in Section 11.

THEOREM 4.3. *If $u\in\mathfrak{H}_1^+(X)\frown\mathfrak{F}(X)$, $u_{\partial\hat{X}+\cup\hat{R}_i}$ is the greatest x_t-harmonic function which does not exceed u, namely, it is nothing but w in the formula* (2.22). *Therefore u is a potential if and only if $u_{\partial\hat{X}+\cup\hat{R}_i}=0$.*

To prove this theorem, we shall prepare two lemmas.

LEMMA 4.2. *If a function u is expressible in the form*

$$u=\int_{\hat{N}}\hat{K}(c,\cdot,\xi)\mu(d\xi) \tag{4.6}$$

by some positive measure μ, u is a potential.

PROOF. Since the restriction θ_N of θ to N is continuous and one-to-one, $\theta_N^{-1}(\hat{N}\to N)$ is measurable. Therefore, noting that $K(c,x,y)=G(x,y)/G(c,y)$ for $y\in N$, we have

$$u=\int_{\hat{N}}\hat{K}(c,\cdot,\xi)\mu(d\xi)=\int_N K(c,\cdot,y)\nu(dy)$$
$$=\sum_{y\in N}\frac{\nu(y)}{G(c,y)}G(\cdot,y)=\int_N f(y)G(\cdot,dy),$$

where $\nu(dy)=\mu(\theta(dy))$, $f(y)=\nu(y)/G(c,y)\geqq 0$.

The next lemma is the maximum principle for x_t-subharmonic functions.

LEMMA 4.3. *If* $u\in\mathfrak{H}^+(X)$ *and* A *is a finite subset of* N, *then* $H_{X-A}u=u$.

PROOF. If we put $v=u-H_{X-A}u$, v is a nonnegative x_t-subharmonic function vanishing over $X-A$. Considering that, if a set E of X contains no traps, $\hat{\Pi}^n(x, E)=\Pi^n(x, E)$ for any n and that the set $X-A$ contains all the traps, we get

$$(4.7)\qquad v(x)\leqq\hat{\Pi}^n v.(x)=\Pi^n v.(x)=\sum_{y\in A}v(y)\Pi^n(x, y).$$

Now assume that v takes the (strictly) positive maximum at the state $a\in A$. But Lemma 1.6 implies that $\Pi^n(a, A)<1$ for some n. Consequently we obtain

$$0<v(a)\leqq v(a)\Pi^n(a, A)<v(a),$$

which is a contradiction.

PROOF OF THEOREM 4.3. First suppose that $u\in\mathfrak{H}^+(X)$. Then our statement is this: $u=u_{\partial\hat{X}+\cup\hat{R}_i}$. For this, by virtue of Theorem 3.5, it suffices to show that $u=u_G=H_{[G]}u$ for any open set G (in M) which includes the set $\partial\hat{X}+\bigcup\hat{R}_i$. Take any sequence $\{A_n\}$ of finite sets approximating $X-[G]$ from below and put $B_n=X-[G]-A_n$. Now we shall prove that $H_{B_n}u\downarrow v\equiv 0$. In fact, in the same way as in Theorem 3.3. (iii), $\bigcap B_n=\emptyset$ implies that $v\in\mathfrak{H}^+(X)$. On the other hand, analogously to the proof of Theorem 4.1, we have

$$(4.8)\qquad u_{\hat{A}}=H_A u=\int_{\bar{\hat{A}}}\hat{K}(c, \cdot, \xi)\mu(d\xi)\qquad\text{for any subset } A \text{ of } X.^{12)}$$

Therefore

$$0\leqq v\leqq H_{B_n}u\leqq H_{X-[G]}u=\int_{M-G}\hat{K}(c, \cdot, \xi)\mu(d\xi).$$

But since $\hat{N}\supset M-G$, by virtue of Lemma 4.2, $H_{X-[G]}u$ is a potential, so that $v\equiv 0$ comes from Corollary of Theorem 2.8. Now applying Lemma 4.3 to each $A_n(\subset N)$, it follows that

$$H_{[G]}u\leqq u=H_{X-A_n}u=H_{[G]+B_n}\leqq H_{[G]}u+H_{B_n}u\rightarrow H_{[G]}u,$$

12) $\bar{\hat{A}}$ denotes the closure of the set $\hat{A}=\theta(A)$. Further, notice that the first equality comes from the fact that $\hat{A}$ is an F_σ set in M.

which proves $u=H_{[G]}u$.

For a general function u of $\mathfrak{H}_1^+(X)\cap\mathfrak{F}(X)$, consider the decomposition (2.22), that is, $u=v+w$. Then since v is a potential and $v_{\partial\hat{X}+\cup\hat{R}_i}(\leqq v)$ is x_t-harmonic, $v_{\partial\hat{X}+\cup\hat{R}_i}=0$. On the other hand, w is x_t-harmonic, so that we have

$$u_{\partial\hat{X}+\cup\hat{R}_i}=v_{\partial\hat{X}+\cup\hat{R}_i}+w_{\partial\hat{X}+\cup\hat{R}_i}=w\,,$$

which completes the proof of our theorem.

The formula (4.3) or (4.8) proves that any function u of $\mathfrak{H}_1^+(X)\cap\mathfrak{F}(X)$ can be represented in the form

$$u=\int_M\hat{K}(c,\,\cdot,\,\xi)\,\mu(d\xi) \tag{4.9}$$

by means of some positive (Radon) measure μ over M. But as will be shown later, such representation is not always uniqne. In the following two sections, we shall treat the uniqueness problem.

10. Minimal x_t-superharmonic functions. The classification of the minimal part M_1 and nonminimal part M_0 of M.

DEFINITION 4.1. A function u of $\mathfrak{H}_1^+(X)\cap\mathfrak{F}(X)$ is *minimal* if it satisfies the condition below: If $u=u_1+u_2$ and $u_i\in\mathfrak{H}_1^+(X)$ for $i=1,2$, then each u_i is a constant multiple of u. Equivalently we can say that u is an extremal element of the function family $\mathfrak{H}_1^+(X)\cap\mathfrak{F}(X)\cap\{v\,;\,v(c)=u(c)\}$ which is convex clearly.

REMARK 4.1. It is easily shown that a function u of $\mathfrak{H}^+(X)$ is minimal if and only if any function v of $\mathfrak{H}^+(X)$ which does not exceed u is a constant multiple of u. Thus our definition is a generalization of the concept 'minimal harmonic' in Martin [**12**].

DEFINITION 4.2. The set of ξ such that $\hat{K}(c,x,\xi)$ is minimal (nonminimal) is called the *minimal* (*nonminimal*) *part of* M and denoted by $M_1(M_0)$. Moreover the set $M_1\cap\partial\hat{X}$ is denoted by $(\partial\hat{X})_1$.

LEMMA 4.4. *Suppose that the nontrivial* ($\not\equiv 0$) *minimal* x_t-*superharmonic function* u *is expressed in the form*

$$u=\int_C\hat{K}(c,\,\cdot,\,\xi)\,\mu(d\xi) \tag{4.10}$$

with some positive measure μ *defined over a Borel subset* C *of* M. *Then* μ *is uniqely determined and the total mass of* μ *concentrates on some point* $\xi_0\in M_1\cap C$. *Therefoer, for any nontrivial minimal*

x_t-superharmonic function u, there exists one and only one $\xi_0 \in M_1$ *such that* $u = u(c)\ \hat{K}(c, \cdot, \xi_0)$.

PROOF. Putting $\mu(M-C)=0$, we take μ for the measure over M. Since u is nontrivial, $\mu(M)=\mu(C)=u(c)>0$. Therefore, there exists at least one point ξ such that $\mu(G)>0$ for any open set $G \ni \xi$. Such point ξ is called a *carrier point of* μ. The set C contains at least one carrier point of μ, because $\mu(D)>0$ for some closed $D \subset C$. Now let ξ_0 be a carrier point of μ in C and G_n, a sequence of open sets such that $G_n \ni \xi_0$ and $\bar{G}_n \downarrow \xi_0$. Then we have

$$u = \int_{G_n} \hat{K}(c, \cdot, \xi)\mu(d\xi) + \int_{M-G_n} \hat{K}(c, \cdot, \xi)\mu(d\xi),$$

in which each term of the right side is a function of $\mathfrak{H}_1^+(X) \frown \mathfrak{F}(X)$ by virtue of Fubini's theorem. Since u is minimal, it follows that

$$v_n \equiv \int_{G_n} \hat{K}(c, \cdot, \xi)\mu(d\xi) = k_n u,$$

$$0 < k_n = \frac{v_n(c)}{u(c)} = \frac{\mu(G_n)}{\mu(M)},$$

so that, putting $\mu_n(d\xi) = k_n^{-1}\mu(d\xi)$ over G_n, we get

$$u = k_n^{-1} v_n = \int_{G_n} \hat{K}(c, \cdot, \xi)\mu_n(d\xi).$$

Noting that $\mu_n(G_n) = u(c) = \mu(M)$ and $\hat{K}(c, x, \xi)$ is ρ-continuous in ξ for each x and letting $n \to \infty$, we have

$$u = u(c)\hat{K}(c, \cdot, \xi_0),$$

which implies that $\xi_0 \in M_1 \cap C$. To prove our statement, take any carrier point ξ_1 of μ. Then the same argument as above proves that

$$u = u(c)\hat{K}(c, \cdot, \xi_1),$$

which shows that $\xi_0 = \xi_1$. Therefore μ is the measure concentrated at ξ_0.

LEMMA 4.5. *If* $\hat{K}_{\{\xi\}}(c, c, \xi) = 1$, *then* $\xi \in M_1$.

PROOF. Let u be any function of $\mathfrak{H}_1^+(X) \frown \mathfrak{F}(X)$. Then according to (4.4), we have

$$(4.11) \quad u_{\{\xi\}} = \int_{\{\xi\}} \hat{K}(c, \cdot, \eta)\mu(d\eta) = \mu(\{\xi\})\hat{K}(c, \cdot, \xi) = u_{\{\xi\}}(c)\hat{K}(c, \cdot, \xi).$$

Putting $u=\hat{K}(c, \cdot, \xi)$ and considering $\hat{K}_{\{\xi\}}(c, c, \xi)=1$, we get

$$\hat{K}_{\{\xi\}}(c, \cdot, \xi) = \hat{K}_{\{\xi\}}(c, c, \xi)\hat{K}(c, \cdot, \xi) = \hat{K}(c, \cdot, \xi) .$$

Therefore, if $\hat{K}(c, \cdot, \xi)=u_1+u_2$ and $u_i \in \mathfrak{H}_1^+(X)$, it holds that

$$\hat{K}(c, \cdot, \xi) = u_1+u_2 \geqq (u_1)_{\{\xi\}}+(u_2)_{\{\xi\}} = \hat{K}_{\{\xi\}}(c, \cdot, \xi) = \hat{K}(c, \cdot, \xi) ,$$

so that we have $u_1=(u_1)_{\{\xi\}}$, $u_2=(u_2)_{\{\xi\}}$. But each $(u_i)_{\{\xi\}}$ is a constant multiple of $\hat{K}(c, \cdot, \xi)$ from (4.11) and hence $\hat{K}(c, \cdot, \xi)$ is minimal.

LEMMA 4.6. *If $\xi \in \hat{X}$, then $\hat{K}_{\{\xi\}}(c, c, \xi)=1$ and hence $\hat{X} \subset M_1$.*

PROOF. It is enough to show that $\hat{K}_G(c, c, \xi) \geqq 1$ for any open set G containing ξ, because $\hat{K}_{\{\xi\}}(c, c, \xi)=\inf_{G \ni \xi} \hat{K}_G(c, c, \xi) \leqq \hat{K}(c, c, \xi)=1$. Take any $y \in X$ such that $\theta(y)=\xi$. It is clear that $[G] \ni y$. Consequently it follows that

$$\begin{aligned}\hat{K}_G(c, c, \xi) &= [H_{[G]}K(c, \cdot, y)].(c) \geqq [H_{\{y\}}K(c, \cdot, y)].(c) \\ &= \frac{1}{p(c, y)}[H_{\{y\}}p(\cdot, y)].(c) = 1\end{aligned}$$

which proves our lemma.

LEMMA 4.7. *Let ξ be a point of $(\partial\hat{X})_1+\bigcup\hat{R}_i$ and C, a Borel subset of $\partial\hat{X}+\bigcup\hat{R}_i$. Then $\hat{K}(c, \cdot, \xi)$ is x_t-harmonic and further*

$$\text{(4.12)} \qquad \begin{aligned}\hat{K}_C(c, \cdot, \xi) &= \hat{K}(c, \cdot, \xi) \qquad \textit{if } \xi \in C , \\ &= 0 \qquad\qquad\quad \textit{if } \xi \notin C .\end{aligned}$$

Therefore $\hat{K}_{\{\xi\}}(c, c, \xi)=1$.

PROOF. To prove the first statement, consider Riesz decomposition (2.22) of $\hat{K}(c, \cdot, \xi)$. Then, if the potential part v of $\hat{K}(c, \cdot, \xi)$ does not vanish, the harmonic part w should vanish, for $\hat{K}(c, \cdot, \xi)$ is minimal and strictly x_t-superharmonic at some state in X. Consequently $\hat{K}(c, \cdot, \xi)$ is a potential, so that we have

$$\begin{aligned}\hat{K}(c, \cdot, \xi) &= \int_N f(y)G(\cdot, dy) = \int_N K(c, \cdot, y)f(y)G(c, dy) \\ &= \int_{\hat{N}} \hat{K}(c, \cdot, \eta)\mu(d\eta) ,\end{aligned}$$

where $\mu(d\eta)=f(\theta^{-1}(\eta))G(c, \theta^{-1}(d\eta)) \geqq 0$. Applying Lemma 4.4, it follows that $\hat{K}(c, \cdot, \xi)=\hat{K}(c, \cdot, \xi_0)$ for some $\xi_0 \in \hat{N}$, i.e. $\xi=\xi_0$. This contradicts to the assumption $\xi \notin \hat{N}$.

To prove the second statement, we shall begin with the case $C \not\ni \xi$. According to Theorem 4.2, we get

$$\hat{K}(c, \cdot, \xi) \geqq \hat{K}_C(c, \cdot, \xi) = \int_C \hat{K}(c, \cdot, \eta)\mu(d\eta). \tag{4.13}$$

Assume that $\hat{K}_C(c, \cdot, \xi) \not\equiv 0$, contrary to our statement. Then, since $\hat{K}_C(c, \cdot, \xi)$ is x_t-harmonic, it follows from the first statement of this theorem and Remark 4.1 that $\hat{K}_C(c, \cdot, \xi) = k\hat{K}(c, \cdot, \xi)$ for $k = \hat{K}_C(c, c, \xi) \neq 0$ and therefore $\hat{K}_C(c, \cdot, \xi)$ is also minimal x_t-harmonic. Applying Lemma 4.4 to the formula (4.13), we have

$$\begin{aligned}\hat{K}(c, \cdot, \xi) &= \frac{1}{k}\hat{K}_C(c, \cdot, \xi) = \frac{1}{k}\hat{K}_C(c, c, \xi)\hat{K}(c, \cdot, \xi_0) \\ &= \hat{K}(c, \cdot, \xi_0) \qquad \text{for some} \quad \xi_0 \in C,\end{aligned}$$

which contradicts to $\xi \notin C$. Thus we have shown that $\hat{K}_C(c, \cdot, \xi) \equiv 0$ if $\xi \notin C$. Next, in the case of $C \ni \xi$, putting $C' = \partial\hat{X} + \bigcup \hat{R}_i - C$ and recalling Theorem 4.3, we have

$$\hat{K}(c, \cdot, \xi) = \hat{K}_{\partial\hat{X}+\cup\hat{R}_i}(c, \cdot, \xi) \leqq \hat{K}_C(c, \cdot, \xi) + \hat{K}_{C'}(c, \cdot, \xi).$$

Since $C' \not\ni \xi$, $\hat{K}_{C'}(c, \cdot, \xi) \equiv 0$ and hence $\hat{K}(c, \cdot, \xi) = \hat{K}_C(c, \cdot, \xi)$.

We shall give a criterion for M_1 and M_0 in the following

THEOREM 4.4.

$$\begin{aligned}\hat{K}_{\{\xi\}}(c, c, \xi) &= 1 \qquad \textit{if} \quad \xi \in M_1, \\ &= 0 \qquad \textit{if} \quad \xi \in M_0.\end{aligned} \tag{4.14}$$

PROOF. Summing up the lemmas above, we have $\hat{K}_{\{\xi\}}(c, c, \xi) = 1$ if and only if $\xi \in M_1 = \hat{X} + (\partial\hat{X})_1$. On the other hand, putting $u = \hat{K}(c, \cdot, \xi)$ in (4.11) and noting that $(u_{\{\xi\}})_{\{\xi\}} = u_{\{\xi\}}$, we get

$$\hat{K}_{\{\xi\}}(c, \cdot, \xi) = (\hat{K}_{\{\xi\}})_{\{\xi\}}(c, \cdot, \xi) = \hat{K}_{\{\xi\}}(c, c, \xi)\hat{K}_{\{\xi\}}(c, \cdot, \xi),$$

so that

$$\hat{K}_{\{\xi\}}(c, c, \xi) = [\hat{K}_{\{\xi\}}(c, c, \xi)]^2.$$

Therefore $\hat{K}_{\{\xi\}}(c, c, \xi) = 1$ or 0, which implies that $\hat{K}_{\{\xi\}}(c, c, \xi) = 0$ is equivalent to $\xi \in M_0$.

THEOREM 4.5. *M_0 is an F_σ set of M. Therefore both M_1 and $(\partial\hat{X})_1$ are Borel subsets in M.*

PROOF. Since the proof is completely analogous to that of Martin **[12]**, Section 4, Theorem II, we shall only give a simple sketch. Let Γ_n be the set of ξ in M satisfying the following condition: If an open set G in M contains ξ and its ρ-diameter $d(G)$ $(=\sup_{\eta,\xi\in G}\rho(\xi,\eta))$ does not exceed $1/n$, then $\hat{K}_G(c,c,\xi)=[H_{[G]}\hat{K}(c,\cdot,\xi)].(c)\leqq 1/2$. That each Γ_n is closed in M comes from the fact that $\hat{K}_G(c,c,\xi)$ is lower semi-continuous as a function of ξ. Moreover it is shown that $\Gamma_n\uparrow$ and $\bigvee_n\Gamma_n=M_0$, which proves our theorem.

11. Main results.

DEFINITION 4.3. A bounded signed (Radon) measure μ over M is *canonical* if $\mu(C)=0$ for any Borel subset C of M_0. The representation

$$\int_M \hat{K}(c,\cdot,\xi)\mu(d\xi)$$

is a *canonical represetation* if μ is a canonical measure.

MAIN THEOREM. *For any function u of $\mathfrak{H}_1^+(X)\frown\mathfrak{F}(X)$, there exists one and only one positive canonical measure μ such that*

$$u=\int_M \hat{K}(c,\cdot,\xi)\mu(d\xi)=\int_{M_1}\hat{K}(c,\cdot,\xi)\mu(d\xi)\,. \tag{4.15}$$

This measure μ is characterized by

$$\begin{aligned}\mu(C)&=u_C(c) && \text{if } C\subset\partial\hat{X}+\bigvee\hat{R}_i\\ &=\int_{\theta^{-1}(C)}[-\mathfrak{G}u.(y)]\,G(c,dy) && \text{if } C\subset\hat{N},\end{aligned}\tag{4.16}$$

where C is a Borel set in M. Therefore, if u is x_t-harmonic, μ vanishes over $\hat{N}$.

Before proving this theorem, we prepare two lemmas.

LEMMA 4.8. *Consider the representation* (4.9) *of a function $u\in\mathfrak{H}_1^+(X)\frown\mathfrak{F}(X)$. Then, for any Borel set C in M, we have*

$$u_C(x)=\int_M \hat{K}_C(c,x,\xi)\mu(d\xi)\,. \tag{4.17}$$

PROOF. If C is an open set in M, our statement is clear from Fubini's theorem. Next, if C is a closed set, (4.17) is a direct consequence of the above result and Lebesgue's theorem. Finally, for a Borel set C, choose the sequences $\{G_n\}$, $\{D_n\}$ which satisfy (3.12) at the state x. Then it follows that

$$u_C(x)=\int_M \lim_{n\to\infty}\hat{K}_{G_n}(c,x,\xi)\mu(d\xi)=\int_M \lim_{n\to\infty}\hat{K}_{D_n}(c,x,\xi)\mu(d\xi)\,,$$

so that $\lim_{n\to\infty} \hat{K}_{G_n}(c, x, \xi) = \hat{K}_C(c, x, \xi) = \lim_{n\to\infty} \hat{K}_{D_n}(c, x, \xi)$ for any $\xi \notin$ an exceptional set of μ-measure 0.

LEMMA 4. 9.

$$(4.18) \qquad u_{M_0} \equiv 0 \qquad \textit{for any function } u \textit{ of } \mathfrak{H}_1^+(X) \cap \mathfrak{F}(X).$$

PROOF. Let Γ_n be the set defined in the proof of Theorem 4. 5. Since $\bigcup \Gamma_n = M_0$, according to the general property of capacity, it is enough to show that $u_{\Gamma_n} \equiv 0$ for every n. Γ_n is compact and therefore it has a finite open covering $\{G_i\}$ each of which satisfies $d(G_i) \leqq 1/n$. Putting $C_i = \Gamma_n \cap G_i$, we have

$$\Gamma_n = \bigcup_{i=1}^{k} C_i, \quad u_{\Gamma_n} \leqq \sum_{i=1}^{k} u_{C_i}.$$

Thus our statement has been reduced to showing $u_{C_i} \equiv 0$. To prove this fact, we shall first notice that

$$\hat{K}_{C_i}(c, c, \xi) \leqq \hat{K}_{G_i}(c, c, \xi) \leqq \frac{1}{2} \qquad \text{for any} \quad \xi \in C_i,$$

which is clear from definition of Γ_n. On the other hand, since C_i is a Borel set in $\partial\hat{X}$, it follows from Theorem 4. 2 that

$$u_{C_i} = \int_{C_i} \hat{K}(c, \cdot, \xi)\mu(d\xi), \qquad u_{C_i}(c) = \mu(C_i).$$

Therefore, applying the preceding lemma, we have

$$u_{C_i}(c) = (u_{C_i})_{C_i}(c) = \int_{C_i} \hat{K}_{C_i}(c, c, \xi)\mu(d\xi) \leqq \frac{1}{2}\mu(C_i) = \frac{1}{2}u_{C_i}(c),$$

which implies that $u_{C_i}(c) = 0$, i.e. $u_{C_i} \equiv 0$.

PROOF OF MAIN THEOREM.

1° *The existence of a canonical representation with a positive measure.* Consider Riesz decomposition (2. 22) of u. Then the potential part v of u can be expressed as follows:

$$v = \int_N f(y) G(\cdot, dy) = \int_N K(c, \cdot, y) f(y) G(c, dy)$$
$$= \int_{\hat{N}} \hat{K}(c, \cdot, \xi)\mu_1(d\xi),$$

where $\mu_1(d\xi) = f(\theta^{-1}(\xi)) G(c, \theta^{-1}(d\xi)) \geqq 0$. On the other hand, as to the harmonic part w of u, it follows from Theorem 4. 2, 4. 3 and the preceding lemma that

$$w = w_{\partial\hat{X}+\cup\hat{R}_i} \leqq w_{(\partial\hat{X})_1+\cup\hat{R}_i} + w_{M_0} = w_{(\partial\hat{X})_1+\cup\hat{R}_i} \leqq w\,,$$

so that

$$w = w_{(\partial\hat{X})_1+\cup\hat{R}_i} = \int_{(\partial\hat{X})_1+\cup\hat{R}_i} \hat{K}(c,\,\cdot,\,\xi)\,\mu_2(d\xi)$$

with some positive measure μ_2. Therefore u is expressible in the form (4.15) with the positive canonical measure μ defined by

$$\mu(C) = \mu_1(C \frown \hat{N}) + \mu_2[C \frown \{(\partial\hat{X})_1 + \bigcup_i \hat{R}\}]$$

for any Borel set C in M.

2° *The uniqueness of the positive canonical measure in* (4.15). Let μ be any positive canonical measure representing u. Put

$$v = \int_{\hat{N}} \hat{K}(c,\,\cdot,\,\xi)\,\mu(d\xi)\,, \quad w = \int_{\partial\hat{X}+\cup\hat{R}_i} \hat{K}(c,\,\cdot,\,\xi)\,\mu(d\xi)\,. \tag{4.19}$$

Then v is a potential of the function $f(y) = \mu(\theta(dy))/G(c, dy)$ (see Lemma 4.2) and w is a nonnegative x_t-harmonic function (see Lemma 4.7). Therefore the decomposition $u = v + w$ is nothing but Riesz decomposition (2.22), so that we have $f(y) = -\mathfrak{G}u.(y)$, which proves the second equality of (4.16). Moreover, recalling Theorem 3.5, 4.3, Lemma 4.7 and 4.8, it follows that, for any Borel set C in $\partial\hat{X} + \bigcup\hat{R}_i$,

$$\begin{aligned} u_C = (u_{\partial\hat{X}+\cup\hat{R}_i})_C = (w)_C &= \int_{\partial\hat{X}+\cup\hat{R}_i} \hat{K}_C(c,\,\cdot,\,\xi)\,\mu(d\xi) \\ &= \int_C \hat{K}(c,\,\cdot,\,\xi)\,\mu(d\xi)\,. \end{aligned} \tag{4.20}$$

Substituting c in place of $\cdot$, we get the first equality of (4.16).

REMARK 4.2. In the end of Section 9, we stated that the representation (4.9) is not always unique. Now our main theorem clarifies the circumstances. In fact, if $\xi_0 \in M_0 \neq \emptyset$, the function $\hat{K}(c,\,\cdot,\,\xi_0)$ is expressible in the form (4.9) by means of both some canonical measure and the unit distribution at ξ_0 which are clearly different. On the other hand, if $M_0 = \emptyset$, (4.9) is nothing but a canonical representation and therefore it is unique for any $u \in \mathfrak{H}_1^+(X) \frown \mathfrak{F}(X)$. In the next chapter, we shall give two examples of $M_0 \neq \emptyset$ (see Examples III, VI).

REMARK 4.3. If $\partial\hat{X}$ is closed in M and θ is the homeomorphism $(X \to \hat{X})$, for example, if the conditions (CMP. 5), (CMP. 6) are satisfied, our arguments will be much more simplified. In fact, then, we need not use Choquet's capacity theorem to obtain the main theorem, and the original proof of Martin [**12**] for ordinary harmonic functions is applicable without any change to our case.

To continue, we shall state some important results derived from the main theorem.

THEOREM 4.6. (i) *The canonical representation of a function* $u \in \mathfrak{F}(X)$ *is unique. Speaking in detail, if u admits of a canonical representation, the canonical measure* μ *representing u is uniquely determined.*

(ii) *In order that a function* $u \in \mathfrak{F}(X)$ *should have a canonical representation, it is necessary and sufficient that u is expressible as the difference of two function of* $\mathfrak{H}_1^+(X) \frown \mathfrak{F}(X)$.

PROOF. For (i), it is enough to show that if

$$0 = \int_M \hat{K}(c, \cdot, \xi)\, \mu(d\xi)$$

with some canonical measure μ, then $\mu = 0$. Let $\mu = \mu_1 - \mu_2$ be Jordan decomposition of μ, that is, $\mu_i \geqq 0$ $(i=1, 2)$ and $\mu_1 \wedge \mu_2 = 0$.[13] Then we have

$$\int_M \hat{K}(c, \cdot, \xi)\, \mu_1(d\xi) = \int_M \hat{K}(c, \cdot, \xi)\, \mu_2(d\xi)\,. \tag{4.21}$$

But since the function defined by (4.21) is a function of $\mathfrak{H}_1^+(X) \frown \mathfrak{F}(X)$ and each μ_i is a positive canonical measure, it follows that $\mu_1 = \mu_2 = \mu_1 \wedge \mu_2 = 0$ and therefore $\mu = 0$.

We shall omit the proof of (ii) which is quite easy.

THEOREM 4.7. *Let u be a function of* $\mathfrak{H}_1^+(X) \frown \mathfrak{F}(X)$ *and* C, C', *Borel sets in* $\partial\hat{X} + \bigcup \hat{R}_i$.

(i) *For any fixed x in X,* $u_C(x)$ *is a positive canonical measure over* $\partial\hat{X} + \bigcup \hat{R}_i$, *namely, it satisfies*

$$u_{C \cup C'}(x) + u_{C \frown C'}(x) = u_C(x) + u_{C'}(x)\,. \tag{4.22}$$

(ii) $u_{C \frown C'}$ *is the greatest one among all the nonnegative* x_t-*harmonic functions which do not exceed both* u_C *and* $u_{C'}$. *Moreover we have*

$$(u_C)_{C'} = (u_{C'})_C = u_{C \frown C'}\,. \tag{4.23}$$

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13) $\mu_1 \wedge \mu_2$ denotes the greatest one among all the measures which do not exceed both μ_1 and μ_2.

PROOF. Let μ be the canonical measure representing u. Then (i) and (4.23) are immediate from the formula (4.20). To prove the first part of (ii), consider any v of $\mathfrak{H}^+(X)$ satisfying $v \leqq u_C$, $v \leqq u_{C'}$. Putting $C'' = \partial \hat{X} + \bigcup \hat{R}_i - C$ and using (4.23), we have

$$0 \leqq v_{C''} \leqq (u_C)_{C''} = u_{C \cap C''} = 0,$$

so that $v = v_{\partial \hat{X} + \cup \hat{R}_i} = v_C + v_{C''} = v_C$. Consequently, $u_{C \cap C'} = (u_{C'})_C \geqq v_C = v$, which proves our statement.

In Theorem 2.1, we have already obtained the facts below: Put $v_1 = u \vee 0$, $v_2 = (-u) \vee 0$, $v_3 = |u|$. Then if $u \in \mathfrak{H}(X)$, each $v_i \in \mathfrak{H}_2^+(X)$ and therefore $\hat{\Pi}^n v_i \uparrow$ with n. Moreover the function u_i defined as

$$u_i = \lim_{n \to \infty} \hat{\Pi}^n v_i \tag{4.24}$$

belongs to $\mathfrak{H}_1^+(X)$ and, if $u_3 \in \mathfrak{F}(X)$, to $\mathfrak{H}^+(X)$.

We shall now give a useful result for an x_t-harmonic function which is not necessarily nonnegative (see [**15**], [**16**]).

THEOREM 4.8. *An x_t-harmonic function u admits of a canonical representation if and only if $u_3(c) < +\infty$, or equivalently, if $\hat{\Pi}^n |u|.(c)$ is bounded in n. In this case, the canonical measure μ for u vanishes over every Borel sets in $\hat{N}$. Moreover, let $\mu = \mu_1 - \mu_2$ be Jordan decomposition of μ and put $\mu_3 = \mu_1 + \mu_2$. Then μ_i is the canonical measure for the function u_i defined by* (4.24).

PROOF. 1° Suppose that $u_3(c) < +\infty$. Since $u_3 \in \mathfrak{H}_1^+(X)$ and $c \to x$ for any x in X, it follows from Corollary 1 of Theorem 2.2 and Theorem 2.1 that $u_3 \in \mathfrak{F}(X)$ and therefore $u_i \in \mathfrak{H}^+(X)$, $u = u_1 - u_2$. Thus, by virtue of Theorem 4.6.(ii), u admits of a canonical representation. Conversely, if u has a canonical representation, we have

$$|u| \leqq \int_M \hat{K}(c, \cdot, \xi)\, \mu_3(d\xi),$$

so that

$$0 \leqq \hat{\Pi}^n |u|.(c) \leqq \int_M [\hat{\Pi}^n \hat{K}(c, \cdot, \xi)].(c) \mu_3(d\xi) \leqq \mu_3(M) < +\infty,$$

which proves our first statement.

2° Consider an x_t-harmonic function admitting of a canonical representation. Then, as was shown in 1°, $u = u_1 - u_2$, $u_i \in \mathfrak{H}^+(X)$. According to the main theorem, the positive canonical measure ν_i for u_i $(i = 1, 2)$ vanishes over $\hat{N}$. But $\mu = \nu_1 - \nu_2$ from the uniqueness of the canonical representation, so that our second statement is proved.

3° Denote, by w_i, the nonnegative x_t-harmonic function defined as the canonical representation with μ_i $(i=1, 2)$. Then, since each μ_i $(i=1,2)$ is a component of Jordan decomposition of μ, we have $\mu_i \leqq \nu_i$, which implies that $w_i \leqq u_i$. On the other hand, since $u=w_1-w_2$, it follows that $w_i \geqq v_i$ $(i=1,2)$. Therefore we get

$$w_i = \hat{\Pi}^n w_i \geqq \hat{\Pi}^n v_i \to u_i ,$$

which proves that $w_i=u_i$ $(i=1,2)$, i.e. μ_i is the canonical measure for u_i. Noting that $u_3=u_1+u_2$, it follows that μ_3 is the canonical measure for u_3.

REMARK 4.4. According to Theorem 2.12 and Remark 2.7, if x_t is a Markov process with a discrete time parameter or with independent increments, the condition in the above theorem can be replaced by the following one: $H^t|u|.(c)$ is bounded in t.

CHAPTER 5. EXAMPLES AND SUPPLEMENTS

12. The discrete boundary and the continuous one. It is clear that the results which were established in the preceding chapters do not depend on q, but only on Π. In other words, it is unessential for the general potential and boundary theory whether our CMP is of a discrete time parameter or a continuous one.

We start with the simplest example of the discrete boundary.

EXAMPLE I. *Random walks and birth-death processes over the set of positive integers.* Let X be the set of positive integers, $\{1, 2, 3, \cdots\}$. Consider the system $\{q, \Pi\}$ satisfying the following conditions: $0<q(x)<+\infty$ for any $x \in X$.

$$\Pi(1, 2) = r_1 > 0 , \quad \Pi(1, \infty) = d_1 \geqq 0 , \quad r_1+d_1 = 1 ,$$

and if $x \geqq 2$,

$$\Pi(x, x+1) = r_x > 0 , \quad \Pi(x, x-1) = l_x > 0 , \quad \Pi(x, \infty) = d_x \geqq 0 , \tag{5.1}$$
$$r_x + l_x + d_x = 1 .$$

The CMP corresponding to the above $\{q, \Pi\}$ is called a *random walk* or a *birth-death process* according as its time parameter is discrete or continuous. Since $p(x, y)>0$ for any $x, y \in X$, according to Lemma 1.5, we have two possible cases below: (a) X is a single indecomposable recurrent set, or else (b) any state in X is nonrecurrent. The condition (CMP. 4) is satisfied evidently. To fix the idea, definiteness, we choose the state 1 as the center.

In the case (a), we have $d_x=0$ for any $x \in X$. The Martin space M consists of a single point ξ and the function $\hat{K}(1, x, \xi)$ is identically equal to 1, so that any nonnegative x_t-superharmonic function is a nonnegative constant.

In the case (b), the conditions (CMP. 5), (CMP. 6) are satisfied. We now construct the Martin space. Noting that $p(x, z)= p(x, y)\,p(y, z)$ for $x \leqq y \leqq z$, we have

$$K(1, x, y) = \frac{p(x, y)}{p(1, y)} = \frac{1}{p(1, x)} \qquad \text{for any pair of } x \leqq y,$$

which implies that the Martin boundary $\partial\hat{X}$ consists of a single point, denoted by $+\infty$, and that

$$\hat{K}(1, x, +\infty) = \frac{1}{p(1, x)}. \tag{5.2}$$

The above function is minimal x_t-harmonic and any nonnegative x_t-harmonic function is a constant multiple of $\hat{K}(1, x, +\infty)$. Moreover it is easily shown that the canonical Martin space is nothing but the usual one-point compactification of X.

If $d_x=0$ for $x \geqq 2$, it is known that the explicit formula of $\hat{K}(1, x, +\infty)$ is given by

$$\hat{K}(1, x, +\infty) = 1+\sum_{i=1}^{x-1} \frac{d_1 l_2 \cdots l_i}{r_1 r_2 \cdots r_i}. \tag{5.3}$$ [14]

Consequently, if $d_x=0$ for every x, we have

$$\hat{K}(1, x, +\infty) \equiv 1, \tag{5.4}$$ [15]

so that any nonnegative x_t-harmonic function is a constant.

Next assume that the condition below is satisfied:

$$P_1(\sigma_{+\infty} < +\infty) > 0. \tag{5.5}$$

As was noted in Theorem 2.9, this is impossible for a random walk and, in our case, it implies that $p(x, +\infty) \equiv P_x(\sigma_{+\infty} < +\infty) > 0$ for every $x \in X$, or equivalently that the equation

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14) This formula is derived from the fact that $\hat{K}(1, x, +\infty)$ is the unique solution of the equation $\Pi u=u$, $u(1)=1$. Also see Karlin and McGregor [**11**].

15) Note that this fact also comes from Lemma 1.6 immediately.

$$(5.6) \qquad (\alpha - \mathfrak{G})u = 0$$

has a bounded (strictly) positive solution for any $\alpha \geqq 0$. $p(x, +\infty)$ represents the probability that the paths starting at x converge to the boundary point $+\infty$ after finite time, so that $+\infty$ is an *exit boundary point in Feller's sense.*[16] The condition (5.5) has a close connection with the choice of q. In fact, if $d_x = 0$, (5.5) is equivalent to

$$(5.7) \qquad \sum_{x \in X} q(x) < +\infty \, .$$

The *random walk* (*or birth-death process*) *over the set of all integers*, $X = \{\cdots, -1, 0, 1, \cdots\}$, is the CMP with a discrete parameter (or continuous parameter) corresponding to the system $\{q, \Pi\}$ which satisfies $0 < q(x) < +\infty$ and (5.1) for any integer. In the same way as before, we have only two possible cases. In the case (b), it follows that $\partial\hat{X}$ consists of two points, say $-\infty$ and $+\infty$, and taking the state 0 as the center, we get

$$(5.8) \qquad \hat{K}(0, x, +\infty) = \frac{1}{p(0, x)} \quad (x \geqq 0), \qquad = p(x, 0) \quad (x < 0),$$

$$\hat{K}(0, x, -\infty) = p(x, 0) \quad (x \geqq 0), \qquad = \frac{1}{p(0, x)} \quad (x < 0),$$

each of which is minimal x_t-harmonic. The canonical Martin space is the usual compactification of X, $\{-\infty, \cdots, -1, 0, 1, \cdots, +\infty\}$. Moreover, in such case, any x_t-harmonic function is expressible as some linear combination of the two functions in (5.8), that is, it admits of a canonical representation. In the case (a), however, there exists an x_t-harmonic function which has no canonical representation. In fact, our bounary theory proves that any x_t-harmonic function having a canonical representation should be a constant. On the other hand, as is well known, the equation

$$\Pi u = u$$

has two independent solutions one of which cannot be a constant. Thus our statement was proved.[17]

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16) It is an interesting problem to extend Feller's boundary classification to the general Martin boundary. We shall discuss this problem in another chance.

17) The simplest example is this: $d_x = 0$, $r_x = l_x = 1/2$ for any x. Then the function $u(x) = x$ is an x_t-harmonic function admiting no canonical representation.

Now we shall introduce a notation. The réduite of the indicator function χ_X of the whole state space is denoted by

$$(\chi_X)_C(x) = h(x, C) \qquad \text{for} \quad C \subset \partial\hat{X}.$$

It has been shown in [4], [16] that, in the canonical Martin space, $h(x, C)$ expresses the probability that the paths from x converge to the boundary set C and palys an important role for the first boundary value problem and the study of bounded x_t-harmonic functions.

In [15], we have proved that the Martin boundary of the space-time Bernoulli process is the interval $[0, 1]$ with the usual topology. Moreover, in [16], by a simple modification of the above process, we have obtained a process for which $h(c, \cdot)$ is Lebesgue measure over $[0, 1]$. We shall give another example of the continuous boundary.

EXAMPLE II. *Feller's dyadic branching scheme* (c.f. Feller [6], Section 17, Example IV). Let X be the countable set which consists of a point c and all the points denoted by $a_1a_2 \cdots a_k$ ($a_i=0$ or 1, $k=1, 2, 3, \cdots$). The point $a_1a_2 \cdots a_k$ is called a *point of the length* k. We now introduce the following semi-order relation: c is the maximal element and, if $x=a_1 \cdots a_k$ and $y=a_1 \cdots a_kb_1 \cdots b_l$, then $x>y$. We consider the Markov process over X defined as follows: $0<q(x)<+\infty$ for any $x \in X$ and

$$\Pi(c, 0) = \Pi(c, 1) = 1/2,$$
$$\Pi(x, x0) = \Pi(x, x1) = 1/2.$$

Clearly c is the unique center and the conditions (CMP. 5), (CMP. 6) are satisfied. Moreover we have

$$K(c, x, y) = \frac{1}{p(c, x)} > 0 \qquad \text{if} \quad x \geqq y, \tag{5.9}$$
$$= 0 \qquad \text{otherwise.}$$

We shall prove that our Martin boundary is Cantor set in the interval $[0, 1]$, namely, the countable direct product, say S, of the compact Hausdorff space consisting of two points 0 and 1.

1° For any $\xi=a_1a_2 \cdots$ of S, define the sequence $\{y_n\}$ of X by $y_n=a_1a_2 \cdots a_n$. Using (5.9), it is easily shown that the above $\{y_n\}$ is a fundamental sequence and the limit function of $K(c, x, y_n)$, denoted by $\hat{K}(c, x, \xi)$, is given by

$$(5.10)\qquad \hat{K}(c, x, \xi) = \frac{1}{p(c, x)} = 2^k \quad \text{if} \quad x = a_1 \cdots a_k$$
$$= 0 \qquad\qquad \text{otherwise.}$$

It is clear that $\hat{K}(c, x, \xi) \neq \hat{K}(c, x, \xi')$ for $\xi \neq \xi'$.

2° We shall show that, for any fundamental sequence $\{z_n\}$, there exists the fundametal sequence which is equivalent to $\{z_n\}$ and satisfies the condition of 1° for some $\xi \in S$. Let y_k be a point of the length k which satisfies the following condition: (α) There exists an infinite subsequence $\{z_{n(i)}\}$ of $\{z_n\}$ each of which is smaller than y_k. The existence of such y_k is immediate from the fact that $\{z_n\}$ has no limit points in X. We now prove the uniqueness. In fact, if the two points y_k, $y_{k'}$ satisfy the condition (α), it follows from (5.9) that

$$(5.11)\qquad \lim_{n\to\infty} K(c, y_k, z_n) = \lim_{i\to\infty} K(c, y_k, z_{n(i)}) = \frac{1}{p(c, y_k)} > 0\,,$$
$$= \lim_{i\to\infty} K(c, y_k, z_{n(i)'}) = 0\,,$$

which is a contradiction. Further we can easily see that $y_1 > y_2 > y_3 > \cdots$. Therefore the sequence $\{y_n\}$ defines a ξ of S and according to 1°, it is a fundamental sequence. The equivalence of $\{y_n\}$ and $\{z_n\}$ is clear from (5.10) and (5.11).

3° We have proved that our Martin boundary $\partial\hat{X}$ coincides with S as a set. We shall show the topological equivalence of the metric ρ with the usual metric d. For this, assume that $\xi^{(n)} = a_1^{(n)} a_2^{(n)} \cdots$, $\xi = a_1 a_2 \cdots$ and $d(\xi^{(n)}, \xi) \to 0$. Then, for any fixed k, we can choose some n_0 such that $a_i^{(n)} = a_i$ $(i = 1, 2, \cdots, k)$ for every $n \geqq n_0$. Therefore we have

$$\hat{K}(c, x, \xi^{(n)}) = \hat{K}(c, x, \xi) \quad \text{if the length of } x \text{ is less than } k,$$

so that

$$\lim_{n\to\infty} \hat{K}(c, x, \xi^{(n)}) = \hat{K}(c, x, \xi) \qquad \text{for any } x \text{ in } X,$$

which means $\rho(\xi^{(n)}, \xi) \to 0$. Thus $\partial\hat{X}$ is homeomorphic to S.

4° Next we prove that $\hat{K}(c, x, \xi)$ is minimal x_t-harmonic for every $\xi \in S$. Let $\xi = a_1 a_2 \cdots$ and consider the sequence $A_n = \{a_1 \cdots a_k\,;\ k \geqq n\}$ of subsets in X. Then, for any open (in M) set $G \ni \xi$, $[G]$ contains A_n after some n_0. Consequently, we get

$$1 \geqq \hat{K}_{\{\xi\}}(c, c, \xi) = \inf_{G \in \mathfrak{U}(\{\xi\})} [H_{[G]} \hat{K}(c, \cdot, \xi)].(c)$$
$$\geqq \lim_{n\to\infty} [H_{A_n} \hat{K}(c, \cdot, \xi)].(c) = 1\,.$$

5° We introduce several conventions. For $x \in X$, $\bar{x}$ and $\underline{x}$ denote, respectively, the points $x11\cdots$ and $x000\cdots$ of S. The *interval* $[\bar{x}, \underline{x}]$ *of* S is defined by the set of all the points of the form, $\xi = xb_1b_2b_3\cdots$ ($b_i=0$ or 1). For any $\xi = a_1a_2\cdots$ of S, $\varphi(\xi)$ is a real number of $[0, 1]$ defined by

$$\varphi(\xi) = \sum_{i=1}^{\infty} \frac{a_i}{2^i}.$$

The point ξ is called a *rational point of* S if there exists an $\eta \neq \xi$ such that $\varphi(\xi) = \varphi(\eta)$. Since each $\hat{K}(c, \cdot, \xi)$ is unbounded and the set of all the rational points of S, say S', is countable, the canonical measure μ representing a bounded x_t-harmonic functions satisfies $\mu(S')=0$. Moreover φ defines the into-homeomorphism from $S-S'$ to $[0, 1]$. Therefore, as was noted by Feller [**6**], we can take the interval $[0, 1]$ with the usual topology as a boundary so far as we treat bounded x_t-harmonic functions. Now we calculate $h(x, C)$. It is easily shown that the canonical measure $h(c, \cdot)$ for χ_X is given by

$$h(c, [\bar{y}, \underline{y}]) = \varphi(\bar{y}) - \varphi(\underline{y}), \tag{5.12}$$

so that, if x is a point of the length k, we have

$$h(x, [\bar{y}, \underline{y}]) = 2^k[\varphi(\bar{y}) - \varphi(\underline{y})] \qquad \text{for any} \quad y \leqq x.$$

Consequently, if the interval $[0, 1]$ is taken as a boundary, the réduite $h(x, \cdot)$, defined over $[0, 1]$, is the uniform probability measure over $[\varphi(\bar{x}), \varphi(\underline{x})]$. In paricular, $h(c, \cdot)$ is Lebesgue measure over $[0, 1]$. Using Theorem 3.2 in [**16**], the bounded x_t-harmonic function u is in one-to-one correspondence with the bounded measurable function f over $[0, 1]$ by the formula

$$u(x) = \int_0^1 f(\xi)\, h(x, d\xi) = 2^k \int_{\varphi(\underline{x})}^{\varphi(\bar{x})} f(\xi)\, d\xi. \tag{5.13}$$

These arguments show that, as the boundary to analyze bounded x_t-harmonic functions, the interval $[0, 1]$ is more convenient than the Martin boundary S.

13. Some singular examples. It is very easy to construct an example of $M_0 \neq \emptyset$ for which the conditions (CMP. 5), (CMP. 6) are not satisfied (see Example VI). But even if (CMP. 5), (CMP. 6) are assumed, we cannot assert $M_0 = \emptyset$. In fact, we have

EXAMPLE III.[18] Let X be the countable set consisting of a point c and the lattice points A_n, B_n, C_n $(n=1, 2, 3, \cdots)$ ordered on three parallel lines A, B, C. Consider now the CMP corresponding to the following system:

$$\begin{aligned}
&q(x)\equiv 1 \qquad \text{for any} \quad x\in X,\\
&\Pi(c, A_1) = \Pi(c, C_1) = 1/2,\\
&\Pi(A_n, A_{n+1}) = a_n, \quad \Pi(A_n, B_n) = 1-a_n,\\
&\Pi(B_n, B_{n+1}) = b_n, \quad \Pi(B_n, \infty) = 1-b_n,\\
&\Pi(C_n, C_{n+1}) = c_n, \quad \Pi(C_n, B_n) = 1-c_n,
\end{aligned}$$

Moreover we assume that a_n, $(1-a_n)$, b_n and $(1-b_n)$ are strictly positive for every n and

$$a_n = c_n \qquad \prod_{n=1}^{\infty} a_n = a > 0, \tag{5.14}$$

$$\frac{n b_{n-1}}{1-a_n} \to 0 \quad (n\to +\infty).$$

It is clear that c is the unique center and the conditions (CMP. 5), (CMP. 6) are satisfied. We can easily calculate $p(x, y)$ as follows:

$$p(0, A_m) = p(0, C_m) = 1/2\cdot \prod_{i=1}^{m-1} a_i,$$

$$p(0, B_m) = \sum_{i=1}^{m} \left(\prod_{j=1}^{i-1} a_j\right)(1-a_i) b_i \cdots b_{m-1},$$

$$p(x, A_m)\begin{cases} = \prod_{i=n}^{m-1} a_i & \text{if } x = A_n,\ n\leqq m\\ = 0 & \text{otherwise,}\end{cases}$$

$$p(x, B_m)\begin{cases} = \prod_{i=n}^{m-1} b_i & \text{if } x = B_n,\ n\leqq m\\ = \sum_{i=n}^{m} \left(\prod_{j=n}^{i-1} a_j\right)(1-a_i) b_i \cdots b_{m-1} & \text{if } x=A_n \text{ or } C_n,\ n\leqq m\\ = 0 & \text{otherwise,}\end{cases}$$

$$p(x, C_m)\begin{cases} = \prod_{i=n}^{m-1} a_i & \text{if } x = C_n,\ n\leqq m\\ = 0 & \text{otherwise,}\end{cases}$$

where we use the conventions, $\prod_{i=m}^{m-1} a_i = 1$ and $(1-a_i)b_i\cdots b_{m-1} = (1-a_m)$ for $i=m$. Therefore we have

18) This is no more than the discrete analogue of Martin's example ([**12**], Section 5, Example 2).

$$K(c, x, A_m)\begin{cases} = 1 & \text{if } x = c \\ = \dfrac{2}{\prod_{i=1}^{n-1} a_i} & \text{if } x = A_n,\ n \leqq m \\ = 0 & \text{otherwise,}\end{cases}$$

so that the sequence $\{A_m\}$ is a fundamental sequence and its corresponding limit function, say $\hat{K}(c, x, \xi_1)$, is given by

$$\hat{K}(c, x, \xi_1)\begin{cases} = 1 & \text{if } x = c \\ = \dfrac{2}{\prod_{i=1}^{n-1} a_i} & \text{if } x = A_n \\ = 0 & \text{otherwise.}\end{cases} \tag{5.15}$$

In the same way, $\{C_m\}$ is the fundamental sequence to which corresponds the following limit function,

$$\hat{K}(c, x, \xi_3)\begin{cases} = 1 & \text{if } x = c \\ = \dfrac{2}{\prod_{i=1}^{n-1} a_i} & \text{if } x = C_n \\ = 0 & \text{otherwise.}\end{cases} \tag{5. 16}$$

Next, take the sequence $\{B_m\}$. Then if $x = B_n$, it follows from (5. 14) that

$$K(c, x, B_m) = \frac{\prod_{i=1}^{m-1} b_i}{\sum_{i=1}^{m} (\prod_{j=1}^{i-1} a_j)(1-a_i) b_i \cdots b_{m-1}} < \frac{b_{m-1}}{(\prod_{i=1}^{m-1} a_i)(1-a_m)}$$

$$< \frac{b_{m-1}}{a(1-a_m)} \to 0 \qquad (m \to +\infty)\,.$$

Also, if $x = A_n$ or C_n, we get

$$K(c, x, B_m) = \frac{\sum_{i=n}^{m} (\prod_{j=n}^{i-1} a_j)(1-a_i) b_i \cdots b_{m-1}}{\sum_{i=1}^{m} (\prod_{j=1}^{i-1} a_j)(1-a_i) b_i \cdots b_{m-1}}$$

$$= \frac{(\prod_{i=n}^{m-1} a_i)(1-a_m) + o(1-a_m)}{(\prod_{i=1}^{m-1} a_i)(1-a_m) + o(1-a_m)} \to \frac{1}{\prod_{i=1}^{n-1} a_i} \qquad (m \to +\infty)\,.$$

Consequently $\{B_m\}$ is the fundamental sequence whose determining limit function is as follows:

$$(5.17)\qquad \hat{K}(c, x, \xi_2)\begin{cases} = 1 & \text{if} \quad x = c \\ = \dfrac{1}{\prod_{i=1}^{n-1} a_i} & \text{if} \quad x = A_n \text{ or } C_n \\ = 0 & \text{otherwise.} \end{cases}$$

Summing up (5.15), (5.16) and (5.17), we know that $\partial\hat{X}$ consists of three points ξ_1, ξ_2, ξ_3 and that

$$\hat{K}(c, x, \xi_2) = 1/2\cdot\{\hat{K}(c, x, \xi_1)+\hat{K}(c, x, \xi_3)\}\ ,$$

which proves $\xi_2 \in M_0$.

To continue, we shall show that the statements in Theorem 3.2 are not always true unless the conditions (CMP. 5), (CMP. 6) are satisfied.

EXAMPLE IV. *X and $\hat{X}$ are not homeomorphic.* We shall give an example in which the mapping $\theta(X\rightarrow\hat{X})$ is one-to-one but not topological, for otherwise our statement is evident. Let X be the set of all nonnegative integers and x_t, the CMP (over X) satisfying the conditions below: $q(x)\equiv 1$ for every x in X, $\Pi(0, y)=1/2^y$ for $y\geqq 1$ and $\Pi(x, \infty)=1$ for $x\geqq 1$. Cleary the state 0 is the unique center and we have

$$K(0, x, y)\begin{cases} = 1 & \text{if} \quad x = 0\,, \\ = 2^x & \text{if} \quad y = x\,, \\ = 0 & \text{otherwise}\,, \end{cases}$$

so that, for any sequence $\{y_n\}$ having no limit points in X, $K(0, x, y_n)$ converges to $K(0, x, 0)$ as $n\rightarrow\infty$. Therefore we have no fundamental sequences, namely, $M=\hat{X}$. Thus $\hat{X}$ is compact and hence is not homeomorphic to X. In particular, the ρ-metric in the canonical Martin space which coincides with X as a set is characterized by $\rho(0, y)\rightarrow 0$ $(y\rightarrow+\infty)$.

EXAMPLE V. *$\hat{X}$ is not open in M.* Let X be the set consisting of a point 0 and all the points (i, j); $i, j=1, 2, 3, \cdots$. Consider the CMP which is determined by the following system: $q(x)\equiv 1$ for any x in X, $\Pi(0, (i, 1))=1/2^i$ for $i\geqq 1$ and $\Pi(i, j), (i, j+1))=1$ for $i, j\geqq 1$. Since 0 is the unique center, we get

$$K(0, x, y)\begin{cases} = 1 & \text{if} \quad x = 0 \\ = 2^i & \text{if} \quad x = (i, j),\ y = (i, k),\ j\leqq k\,, \\ = 0 & \text{otherwise.} \end{cases}$$

Therefore it follows that, for each i, the sequence $\{y_n^{(i)}=(i, n);\ n=1, 2, \cdots\}$ is a fundamental sequence and its determining limit function, denoted by $\hat{K}(0, x, \xi_i)$, is given by

$$\hat{K}(0, x, \xi_i)\begin{cases} = 1 & \text{if} \quad x = 0 \\ = 2^i & \text{if} \quad x = (i, j)\,, \\ = 0 & \text{otherwise,} \end{cases}$$

so that $\hat{K}(0, x, \xi_i)$ converges to $K(0, x, 0)$ with $i\to\infty$, which proves that $\partial\hat{X}$ is not closed in M.

EXAMPLE VI. *$\hat{K}(c, x, \xi)$ is not x_t-harmonic for some $\xi\ni\partial\hat{X}$.* Let x_t be the CMP over $X=\{0, 1, 2, \cdots\}$ as below: $q(x)\equiv 1$ for every x in X, $\Pi(0, y)=1/2^y$ for $y\geqq 1$, $\Pi(1, y)=1/2^{y-1}$ for $y\geqq 2$, and $\Pi(x, \infty)=1$ for $x\geqq 2$. Then it follows that $p(0, y)=p(1, y)=1/2^{y-1}$ for $y\geqq 2$ and therefore

$$K(0, x, 0)\begin{cases} = 1 & \text{for} \quad x = 0 \\ = 0 & \text{otherwise,} \end{cases}$$

$$K(0, x, 1)\begin{cases} = 1 & \text{for} \quad x = 0 \\ = 2 & \text{for} \quad x = 1 \\ = 0 & \text{otherwise,} \end{cases}$$

and if $y\geqq 2$,

$$K(0, x, y)\begin{cases} = 1 & \text{for} \quad x = 0 \ \text{ or } \ 1 \\ = 2^{x-1} & \text{for} \quad x = y \\ = 0 & \text{otherwise.} \end{cases}$$

Letting $y\to\infty$, we get

$$\lim_{y\to\infty} K(0, x, y)\begin{cases} = 1 & \text{for} \quad x = 0 \ \text{ or } \ 1 \\ = 0 & \text{otherwise} \end{cases}$$
$$= 1/2\cdot\{K(0, x, 0)+K(0, x, 1)\}\,.$$

Consequently our Martin boundary consists of one point, say ξ, and the corresponding function $\hat{K}(0, x, \xi)$ is a potential, this is, it is not x_t-harmonic. Clearly this also is another example of $M_0\neq\emptyset$.

14. Supplements. The problem to extend the Martin boundary theory to non-countable Markov processes is very much interesting, but has been not solved yet in the complete form. The typical example is seen in Martin's work [**12**], in which he discussed, to speak probabilistically, the boundary theory for the

Brownian motion over a domain in n-dimensional euclidean space. The main difficulty for the general boundary theory is in proving the analogue of Theorem 2.7 which has played an essential role in our special case. But under some strong conditions on the given process, we can prove the theorem (cited above) in a weaker form and, using it, we can only derive the unique representation theorem of nonnegative x_t-harmonic functions for the class of Markov processes which cover one-dimensional diffusion, n-dimensional Brownian motion (over a domain of R^n), the space-time Poisson process (introduced in [**15**]) and many others. We shall discuss the detail in another chance.

Finally we shall summarize the dual boundary theory. First note that the dual of x_t-superharmonic functions should be taken not relative to Π, but to $\mathfrak{G}$. Strictly speaking,

DEFINITION 5.1. A set function ν over $\tilde{X}$, vanishing on ∞, is *x_t-superharmonic at b* if

(5.18) $$-\infty < \nu(y) \leqq +\infty \quad \text{for each one point set } \{y\},$$

and

(5.19) $$\nu\mathfrak{G}.(b) \leqq 0,$$

that is,

$$\nu q^{-1}\Pi.(b) \equiv \int_{\tilde{X}} \nu(dx) q^{-1}(x) \Pi(x, b) \leqq \nu(b) q^{-1}(b) = \nu q^{-1}.(b).$$

The dual concepts of 'x_t-subharmonic' and 'x_t-harmonic' are introduced in the same way. In particular, the x_t-harmonic set function ν is a σ-finite set function which satisfies

(5.20) $$\nu q^{-1}\Pi.(E) = \int_X \nu(dx) q^{-1}(x) \Pi(x, E) = \int_E \nu(dy) q^{-1}(y) = \nu q^{-1}.(E)$$

for every finite subset E of X. It follows that a positive set function ν is x_t-superharmonic if and only if it is an *excessive measure* in Hunt's sense, namely, it satisfies

(5.21) $$\nu H^t.(E) \leqq \nu(E) \quad \text{for any } t \in T \text{ and any set } E \subset X.$$

To proceed to the dual boundary theory, we shall introduce the dual of (CMP. 4) as follows:

(CMP. 4)* There exists at least one state c^* such that

$$p(x, c^*) > 0 \quad \text{for every} \quad x \in X.$$

Such c^* is called the *dual center* of the process.

Now define

$$(5.22) \qquad K^*(x, y, c^*) = \lim_{\alpha\to 0} \frac{G_\alpha(x, y)}{G_\alpha(x, c^*)} = L^*(x, y, c^*) R(y, c^*) \leq \frac{1}{p(y, c^*)} R(y, c^*),$$

where $L^*(x, y, c^*) = p(x, y)/p(x, c)$ and $R(y, c^*) = \lim_{\alpha\to 0} G_\alpha(y, y)/G_\alpha(c^*, c^*)$. Let X_{c^*} be the family of y-functions, $\{K^*(x, \cdot, c^*) ; x \in X\}$, and M_{c^*}, the set of all the limit functions of X_{c^*}. Then M_{c^*} is compact with the following metric

$$\rho^*(\xi, \eta) = \int_X \frac{|\xi(y) - \eta(y)|}{1 + |\xi(y) - \eta(y)|} m^*(dy) \qquad \text{for} \quad \xi, \eta \in M_{c^*}.$$

Each function of M_{c^*}, consider as a set function, is x_t-superharmonic. The *dual Martin space* M^* and *dual boundary* $\partial\hat{X}^*$ are defined as the spaces homeomorphic to M_{c^*} and $M_{c^*} - X_{c^*}$, respectively. The element of M_{c^*} corresponding to $\xi \in M^*$ is denoted by $\hat{K}^*(\xi, y, c^*)$. In the same way, the limit function of $L^*(x, y, c^*)$ corresponding to $\hat{K}^*(\xi, y, c^*)$ is denoted by $\hat{L}^*(\xi, y, c^*)$. M_1^* is the set of $\xi \in M^*$ such that $\hat{K}^*(\xi, y, c^*)$ is a *minimal x_t-superharmonic set function* whose definition will be clear. Then we can obtain the dual of the main theorem in Chapter 5 as follows:

For any nonnegative x_t-superharmonic set function ν, there exists one and only one positive measure μ^* over M_1^* for which we have

$$(5.23) \qquad \nu(y) = \int_{M_1^*} \hat{K}^*(\xi, y, c^*) \mu^*(d\xi) = R(y, c^*) \int_{M_1^*} \hat{L}^*(\xi, y, c^*) \mu^*(d\xi)$$

for every one point set $\{y\}$. In particular, if ν is an x_t-harmonic set function, the total mass of μ^* is carried on the minimal harmonic part $(\partial\hat{X}^*)_1 \cup \hat{R}_1^*$,[19] *namely, ν is expressible*

$$(5.24) \qquad \nu(y) = R(y, c^*) \int_{(\partial\hat{X}^*)_1 \cup \hat{R}_1^*} \hat{L}^*(\xi, y, c^*) \mu^*(d\xi).$$

If our process is of the discrete time parameter, the formula (1.38) shows that the class of x_t-harmonic set functions coincides

19) According to (CMP. 4)*, X contains at most one indecomposable recurrent set, denoted by R_1. If such R_1 does not exist, $\hat{R}_1^* = \{\phi\}$ conventionally. Otherwise, $\hat{R}_1^*$ is one point set and $\hat{K}^*(\hat{R}_1^*, y, c^*) \equiv K^*(r_1, y, c^*)$ for any $r_1 \in R_1$. Also, of course, $(\partial\hat{X}^*)_1 = \partial\hat{X}^* \cap M_1^*$.

with that of H^t-invariant σ-finite set functions. Therefore the above formula (5.24) determines the class of so-called H^t-invariant measures. This is the generalization of the well known *Doeblin-Lévy formula for the recurrent CMP* (for example, see Derman [2], p. 542) to the non-recurrent case. In fact, if X consists of exactly one indecomposable recurrent set, M^* is one point set and $\hat{L}^*(\xi, y, c^*)=1$ for every $y \in X$, so that any excessive measure (and hence, of course, any H^t-invariant measure) is a constant multiple of $R(y, c^*)$. Clearly $R(y, c^*)$ can be rewritten as

$$R(y, c^*) = \lim_{n\to\infty} \frac{\sum_{t=0}^{n} H^t(y, y)}{\sum_{t=0}^{n} H^t(c^*, c^*)}, \tag{5.25}$$

which is no more than the Doeblin-Lévy formula.

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