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Brownian motions on the 3-dimensional rotation group

By

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1. INTRODUCTION

K. Itô [1950]* and K. Yosida [1952] defined and constructed all the Brownian motions on Lie groups. The purpose of the present paper is to give a new method for constructing the Brownian sample paths on the 3-dimensional rotation group SO (3). The idea is to inject the differentials $\mathfrak{z}(dt)$ of a (skew) Brownian motion on the Lie algebra into SO (3) via the exponential map e and to piece the resulting infinitessimal rotations $e[\mathfrak{z}(dt)]$ into a continuous path (product integral):

1.1
$$\begin{aligned} &\mathfrak{g}_{\infty}(t) \\ &= \bigwedge_{s \leq t} e[\mathfrak{z}(ds)] \\ &= \lim_{n \uparrow \infty} e[\mathfrak{z}(0, 2^{-n})] \cdots e[\mathfrak{z}(2^{-n}[2^n t], t)] \qquad t \geq 0 . \end{aligned}$$

The same trick gives the Brownian motions on all the classical (non-exceptional) simple Lie groups of É. Cartan's list.

F. Perrin [1928] computed the counterparts of the Gauss and Poisson laws on SO (3); for a sketch of Perrin's results, see P. Lévy [1948: 194-203].

I divide the paper into 8 sections: 2 deals with SO (3), its Lie algebra, and its differential operators; 3 with its Brownian motions. 4 states the program of injection. 5 is devoted to sums $\mathfrak{g} = \sum_{n \geq 0} \mathfrak{j}_n$ of stochastic integrals

* K. Itô [1950] means K. Itô's 1950 publication listed at the end of this paper; K. Itô [1950: 6-8] would mean pages 6-8 of that work.

1.2 $$\mathfrak{j}_0 = \mathfrak{e}, \quad \mathfrak{j}_n = \int_0^t \mathfrak{j}_{n-1}\, \mathfrak{j}(ds) \qquad \mathfrak{j}(ds) = \mathfrak{z}(ds) + \frac{1}{2}\mathfrak{z}(ds)^2.$$

6 contains the identification of $\mathfrak{g}$ with $\mathfrak{g}_\infty = \bigwedge_{s\le t} e[\mathfrak{z}(ds)]$. 7 identifies $\mathfrak{g} = \mathfrak{g}_\infty$ with a Brownian motion on SO(3). 8 contains an example which C. D. Gorman [1958] has also treated*. I will suppose, for the purposes of 5, 6, and 7, that the reader is familiar with stochastic integrals as presented, for example, in K. Itô [1951].

I wish to thank H. Trotter who suggested the problem of 8 which was the starting point of this paper. I must also thank K. Itô for helpful talks and for a trick used in 5.

2. ROTATION GROUP

R^3 is the (real) 3-dimensional euclidean space of points $x = (x_1, x_2, x_3)$, *etc.*; $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$; $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$; $\times$ is the outer product for R^3 ($e_1 \times e_2 = e_3$, $e_2 \times e_3 = e_1$, $e_3 \times e_1 = e_2$); small German letters $\mathfrak{f}$, *etc.* stand for (real) 3×3 matrices; $^*\mathfrak{f}$ is the transpose of $\mathfrak{f}$; $\mathfrak{f}^{-1}$ its inverse; $|\mathfrak{f}|$ its norm; SO(3) is the (multiplicative) group of 3×3 orthogonal matrices $\mathfrak{g}$ ($\mathfrak{g}^*\mathfrak{g} = \mathfrak{e} =$ the unit) of determinant $+1$.

Bringing in the infinitessimal rotations

2.1 $$\mathfrak{e}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathfrak{e}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathfrak{e}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the product

2.2 $$[\mathfrak{f}_1, \mathfrak{f}_2] = \mathfrak{f}_1\mathfrak{f}_2 - \mathfrak{f}_2\mathfrak{f}_1,$$

a short computation justifies

2.3 $$[\mathfrak{e}_1, \mathfrak{e}_2] = \mathfrak{e}_3, \quad [\mathfrak{e}_2, \mathfrak{e}_3] = \mathfrak{e}_1, \quad [\mathfrak{e}_3, \mathfrak{e}_1] = \mathfrak{e}_2,$$

which shows that the vector space A of matrices

2.4 $$\mathfrak{a} = \alpha_1\mathfrak{e}_1 + \alpha_2\mathfrak{e}_2 + \alpha_3\mathfrak{e}_3 \qquad \alpha = (\alpha_1, \alpha_2, \alpha_3) \in R^3$$

under the product 2.2 is isomorphic to R^3 under the outer product. A is the so-called Lie algebra of SO(3).

A is connected to SO(3) via the exponential map

2.5 $$e(\mathfrak{f}) = \sum_{n\ge 0} \mathfrak{f}^n / n!$$

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* Note added in proof: for a complete account, see C. D. Gorman, Brownian motion of rotation. Trans. Amer. Math. Soc. 94 (1960), 103–117.

and the logarithm

2.6 $$l(\mathfrak{f}) = \sum_{n\geq 1}(\mathfrak{f}-\mathfrak{e})^n/n \qquad |\mathfrak{f}-\mathfrak{e}|<1:$$

in fact, l maps the neighborhood $|\mathfrak{g}-\mathfrak{e}|<1$ of SO (3) onto a neighborhood of the 0 element of A and e maps the neighborhood $|\mathfrak{a}|<lg2$ of A onto a neighborhood of e in SO (3); both maps are $1:1$; in the first case, the inverse map is $\mathfrak{e}$; in the second, it is l.

SO (3) is homeomorphic to the 3-dimensional projective space P^3: in fact, P^3, viewed as the spherical surface $S^3\subset R^4$ with antipodal identifications, is homeomorphic to the solid 3-dimensional ball of diameter 2π with antipodal surface points identified, and the map taking α ($|\alpha|\leq\pi$) into the rotation $\mathfrak{g}$ of total angle $|\alpha|$ about the axis α in the sense of the right-hand screw rule is a homeomorphism of the solid ball with surface identifications onto SO (3) (for small α, $\alpha\rightarrow\mathfrak{g}$ is just the exponential map).

Consider the class C^2[SO (3)] of functions $f=f(\mathfrak{g})$ defined on SO (3) such that, for $\mathfrak{g}\in$ SO (3), $h(\alpha)=f(\mathfrak{g}e[\alpha_1\mathfrak{e}_1+\alpha_2\mathfrak{e}_2+\alpha_3\mathfrak{e}_3])$ is of class C_2 near $\alpha=0$ and define

2.7 $$(\mathfrak{G}_1 f)(\mathfrak{g}) = h_1(0), \quad (\mathfrak{G}_2 f)(\mathfrak{g}) = h_2(0), \quad (\mathfrak{G}_3 f)(\mathfrak{g}) = h_3(0),$$

where the subscripts stand for partials.

Writing out the power series for f at $\mathfrak{g}=\mathfrak{e}$ up to terms of degree 2, it develops that, with the commutator product 2.2,

2.8 $$[\mathfrak{G}_1, \mathfrak{G}_2] = \mathfrak{G}_3, \quad [\mathfrak{G}_2, \mathfrak{G}_3] = \mathfrak{G}_1, \quad [\mathfrak{G}_3, \mathfrak{G}_1] = \mathfrak{G}_2,$$

so that the algebra of differential operators

2.9 $$\mathfrak{G} = \alpha_1\mathfrak{G}_1+\alpha_2\mathfrak{G}_2+\alpha_3\mathfrak{G}_3 \qquad \alpha = (\alpha_1, \alpha_2, \alpha_3)\in R^3$$

under the commutator product is isomorphic to the Lie algebra; under the usual product, $\mathfrak{G}_1$, $\mathfrak{G}_2$, $\mathfrak{G}_3$ generate the algebra of differential operators on SO (3) commuting with left translations.

Contracting $\mathfrak{G}_1$, $\mathfrak{G}_2$, $\mathfrak{G}_3$ to the class of functions $f\in C^2$[SO (3)] such that $f(\mathfrak{g}_1)=f(\mathfrak{g}_2)$ if $\mathfrak{g}_1 e_3=\mathfrak{g}_2 e_3$ ($e_3=(0, 0, 1)$) and viewing them as differential operators on the coset space SO (3)/SO (2) $=S^2$ of points $\mathfrak{g}e_3$, one gets the following 3 operators:

2.10 $$\sin\phi\frac{\partial}{\partial\psi}-\frac{\cos\psi\cos\phi}{\sin\psi}\frac{\partial}{\partial\phi}, \quad \cos\phi\frac{\partial}{\partial\psi}-\frac{\cos\psi\sin\phi}{\sin\psi}\frac{\partial}{\partial\phi}, \quad \frac{\partial}{\partial\phi},$$

where $0 \leq \psi \leq \pi$ is the colatitude and $0 \leq \phi < 2\pi$ the longitude of $\mathfrak{g}e_3$.

The reader is referred to W. Maak [1950: 161-179, 209-210] and to I. Gelfand and Z. Ya. Šapiro [1956; 207-245] for proofs and more information.

3. BROWNIAN MOTIONS ON SO (3).

Consider the space of all continuous (sample) paths w: $t \in [0, +\infty) \to \mathfrak{g}(t) \in SO(3)$, let $\boldsymbol{B}_t$ be the smallest Borel algebra of subsets of the path space measuring the entries of $\mathfrak{g}(s)$ for each $s \leq t$, let $\boldsymbol{B}$ be the smallest Borel algebra containing all of these, and take non-negative Borel measures $P_{\mathfrak{g}}$ defined on $\boldsymbol{B}$ of total mass $+1$, one to each $\mathfrak{g} \in SO(3)$, such that $P_{\mathfrak{g}}[\mathfrak{g}(0)=\mathfrak{g}]=1$ for each $\mathfrak{g} \in SO(3)$ and $P_{\mathfrak{g}}(B)$ is Borel on SO (3) for each $B \in \boldsymbol{B}$.

$[W, \boldsymbol{B}, P_{\mathfrak{g}} : \mathfrak{g} \in SO(3)]$ is said to be a (left) Brownian motion on SO (3) if it is Markov:

3.1 $$P_{\cdot}[\mathfrak{g}(t+s) \in d\mathfrak{g} \mid \boldsymbol{B}_s] = P_{\mathfrak{h}}[\mathfrak{g}(t) \in d\mathfrak{g}] \mid_{\mathfrak{h}=\mathfrak{g}(s)} \qquad t,\ s \geq 0$$

and if it is also (left) group-invariant:

3.2 $$P_{\mathfrak{g}}(B) = P_{\mathfrak{e}}(\mathfrak{g}^{-1}B) \qquad \mathfrak{g} \in SO(3),\ B \in \boldsymbol{B},$$

where $\mathfrak{g}^{-1}B$ is the set of sample paths such that the translated path $\mathfrak{g}^{-1}\mathfrak{g}(t) : t \geq 0$ lies in B.

Given such a Brownian motion, its generator $\mathfrak{G}$ is defined as

3.3 $$(\mathfrak{G}f)(\mathfrak{g}) = \lim_{t \downarrow 0} t^{-1} E_{\mathfrak{g}}[f(\mathfrak{g}(t)) - f(\mathfrak{g})]$$
$$\mathfrak{g} \in SO(3),\ E_{\cdot}(f) = \int f P_{\cdot}(dw)$$

for the class $C(\mathfrak{G})$ of functions $f \in C[SO(3)]$ such that the limit exists (pointwise) and the resulting $\mathfrak{G}f$ lies again in $C[SO(3)]$.

Because $C(\mathfrak{G})$ is a (left) ideal in C under the convolution

3.4 $$(f_1 \otimes f_2)(\mathfrak{h}) = \int f_1(\mathfrak{h}\mathfrak{g}^{-1}) f_2(\mathfrak{g}) d\mathfrak{g},$$

where $d\mathfrak{g}$ is Haar measure for SO (3), it is clear from the inclusion $C^2 \otimes C(\mathfrak{G}) \subset C^2$ that $C(\mathfrak{G}) \cap C^2$ is well-populated, and it is possible to conclude that $\mathfrak{G}$ is an elliptic differental operator of degree 2 commuting with (left) translations:

3.5 $$\mathfrak{G} = \frac{1}{2} \sum_{i,j \leq 3} l_{ij} \mathfrak{E}_i \mathfrak{E}_j + \sum_{i \leq 3} m_i \mathfrak{E}_i ,$$

where

3.6 $$\mathfrak{l} = \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix} = \lim_{t \downarrow 0} t^{-1} E_{\mathfrak{e}}[\mathfrak{g}(t) - \mathfrak{e}]^2$$

is symmetric ($\mathfrak{l} = {}^*\mathfrak{l}$) and non-negative definite ($\mathfrak{l} \geq 0$), and

3.7 $$\mathfrak{m} = m_1 \mathfrak{e}_1 + m_2 \mathfrak{e}_2 + m_3 \mathfrak{e}_2 = \lim_{t \downarrow 0} t^{-1} E_{\mathfrak{e}}[\mathfrak{g}(t) - \mathfrak{e}] .$$

On the other hand, an elliptic differential operator such as 3.5 with $\mathfrak{l} = {}^*\mathfrak{l} \geq 0$ generates a (left) Brownian motion ; for the proofs, the reader is referred to the articles of K. Itô [1950] and K. Yosida [1952].

4. INJECTION

Consider, now, the standard Brownian motion on R^3 with generator $\frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)$ and sample paths $w: t \to x(t) \in R^3$, write P for the Wiener measure for paths starting at 0, let $E(f) = \int f P(dw)$, select an SO(3) Brownian generator $\mathfrak{G}$ as in 3.5, introduce the (skew) R^3 Brownian motion

4.1 $$z(t) = \mathfrak{l}^{1/2} x(t) + mt \qquad t \geq 0 ,$$

where $\mathfrak{l}^{1/2}$ is the non-negative definite root of $\mathfrak{l}$ and $m = (m_1, m_2, m_3)$, and, making the identification of R^3 and A ($e_1 \to \mathfrak{e}_1$, $e_2 \to \mathfrak{e}_2$, $e_3 \to \mathfrak{e}_3$), think of 4.1 as a Brownian motion $\mathfrak{z} = z_1 \mathfrak{e}_1 + z_2 \mathfrak{e}_2 + z_3 \mathfrak{e}_3$ in the Lie algebra itself.

We inject $\mathfrak{z}$ into SO(3) via the exponental, thus:

4.2 $$\begin{aligned} \mathfrak{g}_n(t) &= \mathfrak{g}(0) \in \mathrm{SO}(3) & t = 0 \\ &= \mathfrak{g}_n(j2^{-n}) e[\mathfrak{z}(\Delta)] & t > 0 \end{aligned}$$
$$\Delta = [j2^{-n}, t), \quad j = [2^n t], \quad \mathfrak{z}(\Delta) = \mathfrak{z}(t) - \mathfrak{z}(j2^{-n})$$

and assert that $\mathfrak{g}_n$ converges to a limit $\mathfrak{g}_\infty$, that the convergence is uniform on compacts, and that $\mathfrak{g}_\infty$ is the Brownian motion on SO(3) with generator $\mathfrak{G}$.

With the help of P. Lévy's Hölder condition [1937 : 168–172]

4.3 $$\limsup_{\substack{t=t_2-t_1\downarrow 0 \\ 0\le t_1\le t_2\le 1}} \frac{|x(t_2)-x(t_1)|}{\sqrt{2t|\lg t|}} = 1$$

and the multiplication table

4.4 $$\mathfrak{e}_1^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathfrak{e}_2^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathfrak{e}_3^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathfrak{e}_1\mathfrak{e}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{e}_2\mathfrak{e}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathfrak{e}_3\mathfrak{e}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

it is clear that, up to terms involving $\mathfrak{z}(\Delta)^3$ (of magnitude $<2^{-4n/3}$),

4.5 $$\mathfrak{g}_n(t)-\mathfrak{g}_n(j2^{-n}) = \mathfrak{g}_n(j2^{-n})(e[\mathfrak{z}(\Delta)]-\mathfrak{e}) = \mathfrak{g}_n(j2^{-n})(\mathfrak{z}(\Delta)+\frac{1}{2}\mathfrak{z}(\Delta)^2),$$

where

4.6 $$\mathfrak{z}(\Delta) = z_1(\Delta)\mathfrak{e}_1+z_2(\Delta)\mathfrak{e}_2+z_3(\Delta)\mathfrak{e}_3 = \begin{pmatrix} 0 & -z_3(\Delta) & z_2(\Delta) \\ z_3(\Delta) & 0 & -z_1(\Delta) \\ -z_2(\Delta) & z_1(\Delta) & 0 \end{pmatrix}$$

and

4.7 $$\frac{1}{2}\mathfrak{z}(\Delta)^2$$
$$= \frac{1}{2}\begin{pmatrix} -[z_2(\Delta)^2+z_3(\Delta)^2] & z_1(\Delta)z_2(\Delta) & z_1(\Delta)z_3(\Delta) \\ z_2(\Delta)z_1(\Delta) & -[z_3(\Delta)^2+z_1(\Delta)^2] & z_2(\Delta)z_3(\Delta) \\ z_3(\Delta)z_1(\Delta) & z_3(\Delta)z_2(\Delta) & -[z_1(\Delta)^2+z_2(\Delta)^2] \end{pmatrix},$$

and the reader who knows stochastic integrals will at once conjecture that $\mathfrak{g}_\infty=\lim_{n\uparrow\infty}\mathfrak{g}_n$ is the solution of

4.8 $$\mathfrak{g}(t) = \mathfrak{g}(0)+\int_0^t \mathfrak{g}(s)\,\mathfrak{j}(ds) \qquad t\ge 0,$$

where

4.9 $$\mathfrak{j}(ds) = \mathfrak{z}(ds)+\frac{1}{2}\,\mathfrak{k}\,ds$$

$$\mathfrak{k} = \frac{1}{2}\begin{pmatrix} -[l_{22}+l_{33}] & l_{12} & l_{13} \\ l_{21} & -[l_{33}+l_{11}] & l_{23} \\ l_{31} & l_{32} & -[l_{11}+l_{22}] \end{pmatrix};$$

$\mathfrak{k}$ is computed from 4.7 using the multiplication table

4.10

	dt	$x_1(dt)$	$x_2(dt)$	$x_3(dt)$
dt	0	0	0	0
$x_1(dt)$	0	dt	0	0
$x_2(dt)$	0	0	dt	0
$x_3(ds)$	0	0	0	dt

and the resulting

4.11

	dt	$z_1(dt)$	$z_2(dt)$	$z_3(dt)$
dt	0	0	0	0
$z_1(dt)$	0	$l_{11}dt$	$l_{12}dt$	$l_{13}dt$
$z_2(dt)$	0	$l_{21}dt$	$l_{22}dt$	$l_{23}dt$
$z_3(dt)$	0	$l_{31}dt$	$l_{32}dt$	$l_{33}dt$

;

and $\int \mathfrak{g}\, d\mathfrak{j}$ means perform the matrix multiplication $\mathfrak{g}\, d\mathfrak{j}$ and compute the 9 resulting stochastic integrals.

The program is to solve 4.8 for $\mathfrak{g}$ and to compare $\mathfrak{g}$ and $\mathfrak{g}_n$: the result will be that $\lim_{n\uparrow\infty} \mathfrak{g}_n = \mathfrak{g}$, permitting the identification of the product integral $\bigwedge_{s\le t} e[\mathfrak{z}(ds)]$ with $\mathfrak{g}$ and, at the same time, proving its existence.

5. SOLVING THE INTEGRAL EQUATION

Given (constant) $\mathfrak{g}(0) \in SO(3)$, consider the continuous SO(3) solutions $\mathfrak{g} = \mathfrak{g}(t)$ of 4.8 such that, for each $t \geq 0$, $\mathfrak{g}(t)$ depends upon $\mathfrak{z}(s) : s \leq t$ alone.

Given 2 such solutions $\mathfrak{g}_1$ and $\mathfrak{g}_2$, their difference $\mathfrak{g} = \mathfrak{g}_2 - \mathfrak{g}_1$ satisfies

5.1 $$\mathfrak{g}(t) = \int_0^t \mathfrak{g}(s)\, \mathfrak{j}(ds) \qquad t \geq 0$$

and, using a formula of K. Itô [1951: 60], it develops that

5.2
$$\begin{aligned}\mathfrak{g}(t)^*\mathfrak{g}(t) &= \int_0^t \mathfrak{g}\,d\mathfrak{j}^*\mathfrak{g} + \int_0^t \mathfrak{g}^*d\mathfrak{j}^*\mathfrak{g} + \int_0^t \mathfrak{g}\,d\mathfrak{j}^*d\mathfrak{j}^*\mathfrak{g} \\ &= \int_0^t g(d\mathfrak{j} + {}^*d\mathfrak{j} + d\mathfrak{j}^*d\mathfrak{j})^*\mathfrak{g}\,.\end{aligned}$$

But, according to the multiplication table 4.11,

5.3
$$d\mathfrak{j} + {}^*d\mathfrak{j} = \mathfrak{k}\,ds\,, \qquad d\mathfrak{j}^*d\mathfrak{j} = d\mathfrak{z}^*d\mathfrak{z} = -\mathfrak{k}\,ds\,,$$

and therefore $\mathfrak{g}^*\mathfrak{g} = 0$, which is impossible unless $\mathfrak{g}_2 = \mathfrak{g}_1$.

Consider, next, the presumptive solution

5.4
$$\mathfrak{g} = \sum_{n\geq 0} \mathfrak{j}_n$$
$$\mathfrak{j}_n(t) = \int_0^t \mathfrak{j}_{n-1}\,d\mathfrak{j} \qquad n \geq 1\,,\quad \mathfrak{j}_0 = \mathfrak{g}(0)$$

and let us check that the sum is convergent using the following trick of K. Itô.

Given $\mathfrak{f}$, if $\gamma_1(=|\mathfrak{f}|^2) \geq \gamma_2 \geq \gamma_3$ are the eigenvalues of $\mathfrak{f}^*\mathfrak{f}$, if $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$ are the corresponding projections $(\gamma_1\mathfrak{p}_1 + \gamma_2\mathfrak{p}_2 + \gamma_3\mathfrak{p}_3 = \mathfrak{f}^*\mathfrak{f})$, and if $\int_{S^2} do$ is the arithmetical average over the spherical surface S^2, then

5.5
$$\int_{S^2} |\mathfrak{p}_1 o|^2 do = \int_{S^2} |\mathfrak{p}_2 o|^2 do = \int_{S^2} |\mathfrak{p}_3 o|^2 do = \frac{1}{3}$$

and

5.6
$$\int_{S^2} o\mathfrak{f}^*\mathfrak{f}o\,do \geq \gamma_1 \int_{S^2} |\mathfrak{p}_1 o|^2 do = \frac{1}{3}|\mathfrak{f}|^2\,,$$

where $o\mathfrak{f}^*\mathfrak{f}o$ is the inner product of $o \in S^2$ and $\mathfrak{f}^*\mathfrak{f}o \in R^3$.

Viewing $\mathfrak{l}^{1/2}x(t): t \geq 0$ as a Brownian motion $\mathfrak{y}$ in the Lie algebra,

5.7
$$\mathfrak{j}_n(t) = \int_0^t \mathfrak{j}_{n-1}\mathfrak{i}\,ds + \int_0^t \mathfrak{j}_{n-1}\mathfrak{y}(ds) \qquad \mathfrak{i} = \frac{1}{2}\mathfrak{k} + \mathfrak{m}\,,$$

and using 5.6 to check

5.8
$$\begin{aligned} &E\left|\int_0^t \mathfrak{j}_{n-1}\mathfrak{y}(ds)\right|^2 \\ &\leq 3\,E\int_{S^2} o\left[\int_0^t \mathfrak{j}_{n-1}\mathfrak{y}(ds)^*\int_0^t \mathfrak{j}_{n-1}\mathfrak{y}(ds)\right] o\,do \\ &= 3\,E\int_{S^2} o\left[\int_0^t \mathfrak{j}_{n-1}\mathfrak{y}(ds)^*\mathfrak{y}(ds)^*\mathfrak{j}_{n-1}\right] o\,do \\ &= -3\,E\int_{S^2} o\left[\int_0^t \mathfrak{j}_{n-1}\mathfrak{k}^*\mathfrak{j}_{n-1}\,ds\right] o\,do \leq 3|\mathfrak{k}|\,E\int_0^t |\mathfrak{j}_{n-1}|\,ds\,,\end{aligned}$$

it is seen that

5.9 $$(E|\mathfrak{j}_n(t)|^2)^{1/2}$$
$$\leq \left(E\left|\int_0^t \mathfrak{j}_{n-1}\mathfrak{i}\,ds\right|^2\right)^{1/2} + \left(E\left|\int_0^t \mathfrak{j}_{n-1}\mathfrak{y}(ds)\right|^2\right)^{1/2}$$
$$\leq (|\mathfrak{i}|t^{1/2} + (3|\mathfrak{k}|)^{1/2})\left(E\int_0^t |\mathfrak{j}_{n-1}|^2 ds\right)^{1/2}.$$

As to $\sum_{n\geq 0}\mathfrak{j}_n$, since $\mathfrak{h}_n = \mathfrak{j}_n - E(\mathfrak{j}_n)$ is a martingale, $|\mathfrak{h}_n|$ is a semi-martingale, so that, using a result of Doob [1953: 314-315] in conjunction with 5.9,

5.10 $$P(\max_{s\leq t}|\mathfrak{h}_n(s)|\geq 2^{-n})$$
$$\leq 2^{2n}E|\mathfrak{h}_n|^2$$
$$\leq 3\,2^{2n}\,E\int_{S^2} o\mathfrak{h}_n{}^*\mathfrak{h}_n o\,do$$
$$\leq 3\,2^{2n}\,E\int_{S^2} o[\mathfrak{j}_n{}^*\mathfrak{j}_n - E(\mathfrak{j}_n)^*E(\mathfrak{j}_n)]o\,do$$
$$\leq 3\,2^{2n}\,E\int_{S^2} o\mathfrak{j}_n{}^*\mathfrak{j}_n o\,do$$
$$\leq 3\,2^{2n}\,E|\mathfrak{j}_n|^2$$

is the general term of a convergent sum, and now the Borel-Cantelli lemma implies that the convergence of

5.11 $$\sum_{n\geq 0}\mathfrak{j}_n = \sum_{n\geq 0}\mathfrak{h}_n + \sum_{n\geq 0}E(\mathfrak{j}_n) = \sum_{n\geq 0}\mathfrak{h}_n + e(\mathfrak{i}t)$$

is uniform on compacts.

$\mathfrak{g}$ is therefore well-defined and continuous; that it solves 4.8 is clear; and that it lies in SO(3) follows from

5.12 $$\mathfrak{g}(t)^*\mathfrak{g}(t)$$
$$= \mathfrak{g}(0)^*\mathfrak{g}(0) + \int_0^t \mathfrak{g}[d\mathfrak{j} + {}^*d\mathfrak{j} + d\mathfrak{j}^*d\mathfrak{j}]^*\mathfrak{g}$$
$$= \mathfrak{e} \qquad\qquad t\geq 0$$

and from the fact that det $(\mathfrak{g})$ $(=\pm 1)$ is continuous and $=+1$ at $t=0$.

6. CONVERGENCE PROOF

Coming to the proof that $\lim_{n\uparrow\infty}\mathfrak{g}_n = \mathfrak{g}$ $(=\sum_{n\geq 0}\mathfrak{j}_n)$, take $t\leq 1$, $j=[2^n t]$, and let us estimate

6.1 $$\mathfrak{g}_n(t)-\mathfrak{g}(0)-\int_0^t \mathfrak{g}_n(s)\mathfrak{j}(ds)$$
$$= \mathfrak{f}_n(t)-\int_0^t \mathfrak{f}_n(s)\mathfrak{j}(ds)$$
$$+ \mathfrak{g}_n(j2^{-n})-\mathfrak{g}(0)-\sum_{i\leq j}\mathfrak{g}_n((i-1)2^{-n})\mathfrak{j}(\Delta_i),$$

where

6.2 $$\mathfrak{f}_n(s) = \mathfrak{g}_n(s)-\mathfrak{g}_n((i-1)2^{-n}) \qquad s\in\Delta_i$$
$$\Delta_i = [(i-1)2^{-n},\ i2^{-n}) \qquad i\leq 2^n.$$

P. Lévy's 4.3 implies that

6.3 $$\max_{t\leq 1}|\mathfrak{f}_n(t)|<\sqrt{3\,2^{-n}\,lg\,2^n} \qquad n\uparrow\infty$$

and that, up to terms of magnitude $2^n[\sqrt{3 2^{-n} lg 2^n}]^3<2^{-n/3}$,

6.4 $$\mathfrak{g}_n(j2^{-n})-\mathfrak{g}(0)-\sum_{i\leq j}\mathfrak{g}_n((i-1)2^{-n})\mathfrak{j}(\Delta_i)$$
$$=\sum_{i\leq j}\mathfrak{g}_n((i-1)2^{-n})(e[\mathfrak{z}(\Delta_i)]-\mathfrak{e}-\mathfrak{j}(\Delta_i))$$
$$=\sum_{i\leq j}\mathfrak{g}_n((i-1)2^{-n})\mathfrak{w}_i$$
$$\mathfrak{w}_i=\frac{1}{2}[\mathfrak{z}(\Delta_i)^2-\mathfrak{k}\,2^{-n}] \qquad i\leq 2^n.$$

Itô's trick (5.6) and the semi-martingale method of 5.11 give

6.5 $$P[\max_{j\leq 2^n}|\sum_{i\leq j}\mathfrak{g}_n((i-1)2^{-n})\mathfrak{w}_i|\geq 2^{-n/3}]$$
$$\leq 2^{2n/3}E|\sum_{j\leq 2^n}\mathfrak{g}_n((i-1)2^{-n})\mathfrak{w}_i|^2$$
$$\leq 3\,2^{2n/3}E\int_{S^2}o[\sum_{i\leq 2^n}\mathfrak{g}_n\mathfrak{w}_i*\sum_{i\leq 2^n}\mathfrak{g}_n\mathfrak{w}_i]o\,do$$
$$=3\,2^{2n/3}E\int_{S^2}o[\sum_{i\leq 2^n}\mathfrak{g}_n\mathfrak{w}_i*\mathfrak{w}_i*\mathfrak{g}_n]o\,do$$
$$=3\,2^{2n/3}E\int_{S^2}o[\sum_{i\leq 2^n}\mathfrak{g}_n(\mathfrak{i}_2 2^{-2n}+\mathfrak{i}_3 2^{-3n}+\mathfrak{i}_4 2^{-4n})*\mathfrak{g}_n]o\,do$$
$$\leq 3(|\mathfrak{i}_2|+|\mathfrak{i}_3|+|\mathfrak{i}_4|)2^{-n/3}$$

with constant $\mathfrak{i}_2$, $\mathfrak{i}_3$, $\mathfrak{i}_4$, so that, thanks to the Borel-Cantelli lemma,

6.6 $$|\mathfrak{g}_n(j2^{-n})-\mathfrak{g}(0)-\sum_{i\leq j}\mathfrak{g}_n((i-1)2^{-n})\mathfrak{j}(\Delta_i)|<2^{-n/3} \qquad n\uparrow\infty.$$

As to $\int_0^t \mathfrak{f}_n d\mathfrak{j}$, if $\varepsilon_n = \varepsilon_n(t)$ is the indicator of the sphere $|\mathfrak{f}_n(t)| < 2^{-n/3}$, then, as is clear from 6.3,

6.7 $$\int_0^t \mathfrak{f}_n d\mathfrak{j} = \int_0^t \mathfrak{f}_n \varepsilon_n d\mathfrak{j} \qquad t \leq 1, \ n \uparrow \infty,$$

and now

6.8 $$\int_0^t \mathfrak{f}_n d\mathfrak{j} = \int_0^t \mathfrak{f}_n \mathfrak{i}\, ds + \int_0^t \mathfrak{f}_n \mathfrak{y}(ds)$$

in conjunction with

6.9 $$\left|\int_0^t \mathfrak{f}_n \mathfrak{i}\, ds\right| \leq |\mathfrak{i}| 2^{-n/3} \qquad t \leq 1, \ n \uparrow \infty,$$

6.10 $$\begin{aligned} &P\left[\max_{t\leq 1}\left|\int_0^t \mathfrak{f}_n \varepsilon_n \mathfrak{y}(ds)\right| \geq 2^{-n/4}\right] \\ &\leq 3\, 2^{n/2} E \int_{S^2} o \left[\int_0^1 \mathfrak{f}_n \varepsilon_n \mathfrak{y}(ds)^* \int_0^1 \mathfrak{f}_n \varepsilon_n \mathfrak{y}(ds)\right] o\, do \\ &= -3\, 2^{n/2} E \int_{S^2} o \int_0^1 \varepsilon_n \mathfrak{f}_n \mathfrak{k}^* \mathfrak{f}_n ds \Big] o\, do \\ &\leq 3\, 2^{n/2} |\mathfrak{k}| 2^{-2n/3} = 3 |\mathfrak{k}| 2^{-n/6}, \end{aligned}$$

and the Borel-Cantelli lemma, justifies

6.11 $$\max_{t\leq 1}\left|\int_0^t \mathfrak{f}_n d\mathfrak{j}\right| < 2^{-n/4} \qquad n \uparrow \infty.$$

Collecting all this, if $\mathfrak{o}_n = \mathfrak{g}_n - \mathfrak{g}(0) - \int_0^t \mathfrak{g}_n d\mathfrak{j}$, then

6.12 $$\max_{t\leq 1} |\mathfrak{o}_n(t)| < 3 \cdot 2^{-n/4} \qquad n \uparrow \infty,$$

and using K. Itô's method of stochastic differential in conjunction with $d\mathfrak{j} + {}^*d\mathfrak{j} + d\mathfrak{j}^* d\mathfrak{j} = 0$ to check

6.13 $$\mathfrak{g}_n^* \mathfrak{g} = \mathfrak{o}_n^* \mathfrak{g} + e - \int_0^t \mathfrak{o}_n^* d\mathfrak{j}^* \mathfrak{g}$$

and estimating $\int_0^t \mathfrak{o}_n^* d\mathfrak{j}^* \mathfrak{g}$ as we estimated $\int_0^t \mathfrak{f}_n d\mathfrak{j}$ justifies

6.14 $$\max_{t\leq 1} |\mathfrak{g} - \mathfrak{g}_n| = \max_{t\leq 1} |\mathfrak{g}_n^* \mathfrak{g} - e| < 2^{-n/5} \qquad n \uparrow \infty,$$

completing the proof.

7. COMPUTING THE GENERATOR

$\mathfrak{g}_\infty = \mathfrak{g} = \sum_{n\geq 0} \mathfrak{j}_n$ is a Brownian motion on SO(3): in fact, it is continuous and its Markovian and (left) group-invariant character is clear from the product integral

7.1 $$\mathfrak{g}_\infty(t_2) = \mathfrak{g}_\infty(t_1) \bigcap_{t_1 \leq t < t_2} e[\mathfrak{z}(ds)] \qquad t_2 \geq t_1 .$$

Coming to its generator, we adapt the stochastic differential of K. Itô [1951; 59-65] to the present setting and find that for $f \in C^2[\mathrm{SO}(3)]$,

7.2 $$\begin{aligned} f(\mathfrak{g}(t)) &= f(\mathfrak{g}(0)) + \int_0^t [z(ds) \cdot grad] f(\mathfrak{g}) \\ &+ \int_0^t \frac{1}{2} [z(ds) \cdot grad]^2 f(\mathfrak{g}) \\ &= f(\mathfrak{g}(0)) + \int_0^t [\mathfrak{l}^{1/2} x(ds) \cdot grad] \mathfrak{f}(\mathfrak{g}) \\ &+ \int_0^t (\mathfrak{G} f)(\mathfrak{g}) ds , \end{aligned}$$

where *grad* is short for $(\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3)$.

But now it is clear that

7.3 $$\begin{aligned} &\lim_{t \downarrow 0} t^{-1} E[f(\mathfrak{g}(t)) - f(\mathfrak{h})] \\ &= \lim_{t \downarrow 0} t^{-1} E \int_0^t (\mathfrak{G} f)(\mathfrak{g}) ds] \\ &= (\mathfrak{G} f)(\mathfrak{h}) \qquad \mathfrak{h} = \mathfrak{g}(0) , \end{aligned}$$

and this completes the identification of $\mathfrak{g}_\infty = \mathfrak{g} = \sum_{n \geq 0} \mathfrak{j}_n$ as the SO (3) BROWNian motion with generator $\mathfrak{G}$.

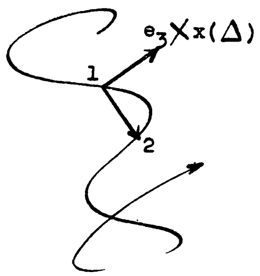
8. ROLLING WITHOUT SLIPPING

Consider the standard Brownian motion on the plane $R^2 \times 0 \subset R^3$ with generator $\frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)$ and sample path $w: t \to x(t) = (x_1(t), x_2(t), 0)$ and let a sphere of diameter 2 roll without slipping on the plane $R^2 \times -1 \subset R^3$ while its center traces out the polygonal line joining the points $x(j2^{-n}): j \geq 0$ of the plane $R^2 \times 0$.

Concentrating on times $t \leq 1$, select $m = m(w)$ such that

8.1 $$|x(t_2) - x(t_1)| < (t_2 - t_1)^{1/3} \qquad 0 \leq t_1 \leq t_2 \leq 1,\ t_2 - t_1 < 2^{-m}$$

and $2^{-m/3} < lg 2$, so that, for $|\alpha| \leq 2^{-m/3}$, $e[\alpha_1 \mathfrak{e}_1 + \alpha_2 \mathfrak{e}_2 + \alpha_3 \mathfrak{e}_3]$ is the counterclockwise rotation of angle $|\alpha|$ about the axis α.



Given $n \geq m$ and $i \leq 2^n$, it is clear from the DIAGRAM that as t grows from $t_1 = (i-1)2^{-n}$ to $t_2 \leq i2^{-n}$ the sphere suffers a rotation

8.2 $$e[-x_2(\Delta)\mathfrak{e}_1 + x_1(\Delta)\mathfrak{e}_2] \qquad \Delta = [t_1, t_2]$$

of angle $|x(\Delta)| < 2^{-n/3}$ counterclockwise about the axis e_3 *cross* $x(\Delta)$.

One sees at once that the total rotation up to time t is identical in law to the $\mathfrak{g}_n(t)$ of 4 computed for

8.3 $$\mathfrak{l} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad m_1 = m_2 = m_3 = 0\,, \quad \mathfrak{g}(0) = \mathfrak{e}\,,$$

and one concludes that $\lim_{n \uparrow \infty} \mathfrak{g}_n = \mathfrak{g}_\infty$ is the SO(3) Brownian motion with generator $\mathfrak{G} = \frac{1}{2}(\mathfrak{E}_1^2 + \mathfrak{E}_2^2)$, permitting the identification of that motion with *the total rotation up to time t of a sphere of diameter 2 rolling without slipping on the plane $R^2 \times -1$ as its center performs a standard Brownian motion on the plane $R^2 \times 0$.* C. D. Gorman [1958] also got a proof that $\lim_{n \uparrow \infty} \mathfrak{g}_n$ exists in the present case.

Consider, now, the path $\mathfrak{g}_\infty e_3$ of the north pole $e_3 = (0, 0, 1)$: this motion is Markov; its generator $\mathfrak{G}_3$ is $\frac{1}{2}(\mathfrak{E}_1^2 + \mathfrak{E}_2^2)$ cut down to the coset space $SO(3)/SO(2) = S^2$:

8.4 $$\mathfrak{G}_3 = \frac{1}{2}\left(\frac{1}{\sin\psi}\frac{\partial}{\partial\psi}\sin\psi\frac{\partial}{\partial\psi} + \cot^2\psi\frac{\partial^2}{\partial\phi^2}\right),$$

where ψ is colatitude and ϕ is longitude on S^2 (see 2.10).

$\mathfrak{G}_3$ splits into the Legendre operator $(2\sin\psi)^{-1}\frac{\partial}{\partial\psi}\sin\psi\frac{\partial}{\partial\psi}$ plus $\frac{1}{2}\cot^2\psi\frac{\partial^2}{\partial\phi^2}$, and this splitting is reflected in the sample path; in fact, *colat* $(\mathfrak{g}_\infty e_3)$ is the process attached to the Legendre operator on $[0, \pi]$ and *longitude* $(\mathfrak{g}_\infty e_3)$ is a standard circular Brownian motion independent of *colat* $(\mathfrak{g}_\infty e_3)$ run with the clock

$$\int_0^t cot^2 [colat(\mathfrak{g}_\infty e_3)] ds .$$

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