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Strongly hyperbolic systems of linear partial differential equations with constant coefficients[1)]

By

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Introduction

As I. G. Petrowsky has defined in his paper [14], the Cauchy problem for a system of linear partial differential equations with constant coefficients, of evolution form

$$(0,1)\qquad \left(\frac{\partial}{\partial t}\right)^{n_i} u_i(x,t)=\sum_{j=1}^{l}\sum_{\substack{|\alpha|\leq M\\ \alpha_0<n_j}} a_{ij}^{\alpha}\left(\frac{\partial}{\partial t}\right)^{\alpha_0}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_m}\right)^{\alpha_m}u_j(x,t)$$

is said to be uniformly correctly posed in the interval $[0, T]$ with respect to the space $(\mathcal{B})$ (for the notations of function spaces, see L. Schwartz [15]), if the following conditions are satisfied.

i) For each $t_0\in[0, T)$ and for each system of functions $\varphi_j^{(k)}(x_1,\cdots,x_m)\in(\mathcal{B})_x$, $k=0, 1, \cdots, n_j-1$, $j=1,\cdots,l$, there is one and only one system of functions $U(x,t)=(u_1(x,t),\cdots,u_l(x,t))\in\overset{l}{\Pi}(\mathcal{B})_{x,t}$ such that this is the solution of (0, 1) for $t_0<t\leq T$ and $\left(\frac{\partial}{\partial t}\right)^{k}u_j(x,t_0)=\varphi_j^{(k)}(x)$.

ii) If $U(x,t_0)$ converges to 0 in $\overset{l}{\Pi}(\mathcal{B})_x$, then $U(x,t)$ converges to 0 uniformly with respect to t_0, $0\leq t_0<T$, in $\overset{l}{\Pi}(\mathcal{B})_{x,t}$.

Petrowsky has shown that the Cauchy problem for (0, 1) is uniformly correctly posed with respect to $(\mathcal{B})$ if and only if the real parts of the zeroes of the polynomial in λ:

$$(0,2)\qquad \det.\,\Big(\lambda^{n_i}\delta_{ij}-\sum_{\substack{|\alpha|\leq M\\ \alpha_0<n_j}}a_{ij}^{\alpha}(i\xi_1)^{\alpha_1}\cdots(i\xi_m)^{\alpha_m}\lambda^{\alpha_0}\Big)\,,$$

1) We have announced the essential part of this paper in [16].

$(\xi_1, \cdots, \xi_n)$ being vector variable of real m-dimensional Euclidean space Ξ^m, do not increase more rapidly than log $(|\xi|+1)$ as $|\xi|$ tends to infinity. L. Gårding, however, has remarked in [6] that this condition is equivalent to the condition that the real parts of the zeroes are bounded when ξ runs over the whole space Ξ^m. He has also noticed that the class of equations satisfying this condition contains other than hyperbolic equations, e.g. the parabolic equation and the Schrödinger equation, and he has found that in order to exclude these equations and pick up the so-called hyperbolic equations, it is necessary to change the function space $(\mathfrak{B})$ to $(\mathfrak{E})$ in the Cauchy problem, that is, to evalute the variation of functions on every compact subset of R^m. His result, in the case of a single equation, is as follows: the uniform correctness of the Cauchy problem with respect to the space $(\mathfrak{E})$ is equivalent to the statements that the equation is of Kowalevsky type (that is to say, the highest degree in t is equal to that in t and ξ) and that the characteristic equation satisfies Petrowsky's condition mentioned above. Hence so far as we consider the equation of Kowalevsky type we need not distinguish between the space $(\mathfrak{B})$ and $(\mathfrak{E})$. We remark here that it is easy to extend Gårding's result to the case of a system of equations in the following form: namely that the uniform correctness of the Cauchy problem for the system (0, 1) of Kowalevsky type with respect to the space $(\mathfrak{E})$ is equivalent to the statement that the polynomial (0,2) is hyperbolic in the sense of Gårding (see also V. M. Borok [1] and P. D. Lax [9]). A system with this property will also be said hyperbolic.

On the other hand, classically, in the case of a single equation, many mathematicians have investigated a little more restricted class called normally (or strictly) hyperbolic equations which are defined by the property that the characteristic polynomial of the principal homogeneous part of the equation has real and distinct roots in λ for any $\xi \neq 0$ of Ξ^m. The equations of this category have one main feature, namely that they are always hyperbolic for any choice of the lower order parts (see Gårding [6] p. 19). Conversely if a single equation is hyperbolic for any choice of the lower order parts, the characteristic equation of the principal homogeneous part must have only real and distinct roots. Thus in the case of a single equation, the property that the hyperbolicity does not disappear by any change of lower order terms is completely characterized by the behaviour of the roots of the characteristic equation of the principal part.

Then there arises a problem: in the case of a system of equations what are the corresponding conditions which characterize the property that the hyperbolicity does not disappear by any change of lower order terms.

It is very important and interesting to investigate the hyperbolic operator from this point of view, because several types of hyperbolic systems treated up to the present have this property. Indeed, equations of second order studied by Friedrichs and Lewy [4], symmetric systems of equations of Friedrichs [5] and Lax [8], and Petrowsky's "hyperbolische Systeme" [13, 14] enter into this category.[2] We call this property strong hyperbolicity as it is more restrictive than the hyperbolicity in the sense of Gårding. In this paper, we shall be concerned with the problem of characterizing strong hyperbolicity in the case of Kowalevsky system, and we shall prove that the class Petrowsky has found in [14] p. 64, is just what we seek for.

1. Notations and the reduction of systems.

In what follows we shall consider linear differential operators with constant coefficients exclusively. Let R^m be real m-dimensional Euclidean space, and Ξ^m be its dual. We shall denote elements of R^m by $x=(x_1, \cdots, x_m)$ and those of Ξ^m by $\xi=(\xi_1, \cdots, \xi_m)$, and by S the unit sphere of Ξ^m, $S=\{\xi: |\xi|=1\}$.

Let us consider a Kowalevsky system of partial differential equations:

$$\text{(1, 1)} \quad \left(\frac{\partial}{\partial t}\right)^{n_i} u_i(x, t) = \sum_{j=1}^{l} \sum_{\substack{|\alpha| \leqq n_j \\ \alpha_0 < n_j}} a_{ij}^{\alpha} \left(\frac{\partial}{\partial t}\right)^{\alpha_0} \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{\alpha_m} u_j(x, t)$$

where $\alpha=(\alpha_0, \cdots, \alpha_m)$ and $|\alpha| = \sum_{k=0}^{m} \alpha_k$, and $n_j > 0$, $i=1, \cdots, l$.

$$\left(\frac{\partial}{\partial t}\right)^{n_i} u_i(x, t) - \sum_{j=1}^{l} \sum_{\substack{|\alpha| = n_j \\ \alpha_0 < n_j}} a_{ij}^{\alpha} \left(\frac{\partial}{\partial t}\right)^{\alpha_0} \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{\alpha_m} u_j(x, t), \quad i=1, \cdots, l,$$

is called the principal part of (1, 1).

2) In the case of variable coefficients, see also Mizohata [11, 12].

By Fourier transformation, this system goes over into a system of ordinary differential equations of the form

$$(1,2)\quad \left(\frac{d}{dt}\right)^{n_i} v_i(t,\xi) = \sum_{j=1}^{l} \sum_{\substack{|\alpha|\leqq n_j \\ \alpha_0<n_j}} a_{ij}^{\alpha}(i\xi_1)^{\alpha_1}\cdots(i\xi_m)^{\alpha_m}\left(\frac{d}{dt}\right)^{\alpha_0} v_j(t,\xi), \qquad i=1,\cdots,l.$$

In this system we put

$$(is)^{n_j-k}\left(\frac{d}{dt}\right)^{k} v_j(t,\xi) = v_{j,k+1}(t,\xi,s), \qquad k=0,1,\cdots,n_j-1,$$

s being a real parameter. Then we get a system of ordinary differential equations of the first order with respect to the unknown functions $v_{11},\cdots,v_{1n_1},v_{21},\cdots,\cdots,v_{l1},\cdots,v_{ln_l}$, of the form

$$(1,3)\quad \frac{d}{dt}v_{ik}(t,\xi,s) - isv_{ik+1}(t,\xi,s) = 0, \qquad k=1,\cdots,n_i-1,$$

$$\frac{d}{dt}v_{in_i}(t,\xi,s) - is\sum_{j=1}^{l}\sum_{|\alpha|=n_j} a_{ij}^{\alpha}\left(\frac{\xi_1}{s}\right)^{\alpha_1}\cdots\left(\frac{\xi_m}{s}\right)^{\alpha_m} v_{j\alpha_0+1}(t,\xi,s)$$

$$-\sum_{j=1}^{l}\sum_{|\alpha|<n_j} a_{ij}^{\alpha}\left(\frac{\xi_1}{s}\right)^{\alpha_1}\cdots\left(\frac{\xi_m}{s}\right)^{\alpha_m}\left(\frac{1}{is}\right)^{n_j-1-|\alpha|} v_{j\alpha_0+1}(t,\xi,s) = 0,$$

or briefly, using a new notation for the unknown functions and $N=\sum_{j=1}^{l} n_j$,

$$(1,4)\quad \frac{d}{dt}v^*(t,\xi,s) - is\sum_{j=1}^{N} a_{ij}\left(\frac{\xi_1}{s},\cdots,\frac{\xi_m}{s}\right)v_j^* - \sum_{j=1}^{N} b_{ij}(\xi,s)v_j^* = 0.$$

We use also the matrix representation

$$(1.5)\quad \left(E\frac{d}{dt} - isA\left(\frac{\xi}{s}\right) - B(\xi,s)\right)V^*(t,\xi,s) = 0,$$

where E is the unit matrix, $A\left(\frac{\xi}{s}\right)=\left(a_{ij}\left(\frac{\xi}{s}\right)\right)$ and $B(\xi,s)=(b_{ij}(\xi,s))$.

Remark 1. It is important to remark that $A\left(\frac{\xi}{s}\right)$ is determined entirely by the coefficients of the principal part of (1, 1). Hence, when (1, 1) is a homogeneous system, $B(\xi,s)=0$.

Remark 2. As is easily seen, every non-zero $\eta\in\Xi^m$ can be written $\eta=s\xi$, $\xi\in S$, s being the length of η. Hence when ξ varies on the whole S and s over the real number field, η varies over the whole Ξ^m, and so we consider $A(\xi)$ only for $\xi\in S$. The elements $a_{ij}(\xi)$ and $b_{ij}(\xi,s)$ are bounded functions in $\xi\in S$ and $|s|\geqq 1$.

The matrix of polynomials

$$(\lambda^{n_i}\delta_{ij}-\sum_{|\alpha|\leqq n_j} a^{\alpha}_{ij}\lambda^{\alpha_0}(i\xi_1)^{\alpha_1}\cdots(i\xi_m)^{\alpha_m}) \tag{1, 6}$$

where δ_{ij} is Kronecker's delta, is called the matrix associated with (1, 1). We study for a while the relation between (1, 5) and (1, 6).

LEMMA 1. *Let there be given a matrix of polynomials in* λ:

$$M(\lambda)=\begin{pmatrix} \lambda^{n_1}+P_{11}(\lambda) & P_{12}(\lambda) & \cdots\cdots & P_{1l}(\lambda) \\ P_{21}(\lambda) & \lambda^{n_2}+P_{22}(\lambda) & \cdots\cdots & P_{2l}(\lambda) \\ \cdots & \cdots & \cdots & \cdots \\ P_{l1}(\lambda) & P_{l2}(\lambda) & \cdots & \lambda^{n_l}+P_{ll}(\lambda) \end{pmatrix}$$

where each $P_{kj}(\lambda)$ *is a polynomial in* λ *of degree at most* n_j-1, *namely*

$$P_{kj}(\lambda)=a^{(1)}_{kj}+a^{(2)}_{kj}\lambda+\cdots+a^{(n_j)}_{kj}\lambda^{n_j-1}, \qquad k, j=1, 2, \cdots, l\,.$$

Imbedding this matrix in an (N,N)*-matrix* $(N=\sum_{j=1}^{l} n_j)$ *of the form*

$$M_1(\lambda)=\left(\begin{array}{c:cccc} M(\lambda) & & & 0 & \\ \hdashline & -1 & & & \\ 0 & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{array}\right),$$

we associate with it another (N,N)*-matrix*

$$M_2(\lambda)=\begin{pmatrix} \lambda & -1 & & & & & & \\ & \lambda & -1 & & & & 0 & \\ & & \ddots & \ddots & & & & \\ & & & -1 & & & & \\ a^{(1)}_{11} & a^{(2)}_{11} & \cdots & \lambda+a^{(n_1)}_{11} & a^{(1)}_{12}\cdots\cdots & a^{(n_2)}_{12} & \cdots\cdots & a^{(n_l)}_{1l} \\ & & & & \lambda & -1 & & \\ & 0 & & & & \ddots\quad\ddots & & 0 \\ & & & & & & -1 & \\ a^{(1)}_{21} & a^{(2)}_{21} & \cdots\cdots & a^{(n_1)}_{21} & a^{(1)}_{22}\cdots & \lambda+a^{(n_2)}_{22} & \cdots\cdots & a^{(n_l)}_{2l} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \ddots & \\ & & & & & & & -1 \\ a^{(1)}_{l1} & a^{(2)}_{l1} & \cdots\cdots & a^{(n_1)}_{l1} & a^{(1)}_{l2}\cdots\cdots & a^{(n_2)}_{l2} & \cdots & \lambda+a^{(n_l)}_{ll} \end{pmatrix}.$$

Then there exist two (N, N)*-matrices* $P(\lambda)$ *and* $Q(\lambda)$ *whose elements are polynomials in* λ *and whose determinants are* ± 1, *such that*

$$P(\lambda)M_1(\lambda)Q(\lambda) = M_2(\lambda)\,.$$

Accordingly, if we denote the elementary divisors of $M(\lambda)$ *by* $e_1(\lambda)$, $\cdots, e_l(\lambda)$, *then the elementary divisors of* $M_2(\lambda)$ *are* $1, \cdots, 1(N-l$ *times*$)$, $e_1(\lambda), \cdots, e_l(\lambda)$.

We omit the proof of this lemma, since it is very easy. In the matrix $M_2(\lambda)$ we call the rows containing $a_{kj}^{(\nu)}$ the *main rows.*

LEMMA 2. *Every* $(N-\nu)$*-rowed minor of* $M_2(\lambda)$ *belongs to the ideal generated by all those* $(N-\nu)$*-rowed minors of* $M_2(\lambda)$ *which are obtained by omitting* ν *rows from among the main rows.*

Hence the greatest common divisor of all $(N-\nu)$*-rowed minors of* $M_2(\lambda)$ *is equal to the greatest common divisor of all those* $(N-\nu)$*-rowed minors which are obtained by omitting* ν *main rows.* (Of course it is also equal to the greatest common divisor of all $(l-\nu)$-rowed minors of $M(\lambda)$.)

PROOF. By lemma 1, it is clear that every $(N-\nu)$-rowed minor of $M_2(\lambda)$ is a linear combination, with polynomial coefficients in λ, of $(l-\nu)$-rowed minors of $M(\lambda)$. On the other hand, each $(l-\nu)$-rowed minor of $M(\lambda)$ appears in the $(N-\nu)$-rowed minors of $M_2(\lambda)$ obtained by omitting ν main rows. In fact, let an $(l-\nu)$-rowed minor of $M(\lambda)$ be $C\begin{pmatrix} k_1, \cdots, k_\nu \\ j_1, \cdots, j_\nu \end{pmatrix}$, where $k_1, \cdots, k_\nu$ and j_1, $\cdots, j_\nu$ denote the numbers of the omitted rows and columns respectively. In the matrix $M_2(\lambda)$, we take the $(N-\nu)$-rowed minor which is obtained by omitting $k_1, \cdots, k_\nu$-th main rows and $\sum_{i=1}^{j_1-1} n_i+1$, $\cdots, \sum_{i=1}^{j_\nu-1} n_i+1$-th columns. Then this minor is clearly equal to $C\begin{pmatrix} k_1, \cdots, k_\nu \\ j_1, \cdots, j_\nu \end{pmatrix}$. The proof is complete.

Remark 3. If each P_{kj} of $M(\lambda)$ in lemma 1 is a homogeneous polynomial in λ and s of degree n_j:

$$P_{kj}(\lambda, s) = a_{kj}^{(1)}s^{n_j}+a_{kj}^{(2)}s^{n_j-1}\lambda+\cdots+a_{kj}^{(n_j)}s\lambda^{n_j-1}, \qquad k, j = 1, \cdots, l\,,$$

then the same results as in lemma 1 and lemma 2 are valid, putting

$$M_1(\lambda, s) = \left(\begin{array}{c:c} \frac{1}{s^{n_j-1}}(\lambda^{n_i}\delta_{ij}+P_{ij}(\lambda, s)) & 0 \\ \hdashline 0 & -sE_{N-l} \end{array}\right)$$

and

$$M_2(\lambda, s) = \begin{pmatrix} \lambda & -s & & & & & & & \\ & \lambda & -s & & & & 0 & & \\ & & \ddots & \ddots & & & & & \\ & & & \ddots & -s & & & & \\ sa_{11}^{(1)} & sa_{11}^{(2)} & \cdots & \lambda+sa_{11}^{(n_1)} & sa_{12}^{(1)} & \cdots\cdots & sa_{12}^{(n_2)} & \cdots\cdots & sa_{1l}^{(n_l)} \\ & & & & \lambda & -s & & & \\ & & 0 & & & \ddots & \ddots & & \\ & & & & & & \ddots\ -s & 0 & \\ sa_{21}^{(1)} & sa_{21}^{(2)} & \cdots\cdots & sa_{21}^{(n_1)} & sa_{22}^{(1)} & \cdots & \lambda+sa_{22}^{(n_2)} & \cdots\cdots & sa_{2l}^{(n_l)} \\ & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & & \ddots & \ddots & \ddots \\ sa_{l1}^{(1)} & sa_{l1}^{(2)} & \cdots\cdots & sa_{l1}^{(n_1)} & sa_{l2}^{(1)} & \cdots\cdots & sa_{l2}^{(n_2)} & \cdots & \lambda+sa_{ll}^{(n_l)} \end{pmatrix}$$

Remark 4. From lemma 1, we see that the reduction of (1, 1) to (1, 5) does not change the determinant of the associated matrix, that is, the following equality holds identically in λ.

(1, 7) $$\det.\,(\lambda^{n_i}\delta_{ij} - \sum_{|\alpha|\leqq n_j} a_{ij}^{\alpha}(i\xi_1)^{\alpha_1}\cdots(i\xi_m)^{\alpha_m}\lambda^{\alpha_0}) \equiv \det.\,(\lambda E - isA(\xi') - B(\xi', s))\,,$$

where $\xi = s\xi'$, $\xi' \in S$.

Remark 5. The matrix $\lambda E - isA(\xi)$ has the same form as $M_2(\lambda, s)$ in remark 3. The elements of $B(\xi, s)$ are 0 except for those of main rows, namely

(1, 8) $$B(\xi, s) = \begin{pmatrix} & & & 0 & & & \\ b_{11}^{(1)} & \cdots & b_{11}^{(n_1)} & b_{12}^{(1)} & \cdots & b_{12}^{(n_2)} & \cdots\cdots & b_{1l}^{(n_l)} \\ & & & 0 & & & \\ b_{21}^{(1)} & \cdots & b_{21}^{(n_1)} & b_{22}^{(1)} & \cdots & b_{22}^{(n_2)} & \cdots\cdots & b_{2l}^{(n_l)} \\ & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \\ b_{l1}^{(1)} & \cdots & b_{l1}^{(n_1)} & b_{l2}^{(1)} & \cdots & b_{l2}^{(n_2)} & \cdots\cdots & b_{ll}^{(n_l)} \end{pmatrix}.$$

2. Strongly hyperbolic systems and the main theorem

The Cauchy problem for the system (1, 1) is said to be uniformly correctly posed with respect to $(\mathcal{E})$ if:

given a system of initial functions $\varphi_j^{(k)}(x) \in (\mathcal{E})_x$, $k=0,1,\cdots$, $n_j-1, j=1,\cdots,l$, there exists a unique solution vector $U(x, t) = (u_1(x, t), \cdots, u_l(x, t)) \in \overset{l}{\Pi}(\mathcal{E})_{x,t}$ of (1, 1) such that

$$\left(\frac{\partial}{\partial t}\right)^{k} u_j(x, 0) = \varphi_j^{(k)}(x)$$

and the linear mapping $\{\varphi_i^{(k)}\} \to U(x, t)$ of the topological vector space $\overset{N}{\Pi}(\mathcal{E})_x$ into the space $\overset{l}{\Pi}(\mathcal{E})_{x,t}$ is continuous.

We call the system (1, 1) hyperbolic if the Cauchy problem for this system is uniformly correctly posed with respect to the space $(\mathcal{E})$.

THEOREM 1. (Gårding [6])

The Kowalevsky system (1, 1) *is hyperbolic if and only if the real parts of the zeroes in* λ *of the determinant of the associated matrix* (1, 6) *are all bounded as* ξ *runs over* Ξ^m.

By virtue of this theorem, in order to examine the hyperbolicity of (1, 1) we have only to consider the behaviour of the zeroes of the determinant of the associated matrix (1, 6).

Remark 5. Writing the system of operators (1, 1) in the matrix form, we can calculate its determinant, in the sense of operator multiplication. Then we obtain a single operator. If we call this operator simply the determinant of the system (1, 1), then theorem 1 reads:

(1, 1) is hyperbolic if and only if its determinant is hyperbolic.

Now we define strong hyperbolicity.

Definition. A homogeneous Kowalevsky system

$$(2, 1) \qquad \boldsymbol{P}(U) = \left(\delta_{ij}\left(\frac{\partial}{\partial t}\right)^{n_i} - \sum_{\substack{|\alpha|=n_j \\ \alpha_0<n_j}} a_{ij}^{\alpha}\left(\frac{\partial}{\partial t}\right)^{\alpha_0}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{\alpha_m}\right) U(x, t)$$

is called *strongly hyperbolic*, if for any system of operators of lower degree

$$(2, 2) \qquad \boldsymbol{R}(U) = \left(\sum_{|\alpha|<n_j} a_{ij}^{\alpha}\left(\frac{\partial}{\partial t}\right)^{\alpha_0}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{\alpha_m}\right) U(x, t)$$

the system of equations

$$(2, 3) \qquad \boldsymbol{P}(U) + \boldsymbol{R}(U) = 0$$

is hyperbolic.

Our purpose is to prove the following

THEOREM 2. *A necessary and sufficient condition for a homogeneous Kowalevsky system* (2, 1) *to be strongly hyperbolic, is that the* *following conditions on the matrix* $A(\xi)$ *in the representation* (1, 5) *are satisfied.*

(*I*) *All characteristic roots of* $A(\xi)$ *are real for any* $\xi \in S$.

(*II*) $A(\xi)$ *is diagonalizable for any* $\xi \in S$.

(*III*) *There exists a positive number* δ *such that for any* $\xi \in S$ *we can find a diagonalizor* $N(\xi)$ *of* $A(\xi)$, (*that is to say,* $N(\xi)A(\xi)N(\xi)^{-1}$ *is a diagonal matrix*), *whose row vectors are of length* 1, *and* $|\det. N(\xi)| \geqq \delta$.

Proof of the Sufficiency

Though Petrowsky has proved the sufficiency of this theorem in [14], we shall give another proof using the characterization of the hyperbolicity in theorem 1. Take an arbitrary system of operators of lower degree (2, 2) and consider the system of equations (2, 3), which we associate with the matrix representation of the form (1, 5). We are going to show that the real parts of the roots of the determinant (1, 6) are all bounded as ξ runs over Ξ^m. Let $\lambda_j(\xi)$, $j=1, \cdots, N$ be the characteristic roots of $A(\xi)$ and $D(\xi)$ be the matrix $(\lambda_j(\xi)\delta_{ij})$. Then we have, by virtue of the equality (1, 7),

$$\begin{aligned}(1,6) &= \det.(\lambda E - isA(\xi') - B(\xi', s)) \\ &= \det.(\lambda E - isD(\xi') - N(\xi')B(\xi', s)N(\xi')^{-1}) \\ &= \prod_{j=1}^{N}(\lambda - is\lambda_j(\xi')) - \sum_{k=1}^{N}\Delta(k)\prod_{j\neq k}^{N}(\lambda - is\lambda_j(\xi')) \\ &\quad - \sum_{k_1,k_2}\Delta(k_1, k_2)\prod_{j\neq k_1,k_2}(\lambda - is\lambda_j(\xi')) - \cdots - \det. B(\xi', s) = 0\end{aligned} \tag{2, 4}$$

where $\Delta(k_1, \cdots, k_\nu)$ denotes the principal minor of $N(\xi')B(\xi', s)N(\xi')^{-1}$ formed by $k_1, \cdots, k_\nu$-th rows and columns. By the condition (III) $\Delta(k_1, \cdots, k_\nu)$ is bounded for $\xi' \in S$ and $|s| \geqq 1$. Let $\sigma_j(\xi)$ $j=1, \cdots, N$ be the roots of (2, 4) and put $\mu_j(\xi) = \mathcal{R}e\sigma_j(\xi)$ and $\nu_j(\xi) = \mathcal{I}m\sigma_j(\xi)$. Dividing (2, 4) by $\prod_{j=1}^{N}(\lambda - is\lambda_j(\xi'))$, we have

$$1 = \sum_{k=1}^{N}\frac{\Delta(k)}{\lambda - is\lambda_k(\xi')} + \sum_{k_1,k_2}\frac{\Delta(k_1,k_2)}{(\lambda - is\lambda_{k_1}(\xi'))(\lambda - is\lambda_{k_2}(\xi'))} + \cdots .$$

So, for any $\sigma_j(\xi) = \mu_j(\xi) + i\nu_j(\xi)$,

$$1 \leqq \sum_{k=1}^{N} \frac{|\Delta(k)|}{|\mu_j + i(\nu_j - s\lambda_k)|} + \sum_{k_1, k_2} \frac{|\Delta(k_1, k_2)|}{|(\mu_j + i(\nu_j - s\lambda_{k_1})| \cdot |\mu_j + i(\nu_j - s\lambda_{k_2})|} + \cdots$$

$$\leqq \sum_{k=1}^{N} \frac{|\Delta(k)|}{|\mu_j(\xi)|} + \sum_{k_1, k_2} \frac{|\Delta(k_1, k_2)|}{|\mu_j(\xi)|^2} + \cdots .$$

Since all the minors $\Delta(k)$, $\Delta(k_1, k_2)$, $\cdots$, are bounded, $\mu_j(\xi)$ must be also bounded for $\xi' \in S$ and $|s| \geqq 1$. $\mu_j(\xi)$ is clearly bounded for $\xi' \in S$ and $|s| < 1$, and hence the proof is complete.

Proof of the Necessity

The condition (I) is necessary because if we choose the zero operator for (2, 2), then (2, 1) itself must be hyperbolic.

Necessity of (II). We are going to show that if the condition (II) is not satisfied, then there exists a system of the form (2, 2) for which (2, 3) is not hyperbolic.

LEMMA 3. *If the equation*

$$(2, 5) \quad \left(\frac{\partial}{\partial t}\right)^N u(x, t) = \sum_{\substack{|\alpha| \leqq N \\ \alpha_0 < N}} a^{\alpha} \left(\frac{\partial}{\partial t}\right)^{\alpha_0} \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{\alpha_m} u(x, t)$$

is hyperbolic, then for any fixed $\xi \in S$,

$$(2, 6) \quad \left(\frac{\partial}{\partial t}\right)^N v(\sigma, t) = \sum_{\substack{|\alpha| \leqq N \\ \alpha_0 < N}} a^{\alpha} \xi_1^{\alpha_1} \cdots \xi_1^{\alpha_m} \left(\frac{\partial}{\partial t}\right)^{\alpha_0} \left(\frac{\partial}{\partial \sigma}\right)^{\alpha_1 + \cdots + \alpha_m} v(\sigma, t)$$

is a hyperbolic equation with respect to the two variables t and σ.

PROOF. Since (2, 5) is hyperbolic, the real parts of the roots of

$$\lambda^N - \sum_{|\alpha| \leqq N} a^{\alpha} (i\xi_1)^{\alpha_1} \cdots (i\xi_m)^{\alpha_m} \lambda^{\alpha_0} = 0$$

are bounded as ξ runs over Ξ^m. Now fix an element $\xi \in S$ and consider the straight line $\{s\xi : -\infty < s < \infty\}$. Then on this line, the equation is

$$\lambda^N - \sum_{|\alpha| \leqq N} a^{\alpha} \xi_1^{\alpha_1} \cdots \xi_m^{\alpha_m} (is)^{\alpha_1 + \cdots + \alpha_m} \lambda^{\alpha_0} = 0$$

and the real parts of the roots are of course bounded. Hence (2, 6) is hyperbolic.[3]

LEMMA 4. (A. Lax [7]) *For an equation of Kowalevsky type in two variables t and σ*

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3) The converse of this lemma is not true (see Courant-Lax [2]).

$$\left(\frac{\partial}{\partial t}\right)^N v(\sigma, t) = \sum_{\substack{i+j\leqq N \\ i<N}} a^{ij}\left(\frac{\partial}{\partial t}\right)^i\left(\frac{\partial}{\partial \sigma}\right)^j v(\sigma, t)$$

to be hyperbolic, it is necessary and sufficient that, if we arrange the equation as the sum of operators of homogeneous degree,

$$P_N\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \sigma}\right)v(\sigma, t)+P_{N-1}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \sigma}\right)v(\sigma, t)+\cdots+P_0 v(\sigma, t) = 0$$

where $P_\nu\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \sigma}\right)$ *is a homogeneous polynomial in* $\frac{\partial}{\partial t}$ *and* $\frac{\partial}{\partial \sigma}$ *of degree* ν, *and if we have*

$$P_N(\lambda, s) = (\lambda-\lambda_1 s)^{m_1}\cdots(\lambda-\lambda_p s)^{m_p}$$

then $\lambda_1, \cdots, \lambda_p$ *are real numbers and* $P_{N-\nu}(\lambda, s)$ *contains the factors of the form* $(\lambda-\lambda_i s)^{m_i-\nu}$ *when* $m_i > \nu$.

We omit the proof.

Now we return to the proof of the necessity of the condition (II). Suppose that the homogeneous Kowalevsky system (2, 1) in question satisfies the condition (I) but not (II). Then there exists an element $\xi^0=(\xi_1^0, \cdots, \xi_m^0)\in S$ such that $A(\xi^0)$ is not diagonalizable, that is to say, the minimum polynomial of $A(\xi^0)$ has a multiple root.

Factorizing det. $(\lambda E-sA(\xi^0))$, we have

$$\det.(\lambda E-sA(\xi^0)) = (\lambda-\lambda_1(\xi^0)s)^{m_1}\cdots(\lambda-\lambda_p(\xi^0)s)^{m_p}.$$

Then the minimum polynomial of $sA(\xi^0)$ is of the form

$$(\lambda-\lambda_1(\xi^0)s)^{k_1}\cdots(\lambda-\lambda_p(\xi^0)s)^{k_p}$$

where $k_1, \cdots, k_p$ are positive integers and one of them is larger than 1, and without loss of generality we can suppose $k_1>1$. It follows that the greatest common divisor of all $(N-1)$-rowed minors of $\lambda E-sA(\xi^0)$ is

$$(\lambda-\lambda_1(\xi^0)s)^{m_1-k_1}\cdots(\lambda-\lambda_p(\xi^0)s)^{m_p-k_p}.$$

Now it is clear from the construction of the system (1, 3) that $\lambda E-sA(\xi^0)$ has the same form as $M_2(\lambda)$ in lemma 1. Therefore by lemma 2, all $(N-1)$-rowed minors obtained by omitting a main row, have the form

$$(\lambda-\lambda_1(\xi^0)s)^{m_1-k_1}\cdots(\lambda-\lambda_p(\xi^0)s)^{m_p-k_p}\varphi_{ij}^{(\nu)}(\lambda, s)$$

where i, j and ν denote the same suffixes as those of the elements of the main rows, and the polynomials $\varphi_{ij}^{(\nu)}(\lambda, s)$ have no common factor. Hence there exists a polynomial $\varphi_{i_0j_0}^{(\nu_0)}(\lambda, s)$ for which we have $\varphi_{j_0j_0}^{(\nu_0)}(\lambda_1 s, s) \neq 0$.

Let $B(\xi)$ be an (N, N)-matrix whose elements are 0 except for the element $b_{i_0j_0}^{(\nu_0)}$ in the same position as $\varphi_{i_0j_0}^{(\nu_0)}$, where we put $b_{i_0j_0}^{(\nu_0)}=\left(\frac{\xi_\rho}{s}\right)^{n_{j_0}-\nu_0}$, ξ_ρ being a coordinate for which $\xi_\rho^0 \neq 0$. Then $B(\xi)$ has the same form as (1, 8). Clearly the matrix $\lambda E - isA(\xi) - B(\xi)$ corresponds to the following Kowalevsky system:

$$(2, 7) \qquad \left(\frac{\partial}{\partial t}\right)^{n_i} u_i = \sum_{j=1}^{l} \sum_{|\alpha|=n_j} a_{ij}^{\alpha} \left(\frac{\partial}{\partial t}\right)^{\alpha_0} \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{\alpha_m} u_j, \quad \text{for } i \neq i_0,$$

$$\left(\frac{\partial}{\partial t}\right)^{n_{i_0}} u_{i_0} = \sum_{j=1}^{l} \sum_{|\alpha|=n_j} a_{i_0 j}^{\alpha} \left(\frac{\partial}{\partial t}\right)^{\alpha_0} \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{\alpha_m} u_j + \left(\frac{\partial}{\partial t}\right)^{\nu_0 - 1} \left(\frac{\partial}{\partial x_\rho}\right)^{n_{j_0}-\nu_0} u_{j_0}.$$

On the other hand, as a polynomial in two variables λ and s,

$$\begin{aligned} &\det. (\lambda E - sA(\xi^0) - B(\xi^0)) \\ &= (\lambda - \lambda_1(\xi^0)s)^{m_1} \cdots (\lambda - \lambda_p(\xi^0)s)^{m_p} - b_{i_0j_0}^{(\nu_0)} (\lambda - \lambda_1(\xi^0)s)^{m_1-k_1} \cdots \\ &\quad (\lambda - \lambda_p(\xi^0)s)^{m_p-k_p} \varphi_{i_0j_0}^{(\nu_0)}(\lambda, s) \end{aligned}$$

does not satisfy the condition of lemma 4, since the second term of the right hand side of this equality is the homogeneous part of degree $N-1$ and $k_1 > 1$, while $\varphi_{i_0j_0}^{(\nu_0)}(\lambda, s)$ has not the factor $\lambda - \lambda_1(\xi^0)s$. Then it follows from lemma 3 and lemma 4 that the determinant of (2, 7) is not hyperbolic, that is, by virtue of remark 5, (2, 7) is not hyperbolic.

Necessity of the condition (III). Supposing that a Kowalevsky system of the form (2, 1) satisfies the conditions (I) and (II) but not (III), we shall again add to it a system of operators of lower degree (2, 2) such that the whole system (2, 3) is not hyperbolic.

Before we proceed to the proof, we describe its outline. First, we take a sequence ξ_n, $n=1, 2, \cdots$ in S such that the determinant of $N(\xi_n)$ tends to zero as $n \to \infty$. Let B be a matrix whose elements are 0 except for (i_0, j_0)-th element. Then, denoting the (i, j)-th element of $N(\xi)$ by $n_{ij}(\xi)$ and that of $N(\xi)^{-1}$ by $n_{ij}^{-1}(\xi)$, we have $N(\xi_n)BN(\xi_n)^{-1} = (n_{li_0}(\xi_n) b_{i_0j_0} n_{j_0k}^{-1}(\xi_n))$, and so

$$(2, 8) \qquad \begin{aligned} \det. (\lambda E - isA(\xi_n) - B) &= \det. (\lambda E - isD(\xi_n) - N(\xi_n)BN(\xi_n)^{-1}) \\ &= \prod_{j=1}^{N} (\lambda - is\lambda_j(\xi_n)) - \sum_{k=1}^{N} n_{ki_0}(\xi_n) b_{i_0j_0} n_{j_0k}^{-1}(\xi_n) \prod_{j \neq k} (\lambda - is\lambda_j(\xi_n)). \end{aligned}$$

In order to make the real part of a zero of this polynomial in λ tend to infinity as $n\to\infty$, we should modify the suffix (i_0, j_0) so that a coefficient $a_k(\xi_n)=n_{ki_0}(\xi_n)b_{i_0j_0}n^{-1}_{j_0k}(\xi_n)$ tends to infinity, since if not the real parts of zeroes would remain bounded, by the same reason as in the proof of the sufficiency.

This modification is easy when $\lambda_1(\xi_n), \cdots, \lambda_N(\xi_n)$ are distinct. We merely find a non-bounded $n^{-1}_{j_0k}(\xi_n)$, and among the coordinates of k-th row vector of $N(\xi_n)$ we take an $n_{ki_0}(\xi_n)$ which does not tend to zero as $n\to\infty$. This is possible since the length of each row vector of $N(\xi_n)$ is 1. Then $|a_k(\xi_n)|=|n_{ki_0}(\xi_n)b_{i_0j_0}n^{-1}_{j_0k}(\xi_n)|$ tends to infinity. But when $\lambda_1(\xi_n), \cdots, \lambda_N(\xi_n)$ are not distinct, say $\lambda_1(\xi_n)=\lambda_2(\xi_n)$, the equation is

$$\det.(\lambda E-isA(\xi_n)-B)=\prod_{j=1}^{N}(\lambda-is\lambda_j(\xi_n))$$
$$-\{a_1(\xi_n)+a_2(\xi_n)\}\prod_{j=2}^{N}(\lambda-is\lambda_j(\xi_n))-\sum_{k=3}^{N}a_k(\xi_n)\prod_{j\neq k}(\lambda-is\lambda_j(\xi_n))$$

and hence, even if we may modify the suffix (i_0, j_0) so that $a_1(\xi_n)$ tends to infinity, the value $a_1(\xi_n)+a_2(\xi_n)$ may not tend to infinity. We can get over this difficulty by modifying $N(\xi_n)$ so that the sum $a_1(\xi_n)+a_2(\xi_n)$ has only one non-zero term.

There is another difficulty: the matrix B which we wish to find must be of the form (1, 8), that is, the unique non-zero element $b_{i_0j_0}$ of B must belong to a *main row*. This problem will be solved by virtue of lemma 2.

After these modifications, we shall prove that the real part of a zero of (2, 8) tends to infinity if we choose a sequence of vectors $\eta_n=\sigma_n\xi_n$ in Ξ^n and let $n\to\infty$.

Now we show the detail of the proof. Since the condition (III) is not satisfied, there exists an infinite sequence $\xi_n\in S$, $n=1, 2, \cdots$, such that for any method of construction of $N(\xi_n)$, $|\det. N(\xi_n)|$ tends to zero as $n\to\infty$. By taking a subsequence, we can assume that ξ_n converges to an element $\xi^0=(\xi^0_1, \cdots, \xi^0_m)\in S$ and the multiplicity of the characteristic roots of $A(\xi_n)$ remains invariant for all n. Hence we can write

$$\det.(\lambda E-isA(\xi_n))=(\lambda-is\lambda_1(\xi_n))^{m_1}\cdots(\lambda-is\lambda_p(\xi_n))^{m_p}$$

where $m_1, \cdots, m_p$ are independent of n.

As is well-known, a row vector of the diagonalizor $N(\xi)$ is an eigenvector of $A(\xi)$. Therefore in order to construct a diagonalizor we must solve the equations

$$(2, 9) \qquad (x_1, \cdots, x_N)A(\xi_n) = \lambda_i(\xi_n)(x_1, \cdots, x_N), \qquad i = 1, \cdots, p.$$

We construct its solution vectors as follows. Take a characteristic root of $A(\xi_n)$, say $\lambda_1(\xi_n)$, then again choosing a subsequence of ξ_n if necessary, we can find an $(N-m_1)$-rowed minor $\Delta_0(\xi_n)$ of $\lambda_1(\xi_n)E - A(\xi_n)$ such that for any other $(N-m_1)$-rowed minor $\Delta(\xi_n)$, $\dfrac{\Delta(\xi_n)}{\Delta_0(\xi_n)}$ is bounded as $n\to\infty$. Moreover, by lemma 2, we can assume that $\Delta_0(\xi_n)$ is constructed by omitting m_1 main rows, say ν_1-th, $\cdots$, ν_{m_1}-th rows of $\lambda_1(\xi_n)E-A(\xi_n)$. Then an easy calculation using the well-known Cramér's method shows that m_1 linearly independent solution vectors of (2, 9) are given in the following form:

(2, 10)

$$X_1 = \left(\pm\frac{\Delta_{11}(\xi_n)}{\Delta_0(\xi_n)},\ \pm\frac{\Delta_{12}(\xi_n)}{\Delta_0(\xi_n)}, \cdots\cdots, 1, \cdots, 0, \cdots\cdots, 0, \cdots,\ \pm\frac{\Delta_{1N}(\xi_n)}{\Delta_0(\xi_n)}\right)$$

$$X_2 = \left(\pm\frac{\Delta_{21}(\xi_n)}{\Delta_0(\xi_n)},\ \pm\frac{\Delta_{22}(\xi_n)}{\Delta_0(\xi_n)}, \cdots\cdots, 0, \cdots, 1, \cdots\cdots, 0, \cdots,\ \pm\frac{\Delta_{2N}(\xi_n)}{\Delta_0(\xi_n)}\right)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$X_{m_1} = \left(\pm\frac{\Delta_{m_11}(\xi_n)}{\Delta_0(\xi_n)},\ \pm\frac{\Delta_{m_12}(\xi_n)}{\Delta_0(\xi_n)}, \cdots, \underset{\substack{\updownarrow \\ \nu_1\text{-th}}}{0}, \cdots, \underset{\substack{\updownarrow \\ \nu_2\text{-th}}}{0}, \cdots\cdots, \underset{\substack{\updownarrow \\ \nu_{m_1}\text{-th}}}{1}, \cdots,\ \pm\frac{\Delta_{m_1N}(\xi_n)}{\Delta_0(\xi_n)}\right) \text{ coordinate}$$

where $\Delta_{ij}(\xi_n)$ are $(N-m_1)$-rowed minors of $\lambda_1(\xi_n)E-A(\xi_n)$. The length of X_i,

$$|X_i| = \left\{\sum_{j\neq\nu_1,\cdots,\nu_{m_1}}^{N}\left|\frac{\Delta_{ij}(\xi_n)}{\Delta_0(\xi_n)}\right|^2+1\right\}^{1/2}$$

is bounded as $n\to\infty$, so when we normalize these vectors as

$$(2, 11) \qquad X_i' = \frac{1}{|X_i|}X_i, \qquad i = 1, 2, \cdots, m_1,$$

the ν_i-th coordinate of the vector X_i does not tend to zero as $n\to\infty$, while the other ν_1-th, $\cdots$, ν_{m_1}-th ones are 0. We construct all eigenvectors of the other characteristic roots $\lambda_2, \cdots, \lambda_p$ in the same manner. A diagonalizor $N(\xi_n)$ is constructed by rearranging these vectors. We use $N(\xi_n)$ thus constructed in the sequel exclusively.

Since det. $N(\xi_n)^{-1}$ is not bounded by the hypothesis that det. $N(\xi_n)$ tends to zero as $n\to\infty$, there exists an element $n_{j_0k_0}^{-1}(\xi_n)$ which tends to infinity as $n\to\infty$. Take the k_0-th row vector of $N(\xi_n)$. This is an eigenvector of some characteristic root of $A(\xi_n)$, say $\lambda_1(\xi_n)$. We can suppose then that it is X'_{k_0} in (2, 11), whose ν_{k_0}-th coordinate does not tend to zero, and whose other ν_1-th, $\cdots$, ν_{m_1}-th ones are 0. Put $i_0=\nu_{k_0}$ and let $B(\xi)$ be the matrix whose elements are 0 except for the (i_0, j_0)-element which takes the value $b\cdot\left(\frac{\xi_\rho}{s}\right)^\sigma$, where b is a constant to be determined later, ξ_ρ is a coordinate for which ξ_ρ^0(ρ-th coordinate of ξ^0) $\neq 0$ and σ is an integer to be determined so that the matrix

$$\lambda E-isA(\xi)-B(\xi), \qquad \xi\in S, \tag{2, 12}$$

corresponds to a system of partial differential equations, as was done in the proof of the necessity of the condition (II), which is possible since $B(\xi)$ has the form of (1, 8)

Now we shall prove that this system is not hyperbolic. First we calculate the determinant of this system.

$$\begin{aligned}(2, 13)\qquad &\det.(\lambda E-isA(\xi)-B(\xi))\\ &=\det.(N(\xi)(\lambda E-isA(\xi)-B(\xi))N(\xi)^{-1})\\ &=\det.(\lambda E-isD(\xi)-N(\xi)B(\xi)N(\xi)^{-1})\,.\end{aligned}$$

From the construction of $N(\xi_n)$, it follows that $n_{ki_0}(\xi_n)=0$ for $1\leq k\leq m_1$, $k\neq k_0$, and so the first m_1 diagonal elements of $N(\xi_n)B(\xi_n)N(\xi_n)^{-1}$ are zero except for the k_0-th element and $|n_{k_0i_0}(\xi_n)n^{-1}_{j_0k_0}(\xi_n)|$ tends to infinity as $n\to\infty$. Moreover, since $N(\xi_n)B(\xi_n)N(\xi_n)^{-1}$ is a matrix of rank 1, any minor of degree larger than 1 is 0. Therefore

(2, 14)

$$\begin{aligned}(2, 13) &= \prod_{j=1}^{p}(\lambda-is\lambda_j(\xi))^{m_j}-b\xi_\rho^\sigma\cdot n_{k_0i_0}(\xi)n^{-1}_{j_0k_0}(\xi)\frac{\prod_{j=1}^{p}(\lambda-is\lambda_j(\xi))^{m_j}}{\lambda-is\lambda_1(\xi)}\\ &\quad -b\xi_\rho^\sigma\sum_{j=2}^{p}\left\{\sum_{k=\sum_1^{j-1}m_i+1}^{\sum_1^{j}m_i} n_{ki_0}(\xi)n^{-1}_{j_0k}(\xi)\right\}\frac{\prod_{\nu=1}^{p}(\lambda-is\lambda_\nu(\xi))^{m_\nu}}{\lambda-is\lambda_j(\xi)}\\ &=\left\{\prod_{j=1}^{p}(\lambda-is\lambda_j(\xi))^{m_j-1}\right\}\left\{\prod_{j=1}^{p}(\lambda-is\lambda_j(\xi))-\sum_{k=1}^{p}a_k(\xi)\prod_{j\neq k}(\lambda-is\lambda_j(\xi))\right\}\end{aligned}$$

where we put

$$a_1(\xi) = b\cdot\xi_p^\sigma n_{k_0 i_0}(\xi) n_{j_0 k_0}^{-1}(\xi),$$

$$a_j(\xi) = b\cdot\xi_p^\sigma \sum_{k=\sum_1^{j-1} m_i+1}^{\sum_1^j m_i} n_{k i_0}(\xi) n_{j_0 k}^{-1}(\xi), \qquad j = 2, \cdots, p.$$

It is clear from the construction of $n_{ij}(\xi)$ that $a_k(\xi)$, $k=1, 2, \cdots, p$, are homogeneous functions in $\xi\in\Xi^n$ of order 0, and $|a_1(\xi_n)|$ tends to infinity as $n\to\infty$. But there may be another $a_j(\xi_n)$ which tends to infinity more rapidly than $a_k(\xi_n)$. Let $a_{j_0}(\xi_n)$ be the most rapidly increasing term of all $a_k(\xi)$, $k=1, 2, \cdots, p$. Here we determine the value of the coefficient b so that the real part of $\dfrac{a_{j_0}(\xi_n)}{|a_{j_0}(\xi_n)|}$ tends to 1.[4,5] Consider the equation

$$(2, 15) \qquad \prod_{j=1}^{p} (\lambda - is\lambda_j(\xi)) - \sum_{k=1}^{p} a_k(\xi) \prod_{j\neq k} (\lambda - is\lambda_j(\xi)) = 0.$$

We are going to show that if we take some sequence $s\xi=\eta_n$, $n=1, 2, \cdots$ in Ξ^m, the real part of a root of (2, 15) is not bounded as $n\to\infty$. Put

$$t_n = |a_{j_0}(\xi_n)| \text{ and } s_n = \min_{j,k,j\neq k} |\lambda_j(\xi_n) - \lambda_k(\xi_n)|.$$

Then $t_n\to\infty$ and s_n is bounded as $n\to\infty$.[6] Next, consider the sequence of vectors

$$\eta_n = \frac{t_n^2}{s_n}\xi_n, \qquad n = 1, 2, \cdots$$

in Ξ^m, which means that η_n has the same direction as ξ_n and its length is $\dfrac{t_n^2}{s_n}$. Then (2, 15) is

$$\prod_{j=1}^{p} (\lambda - i\frac{t_n^2}{s_n}\lambda_j(\xi_n)) - \sum_{k=1}^{p} a_k(\xi_n) \prod_{j\neq k} (\lambda - i\frac{t_n^2}{s_n}\lambda_j(\xi_n)) = 0.$$

We denote the roots of this equation by $\sigma_j(\eta_n)=\mu_j(\eta_n)+i\nu_j(\eta_n)$,

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4) Of course $\dfrac{a_{j0}(\xi_n)}{|a_{j0}(\xi_n)|}$, $n=1, 2, \cdots$ is not a convergent sequence on the unit circle in the complex plane in general, but we can assume its convergence by taking a subsequence of ξ_n.

5) If all the coefficients of $A(\xi)$ are real, then we merely put $b=1$.

6) Choosing again a subsequence of ξ_n if necessary, we can assume that t_n and s_n are monotone sequences, and $|a_j(\xi_n)|\leqq M\cdot t_n$, $j=1, \cdots, p$.

$j=1, 2, \cdots, p$. Dividing this equation by t_n^p, we have

$$(2, 16) \qquad \prod_{j=1}^{p}(\tilde{\lambda}-i\frac{t_n}{s_n}\lambda_j(\xi_n))-\sum_{k=1}^{p}\tilde{a}_k(\xi_n)\prod_{j\neq k}(\tilde{\lambda}-i\frac{t_n}{s_n}\lambda_j(\xi_n)) = 0\,,$$

where $\tilde{\lambda}=\frac{\lambda}{t_n}$ and $\tilde{a}_k(\xi_n)=\frac{a_k(\xi_n)}{t_n}$, $|\tilde{a}_k(\xi_n)|\leqq M$. the roots of this equation are $\tilde{\sigma}_j(\eta_n)=\tilde{\mu}_j(\eta_n)+i\tilde{\nu}_j(\eta_n)=\frac{\mu_j(\eta_n)}{t_n}+i\frac{\nu_j(\eta_n)}{t_n}$.

PROPOSITION. There exists a one-to-one correspondence between $i\frac{t_n}{s_n}\lambda_j(\xi_n)$, $j=1, 2, \cdots, p$ and $\tilde{\sigma}(\eta_n)$, $j=1, 2, \cdots, p$, such that, denoting the corresponding values by the same suffixes, $\tilde{\sigma}_j(\eta_n)-i\frac{t_n}{s_n}\lambda_j(\xi_n)$ are bounded as $n\rightarrow\infty$.

PROOF. There exists n_0 such that $M\cdot p<\frac{t_n}{2}$ for all $n\geqq n_0$. Hence we can find a number ρ, $M\cdot p<\rho<\frac{t_n}{2}$ for all $n\geqq n_0$, which implies $|\tilde{a}_k(\xi_n)|\leqq M<\frac{\rho}{p}$, and

$$\rho<\frac{1}{2}\left|i\frac{t_n}{s_n}\lambda_j(\xi_n)-i\frac{t_n}{s_n}\lambda_k(\xi_n)\right|, \qquad j\neq k\,,$$

for all $n\geqq n_0$, since

$$\left|i\frac{t_n}{s_n}\lambda_j(\xi_n)-i\frac{t_n}{s_n}\lambda_k(\xi_n)\right| = \frac{t_n}{s_n}|\lambda_j(\xi_n)-\lambda_k(\xi_n)|\geqq t_n$$

for all j and k, $j\neq k$.

Put

$$f_n(z) = \prod_{j=1}^{p}\left(z-i\frac{t_n}{s_n}\lambda_j(\xi_n)\right)$$

and

$$g_n(z) = -\sum_{k=1}^{p}\tilde{a}_k(\xi_n)\prod_{j\neq k}\left(z-i\frac{t_n}{s_n}\lambda_j(\xi_n)\right).$$

We compare their values on the circle C_ν with centre $i\frac{t_n}{s_n}\lambda_\nu(\xi_n)$ and radius ρ in the complex plane, $\nu=1, 2, \cdots, p$. On this circle

$$|f_n(z)| = \rho\prod_{j\neq\nu}\left|z-i\frac{t_n}{s_n}\lambda_j(\xi_n)\right|$$

and

$$|g_n(z)| \leqq \rho \sum_{k \neq \nu}^{p} |\tilde{a}_k(\xi_n)| \prod_{\substack{j \neq k \\ \neq \nu}} \left| z - i\frac{t_n}{s_n}\lambda_j(\xi_n) \right| + |\tilde{a}_\nu(\xi_n)| \prod_{j \neq \nu} \left| z - i\frac{t_n}{s_n}\lambda_j(\xi_n) \right|$$

$$< \rho \sum_{k \neq \nu} \frac{\rho}{p} \prod_{\substack{j \neq k \\ \neq \nu}} \left| z - i\frac{t_n}{s_n}\lambda_j(\xi_n) \right| + \frac{\rho}{p} \prod_{j \neq \nu} \left| z - i\frac{t_n}{s_n}\lambda_j(\xi_n) \right|$$

$$< \rho \prod_{j \neq \nu} \left| z - i\frac{t_n}{s_n}\lambda_j(\xi_n) \right| = |f_n(z)|.$$

Hence by Rouché's theorem in function theory, the number of zeroes of $f_n(z)$ and that of $f_n(z)+g_n(z)$ coincide in the interior of C_ν. But the only zero of $f_n(z)$ in this domain is $i\frac{t_n}{s_n}\lambda_j(\xi_n)$. Therefore there exists one and only one root $\tilde{\sigma}_\nu(\eta_n)$ of (2, 16) such that

$$\left|\tilde{\sigma}_\nu(\eta_n) - i\frac{t_n}{s_n}\lambda_\nu(\xi_n)\right| < \rho, \quad \text{while} \quad \left| i\frac{t_n}{s_n}\lambda_j(\xi_n) - i\frac{t_n}{s_n}\lambda_k(\xi_n)\right| \to \infty .$$

The proof of the proposition is thus complete.

Put $\beta_\nu(\eta_n) = \tilde{\sigma}_\nu(\eta_n) - i\frac{t_n}{s_n}\lambda_\nu(\xi_n)$, $\nu = 1, 2, \cdots, p$. Since

$$\prod_{j=1}^{p}(\tilde{\lambda} - i\frac{t_n}{s_n}(\lambda_j(\xi_n)) - \sum_{k=1}^{p}\tilde{a}_k(\xi_n)\prod_{j \neq k}\left(\tilde{\lambda} - i\frac{t_n}{s_n}\lambda_j(\xi_n)\right) = \prod_{j=1}^{p}(\tilde{\lambda} - \tilde{\sigma}_j(\eta_n))$$

is an identity in $\tilde{\lambda}$, we can substitute $i\frac{t_n}{s_n}\lambda_{j_0}(\xi_n)$ for $\tilde{\lambda}$ (j_0 being the suffix $|a_{j_0}(\xi_n)| = t_n$), and then we have

$$-\tilde{a}_{j_0}(\xi_n)\prod_{j \neq j_0}\left(i\frac{t_n}{s_n}\lambda_{j_0}(\xi_n) - i\frac{t_n}{s_n}\lambda_j(\xi_n)\right) = -\beta_{j_0}(\eta_n)\prod_{j \neq j_0}\left(i\frac{t_n}{s_n}\lambda_{j_0}(\xi_n) - \tilde{\sigma}_j(\eta_n)\right).$$

Thus

$$\frac{\tilde{a}_{j_0}(\xi_n)}{\beta_{j_0}(\eta_n)} = \prod_{j \neq j_0}\left(1 - \frac{\beta_j(\eta_n)}{i\frac{t_n}{s_n}\lambda_{j_0}(\xi_n) - i\frac{t_n}{s_n}\lambda_j(\xi_n)}\right). \tag{2.17}$$

In this relation, $|\beta_j(\eta_n)| < \rho$, $|\tilde{a}_{j_0}(\xi_n)| = 1$ and

$$\left| i\frac{t_n}{s_n}\lambda_{j_0}(\xi_n) - i\frac{t_n}{s_n}\lambda_j(\xi_n)\right| \geqq t_n \to \infty \qquad \text{as } n \to \infty .$$

Therefore the right hand term of (2, 17) tends to 1, and so we have

$$\lim_{n \to \infty} \frac{\beta_{j_0}(\eta_n)}{\tilde{a}_{j_0}(\xi_n)} = 1$$

Since the imaginary part of $\tilde{a}_{j_0}(\xi_n)$ tends to zero, the real part of $\beta_{j_0}(\eta_n)$ cannot tend to zero, which means that the real part of $\tilde{\sigma}_{j_0}(\eta_n)$ does not approach zero, that is, $|\tilde{\mu}_{j_0}(\eta_n)| = \left|\frac{1}{t_n}\mu_{j_0}(\eta_n)\right| \geqq c > 0$. Hence $|\mu_{j_0}(\eta_n)| \to \infty$ as $n \to \infty$. This completes the proof of the necessity of the condition (III).

3. Examples

1) Single operators

THEOREM 3. *A homogeneous equation of Kowalevsky type*

$$\left(\frac{\partial}{\partial t}\right)^n u(x, t) = \sum_{\substack{|\alpha|=n \\ \alpha_0<n}} a^\alpha \left(\frac{\partial}{\partial t}\right)^{\alpha_0} \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{\alpha_m} u(x, t)$$

is strongly hyperbolic if and only if the polynomial

$$\lambda^n - \sum_{\substack{|\alpha|=n \\ \alpha_0<n}} a^\alpha \xi_1^{\alpha_1} \cdots \xi_m^{\alpha_m} \lambda^{\alpha_0} = 0 \tag{3, 1}$$

has real and distinct roots in λ *for any* $\xi \in S$.

PROOF. Necessity. Reducing the equation to the form of (1, 5) we have

$$A(\xi) = \begin{pmatrix} 0 & 1 & & & \\ & & & 0 & \\ & 0 & 1 & & \\ 0 & & \ddots & \ddots & \\ & & & \ddots & 1 \\ A_1(\xi) & A_2(\xi) & \cdots & \cdots & A_n(\xi) \end{pmatrix}, \quad A_i(\xi) = \sum_{\alpha_1+\cdots+\alpha_m=i} a^\alpha \xi_1^{\alpha_1} \cdots \xi_m^{\alpha_m}.$$

From the condition (II) of theorem 2, $A(\xi)$ must be diagonalizable, that is, the minimum polynomial of $A(\xi)$ has no multiple root. But the minimum polynomial of $A(\xi)$ is equal to (3, 1). Hence the roots of the equation (3, 1) must be distinct.

Sufficiency.[7] It will suffice to show that the conditions in theorem 3 imply the condition (III) in theorem 2. Let us denote the distinct real roots by $\lambda_1(\xi), \cdots, \lambda_n(\xi)$. Then, since they are continuous in ξ, there exists a positive number c such that for any $\xi \in S$,

$$|\lambda_i(\xi) - \lambda_j(\xi)| \geqq c \quad \text{if} \quad i \neq j.$$

7) See also Leray [10].

A diagonalizor $N(\xi)$ is given as follows:

$$N(\xi)=\begin{pmatrix} \lambda_1(\xi)^{n-1} & \lambda_1(\xi)^{n-2} & \cdots\cdots & 1 \\ \lambda_2(\xi)^{n-1} & \lambda_2(\xi)^{n-2} & \cdots\cdots & 1 \\ & \cdots\cdots\cdots\cdots & & \\ \lambda_n(\xi)^{n-1} & \lambda_n(\xi)^{n-2} & \cdots\cdots & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ A_1(\xi) & 1 & & 0 \\ A_2(\xi) & A_1(\xi) & 1 & \\ & \cdots\cdots\cdots\cdots & & \\ A_{n-1}(\xi) & A_{n-2}(\xi) & \cdots\cdots\cdots & 1 \end{pmatrix}.$$

Since each term is bounded as ξ runs over S, we need not normalize row vectors of $N(\xi)$. We see immediately

$$|\det. N(\xi)| = \prod_{i<j} |\lambda_j(\xi)-\lambda_i(\xi)| \geqq c^{\frac{n(n-1)}{2}} > 0 .$$

2) Hermitian operators

Suppose that we are given a system of first order operators

$$(3, 2) \qquad \sum_{j=1}^{n} b_{ij} \frac{\partial}{\partial t} u_j(x, t) = \sum_{j=1}^{n}\sum_{k=1}^{m} a_{ij}^k \frac{\partial}{\partial x_k} u_j(x, t), \qquad i = 1, \cdots, n .$$

where the constant matrix $B=(b_{ij})$ is hermitian positive and $A(\xi) =(\sum_{k=1}^{m} a_{ij}^k \xi_k)$ is hermitian for any $\xi \in S$. One of the most familiar examples of this type is the system of equations of Maxwell (see Courant-Hilbert [3] p. 377):

$$\begin{aligned} \sigma_1 \frac{\partial u_1}{\partial t} &= -\frac{\partial u_5}{\partial x_3}+\frac{\partial u_6}{\partial x_2}, & \frac{\partial u_4}{\partial t} &= \frac{\partial u_2}{\partial x_3}-\frac{\partial u_3}{\partial x_2}, \\ \sigma_2 \frac{\partial u_2}{dt} &= \frac{\partial u_4}{\partial x_3}-\frac{\partial u_6}{\partial x_1}, & \frac{\partial u_5}{\partial t} &= -\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}, \\ \sigma_3 \frac{\partial u_3}{\partial t} &= -\frac{\partial u_4}{\partial x_2}+\frac{\partial u_5}{\partial x_1}, & \frac{\partial u_6}{\partial t} &= \frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}. \end{aligned}$$

Now (3, 2) can be written

$$B\frac{\partial}{\partial t} U(x, t) = A\Big(\frac{\partial}{\partial x}\Big) U(x, t)$$

and so we have

$$\frac{\partial}{\partial t} U(x, t) = B^{-1} A\Big(\frac{\partial}{\partial x}\Big) U(x, t) .$$

We shall show that $B^{-1}A(\xi)$ satisfies the conditions of theorem 2. Since B is hermitian positive, there exists a unitary matrix V such that

$$VBV^* = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix},$$

where $\sigma_1, \cdots, \sigma_n$, the characteristic roots of B, are real and positive. Put

$$Q = \begin{pmatrix} \frac{1}{\sqrt{\sigma_1}} & & \\ & \ddots & \\ & & \frac{1}{\sqrt{\sigma_n}} \end{pmatrix}.$$

Then we have clearly $Q^* = Q$ and $QVBV^*Q^* = E$. Since the matrix $QVA(\xi)V^*Q^*$ is also hermitian, we can find a unitary matrix $U(\xi)$ such that $U(\xi)QVA(\xi)V^*Q^*U(\xi)^*$ is a diagonal matrix whose elements are all real. Put $N(\xi) = U(\xi)Q^{-1}V$. Then

$$\begin{aligned} N(\xi)(B^{-1}A(\xi))N(\xi)^{-1} &= (U(\xi)QVBV^*Q^*U(\xi)^*)^{-1}(U(\xi)QVA(\xi)V^*Q^*U(\xi)^*) \\ &= E\cdot U(\xi)QVA(\xi)V^*Q^*U(\xi)^* \end{aligned}$$

is a diagonal matrix whose elements are all real. So $B^{-1}A(\xi)$ satisfies the conditions (I) and (II) of theorem 2. The condition (III) is clearly satisfied since each element of $U(\xi)$ is bounded and

$$|\det. N(\xi)| = |\det. Q^{-1}| = |\sigma_1 \cdots \sigma_n|^{\frac{1}{2}}.$$

Hence the system (3, 2) is strongly hyperbolic.

3) Petrowsky's example

Petrowsky has shown an example which satisfies the conditions (I) and (II) but is not strongly hyperbolic. (see [14], p. 67)

$$\begin{aligned} \frac{\partial}{\partial t}u_1 &= \frac{\partial}{\partial x_1}u_1 - \frac{\partial}{\partial x_2}u_2, \\ \frac{\partial}{\partial t}u_2 &= -\frac{\partial}{\partial x_2}u_1 - \frac{\partial}{\partial x_2}u_3, \\ \frac{\partial}{\partial t}u_3 &= 0. \end{aligned}$$

The associated matrix is

$$\begin{pmatrix} \lambda-\xi_1 & \xi_2 & 0 \\ \xi_2 & \lambda & \xi_2 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix} - \begin{pmatrix} \xi_1 & -\xi_2 & 0 \\ -\xi_2 & 0 & -\xi_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The characteristic equation takes the form

$$\lambda(\lambda^2-\xi_1\lambda-\xi_2^2) = 0 .$$

The roots $\lambda_1=\frac{1}{2}(\xi_1+\sqrt{\xi_1^2+4\xi_2^2})$, $\lambda_2=\frac{1}{2}(\xi_1-\sqrt{\xi_1^2+4\xi_2^2})$ and $\lambda_3=0$ are real. If $\xi_2 \neq 0$, these three roots are distinct, so $A(\xi)$ is diagonalizable. If $\xi_2=0$, $A(\xi)$ is itself diagonal. Hence the conditions (I) and (II) are satisfied. We calculate the diagonalizor $N(\xi)$ for every $\xi \in S$. If $\xi_2=0$, we can take $N(\xi)=E$. Assume that $\xi_2 \neq 0$ and we have

$$N(\xi) = \begin{pmatrix} -\frac{\lambda_1}{\varepsilon} & \frac{\xi_2}{\varepsilon} & -\frac{\xi_2^2}{\lambda_1\varepsilon} \\ -\frac{\lambda_2}{\varepsilon} & \frac{\xi_2}{\varepsilon} & -\frac{\xi_2^2}{\lambda_2\varepsilon} \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{where} \quad \varepsilon=\sqrt{\xi_1^2+3\xi_2^2}\,.$$

Hence $|\det. N(\xi)| = \dfrac{|\xi_2|\sqrt{\xi_1^2+4\xi_2^2}}{\xi_1^2+3\xi_2^2}$ tends to zero as $\xi_2 \to 0$.

Remark. As was seen in 1), there are some cases where the conditions (I) and (II) imply (III). For example, systems of two equations of first order, namely

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{m} a_i \frac{\partial u}{\partial x_i} + \sum_{i=1}^{m} b_i \frac{\partial v}{\partial x_i}$$

$$\frac{\partial v}{\partial t} = \sum_{i=1}^{m} c_i \frac{\partial u}{\partial x_i} + \sum_{i=1}^{m} d_i \frac{\partial v}{\partial x_i}$$

come into this case. Another example: in the case of a system of equations of first order, if m, the dimension of R^m, is 2, and if there exists a $\xi=(\xi_1, \xi_2)$ such that $\det.(\lambda E-A(\xi))$ has only one N-ple root, then the conditions (I) and (II) imply (III). The details are left to the reader.

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