MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES A
Vol. XXXIII, Mathematics No. 2, 1960.

On some ergodic transformations in metric spaces

By

Yoshimichi MIBU

(Communicated by Prof. A. Komatu)

(Received September 10, 1960)

Introduction. Let $(\Omega, \mathfrak{S}, \mu)$ be a measure space and T an invertible measure-preserving transformation on Ω. Let $\mathfrak{B}(\Omega, \mathfrak{S}, \mu)$ be the set of all equivalence classes of measurable sets of Ω, where two measurable sets E and F are called equivalent if and only if their symmetric difference $E \ominus F$ has measure 0. Then the set $\mathfrak{B}(\Omega, \mathfrak{S}, \mu)$ is a Boolean algebra under the natural Boolean operations and the measure μ can be considered as a measure on this Boolean algebra. (This Boolean algebra $\mathfrak{B}(\Omega, \mathfrak{S}, \mu)$ is called the measure algebra associated with the measure space $(\Omega, \mathfrak{S}, \mu)$.) The transformation T induces in a natural way a measure-preserving automorphism of the measure algebra $\mathfrak{B}(\Omega, \mathfrak{S}, \mu)$. The set of all complex valued μ-measurable functions $f(x)$ for which $\int_\Omega |f(x)|^2 d\mu(x) < \infty$ forms a Hilbert space $L_2(\Omega, \mathfrak{S}, \mu)$, and the transformation T induces a unitary operator on $L_2(\Omega, \mathfrak{S}, \mu)$ if we correspond to every $f(x) \in L_2(\Omega, \mathfrak{S}, \mu)$ a function $g(x)$ such that $g(x) = f(T(x))$. In this way an invertible measure-preserving transformation T on Ω can be regarded as a measure-preserving automorphism of $\mathfrak{B}(\Omega, \mathfrak{S}, \mu)$ or a unitary operator on $L_2(\Omega, \mathfrak{S}, \mu)$.

Let us suppose that S and T are invertible measure-preserving transformations on Ω. To discuss relations between such two transformations there has been introduced the concepts "similarity", "conjugacy" and "equivalence". First, S and T will be called (geometrically) similar if there exists an invertible measure-preserving transformation Q on Ω such that $S = Q^{-1}TQ$. (In this case we say that two transformations S and T are essentially the same as transformations on a measure space Ω.) Next, S and T are called (algebraically) conjugate if there exists a measure-preserving automorphism Q of $\mathfrak{B}(\Omega, \mathfrak{S}, \mu)$ such that $S=Q^{-1}TQ$, where S and T are regarded as measure-preserving automorphisms of $\mathfrak{B}(\Omega, \mathfrak{S}, \mu)$ respectively. (In this case we say that two transformations S and T are essentially the same as automorphisms of the measure algebra $\mathfrak{B}(\Omega, \mathfrak{S}, \mu)$.) Finally, S and T will be called (spectrally) equivalent if there exists a unitary operator Q on $L_2(\Omega, \mathfrak{S}, \mu)$ such that $S=Q^{-1}TQ$, where S and T are regarded as unitary operators on $L_2(\Omega, \mathfrak{S}, \mu)$ respectively. (In this case we say that two transformations S and T are essentially the same as transformations on the Hilbert space $L_2(\Omega, \mathfrak{S}, \mu)$.) It is easy to observe that similarity, conjugacy and equivalence can also be defined for pair of transformations that do not act on the same domain; in this case the implementing transformation Q will map one of the two domains onto the other. In an obvious sense similarity implies conjugacy, and conjugacy implies equivalence. But the converse is false for both these implications.

In the theory of ergodic transformations it is an interesting problem to discuss the relation between two transformations from viepoints of these concepts. J. von Neumann proved that an ergodic measure-preserving transformation with discrete spectrum* is conjugate to a rotation** on a compact abelian group. This is an interesting consequence which is called the representation theorem. And this theorem reminds us of the following question.

"*Under what conditions does it follow that an invertible measure-preserving transformation T on a measure space $(\Omega, \mathfrak{S}, \mu)$ is similar to a rotation on a compact abelian group?*"

The purpose of the present paper is to give an answer to the above question by proving three Representation Theorems (see below). In the proofs of these theorems the results in [4] employ an important role. Incidentally we shall prove a theorem that if a Riemanian manifold Ω admits an ergodic measure-preserving transformation which is also an isometric transform-

* A transformation T on a measure space $(\Omega, \mathfrak{S}, \mu)$ is said to have discrete spectrum if there exists a basic $\{f_j\}$ of $L_2(\Omega, \mathfrak{S}, \mu)$ each of which is a proper vector of the induced unitary operator U on $L_2(\Omega, \mathfrak{S}, \mu)$.

** Let G be a compact group. A mapping $x\rightarrow ax$ (where a is an element of G) is called a rotation on the group G.

ation of Ω onto itself, then Ω is a torus. This might be an interesting consequence.

§ 1. Group of isometric transformations.

The following lemmas are easily deduced from the theorems in [4].

LEMMA 1.1. *Let Ω be a metric space and $\mathfrak{H}$ the group of all isometric transformations of Ω onto itself. For every $\varepsilon > 0$, an arbitrary positive integer n and any n elements $x_1, x_2, \cdots, x_n$ of Ω we set*

$$(1.1)\quad U(\varepsilon\,; x_1, x_2, \cdots, x_n) = \{\sigma\,; \rho(\sigma x_i, x_i) < \varepsilon,\ i = 1, 2, n, \sigma \in \mathfrak{H}\}$$

If the set of all such $U(\varepsilon\,; x_1, x_2, \cdots, x_n)$ is taken as a complete system of neighborhoods of the identity of $\mathfrak{H}$, then $\mathfrak{H}$ becomes a topological group. (*See Theorem* 3.7 *in* [4])

It is easy to see that the topology which is introduced in our lemma coincides with the topology which is defined in Theorem 3.7 in [4].

LEMMA 1.2. *Let Ω be a compact metric space and $\mathfrak{H}$ the group af all isometric transformations of Ω onto itself. Let us suppose that $\mathfrak{H}$ is topologized by the method of Lemma* 1.1. *Then $\mathfrak{H}$ is a compact topological group.* (*see Theorem* 4.4, *Theorem* 4.8 *in* [4])

LEMMA 1.3. *Let Ω be a locally compact and connected metric space and $\mathfrak{H}$ the group of all isometric transformations of Ω onto itself. If $\mathfrak{H}$ is topologized by the method of Lemma* 1.1, *then $\mathfrak{H}$ is a locality compact and σ-compact topological group. Moreover, there exises an $\varepsilon > 0$ such that the neighborhood*

$$(1.2)\qquad U(\varepsilon\,; x) = \{\sigma\,; \sigma \in \mathfrak{H},\ \rho(\sigma x, x) < \varepsilon\}$$

has compact closure for every $x \in \Omega$. (*See Theorem* 4.4, *Theorem* 4.5 *and Theorem* 4.8 *in* [4])

LEMMA 1.4. *Let Ω be a metric space whose bounded set is always compact and $\mathfrak{H}$ the group of all isometric transformations of Ω onto itself. We introduce a topology in $\mathfrak{H}$ as in Lemma* 1.1. *Then $\mathfrak{H}$ becomes a locally compact and σ-compact topological group, and for every $\varepsilon > 0$ and every point $x \in \Omega$ the neighborhood*

$$(1.3)\qquad U(\varepsilon\,; x) = \{\sigma\,; \sigma \in \mathfrak{H},\ \rho(\sigma x, x) < \varepsilon\}$$

has compact closure. (*See Theorem* 4.7 *and Theorem* 4.8 *in* [4])

REMARK. In this § we had not assumed that the group $\mathfrak{H}$ satisfies the condition (C_{II}) of Theorem 3.4 in [4]. But our lemmas are proved quite similary as in [4].

§ 2. Some lemmas.

LEMMA 2.1. *Let G be a locally compact topological group and H an abstract subgroup of G such that $\bar{H}=G$. If a measure μ in G is invariant under H, that is, the relation $\mu(aA)=\mu(A)$ holds for every $a\in H$ and every Baire set $A\subseteqq G$, then μ is a Haar measure in G.*

The proof is easy and is omitted.

LEMMA 2.2. *Let G be a locally compact topological group. If there exists an element $a\in G$ such that the cyclic group $H=\{a^n; n=0, \pm 1, \pm 2, \cdots\}$ is dense in G, then G is a compact abelian group or a discrete cyclic group.*

For a proof of this lemma see Lemma 2 in [2], p. 96.

LEMMA 2.3. *Let Ω be a Hausdorff space and $\mathfrak{H}$ a group of homeomorphisms of Ω which satisfies the condition that*

(2.1) *for any two points $x, y\in\Omega$ there exists a homeomorphism $\sigma\in\mathfrak{H}$ such that $\sigma x=y$.*

Let us suppose that $\mathfrak{H}$ is topologized in such a manner that $\mathfrak{H}$ becomes a locally compact and σ-compact topological group. For any point $x\in\Omega$ we define a mapping ψ_x of $\mathfrak{H}$ onto the space Ω such that

$$\psi_x(\sigma) = \sigma x. \tag{2.2}$$

If the mapping ψ_x is continuous and the space Ω is of the second category, then the mapping ψ_x is open.

PROOF. Let W be an arbitrary open set of $\mathfrak{H}$. First, we shall show that $\psi_x(W)$ contains an open set. We select an open set V such that its closure $\bar{V}$ is compact and $\bar{V}\subseteqq W$. Since the topological group $\mathfrak{H}$ is σ-compact, there exists a countable set of elements σ_n, $n=1, 2, \cdots$ of $\mathfrak{H}$ such that the system of open sets $\sigma_n V$, $n=1, 2, \cdots$, covers the topological group $\mathfrak{H}$. Suppose that $\psi_x(\sigma_n\bar{V})=F_n$, $n=1, 2, \cdots$. Since every F_n is a continuous image of a compact set, it is also a compact closed set. From the condition (2.1) of the lemma we have $\bigcup_{n=1}^{\infty}F_n=\Omega$. Since the space Ω

is of the second category, there exists at least one set F_n containing an open set. Then the set $F=\psi_x(\bar{V})=\sigma_n^{-1}\psi_x(\sigma_n\bar{V})=\sigma_n^{-1}F_n$ contains also an open set. Since $\bar{V}\subseteqq W$, $\psi_x(W)$ contains an open set.

Let U be a neighborhood of the identity $e\in\mathfrak{H}$. Then there exists a neighborhood W of the identity such that $W^{-1}W\subseteqq U$. From what we have just proved $\psi_x(W)$ contains an open set W^*. Let $p\in W^*$ and σ an element of W such that $\psi_x(\sigma)=p$ $(\sigma x=p)$. Then $\sigma^{-1}W$ is a neighborhood of the identity e which is contained in U. Consequently we have $\psi_x(U)\supseteqq\psi_x(W^{-1}W)\supseteqq\psi_x(\sigma^{-1}W)=\sigma^{-1}\psi_x(W)\supseteqq\sigma^{-1}W^*$. Since W^* contains the point $p=\psi_x(\sigma)(=\sigma x)$, the set $\sigma^{-1}W^*$ is an open set containing the point $\sigma^{-1}p=\sigma^{-1}\sigma x=x$.

The above argument asserts that if U is an arbitrary neighborhood of the identity $e\in\mathfrak{H}$, then $\psi_x(U)$ contains an open set which contains the point x. Let D be an open set of $\mathfrak{H}$ and σ an arbitrary element of D. Then $\sigma^{-1}D$ is clearly a neighborhood of the identity $e\in\mathfrak{H}$. From what we have just mentioned the point x is an inner point of the set $\psi_x(\sigma^{-1}D)=\sigma^{-1}\psi_x(D)$. Hence $\sigma x=\psi_x(\sigma)$ is an inner point of the set $\sigma\sigma^{-1}\psi_x(D)=\psi_x(D)$. This shows that $\psi_x(D)$ is open. Our lemma is completely proved.

§3. Ergodic transformations.

LEMMA 3.1. *Let Ω be a metric space and μ a measure in Ω such that every non-empty open set has positive measure. If an isometric transformation σ of Ω onto itself is an ergodic measure-preserving transformation on Ω, then for every point x of Ω the set $\{\sigma^n x\,;\,n=0,\pm1,\pm2,\cdots\}$ is everywhere dense in Ω.*

The proof is easy and is omitted.

LEMMA 3.2. *Let Ω be a locally compact metric space and $\mathfrak{H}$ the group of all isometric transformations of Ω onto itself. We shall assume that the following three conditions are satisfied*:

1) *If we introduce a topology in Ω as in Lemma* 1.1, *then $\mathfrak{H}$ becomes a locally compact and σ-compact topological group.*

2) *There exists an element $\sigma_0\in\mathfrak{H}$ and a point $x_0\in\Omega$ such that the set $\{\sigma_0^n x_0\,;\,n=0,\pm1,+2,\cdots)$ is everywhere dense in Ω.*

3) *There exists a positive number ε such that a neighborhood $U(\varepsilon\,;x)=\{\sigma\,;\sigma\in\mathfrak{H},\ \rho(\sigma x,\ x)<\varepsilon\}$ has compact closure for every point $x\in\Omega$.*

Then we have

(α) *The space* Ω *is compact or discrete.*

(β) *We can define an operation in* Ω *which associates with each pair of elements* x, y *of* Ω *a third element* z *of* Ω, *written as* $z=x\circ y$, *satisfying the following conditions*:

(i) Ω *becomes an abelian topological group by the product* $x\circ y$ *and the original topology of* Ω.

(ii) *The relation* $\sigma_0 x=\sigma_0 x_0\circ x$ *holds for every* $x\in\Omega$.

PROOF. Let H be the cyclic group which is generated by the element $\sigma_0\in\mathfrak{H}$. Then $\bar{H}$ is clearly a locally compact and σ-compact topological group and H is everywhere dense in $\bar{H}$. Hence by Lemma 2. 2 the group $\bar{H}$ is a compact abelian group or a discrete cyclic group.

First, we shall assume that $\bar{H}$ is a compact abelian group. We define a mapping ψ_{x_0} of $\mathfrak{H}$ into Ω such that $\psi_{x_0}(\sigma)=\sigma x_0$ for every $\sigma\in\mathfrak{H}$. It is easy to see that ψ_{x_0} is continuous. Since $\bar{H}$ is compact, $\psi_{x_0}(\bar{H})=\{\sigma x_0\,;\,\sigma\in\bar{H}\}$ is also compact and closed. Hence we have $\Omega\supseteqq\psi_{x_0}(\bar{H})=\overline{\psi_{x_0}(\bar{H})}\supseteqq\overline{\psi_{x_0}(H)}=\Omega$, that is, $\psi_{x_0}(\bar{H})=\Omega$. Let N be the set of all elements $\sigma\in\bar{H}$ such that $\sigma x_0=x_0$. And let $\bar{H}/N$ be the factor group of $\bar{H}$ by the closed normal subgroup N. Then it is easily seen that the mapping ψ_{x_0} can be regarded as a mapping of $\bar{H}/N$ onto the space Ω. Moreover, ψ_{x_0} is a one-to-one continuous mapping of the compact space $\bar{H}/N$ onto the space Ω and hence a topological mapping. We associate with each pair of elements $x, y\in\Omega$ a third element $z\in\Omega$ such that

$$z = \psi_{x_0}(\psi_{x_0}^{-1}(x)\cdot\psi_{x_0}^{-1}(y))\,. \tag{3.1}$$

Then it is evident that Ω becomes a compact abelian group which is isomorphic with the topological group $\bar{H}/N$. Let x be an arbitrary point in Ω and σ an element of $\bar{H}$ such that $\sigma x_0=x$ (notice that $\psi_{x_0}(\bar{H})=\Omega$). Then it is easy to see that $\psi_{x_0}^{-1}(x)=\sigma N$. On the other hand $\psi_{x_0}^{-1}(\sigma_0 x_0)=\sigma_0 N$. Hence by the definition of multiplication of the group Ω we have $\sigma_0 x_0\circ x=\psi_{x_0}(\psi_{x_0}^{-1}(\sigma_0 x_0)\cdot\psi_{x_0}^{-1}(x))=\psi_{x_0}(\sigma_0 N\cdot\sigma N)=\psi_{x_0}(\sigma_0\sigma N)=\sigma_0\sigma x_0=\sigma_0 x$. The assertions ($\alpha$) and ($\beta$) are thereby proved.

Next, we shall assume that $\bar{H}$ is a discrete cyclic group, that is, $\bar{H}=H$. In the first place, we shall show that $\psi_{x_0}(\mathfrak{H})=\Omega$. Let x be an arbitrary point in Ω. Since the set $\{\sigma_0^n x_0\,;\,n=0,\ \pm1,\ \pm2,\cdots\}$ is everywhere dense in Ω, we can select from the set $\{\sigma_0^n x_0\,;\,n$

$=0, \pm 1, \cdots\}$ a subsequence $\sigma_0^{n_1}x_0, \sigma_0^{n_2}x_0, \cdots, \sigma_0^{n_i}x_0, \cdots$ which converges to the point x. Let ε be a positive number as in the condition 3) of the present lemma. There exists a positive integer J such that $\rho(\sigma_0^{n_J}x_0, \sigma_0^{n_i}x_0) < \varepsilon$ for every $i \geqq J$. Hence it holds that $\rho(x_0, \sigma_0^{(n_i - n_J)}x_0) < \varepsilon$ for every $i \geqq J$. This implies that $\sigma_0^{(n_i - n_J)} \in U(\varepsilon; x_0)$, that is, $\sigma_0^{n_i} \in \sigma_0^{n_J} U(\varepsilon; x_0)$ for every $i \geqq J$. Since the neighborhood $U(\varepsilon; x_0)$ has compact closure we can easily see that $\psi_{x_0}(\sigma_0^{n_J}\overline{U(\varepsilon; x_0)}) \supseteqq \overline{\psi_{x_0}(\sigma_0^{n_J}U(\varepsilon; x_0))} \supseteqq$ [closure of the set $\{\sigma_0^{n_i}x_0; i \geqq J\}] \ni x$.

Hence we have $\psi_{x_0}(\mathfrak{H}) = \Omega$. Let N be the set of those elements $\sigma \in \mathfrak{H}$ for which $\sigma x_0 = x_0$. From the condition 3) of the lemma it is easy to see that N is a compact subgroup of $\mathfrak{H}$. If there exists an integer $n > 0$ such that $\sigma_0^n x_0 = x_0$, then Ω consists of at most n points $x_0, \sigma_0 x_0, \cdots, \sigma_0^{n-1}x_0$. In this case our assertion is trivial. Hence we assume that

$$\sigma_0^n x_0 \neq x_0 \quad \text{for} \quad n \neq 0. \tag{3.2}$$

Since $N \frown H = e$ (identity of $\mathfrak{H}$), N is compact and H is discrete, we can easily prove that there exists a neighborhood V of the identity $e \in \mathfrak{H}$ such that

$$VNV^{-1} \frown H = e. \tag{3.3}$$

Let $D_n = \psi_{x_0}(\sigma_0^n V)$, $n = 0, \pm 1, \cdots$. By Lemma 2.3 the mapping ψ_{x_0} is an open mapping of $\mathfrak{H}$ onto Ω and hence every D_n is an open set containing the point $\sigma_0^n x_0$. We shall show that $D_n \frown D_m = 0$, $n \neq m$. Assume the contrary. Then there exist two elements $\tau_1, \tau_2 \in V$ such that $\sigma_0^n \tau_1 x_0 = \sigma_0^m \tau_2 x_0$. This implies that $\tau_2^{-1}\sigma_0^{n-m}\tau_1 x_0 = x_0$. Hence we have $\tau_2^{-1}\sigma_0^{n-m}\tau_1 \in N$, that is, $\sigma_0^{n-m} \in \tau_2 N \tau_1^{-1} \subset VNV^{-1}$. Thus we have arrived at a contradiction (see (3.2), (3.3)). Consequently, we have $D_n \frown D_m = 0$ for $n \neq m$ and $D_n \ni \sigma_0^n x_0$. Since $\psi_{x_0}(V) = D_0$ is an open set containing the point x_0, there exists a positive number δ such that $\{x; \rho(x_0, x) < \delta\} \subseteqq D_0$. According to the isometry of every transformation σ_0^n we can easily see that $\{x; \rho(\sigma_0^n x_0, x) < \delta\} = \sigma_0^n\{x; \rho(x_0, x) < \delta\} \subseteqq \sigma_0^n D_0 = D_n$. From this we can easily prove that the closure of the set $\{\sigma_0^n x_0; n = 0, \pm 1, \pm 2, \cdots\}$ coincides with itself. Hence we have $\Omega = \{\sigma_0^n x_0; n = 0, \pm 1, \cdots\}$. In this case it is quite easy to see that the assertions (α) and (β) are valid. Our lemma is thereby proved.

Representation Theorem 1. *Let Ω be a compact metric space* *and T an ergodic measure-preserving transformation on a measure space* $(\Omega, \mathfrak{S}, \mu)$ *which satisfies the condition that*

(*A*) $\mathfrak{S}$ *is the class of all Borel sets of* Ω *and* μ *is a countably additive measure on* $\mathfrak{S}$ *such that* $\mu(D)>0$ *for every non-empty open subset D of* Ω.

If T is an isometric transformation of Ω *onto itself, then T is similar to a rotation on a compact abelian group.*

PROOF. Let $\mathfrak{H}$ be the group of all isometric transformations of Ω onto itself. We introduce in $\mathfrak{H}$ a topology as in Lemma 1.1. By Lemma 1.2 $\mathfrak{H}$ is a compact group. Hence the conditions 1) and 3) of Lemma 3.2 are satisfied. We select an arbitrary point $x_0 \in \Omega$. By Lemma 3.1 the set $\{T^n x_0; n=0, \pm 1, \pm 2, \cdots\}$ is everywhere dense in Ω. Hence the condition 2) of Lemma 3.2 is also satisfied. By Lemma 3.2 we can define the product $x \circ y$ for every pair of points x, y of Ω satisfying the conditions (i) and (ii) in Lemma 3.2. In this way Ω can be regarded as a compact abelian group and the mapping T is a rotation on the compact abelian group Ω such that $Tx = Tx_0 \circ x$. Let H be the cyclic subgroup of Ω which is generated by the element Tx_0 of the group Ω. Then it is easy to see that the measure μ is invariant under H and H is everywhere dense in Ω. Hence by Lemma 2.1 the measure μ is a Haar measure in the compact abelian group Ω. Our theorem is thereby proved.

REPRESENTATION THEOREM 2. *Let* Ω *be a locally compact and connected metric space and T an ergodic measure-preserving transformation on a measure space* $(\Omega, \mathfrak{S}, \mu)$ *which satisfies the condition* (*A*). *If T is an isometric transformation of* Ω *onto itself, then T is similar to a rotation on a compact abelian group.*

PROOF. By using Lemma 1.3, Lemma 2.1 and Lemma 3.2 we can prove quite similarly as in Representation Theorem 1 that T is similar to a rotation on a compact abelian group or a translation* on a discrete cyclic group. Since Ω is connected, T can not similar to a translation on a discrete cyclic group. Hence we have our theorem.

REPRESENTATION THEOREM 3. *Let* Ω *be a metric space whose bounded set is always compact and T an ergodic measure-preserving*

* Let G be a discrete group. A mapping $x \rightarrow ax$ (where a is an element of G) is called a translation on G.

transformation on a measure space (Ω, $\mathfrak{S}$, μ) *which satisfies the condition* (A). *If* T *is an isometric transformation of* Ω *onto itself, then* T *is similar to a rotation on a compact abelian group or a translation on a discrete cyclic group.*

This theorem is proved quite similarly as in Representation Theorem 1 by using Lemma 1.4, Lemma 2.1 and Lemma 3.2.

THEOREM. *Let* Ω *be a Riemanian manifold and* μ *a measure in* Ω *such that every non-empty open set has positive measure. If there exists an ergodic measure-preserving transformation which is also an isometric transformation of* Ω *onto itself, then* Ω *is a torus.*

PROOF. By using Lemma 1.3, Lemma 2.2 and Lemma 3.2 we can easily prove that Ω is regarded as a comact abelian group or a discrete cyclic group. On the other hand Ω is connected and locally connected. Hence we have our theorem.

REFERENCES

[1] L. Pontrjagin, Topological groups, Princeton (1939).

[2] A. Weil, L'intégration dans les groupes topologiques et ses applications, Paris, 1953.

[3] P. R. Halmos, Lectures on ergodic theory, Publication of the Mathematical Society of Juapan, **3** (1956).

[4] Y. Mibu, On measure invariant under given homeomorphism group of a uniform space, J. Math. Soc. Japan, **10** (1958) pp. 405-429.