MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES A
Vol. XXXIII, Mathematics No. 2, 1960.

A theorem on closed mapping

By

Hisahiro TAMANO

(Received September 10, 1960)

Let f be a closed continuous mapping of a space X onto a paracompact space Y. It is well known that X is again paracompact if $f^{-1}(y)$ is compact for each $y \in Y$, (c.f. [3] and [5], p. 81). A similar result on the normality of X will be obtained, and we shall show a necessary and sufficient condition for the normality of X in terms of the properties of point inverse $f^{-1}(y)$, which is the purpose of this note. We shall show that if f is a closed continuous mapping of a space X onto a paracompact space Y, then X is normal if and only if $f^{-1}(y)$ is normal and normally embedded in X (that is, every bounded continuous function on $f^{-1}(y)$ has a continuous extension over X) for each $y \in Y$. As a direct consequence of this, it will be proved that the product $X \times Y$ of a paracompact space X with a normal space Y is normal if the projection mapping $p: X \times Y \rightarrow X$ is closed (c.f. [3]).

All spaces mentioned here will be assumed to be completely regular T_1-spaces and all functions to be real valued.

§1. **Preliminary.** The Stone-Čech compactification βX of a space X is a compact (Hausdorff) space containing X as a dense subspace such that every bounded continuous function on X has a continuous extension over βX. It is easy to see that if F, G are closed subsets of X which are functionally separated (that is, there is a continuous function h on X such that $h=0$ on F and $h=1$ on G), then $Cl_{\beta X}(F) \cap Cl_{\beta X}(G) = \phi$, where $Cl_{\beta X}(F)(Cl_{\beta X}(G))$ denotes the closure of F (resp. G) taken in βX. Therefore we have:

Lemma 1.[1] X is normal if and only if $Cl_{\beta X}(F) \cap Cl_{\beta X}(G) = \phi$

1) This result is due to Čech [1].

for each pair of disjoint closed sets F, G of X.

The following is the principal theorem on the Stone-Čech compactification and will play an important role in the present note.

Lemma 2. Let f be a continuous mapping of a space X into a compact space Y, then there is a continuous extension $f*$ of f which carries βX into Y.

For the proof, see [7], p. 153.

A mapping $f: X \rightarrow Y$ is said to be closed if the image of a closed subset of X is closed. Let f be a continuous mapping of a space X onto another space Y, then f can be considered as a continuous mapping of X into βY. Accordingly, there is by Lemma 2 a continuous extension $f*$ of βX onto βY (c.f. [8], p. 476).

Proposition 1. Let f be a continuous mapping of X onto Y and let $f*$ be the continuous extension of f over βX. Then f is closed if and only if $f^{-1}(y) \cap F = \phi$ implies $f*^{-1}(y) \cap Cl_{\beta X}(F) = \phi$ for any point $y \in Y$ and for any closed set $F \subset X$.

Proof. Suppose that there is a closed set $F \subset X$ such that $f^{-1}(y) \cap F = \phi$ and $f*^{-1}(y) \cap Cl_{\beta X}(F) \neq \phi$ for some $y \in Y$. Let z be a point of $f*^{-1}(y) \cap Cl_{\beta X}(F)$, then z is an accumulation point of F and therefore $f*(z) = y$ is an accumulation point of $f*(F) = f(F)$, by virtue of the continuity of $f*$. Obviously, $f^{-1}(y) \cap F = \phi$ implies $y \in f(F)$. It follows that $f(F)$ is not closed and hence f is not closed. Conversely, if f is not closed, then there is a closed subset F of X such that $f(F)$ is not closed in Y. Let y be a point of $Cl_Y(f(F)) - f(F)$, then $F \cap f^{-1}(y) = \phi$. We shall show that $Cl_{\beta X}(F) \cap f*^{-1}(y) \neq \phi$ which will complete the proof. Suppose, on the contrary, that $Cl_{\beta X}(F) \cap f*^{-1}(y) = \phi$, then we have $f*(Cl_{\beta X}(F)) \not\ni y$. On the other hand, $f*(Cl_{\beta X}(F)) \cap Y$ is a closed set of Y containing $f(F)$, since $f*(Cl_{\beta X}(F))$ is compact and since $f*(Cl_{\beta X}(F)) \cap Y \supset f*(Cl_{\beta X}(F) \cap X) = f*(F) = f(F)$. It follows that $y \in Cl_Y(f(F)) \subset f*(Cl_{\beta X}(F))$, which is contradictory.

A partition of unity on a space X is a family $\{\varphi_\lambda\}$ of continuous function on X such that $\Sigma \varphi_\lambda = 1$ for each $x \in X$ and all but a finite numbers of φ_λ's vanish outside some neighborhood of each point of X. For the sake of convenience, we shall designate by $0(\varphi_\lambda)$ the elementary open set defined by $|\varphi_\lambda|$. That is $0(\varphi_\lambda) = \{x \in X; \varphi_\lambda(x) \neq 0\}$.

Lemma 3. A space X is paracompact if and only if for any compact set $C \subset \beta X - X$ there is a partition of unity $\{\varphi_\lambda\}$ on X such that $Cl_{\beta X}(0(\varphi_\lambda)) \cap C = \phi$ for each λ.

Proof. (Necessity) For each $x \in X$, take an open neighborhood U_x of x such that $Cl_{\beta X}(U_x) \cap C = \phi$ and consider a covering $\{U_x\}_{x \in X}$ of X. If X is paracompact, then there is a locally finite refinement $\{U_\lambda\}$ of $\{U_x\}_{x \in X}$. Furthermore, there is a partition of unity $\{\varphi_\lambda\}$ which is subordinate to $\{U_\lambda\}$ (c.f. [2], p. 71). It is evident that $Cl_{\beta X}(0(\varphi_\lambda)) \cap C = \phi$, and the necessity of the condition is proved. (Sufficiency) Let $\{U_\alpha\}$ be any open covering of X. For each U_α, we take and fix one open set $U*_\alpha$ of βX such that $U*_\alpha \cap X = U_\alpha$, Put $C_\alpha = [U^*_\alpha]^C$, where $[U*_\alpha]^C$ denotes the complementary set of $U*_\alpha$, and put $C = \bigcap_\alpha C_\alpha$, then C is a compact set contained in $\beta X - X$. From the hypothesis of the lemma, there is a partition of unity $\{\varphi_\lambda\}$ such that $Cl_{\beta X}(0(\varphi_\lambda)) \cap C = \phi$. Since $\bigcup_\alpha U*_\alpha = \beta X - C$, $\{U*_\alpha\}$ covers $Cl_{\beta X}(0(\varphi_\lambda))$ for each λ and consequently there is a finite number of $U*_\alpha$'s, say $U*_{\alpha_1}, \cdots, U*_{\alpha_n}$, such that $\bigcup_{k=1}^{n} U*_{\alpha_k} \supset Cl_{\beta X}(0(\varphi_\lambda))$. Put $H_{\lambda,k} = 0(\varphi_\lambda) \cap U*_{\alpha_k}$, then $0(\varphi_\lambda) = \bigcup_{k=1}^{n} H_{\lambda,k}$. Thus, each $0(\varphi_\lambda)$ can be represented as a finite union of open sets of the form $H_{\lambda,k}$. Constructing $H_{\lambda,k}$ for each $0(\varphi_\lambda)$ in this way, we have an open refinement $\{H_{\lambda,k}\}$ of $\{U_\alpha\}$, which is obviously locally finite. It follows that X is paracompact.

Proposition 2. Let C be a compact set which is contained in $\beta X - X$. Let F, G be two closed sets of X for which $Cl_{\beta X}(F) \cap Cl_{\beta X}(G) \neq \phi$. If there is a partition of unity $\{\varphi_\lambda\}$ such that $Cl_{\beta X}(0(\varphi_\lambda)) \cap C = \phi$ for each λ, then $Cl_{\beta X}(F) \cap Cl_{\beta X}(G) \not\subset C$.

Proof. Suppose that $Cl_{\beta X}(F) \cap Cl_{\beta X}(G) \subset C$, and put $A_\lambda = Cl_{\beta X}(0(\varphi_\lambda)) \cap Cl_{\beta X}(F)$, $B_\lambda = Cl_{\beta X}(0(\varphi_\lambda)) \cap Cl_{\beta X}(G)$. For each λ, let us define a continuous function h_λ as follows. Set

$$h_\lambda = 0 \quad \text{if} \quad A_\lambda = \phi, \qquad \text{and}$$
$$h_\lambda = 1 \quad \text{if} \quad B_\lambda = \phi.$$

In another case, it is true that both A_λ and B_λ are non-void compact set of βX and $A_\lambda \cap B_\lambda = \phi$, because $A_\lambda \cap B_\lambda \subset C \cap Cl_{\beta X}(0(\varphi_\lambda)) = \phi$. There is a continuous function $h*_\lambda$ on βX such that $h*_\lambda = 1$ on A_λ and $h*_\lambda = 0$ on B_λ. Let us define h_λ to be the restriction of $h*_\lambda$ on X, in this case. Thus,

$$h_\lambda = 1 \quad \text{on} \quad A_\lambda \cap X \qquad \text{and}$$
$$h_\lambda = 0 \quad \text{on} \quad B_\lambda \cap X, \quad \text{if} \quad A_\lambda \neq \phi \quad \text{and} \quad B_\lambda \neq \phi .$$

It is easy to verify that $f = \Sigma h_\lambda \cdot \varphi_\lambda$ is a continuous function on X and that $f=1$ on F and $f=0$ on G. Therefore F, G are functionally separated closed sets of X, and it follows that $Cl_{\beta X}(F) \cap Cl_{\beta X}(G) = \phi$ which is contradictory.

§2. **The Theorem.** We first introduce the notion of normally embedded subspace. We shall say that a subspace E of X is normally embedded in X if every bounded continuous function on E has a continuous extension over X. It is easy to see that E is normally embedded in X if and only if $Cl_{\beta X}(E) = \beta E$ (more precisely, $Cl_{\beta X}(E)$ is homeomorphic with βE). In fact, if E is normally embedded in X, then every bounded continuous function $f \in C^*(E)$ has a continuous extension over X and hence over βX. Taking restriction on $Cl_{\beta X}(E)$ of the continuous extension of f over βX, we have a continuous extension of f over $Cl_{\beta X}(E)$. Thus, $Cl_{\beta X}(E)$ is a compact space containing E as a dense subspace such that every bounded continuous function on E has a continuous extension over $Cl_{\beta X}(E)$. Therefore $Cl_{\beta X}(E) = \beta E$, by virtue of the uniqueness of the Stone-Čech compactification (within homeomorphism). Conversely, if $Cl_{\beta X}(E) = \beta E$, then every bounded continuous function $f \in C^*(E)$ has a continuous extension f' over $Cl_{\beta X}(E)$, and, since $Cl_{\beta X}(E)$ is a closed subspace of a normal space βX, f' has a continuous extension f^* over βX. Clearly, the restriction on X of f^* is the desired extension of f over X.

For any space X, every compact subspace is normally embedded in X and every subspace having unique uniform structure (or equivalently, having unique compactification) is normally embedded in X.

In a normal space X, every closed subspace is normally embedded in X. Moreover, it is true that X is normal if and only if every closed subspace of X is normally embedded in X. If f is a projection mapping of a product space $X \times Y$ onto Y, then $f^{-1}(y)$ is normally embedded in $X \times Y$ for each $y \in Y$.

Theorem. Let f be a closed continuous mapping of a space X onto a paracompact space Y. Then, X is normal if and only if $f^{-1}(y)$ is normal and normally embedded in X for each $y \in Y$.

Proof. The necessity of the condition is clear, therefore we have only to prove the sufficiency. Suppose that X is not normal, then there are two disjoint closed sets F, G of X such that $Cl_{\beta X}(F) \cap Cl_{\beta X}(G) \neq \phi$, by virtue of Lemma 1. Put $C = Cl_{\beta X}(F) \cap Cl_{\beta X}(G)$, then it is clear that $C \subset \beta X - X$. Let $f*$ be the continuous extension of f over βX. ($f*$ is a continuous mapping of βX onto βY.) We shall show firstly that $f*(C) \not\subset \beta Y - Y$. If this is not the case, then $C' = f*(C)$ is a compact set contained in $\beta Y - Y$. Since Y is paracompact, there is a partition of unity $\{\psi_\lambda\}$ on Y such that $Cl_{\beta Y}(0(\psi_\lambda)) \cap C' = \phi$ for each λ, by virtue of Lemma 3. Letting $\varphi_\lambda(x) = \psi_\lambda(f(x))$, we have a partition of unity $\{\varphi_\lambda\}$ on X. In fact, the local finiteness of $\{0(\varphi_\lambda)\}$ follows from the local finiteness of $\{0(\psi_\lambda)\}$ and the continuity of f. Since $Cl_{\beta Y}(0(\psi_\lambda)) \cap C' = \phi$, we have $Cl_{\beta X}(0(\varphi_\lambda)) \cap f*^{-1}(C') = \phi$. It follows from Proposition 2 that $C \not\subset f*^{-1}(C')$. On the other hand, $f*(C) = C'$ and hence $C \subset f*^{-1}(C')$. We have thus a contradiction. Consequently, we see that there is a point $z \in C$ such that $f*(z) = y \in Y$.

If it is true that $F' = f*^{-1}(y) \cap F \neq \phi$ and $G' = f*^{-1}(y) \cap G \neq \phi$, then F', G' are disjoint closed sets of $f^{-1}(y)$, and, since $f^{-1}(y)$ is normal, there is a continuous function g on $f^{-1}(y)$ such teat $g=1$ on F' and $g=0$ on G'. Since $f^{-1}(y)$ is normally embedded in X, there is a continuous extension $g*$ of g over X and consequently F', G' are functionally separated closed sets of X. Therefore we have $Cl_{\beta X}(F') \cap Cl_{\beta X}(G') = \phi$. It follows that at least one of $Cl_{\beta X}(F')$ and $Cl_{\beta X}(G')$ does not contain z. Assume that $z \notin Cl_{\beta X}(F')$ and let V be an open set of βX containing $Cl_{\beta X}(F')$ such that $Cl_{\beta X}(V) \not\ni z$. Put $H = F \cap [V]^c$, then H, G are disjoint closed sets of X for which $Cl_{\beta X}(H) \cap Cl_{\beta X}(G) \ni z$. Furthermore, it follows immediately that $f^{-1}(y) \cap H = \phi$ and $f*^{-1}(y) \cap Cl_{\beta X}(H) \ni z$, by virtue of the definition of H. Thus, we see that there exists a closed set H of X such that $f^{-1}(y) \cap H = \phi$ and $f*^{-1}(y) \cap Cl_{\beta X}(H) \neq \phi$ for some $y \in Y$. It follows that f is not closed, by virtue of Proposition 1. But this contradicts the hypothesis of the theorem, therefore X must be normal.

Remark 1. We cannot replace the paracompactness of Y by the normality of Y, in the preceding theorem, as the following example shows: Let Ω denote the set of all ordinals less than the first uncountable ordinal and let Ω_0 denote the set of all ordinals less than or equal to the first uncountable ordinal, each with the order topology. Let $X=\Omega\times\Omega_0$, and let $X=\Omega$, and let f be the projection mapping of X onto Y. Then f is closed, and $f^{-1}(y)$ is compact for each $y\in Y$. As is well known, X is not normal while Y is normal (c.f. [5], p. 93 and [7], p. 132).

Corollary. Let $X\times Y$ be a product space of a paracompact space X with a normal space Y. If the projection mapping $p: X\times Y\to X$ is closed, then $X\times Y$ is normal.

Proof. This is an immediate consequence of the preceding theorem, in view of the fact that $f^{-1}(x)$ is normally embedded in $X\times Y$.

Remark 2. A slightly stronger result may be proved: If E is a subspace of $X\times Y$ in the preceding corollary such that $E\cap\{x\}\times Y$ is closed for each $x\in X$, and if the restriction on E of the projection mapping is closed, then E is normal.

Remark 3. If the projection mapping $p: X\times Y\to X$ is closed, then Y is pseudocompact ([6], p. 67) provided that X is not a finite set. In fact, if Y is not pseudocompact, then there is an unbounded continuous function $h(y)$ on Y. Let $f(x)$ be the continuous function on X such that the elementary open set $\{x\in X; f(x)>0\}$ is not closed (c.f. footnote (10) of the preceding paper "A note on the Stone-Čech compactification of a product of two spaces"), then the projection of the closed set $\{(x, y)\in X\times Y; f(x)h(y)=1\}$ is not closed in X. Accordingly, if X is not a finite set, then the space $X\times Y$ in the above corollary is countably paracompact. To prove this, let us recall Dowker's theorem [4] which states that a space Z is normal and countably paracompact if and only if $Z\times I$ is normal, where I is the closed unit interval. As is well known, a normal pseudocompact space is countably compact [6] and hence countably paracompact, therefore $Y\times I$ is normal. On the other hand, the projection mapping $p*: X\times(Y\times I)\to X$ is closed, because $p*=p\circ p'$ and since I is compat $p': X\times Y\times I\to X\times Y$ is closed. Applying the above corollary to $X\times(Y\times I)$, we see that $X\times Y\times I$ is normal. Therefore $X\times Y$ is a normal countably paracompact space. The same is valid for the space E in Remark 2.

REFERENCES

[1] E. Čech: On bicompact spaces, Ann. of Math. (2) 38 (1937), 823-844.

[2] J. Dieudonné: Une généralization des espaces compacts, J. Math. Pures Appl. 23 (1944), 65-76.

[3] ——————: Un critère de normalité pour les espaces produits. Colloq. Math. 6 (1958), 29-32. (Math. Review Vol. 21, 4 (1960))

[4] C. H. Dowker: On countably paracompact spaces, Canadian J. Math. 3 (1951), 219-244.

[5] M. Henriksen and J. R. Isbell: Some properties of compactifications, Duke Math. J. 25 (1958), 83-105.

[6] E. Hewitt: Rings of real-valued continuous function I, Trans. Amer. Math. Soc. 64 (1948), 45-99.

[7] J. L. Kelley: General topology, New York, 1955.

[8] M. H. Stone: Application of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc. 41 (1937), 375-481.