Memoirs of the College of Science, University of Kyoto, Series A
Vol. XXXIII, Mathematics No. 2, 1960.

Existence of a bounded solution and existence of a periodic solution of the differential equation of the second order

By

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(Received September 8, 1960)

1. Introduction. We consider a differential equation of the second order

$$x'' = F(t, x, x'), \tag{1}$$

where $F(t, x, x')$ is periodic of t. Massera has proved that if all the solutions exist in the future and if one of them is bounded in the future, then a periodic solution exists [1]. Therefore, even if all the solutions are not bounded, when we see the existence of a bounded solution, we can prove the existence of a periodic solution in some cases.

In this paper we discuss the existence of a bounded solution and we apply it to the existence of a periodic solution.

Now we assume that $F(t, x, x')$ is continuous in $I\times R_x^1\times R_y^1$, where I is the interval $0\leqq t<\infty$ and R^n is the n-dimensional Euclidean space. For Theorem 1, the periodicity of $F(t, x, x')$ is not necessary.

2. Existence of a bounded solution. We shall obtain an existence theorem of a bounded solution by considering the boundary value problem.

Theorem 1. *Suppose that two functions* $\bar{\omega}(t)$ *and* $\underline{\omega}(t)$ *are*

* This research was partially supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under Contract Number AF 49 (638)-382. Reproduction in whole or in part is permitted for any purpose of the United States Government.

defined on I, twice differentiable and bounded on I with their derivatives and that they satisfy the following inequalities;

$$\underline{\omega}(t) \leqq \bar{\omega}(t)$$
$$\bar{\omega}''(t) \leqq F(t, \bar{\omega}(t), \bar{\omega}'(t))$$
$$\underline{\omega}''(t) \geqq F(t, \underline{\omega}(t), \underline{\omega}'(t)) .$$

Let D be the domain such that $0 \leqq t < \infty$, $\underline{\omega}(t) \leqq x \leqq \bar{\omega}(t)$. *And we represent two domains* $(t, x) \in D$, $y \geqq K$ *and* $(t, x) \in D$, $y \leqq -K$ *by* D_1 *and* D_2 *respectively, where K is a positive number and may be sufficiently large.*

We assume that there exist two positive continuous functions $V_1(t, x, y)$ *defined in* D_1 *and* $V_2(t, x, y)$ *defined in* D_2 *satisfying the following conditions;*

1° $V_i(t, x, y) \leqq a(|y|)$ $(i=1, 2)$, *where* $a(r)$ *is a positive continuous function,*

2° $V_i(t, x, y)$ *tend to infinity uniformly as* $|y| \to \infty$,

3° $V_i(t, x, y) \in C_0(x, y)$ (*cf.* [2]) *and we have in the interiors of* D_1 *and* D_2

$$V_1'(t, x, y) = \varliminf_{h \to +0} \frac{1}{h} \{V_1(t+h, x+hy, y+hF(t, x, y)) - V_1(t, x, y)\} \geqq 0$$

$$V_2'(t, x, y) = \varlimsup_{h \to +0} \frac{1}{h} \{V_2(t+h, x+hy, y+hF(t, x, y)) - V_2(t, x, y)\} \leqq 0 .$$

Then the equation (1) *has a bounded solution, where a bounded solution means a solution* $x(t)$ *such that* $x(t)^2 + x'(t)^2$ *is bounded for all* $t \geqq t_0$.

Proof. Let n be an arbitrary integer and let D_n be the domain such that

$$0 \leqq t \leqq n , \quad \underline{\omega}(t) \leqq x \leqq \bar{\omega}(t) , \quad |y| < \infty .$$

First of all, we show that there is a solution $x_n(t)$ of (1) such that

$$x_n(0) = \underline{\omega}(0) , \quad x_n(n) = \underline{\omega}(n)$$

and that for $0 \leqq t \leqq n$ we have

$$\underline{\omega}(t) \leqq x_n(t) \leqq \bar{\omega}(t)$$

and

$$|x_n'(t)| < M ,$$

where M is a positive number independent of n.

By the assumptions for $\overline{\omega}(t)$ and $\underline{\omega}(t)$, we can assume that $|\overline{\omega}'(t)|<K$ and $|\underline{\omega}'(t)|<K$, because K may be sufficiently large. We can choose K independent of n. By the condition 1°, we have

$$V_1(t, x, K) \leqq a(K), \quad V_2(t, x, -K) \leqq a(K).$$

Since $V_i(t, x, y)$ tend to infinity uniformly as $|y| \to \infty$, we can choose a positive number M such that for $(t, x) \in D$,

$$a(K) < V_1(t, x, M), \quad a(K) < V_2(t, x, -M).$$

And this M is independent of n.

Then considering the function $H(t, x, y)$ such that

$$H(t, x, y) = \begin{cases} F(t, x, M) & (y > M) \\ F(t, x, y) & (|y| \leqq M) \\ F(t, x, -M) & (y < -M), \end{cases}$$

we define the function $F^*(t, x, y)$ as follows;

$$F^*(t, x, y) = \begin{cases} H(t, \overline{\omega}(t), y) + \dfrac{x - \overline{\omega}(t)}{x - \overline{\omega}(t) + 1} & (x > \overline{\omega}(t)) \\ H(t, x, y) & (\underline{\omega}(t) \leqq x \leqq \overline{\omega}(t)) \\ H(t, \underline{\omega}(t), y) - \dfrac{\underline{\omega}(t) - x}{\underline{\omega}(t) - x + 1} & (x < \underline{\omega}(t)). \end{cases}$$

This function $F^*(t, x, y)$ is defined, continuous and bounded on $0 \leqq t \leqq n$, $|x| < \infty$, $|y| < \infty$. Therefore the equation

$$(2) \qquad x'' = F^*(t, x, x')$$

has at least a solution $x_n(t)$ such that $x_n(0) = \underline{\omega}(0)$, $x_n(n) = \underline{\omega}(n)$. From the assumption for $\overline{\omega}(t)$ and $\underline{\omega}(t)$, we can see that we have $\underline{\omega}(t) \leqq x_n(t) \leqq \overline{\omega}(t)$ for $0 \leqq t \leqq n$.

Now we show that $|x_n'(t)| < M$. Since we have $\underline{\omega}(t) \leqq x_n(t) \leqq \overline{\omega}(t)$, we have $-K < x_n'(0)$. Suppose that at some t, say t_1, we have $x_n'(t_1) \leqq -M$. Then there exist t_2 and t_3 such that $t_2 < t_3$, $x_n'(t_2) = -K$, $x_n'(t_3) = -M$ and that for $t_2 < t < t_3$,

$$-M < x_n'(t) < -K.$$

Now we consider the function $V_2(t, x_n(t), x_n'(t))$. This function is non-increasing along the solution by the condition 3°. Hence there arises a contradiction. Therefore we have

$$x_n'(t) > -M.$$

By considering the function $V_1(t, x_n(t), x_n'(t))$, we can see that we have $x_n'(t) < M$.

Since in the region $0 \leqq t \leqq n$, $\underline{\omega}(t) \leqq x \leqq \overline{\omega}(t)$, $|y| \leqq M$, $F^*(t, x, y)$ is equal to $F(t, x, y)$, the solution $x_n(t)$ of (2) becomes the desired solution of (1).

Now we consider the sequence of functions $\{\xi_n(t)\}$ such that

$$\xi_n(t) = \begin{cases} x_n(t) & (0 \leqq t \leqq n) \\ \underline{\omega}(t) & (n < t < \infty). \end{cases}$$

Since this sequence of functions is uniformly bounded and equicontinuous, we can choose a uniformly convergent subsequence and let $x(t)$ be its limiting function. It is clear that we have

$$\underline{\omega}(t) \leqq x(t) \leqq \overline{\omega}(t) \qquad (0 \leqq t < \infty),$$
$$|x'(t)| \leqq M$$

and that $x(t)$ is a solution of (1).

When we have $\underline{\omega}(0) = \overline{\omega}(0)$, in place of the condition 3°, it is sufficient that we have $V_1'(t, x, y) \leqq 0$, $V_2'(t, x, y) \leqq 0$.

3. Existence of a periodic solution. For the continuability of solutions, we have the following theorem. More generally, we consider a system

$$(3) \qquad x' = F(t, x),$$

where x is an n-dimensional vector and $F(t, x)$ is a continuous vector field defined on $I \times R^n$.

Theorem 2. *If corresponding to each T there exists a positive continuous function $W(t, x)$ satisfying the following conditions in the domain*

$$0 \leqq t \leqq T, \quad ||x|| \geqq R_0 \qquad (R_0 \textit{ may be sufficiently large});$$

1° *$W(t, x)$ tends to infinity uniformly as $||x|| \to \infty$,*
2° *$W(t, x) \in C_0(x)$ and*

$$W'(t, x) = \overline{\lim_{h \to +0}} \frac{1}{h} \{W(t+h, x+hF(t, x)) - W(t, x)\} \leqq 0,$$

then every solution of (3) *exists in the future.*

In some cases, the following theorem is more convenient. Namely we consider a system

$$(4)\qquad \begin{cases} x' = F(t, x, y) \\ y' = G(t, x, y), \end{cases}$$

where x is an n-dimensional vector, y is an m-dimensional vector and $F(t, x, y)$, $G(t, x, y)$ are continuous on $I\times R_x^n\times R_y^m$.

Theorem 3. *We assume that corresponding to each T there exists a positive continuous function $W_1(t, x, y)$ satisfying the following conditions in the domain*

$0\leqq t\leqq T$, $||x||^2+||y||^2\geqq R_0^2$ (*R_0 may be sufficiently large*);

1° *$W_1(t, x, y)$ tends to infinity uniformly as $||y||\to\infty$,*

2° *$W_1(t, x, y)\in C_0(x, y)$ and*

$$W_1'(t, x, y) = \overline{\lim_{h\to+0}}\frac{1}{h}\{W_1(t+h,\ x+hF(t, x, y),\ y+hG(t, x, y))-W_1(t, x, y)\}\leqq 0.$$

Moreover we assume that corresponding to each K and each T there exists a positive continuous function $W_2(t, x, y)$ satisfying the following conditions in the domain

$0\leqq t\leqq T$, $||x||\geqq R_1$, $||y||\leqq K$ (*R_1 may be sufficiently large*);

1° *$W_2(t, x, y)$ tends to infinity uniformly as $||x||\to\infty$,*

2° *$W_2(t, x, y)\in C_0(x, y)$ and $W_2'(t, x, y)\leqq 0$.*

Then all the solutions of (4) *exist in the future.*

Proof. We show that for any $\alpha>R_0$ and any $T>0$, all the solutions starting from the domain $D_\alpha[0\leqq t\leqq T,\ ||x||^2+||y||^2\leqq\alpha^2]$ are continuable to $t=T$. Now let $W_1(t, x, y)$ be the one corresponding to T. We put

$$M(\alpha) = \max_{\substack{0\leqq t\leqq T \\ ||x||^2+||y||^2=\alpha^2}} W_1(t, x, y).$$

Since $W_1(t, x, y)$ tends to infinity uniformly as $||y||\to\infty$, we can choose a positive number β such that $\beta>\alpha$ and

$$m(\beta) = \inf_{\substack{0\leqq t\leqq T \\ ||y||=\beta \\ ||x||<\infty}} W_1(t, x, y) > M(\alpha).$$

Now we suppose that at some t, say t_1, we have $||y(t_1)||=\beta$, where

$x=x(t)$, $y=y(t)$ is a solution of (4) starting from D_α. Then there exist t_2 and t_3 such that

$$\|x(t_2)\|^2+\|y(t_2)\|^2=\alpha^2,\ \|y(t_3)\|=\beta$$

and that for $t_2<t<t_3$, we have $\|x(t)\|^2+\|y(t)\|^2>\alpha^2$. Considering the function $W_1(t, x(t), y(t))$, we have

$$M(\alpha)\geqq W_1(t_2, x(t_2), y(t_2))\geqq W_1(t_3, x(t_3), y(t_3))\geqq m(\beta).$$

This contradicts $m(\beta)>M(\alpha)$. Therefore we have $\|y(t)\|<\beta$ for $t\leqq T$.

Next let $W_2(t, x, y)$ be the one corresponding to T and β. We can assume that $\alpha>R_1$. We put

$$M(\alpha, \beta)=\max_{\substack{0\leqq t\leqq T\\ \|x\|=\alpha\\ \|y\|\leqq\beta}} W_2(t, x, y).$$

We can choose a positive number $\gamma(>\alpha)$ such that

$$m(\gamma, \beta)=\min_{\substack{0\leqq t\leqq T\\ \|x\|=\gamma\\ \|y\|\leqq\beta}} W_2(t, x, y)>M(\alpha, \beta).$$

In the same way as the above, considering the function $W_2(t, x(t), y(t))$, if we suppose that we have $\|x(t)\|=\gamma$ at some t, there arises a contradiction. Therefore we can see that $\|x(t)\|<\gamma$ for $t\leqq T$.

From the above-mentioned, we have $\|x(t)\|<\gamma$, $\|y(t)\|<\beta$ for $t\leqq T$. Hence this solution is continuable to $t=T$. Since T is arbitrary, we can see that this solution exists in the future.

Therefore by Theorem 1 with Theorem 2 or Theorem 3, we can prove the existence of a periodic solution of the equation (1). For example we consider the equation

$$(5)\qquad \ddot{x}+f(x, \dot{x})+g(t, x)=p(t),$$

where $f(x, y)$ and $g(t, x)$ are continuous and locally Lipschitzian with respect to x and y, $p(t)$ and $g(t, x)$ are periodic in t and $p(t)$ is continuous. We assume that $f(x, y)\,y\geqq 0$ and $-\infty<G(t, x)$ for all (t, x), where $G(t, x)=\int_0^x g(t, s)\,ds$ and that $\dfrac{|G_t|}{\sqrt{G(t, x)+C}}$ is bounded, where $G_t=\dfrac{\partial G(t, x)}{\partial t}$ and C is a constant such that

$G(t, x)+C>0$. Moreover we assume that there exist two constants a, b such that $a<b$ and

$$(6) \qquad \begin{cases} 0 \leqq f(a, 0)+g(t, a)-p(t) \\ 0 \geqq f(b, 0)+g(t, b)-p(t) \end{cases}$$

and that there exists a positive continuous function $\varphi(u)$ for $-\infty<u<+\infty$ such that $|f(x, y)| \leqq \varphi(y)$ and

$$\int^{\infty} \frac{u}{\varphi(u)+c}\,du = \int^{-\infty} \frac{u}{\varphi(u)+c}\,du = +\infty,$$

where $|g(t, x)|+|p(t)| \leqq c$ for $0 \leqq t<\infty$, $a \leqq x \leqq b$.

For an arbitrary $T>0$, in the domain $0 \leqq t \leqq T$, $|x|<\infty$, $|y|<\infty$, we consider a system

$$(7) \qquad \dot{x} = y, \quad \dot{y} = -f(x, y)-g(t, x)+p(t)$$

which is equivalent to the equation (5). Since $\dfrac{|G_t|}{\sqrt{G(t, x)+C}}$ is bounded, there is a positive constant k such that

$$\frac{|G_t|}{\sqrt{2(G(t, x)+C)}} \leqq k.$$

In the domain $0 \leqq t \leqq T$, $x^2+y^2 \geqq R_0^2$, we put

$$W_1(t, x, y) = \exp\left\{\sqrt{2(G(t, x)+C)+y^2}-\int_0^t |p(t)|\,dt-kt\right\}.$$

Then we have

$$\begin{aligned} W_1'(t, x, y) &= W_1(t, x, y)\left\{\frac{G_t+g(t, x)y-f(x, y)y-g(t, x)y+p(t)y}{\sqrt{2(G(t, x)+C)+y^2}} \right. \\ &\qquad \left. -|p(t)|-k\right\} \\ &\leqq W_1(t, x, y)\left\{\frac{|G_t|}{\sqrt{2(G(t, x)+C)}}+|p(t)|-|p(t)|-k\right\} \\ &\leqq 0. \end{aligned}$$

Therefore $W_1(t, x, y)$ is non-increasing along the solution of (7). The function $W_1(t, x, y)$ is the one in Theorem 3. In the system (7), the boundedness of $|y(t)|$ implies the boundedness of $|x(t)|$, because of $\dot{x}=y$. Namely both $|x(t)|$ and $|y(t)|$ are bounded on $0 \leqq t \leqq T$ and hence all the solutions of (7) are continuable to $t=T$. Since T is arbitrary, all the solutions exist in the future.

Now if we put $\bar{\omega}(t)=b$ and $\underline{\omega}(t)=a$, these satisfy the conditions for $\bar{\omega}(t)$ and $\underline{\omega}(t)$ by (6). In the domains $D_1[0\leqq t<\infty,\ a\leqq x\leqq b,\ y\geqq K]$ and $D_2[0\leqq t<\infty,\ a\leqq x\leqq b,\ y\leqq -K]$, we define $V_1(t, x, y)$ and $V_2(t, x, y)$ respectively as follows;

$$V_1(t, x, y) = \exp\left\{x+\int_K^y \frac{u}{\varphi(u)+c}\,du\right\},$$

$$V_2(t, x, y) = \exp\left\{x+\int_{-K}^y \frac{u}{\varphi(u)+c}\,du\right\}.$$

Then we have

$$\begin{aligned} V_1'(t, x, y) &= V_1(t, x, y)\left[y+\frac{y}{\varphi(y)+c}\{-f(x, y)-g(t, x)+p(t)\}\right] \\ &\geqq V_1(t, x, y)\left[y-\frac{y}{\varphi(y)+c}(\varphi(y)+c)\right] \\ &\geqq 0 \end{aligned}$$

and in the same way, we have $V_2'(t, x, y)\leqq 0$. Since $V_1(t, x, y)$ and $V_2(t, x, y)$ satisfy the conditions in Theorem 1, we can see that there exists a bounded solution. Therefore, by Massera's theorem, we can see that there exists a periodic solution.

For example, the equation $\ddot{x}+k\sin x=p(t)$ $(k>0)$ has a periodic solution if $p(t)$ is periodic and $|p(t)|\leqq k$.

When we assume that in place of $-\infty<G(t, x)$, we have $G(t, x)<\infty$ and $|g(t, x)|<\infty$, we can also see that the equation (5) has a periodic solution.

Since the equation $\ddot{x}+f(x)\dot{x}+g(x)=p(t)$ is a special case of the above-mentioned, we can see the existence of a periodic solution under the condition

$$(8)\qquad \begin{cases} 0\leqq g(a)-p(t) \\ 0\geqq g(b)-p(t). \end{cases}$$

But Seifert showed the author that in this case he can prove the existence of a periodic solution only under the condition (8) without the condition such that $-\infty<G(x)$ and $f(x)\geqq 0$ by seeing the index of the bounding curve of a simply-connected region relative to a vector field induced by the mapping.

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