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Some remarks on prime divisors

By

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A ring will mean always a commutative ring with a unit.

When R is a Noetherian ring, a prime ideal $\mathfrak{p}$ of R is a prime divisor of an ideal $\mathfrak{a}$ if and only if $\mathfrak{p}R_{\mathfrak{p}}$ is a prime divisor of $\mathfrak{a}R_{\mathfrak{p}}$ and it is true that every semi-prime ideal of R has no imbedded prime divisor.

In the present paper, we give an example of non-Noetherian ring in which the above two are not true. We prove more generally the following existence theorem:

THEOREM. *Let* $\{R_\lambda\}$ *be a set of quasi-local rings which contain a common field* K. *Then there is a ring* T *such that* (1) *for each* R_λ, *there is a maximal ideal* $\mathfrak{m}_\lambda$ *of* T *such that* $T_{\mathfrak{m}_\lambda} \cong R_\lambda$, (2) *if* $\mathfrak{m}$ *is a maximal ideal of* T, *then* $T_{\mathfrak{m}}$ *is isomorphic to either* K *or one of* R_λ *and* (3) *the total quotient ring of* T *is* T *itself.*

Consider the case where R_λ are integral domains which are not fields. By (3), every non-unit of T is a zero-divisor, whence every maximal ideal of T is a prime divisor of zero. But, $\mathfrak{m}_\lambda T_{\mathfrak{m}_\lambda}$ is not a prime divisor of zero. Furthermore, that $T_{\mathfrak{m}}$ has no nilpotent element for every maximal ideal $\mathfrak{m}$ of T by (2) implies that T itself has no nilpotent elements, and the zero ideal of T is semi-prime. But, each $\mathfrak{m}_\lambda$ is not minimal, hence is an imbedded prime divisor of zero.

Thus we are to prove the theorem. Let A be an infinite set from which there is a map ϕ onto the set $\{R_\lambda\}$ and let B be another infinite set. Set $C = A \times B$. Let Ω be the set of functions f defined on $A \cup C$ (disjoint union) such that (i) if $a \in A$, then $f(a) \in \phi(a)$ and (ii) if $c \in C$, then $f(c) \in K$. Let M be the subset of Ω consisting of those f such that (i) $f(c) = 0$ for every $c \in C$, (ii) $f(a) = 0$ for all but a finite number of elements a of A and (iii)

$f(a)$ is in the maximal ideal of $\phi(a)$. Let $K*$ be the subset of Ω consisting of those f such that $f(a)=0$ for all $a\in A$ and $f(c)=0$ for all but a finite number of elements c of C. Elements k of K are identified with elements f of Ω such that $f(x)=k$ for every $x\in A\cup C$. e_a is, for each $a\in A$, an element of Ω such that $e_a(x)$ is 1 or zero according to whether $x\in \{a\}\cup a\times B$ or not. e_c is, for each $c\in C$, such that $e_c(x)$ is 1 or zero according to whether $x=c$ or not. Now let T be the subring of Ω generated by M, $K*$, K and the Ke_a. Note that $e_c\in K*$ for every $c\in C$. Since M and $K*$ are ideals of Ω, they are ideals of T, and we have $T=K+K*+M+\Sigma Ke_a$. Assume that $f\in T$ is a non-unit in T and we are to prove that f is a zero-divisor. $f=k+k*+m+\Sigma k_a e_a$ with $k\in K$, $k*\in K*$, $m\in M$, $k_a\in K$ ($k_a=0$ except for a finite number of a). If $f(x)$ is a unit for every $x\in A\cup C$, then f is a unit in Ω. From our expression of f, we see easily that f^{-1} is in T, and f is a unit in T which is a contradiction. Therefore $f(x)$ is a non-unit for some x. If $x\in C$, then $f(x)=0$ and f is a zero divisor (for $fe_x=0$). Assume that $x\in A$. Then $k+k_x=0$. Since B is an infinite set, there is a $b\in B$ such that, for $c=x\times b$, $k*(c)=0$, whence $f(c)=0$, and f is a zero divisor. Thus (3) is proved. Let $\mathfrak{m}$ be a maximal ideal of T and let σ be the natural homomorphism from T into $T_{\mathfrak{m}}$. Since a quasi-local ring has no non-trivial idempotent element, we see that $\sigma(e_x)$ is either 1 or zero for each $x\in A\cup C$.

(い) The case where $e_c\notin\mathfrak{m}$ for a $c\in C$: The set $\mathfrak{q}$ of $f\in T$ such that $f(c)=0$ is an ideal of T and $e_c\mathfrak{q}=0$, whence $\mathfrak{q}$ is contained in the kernel of σ. Since $T/\mathfrak{q}\cong K$, we see that $T_{\mathfrak{m}}=T/\mathfrak{q}\cong K$.

(ろ) The case where $e_c\in\mathfrak{m}$ for all $c\in C$ but there is an $a\in A$ such that $e_a\notin\mathfrak{m}$: The kernel $\mathfrak{q}'$ of σ contains $1-e_a$ and all the e_x ($a\neq x\in A\cup C$). Let $\mathfrak{q}$ be the set of $f\in T$ such that $f(a)=0$. We want to show that $\mathfrak{q}\subseteq\mathfrak{q}'$. Let f be an arbitrary element of $\mathfrak{q}$. $f=k+k*+m+\Sigma k_{a'}e_{a'}$ ($k\in K$, $k*\in K*$, $m\in M$, $k_{a'}\in K$, $a'\in A$). Since $1-e_a$ and e_x ($x\neq a$) are in $\mathfrak{q}'$, $f\equiv k+m(a)+k_a$ modulo $\mathfrak{q}'$. On the other hand, since $f(a)=0$, we have $k+m(a)+k_a=0$, whence $f\equiv 0$ modulo $\mathfrak{q}'$, i.e., $f\in\mathfrak{q}'$. Thus $\mathfrak{q}\subseteq\mathfrak{q}'$. Since $T/\mathfrak{q}\cong\phi(a)$ which is a quasi-local ring, we see that $T_{\mathfrak{m}}=T/\mathfrak{q}\cong\phi(a)$.

(は) The remaining case is the one where $\mathfrak{m}$ contains all the e_x ($x\in A\cup C$). Then $\sigma(e_x)=0$. Therefore, $\sigma(K*+\Sigma Ke_a)=0$. Furthermore, if $c\in C$, then $1-e_c\notin\mathfrak{m}$ and $(1-e_c)M=0$. Therefore

$\sigma(M)=0$. Since $T/(K*+M+\Sigma Ke_a)\cong K$, $\sigma(T)\cong K$ and therefore $T_{\mathfrak{m}}\cong K$.

Since the above three cases exist and since ϕ is surjective, we see that (1) and (2) are true, and the theorem is proved completely.

Remark. With the same notations as above, $M+K$ becomes a quasi-local ring with maximal ideal M. The total quotient ring of $M+K$ is $M+K$ itself. If the R_λ are integral domains which are not fields, then the zero of $M+K$ is semi-prime and M is an imbedded prime divisor of zero.