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# Remark on a note by Mr. M. Nagata on coefficient fields of complete local rings

By

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Let $R$ and $R'$ be complete local rings (which may not be Noetherian) and $R \subseteq R'$. In his "Note on coefficient fields of complete local rings"[1] Mr. Masayoshi Nagata stated a simple condition under which *at least one* coefficient ring of $R$ may be extended to a coefficient ring of $R'$—while the corresponding extension of the residue class fields is inseparable. I have proved his theorem without the assumption that $R$ contains a field. Thus it is shown to hold in the case of unequal characteristics as well.

*Theorem.* Assume that the residue class fields $K$ and $K'$ of $R$ and $R'$ respectively are of characteristics $p \neq 0$ and that $R'^p \subseteq R$. Then there is a coefficient ring of $R$ which is extendable to a coefficient ring of $R'$.

*Proof.* Since $R'^p \subseteq R$, we have the chain $K^p \subseteq K'^p \subseteq K \subseteq K'$. Let $\{\bar{c}_\sigma\}$ be a maximal subset of $K$ among those which are $p$-independent over $K'^p$ and let $\{\bar{c}'_{\sigma'}\}$ be such that $\{\bar{c}_\sigma, \bar{c}'_{\sigma'}\}$ forms a $p$-basis of $K'$. Then $K = K'^p(\bar{c}_\sigma)$, $K' = K(\bar{c}'_{\sigma'})$, and $K'^p = K^p(\bar{c}'^p_{\sigma'})$, whence $\{\bar{c}'^p_{\sigma'}, \bar{c}_\sigma\}$ is obviously a $p$-basis of $K$. Let $c_\sigma$, $c'_{\sigma'}$ be representatives of $\bar{c}_\sigma$, $\bar{c}'_{\sigma'}$ in $R$, $R'$ respectively. There is one and only one coefficient ring $E'$ in $R'$ containing $\{c_\sigma, c'_{\sigma'}\}$. $\{c'^p_{\sigma'}, c_\sigma\}$ is a set of representatives of the $p$-basis $\{\bar{c}'^p_{\sigma'}, \bar{c}_\sigma\}$ of $K$; it is contained in $R$ (since $R'^p \subseteq R$) and also in $E'$. There is one and only one coefficient ring $E$ in $R$ containing $\{c'^p_{\sigma'}, c_\sigma\}$. $E'$ must contain $E$, because it includes the set $\{c'^p_{\sigma'}, c_\sigma\}$ of representatives *in* $E$ of a $p$-basis of $K$.

In this proof repeated use was made of the following

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1) These Memoirs, Series A, Vol. XXXII, Mathematics No. 1, 1959, pp. 91–92.

*Lemma.* In a complete local ring $R'$ there exists one and only one coefficient ring containing an arbitrarily given set of representatives of a $p$-basis of the residue class field $K'$, which is assumed to be of characteristic $p \neq 0$. If $E$ is a local subring of $R'$ and $E_p$ its maximal ideal, then a coefficient ring of $R'$ contains $E$ if and only if it contains a set of representatives in $E$ of a $p$-basis of the residue class field of $E$.

This lemma was proved in an unpublished paper of mine. It may be obtained by combining Cohen's existence lemma (cited in Mr. Nagata's note) with the stepwise construction of coefficient rings as proposed for the first time by Mr. Nagata.[2] The second assertion results from an easy application of the first one to the complete local ring $f^{-1}(f(E))$, where $f$ is the canonical homomorphism of $R'$ onto $K'$.

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2) M. Nagata, On the structure of complete local rings, Nagoya Mathematical Journal, vol. 1 (1950), pp. 63-70; Corrections, vol. 5 (1953), pp. 145-147. See also P. Roquette, Abspaltung des Radikals in vollständigen lokalen Ringen, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, vol. 23 (1959), pp. 75-113.