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# On rational surfaces, II

By

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In §1 of the present paper, we introduce the notion of a virtual linear system on a non-singular projective surface and we clarify the theories of infinitely near points, of divisors and of linear system with preassigned base conditions.

We introduce in §2 the notions of a numerical types and of non-special points with respect to Cremona transformations. They play important roles in §3 in order to prove characterizations and existence theorems of exceptional curves of the first kind and of Cremona transformations. In §4, we introduce the notion of an abnormal curve, and in §5 we give some remarks on superabundance of a complete virtual linear system on a projective plane $S$. We add some remarks in §6 on the case where the number of base points is at most 9.

The recent paper "On rational surfaces, I" in the last volume of our memoirs is quoted as Part I in the present paper. The notations and terminology in Part I are preserved in this paper, except for that the symbol $\{\}$ for the total transform of a divisor is changed to $()$; see §1. We recall here that an $S$ denotes always a projective plane. A curve will mean a positive divisor on a surface. A divisor $c$ on a surface $F$ is identified with a divisor $c'$ on a surface $F'$ if $c=\sum m_i c_i$ and $c'=\sum m_i c'_i$ and if $c_i$ and $c'_i$ are irreducible and are identical with each other as point sets (identification of points is made by natural birational transformations).

## 1. Virtual linear system.

Let $F$ be a non-singular projective surface and let $B$ be the family of non-singular projective surfaces which are birational with $F$ by natural transformations.

(い) A linear combination of curves on $F$ and points over $F$ with integral coefficients (a point over $F$ means a point which is an infinitely near point of a point of $F$) is called a *cycle* over $F$. A cycle is divided into two parts; one is the *divisorial part* and the other is the *zero-dimensional part*. We say that two cycles $c$ and $c^*$ over $F$ are *linearly equivalent* to each other if (i) the divisorial parts of $c$ and $c^*$ are linearly equivalent to each other and (ii) the zero-dimensional parts of $c$ and $c^*$ are identical with each other.

(ろ) Cycles over $F$ are classified by the linear equivalence relation. Each linear equivalence class is called a *complete virtual linear system* on $F$. If $c$ is a cycle over $F$, then the linear equivalence class containing $c$ is called the complete virtual linear system containing $c$ and is denoted by $\mathcal{L}(c)$.

A set $L$ of cycles over $F$ is called a *virtual linear system* on $F$ if, for any two members $c$ and $c'$ of $L$, there are a cycle $c^*$ over $F$ and a linear system $L^*$ on $F$ such that (i) $c-c^*$ and $c'-c^*$ are in $L^*$ and (ii) if $c''\in L^*$ then $c''+c^*\in L$.

An important example of a virtual linear system is a complete virtual linear system. Another example of a virtual linear system is a *fractional linear system* which is defined to be a set $L$ of cycles over $F$ for which there are a cycle $c$ over $F$ and a linear system $L^*$ on $F$ such that $L$ is the set of cycles $c+l^*$ with $l^*\in L^*$. We note that a virtual linear system $L$ is characterized by the property that for any two members of $L$ (or equivalently, for any finite number of members of $L$) there is a fractional linear system contained in $L$ and containing the given members.

Let $L$ be a virtual linear system on $F$. The set $L'$ of divisorial parts of members of $L$ is again a virtual linear system and is called the *divisorial part* of $L$. The zero-dimensional part of a member of $L$ is called the *zero-dimensional part* of $L$.

(は) Let $F'$ be a member of $B$ which dominates $F$. Then the antiregular transform $T$ from $F$ onto $F'$ defines by obvious manner the *total transforms* $T(c)$ ($T\{c\}$ in Zariski's notation) of cycles $c$ over $F$ and $T(c)$ become cycles over $F'$. Thus $T$ becomes a map from the set of cycles over $F$ onto the set of cycles over $F'$. The inverse map of $T$ is called the *projection* from $F'$ onto $F$ and is denoted by $\mathrm{proj}_F$.

The definition of linear equivalence implies that:

LEMMA 1.1. *Two cycles $c$ and $c^*$ over $F$ are linearly equivalent to each other if and only if $T(c)$ and $T(c^*)$ are linearly equivalent to each other.*

(に) Let $F'$ be an arbitrary member of $B$. There is a surface $F''$ in $B$ which dominates both $F$ and $F'$. Let $T$ be the transformation from $F$ onto $F''$. Then the map $T^*=\mathrm{proj}_{F'}\cdot T$ is well defined and is independent of the particular choice of $F''$. This $T^*$, which is obviously a map from the set of cycles over $F$ onto the one over $F'$, is called the *transformation* from $F$ onto $F'$ and is denoted by $T_{F\to F'}$.

By virtue of Lemma 1.1, we have easily the following result:

LEMMA 1.2. *If $L$ is a complete virtual linear system, or a virtual linear system, or a fractional linear system on $F$, then so is $T_{F\to F'}(L)$ respectively on $F'$.*

(ほ) Let $c$ and $c'$ be divisors on $F$. Then the intersection number $(c, c')$ of $c$ and $c'$ is well defined. Let $P_1, \cdots, P_n$ be points over $F$. For cycles $c+\Sigma n_iP_i$ and $c'+\Sigma n_i'P_i$, we define the *intersection number* $(c+\Sigma n_iP_i,\ c'+\Sigma n_i'P_i)$ to be $(c, c')-\Sigma n_in_i'$. If $L$ and $L'$ are virtual linear systems on $F$ and if $d\in L$, and $d'\in L'$, then $(d, d')$ is independent of the particular choice of $d$, $d'$. The intersection number $(d, d')$ is called the *intersection number* of $L$ and $L'$ or of $d$ and $L'$ and is denoted by $(L, L')$ or $(d, L')$.

If $d$ is either a cycle over $F$ or a virtual linear system on $F$, then $(d, d)$ is called the *grade* of $d$ and is denoted by $I(d)$.

LEMMA 1.3. *If $F'\in B$, then $(d, d')=(T_{F\to F'}(d), T_{F\to F'}(d'))$ for any two cycles $d$ and $d'$ over $F$.*

The proof is easy be virtue of the following well known, easy lemma:

LEMMA 1.4. *If $c$ is a divisor on $F$ and if $P$ is a point on $F$, then* $(\mathrm{dil}_P(P), \mathrm{dil}_P(P))=-1$ *and* $(\mathrm{dil}_P(c), \mathrm{dil}_P(P))=0$.

In the notations in Lemma 1.4, we should remark

COROLLARY. $\mathrm{dil}_P(c)=\mathrm{dil}_P[c]+m(P; c)\,\mathrm{dil}_P(P)$ (*where* $\mathrm{dil}_P[c]$ *denotes the proper transform of* $c$) *and therefore* $(\mathrm{dil}_P[c], \mathrm{dil}_P(P))=m(P; c)$.

(へ) For a given cycle over $F$, we consider an $F'\in B$ such that $T_{F\to F'}(c)$ has no zero-dimensional part. We say that $c$ is *virtually positive* ($c\triangleright 0$ in symbol) if $T_{F\to F'}(c)$ is positive; this is defined independently of the particular choice of $F'$ as is easily seen.

When $L$ is a virtual linear system, then $L^+$ denotes that set of $c$ in $L$ such that $c \geqq 0$. $L^+$ is obviously a fractional linear system, and the divisorial part of $L^+$ is a linear system on $F$. This last linear system is denoted by $L^{\oplus}$, and is called the *effective linear system associated* with $L$. The dimension of $L^{\oplus}$ is called the *effective dimension* of $L$ and is denoted by effdim $L$.

LEMMA 1.5. *If* $F'' \in B$, *then* $(T_{F\to F''}(L))^+ = T_{F\to F''}(L^+)$, *and on the other hand,* $(T_{F\to F''}(L))^{\oplus}$ *and* $T_{F\to F''}(L^{\oplus})$ *coincides with each other up to fixed components and the zero-dimensional parts. In particular, we have* $\mathrm{effdim}\, L = \mathrm{effdim}\, T_{F\to F''}(L)$.

(と) When $P_1, \cdots, P_r$ are mutually distinct points over $F$, a curve $c$ on $F$ is said to *go through the points* $P_i$ *with virtual multiplicities at least* $m_i$ if $c - \Sigma m_i P_i \geqq 0$. It must be observed that the above condition does not mean that $m(P_i; c) \geq m_i$. For instance, when $P_0$ is a point on $F$ and when $P_1, \cdots, P_r$ are mutually distinct infinitely near points of $P_0$ of the first order, then it holds that

LEMMA 1.6. *If* $m \leq r-1$, *then a curve* $c$ *goes through* $P_0$ *with virtual multiplicity at least* $m$ *and the points* $P_1, \cdots, P_r$ *with virtual multiplicities at least* 1 *if and only if* $m(P_i; c) \geq m+1$.

The proof is straightforward by virtue of the corollary to Lemma 1.4.

Let $L$ be a virtual linear system whose divisorial part and the zero-dimensional part are $L''$ and $-\Sigma m_i P_i$ respectively. Then the above definition justifies to call $L^{\oplus}$ the *linear system of curves in* $L''$ *which goes through the points* $P_i$ *with virtual multiplicities at least* $m_i$. As is well known and as is easily seen, the following inequality holds good:

LEMMA 1.7. $\mathrm{effdim}\, L \geq \mathrm{effdim}\, L'' - \Sigma_{m_i>0} m_i(m_i+1)/2$.

(ち) If $L$ and $L'$ are virtual linear systems on $F$, then the set $M = \{c + c'; c \in L, c' \in L'\}$ is contained in a complete virtual linear system. This complete virtual linear system is called the *complete sum* of $L$ and $L'$ and is denoted by $[L+L']$. The smallest virtual linear system containing $M$ is called the *minimal sum* of $L$ and $L'$ and is denoted by $L+L'$. It is obvious that if $F' \in B$, then $T_{F\to F'}([L+L']) = [T_{F\to F'}(L) + T_{F\to F'}(L')]$ and $T_{F\to F'}(L+L') = T_{F\to F'}(L) + T_{F\to F'}(L')$.

(り) We say that a curve on $F$ is *virtually connected* if, for any curves $c'$ and $c''$ such that $c = c' + c''$, the intersection number

$(c', c'')$ is positive. A virtually connected curve is obviously connected (but, not conversely).

PROPOSITION 1. *Let $c$ be a curve on $F$ and let $F'$ be a member of $B$ which dominates $F$. Then $c$ is virtually connected if and only if so is $T_{F\to F'}(c)$.*

Proof. The if part is obvious by the definition. Assume that there are curves $c'$ and $c''$ on $F'$ such that $T_{F\to F'}(c)=c'+c''$ and such that $(c', c'')\leq 0$. Let the divisorial part and the zero dimensional part of $T_{F'\to F}(c')$ be $c^*$ and $\Sigma m_iP_i$ respectively. Then we have $T_{F'\to F}(c'')=(c-c^*)-\Sigma m_iP_i$. Since $c''$ is a curve, $c^{**}=c-c^*$ must be either a curve or zero. $0\geq(c', c'')=(c^*+\Sigma m_iP_i, c^{**}-\Sigma m_iP_i)=(c^*, c^{**})+\Sigma m_i^2$, from which the only if part follows. Thus Proposition 1 is proved.

A virtual linear system $L$ on $F$ is said to be *virtually connected* if for an $F'\in B$ such that $T_{F\to F'}(L)$ has no zero dimensional part, $(T_{F\to F'}(L))^+$ is not empty and at least one member of it is virtually connected. Proposition 1 above shows that the above definition does not depend on the particular choice of $F'$.

An easy example of a connected curve which is not virtually connected is given as follows: Let $c$ be a connected curve on $F$ and let $P$ be a point on $c$. On the surface $F'=\mathrm{dil}_P F$, the curve $c'=\mathrm{dil}_P(c+P)$ is the required example.

(ぬ) For a given virtually positive cycle $c$ on $F$, we consider the set $C$ of $T_{F\to F'}(c)$ $(F'\in B)$ such that the cycles $T_{F\to F'}(c)$ are curves. A minimal member in $C$ (in the sence of domination) is called a *minimal curve* of $c$.

THEOREM 1. *Assume that a curve $c$ on $F$ is connected. Then $c$ has at least two minimal curve if and only if a minimal curve of $c$ is a multiple of a non-singular rational curve of grade zero.*

Proof. The if part is obvious. In order to prove the only if part, we may assume that there is a minimal curve $c'$ of $c$ which is not dominated by $c$ but there is a point $P$ on $c$ such that $\mathrm{dil}_P(c)$ dominates $c'$. Let $F'$ be a member of $B$ which carries $c'$, and let $P_1', \cdots, P_s'$ be the fundamental points over $F'$ with respect to $F$; we may renumber them so that $\mathrm{dil}_{(P'_1,\cdots,P'_s)}$ is well defined on $F$ and that $c'-P_i'\rhd 0$ if and only if $i\leq r$. Then, with $D=\mathrm{dil}_{(P'_1,\cdots,P'_r)}$, $D(c')=\mathrm{dil}_P(c)$. Since $c'$ is a minimal curve, we have $r\geq 1$. Set $p=\mathrm{dil}_P(P)$ and let $d'-\Sigma m_iP_i'$ be $D^{-1}(p)$ ($d'$ being a curve on $F'$). Since $p$ is irreducible, $m_i = m(P'_i ; d')$. Since $(p, \mathrm{dil}_P(c)) = 0$, we have $0 = (d' - \sum m_i P', c') = (d', c')$. Set $m = m(P; c)$. Then $p$ is contained in $\mathrm{dil}_P(c)$ exactly $m$-times and $c' - md' \geq 0$ and furthermore $d'$ and $c' - md'$ have no common component. Therefore $0 \leq (d', c' - md') = -m(d', d')$, and $I(d') \leq 0$. Since $d' - \sum m_i P_i = D^{-1}(p)$, $-1 = I(d') - \sum m_i^2$, and since $r \geq 1$, we must have $I(d') = 0$, $r = 1$, $m_1 = 1$. Since $I(d') = 0$, and since $c'$ is connected, we have $c' - md' = 0$, i.e., $c' = md'$. Since $D(d' - P'_1) = p$, which is an irreducible exceptional curve of the first kind, and since $m_1 = 1$, we see that $d'$ is an irreducible non-singular rational curve of grade 0, and the assertion is proved completely.

(ゐ) Let $c - \sum m_i P_i$ be a cycle over $F$, where $c$ is a divisor and the $P_i$ are mutually distinct points over $F$. The arithmetic genus $p_a(c)$ is well defined. $p_a(c) - \sum m_i(m_i - 1)/2$ is called the *virtual genus* of $c - \sum m_i P_i$, and is denoted by $vg(c - \sum m_i P_i)$. The *virtual genus* of a virtual linear system $L$ is defined to be the virtual genus of a member of $L$ and is denoted by $vg(L)$.

LEMMA 1.8. *If $F' \in B$, then $vg(L) = vg(T_{F \to F'}(L))$.*

As for the proof, we may assume that $F'$ dominates $F$. Then the proof is easy[1] by virtue of the following well known formula:

LEMMA 1.9. *$p_a(c+d) = p_a(c) + p_a(d) + (c, d) - 1$ for any two divisors $c$ and $d$ on $F$. Consequently, $p_a(mc) = [m(m-1)/2] \cdot (c, c) + m \cdot p_a(c) - m + 1$ for any rational integer $m$.*

As for Lemma 1.9, see Zariski [6].

By virtue of Lemma 1.8, Lemma 1.9 can be generalized to cycles $c$ and $d$ over $F$ with $vg$ instead of $p_a$. Hence

LEMMA 1.10. *If $L$ and $L'$ are virtual linear system on $F$, then $vg(L + L') = vg([L + L']) = vg(L) + vg(L') + (L, L') - 1$.*

(ゑ) A virtual linear system $L$ on $F$ is called *irreducible* if there is an $F' \in B$ such that $T_{F \to F'}(L)$ contains an irreducible curve or equivalently, if there exists a virtually positive member $c$ of

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1) The first step is to prove that if $c$ is a divisor on $F$ and if $F' \in B$ dominates $F$, then $p_a(c) = p_a(T_{F \to F'}(c))$, which is proved as follows: Let $d$ be a hypersurface section of order high enough such that $d$ and $c+d$ are linearly equivalent to non-singular curves $d'$ and $c'$ respectively which do not go through any fundamental points. For $c'$ and $d'$, the above is true because the arithmetic genus of a non-singular curve coincides with the geometric genus of the curve. Since the arithmetic genus is an invariant of a linear equivalence class, we have $p_a(c) = p_a(c') - p_a(d') - (c, d) + 1 = p_a(T_{F \to F'}(c))$ (by Lemma 1.9).

$L$ such that a minimal curve of $c$ is irreducible. $L$ is said to be *irreducible over* $F$ if we can choose such an $F'$ so that $F \leq F'$.

(お) Let $P_1, \cdots, P_r$ be points over $F$ such that $D = \mathrm{dil}_{(P_1, \cdots, P_r)}$ is well defined on $F$ and let $c$ be an irreducible curve on $F$. Set $m_i = m(P_i; c)$. Then $c' = D(c - \Sigma m_i P_i)$ is an irreducible curve. When $L$ is a virtual linear system on $F$, then $(D(L))^{\oplus}$ cut out on $c'$ a linear system of divisors of $c'$. This last linear system is called the *trace* of $L$ on $c$ with respect to the points $P_i$. It should be noted that if $c'$ is on another $F' \in B$, then $(T_{F \to F'}(L))^{\oplus}$ cut out the same trace on $c'$.

(か) We say that virtual linear system $L$ on $F$ is *exceptional* if there is an $F' \in B$ such that a single point is a member of $T_{F \to F'}(L)$, or equivalently, if for a $F'' \in B$, $(T_{F \to F''}(L))^{+}$ consists of an exceptional curve of the first kind. It is well known that

PROPOSITION 2. *A virtual linear system* $L$ *is exceptional if and only if* $L$ *is irreducible,* $vg(L) = 0$ *and* $I(L) = -1$.

## 2. Numerical types.

We consider complete virtual linear systems on projective planes $S$. $B$ denotes from now on the $B$ in §1 in the case where $F$ is an $S$.

(い) Let $L$ be a complete virtual linear system on a projective plane $S$ and let $c - \Sigma m_i P_i$ be a member of $L$, where $c$ is a divisor on $S$ and the $P_i$ are mutually distinct points over $S$. Let the degree of $c$ be $d$. The $L$ is characterized by $d$ and $\Sigma m_i P_i$, and $L$ is denoted by $\mathcal{L}(d; \Sigma m_i P_i)$. The effective linear system associated with $L$ is denoted by $\mathcal{L}^{\oplus}(d; \Sigma m_i P_i)$.

$[d(d+3) - \Sigma m_i(m_i+1)]/2$ is called the *virtual dimension* of $L$ or of $L^{\oplus}$, and is denoted by $\mathrm{vdim}\, L$ or by $\mathrm{vdim}\, L^{\oplus}$. $\mathrm{effdim}\, L - \mathrm{vdim}\, L$ is called the *superabundance* of $L$ or of $L^{\oplus}$ and is denoted by $\mathrm{supab}\, L$ or by $\mathrm{supab}\, L^{\oplus}$. Lemma 1.7 implies that if $d \geq 0$, then $\mathrm{supab}\, \mathcal{L}(d; \Sigma m_i P_i) \geq 0$.

(ろ) It is obvious that $vg(\mathcal{L}(d; \Sigma m_i P_i)) = (d^2 - 3d + 2 - \Sigma m_i(m_i - 1))/2$, and therefore we have the following formulas, where $L$ and $L'$ are complete virtual linear systems on an $S$.

LEMMA 2.1. (i) $\mathrm{vdim}\, L = I(L) - vg(L) + 1$, (ii) $\mathrm{vdim}\, [L + L'] = \mathrm{vdim}\, L + \mathrm{vdim}\, L' + (L, L')$, (iii) *if* $L = \mathcal{L}(d; \Sigma m_i P_i)$, *then* $3d - \Sigma m_i = I(L) - 2 \cdot vg(L) + 2 = 2 \cdot \mathrm{vdim}\, L - I(L)$.

(は) Let $P_1, \cdots, P_r$ be points over an $S$ such that $\mathrm{dil}_{(P_1,\cdots,P_r)}$ is well defined on the $S$. Set $F=\mathrm{dil}_{(P_1,\cdots,P_r)}(S)$. Though every natural Cremona transformation is the product of quadratic Cremona transformations, it is not true in general that every $F$-admissible Cremona transformation is the product of $F$-admissible quadratic Cremona transformations. A Cremona transformation $T$ defined on the $S$ is called a *Cremona transformation with centers within* $P_i$ if it is the product of $F$-admissible quadratic Cremona transformations.

LEMMA 2.2. *If $T$ is an $F$-admissible Cremona transformation, then there are exactly $r$ fundamental points, say $P'_1, \cdots, P'_r$ over $T(S)$ with respect to $F$ and $F=\mathrm{dil}_{(P'_1,\cdots,P'_r)}(T(S))$.*

Proof. If $T$ is quadratic, then the assertion is obvious, hence the assertion is proved easily if $T$ is a Cremona transformation with centers within $P_i$. There are points $P_{r+1}, \cdots, P_s$ such that $F'=\mathrm{dil}_{(P_1,\cdots,P_s)}(S)$ is well defined and such that $T$ is a Cremona transformation with centers within $P_1, \cdots, P_s$. On the other hand, since $F$ dominates $T(S)$, $F=\mathrm{dil}_{(P'_1,\cdots,P'_{r'})}(T(S))$ with the fundamental points $P'_i$ over $T(S)$ with respect to $F$. Then $F'=\mathrm{dil}_{(P'_1,\cdots,P'_{r'},P_{r+1},\cdots,P_s)}(T(S))$, and therefore $P'_1, \cdots, P'_{r'}, P_{r+1}, \cdots, P_s$ is the set of fundamental points over $T(S)$ with respect to $F'$. It follows from the first remark that $r'=r$.

These fundamental points $P'_1, \cdots, P'_r$ in Lemma 2.2 are called the *corresponding base points* to the $P_i$ under $T$.

(に) We call a vector $(d, m_1, \cdots, m_r)$, with $r+1$ integers $d, m_i$, a *numerical type* with $r$ base points; $d$ is called the *degree* and the $m_i$ are called *multiplicities* of the numerical type. Two numerical types $(d, m_1, \cdots, m_r)$ and $(d', m'_1, \cdots, m'_r)$ are said to be *similar* to each other if there are mutually distinct points $P_1, \cdots, P_r$ on an $S$ and a Cremona transformation $T$ with centers within $P_i$ such that $T(\mathcal{L}(d; \Sigma m_iP_i))$ is expressed as $\mathcal{L}(d'; \Sigma m'_iP'_i)$ on $T(S)$, where $P'_i$ are the corresponding base points. It should be remarked here the following easy fact:

LEMMA 2.3. *If $T$ is a quadratic Cremona transformation defined on an $S$ with centers $P_1, P_2, P_3$, then $T(\mathcal{L}(d; \Sigma_1^r m_iP_i))=\mathcal{L}(d+a; \Sigma_1^3(m_i+a)P_i^*+\Sigma_4^r m_iP_i)$, where $a=d-(m_1+m_2+m_3)$, the $P_i$ are mutually distinct points over the $S$, and the $P_j^*$ $(j=1, 2, 3)$ are the fundamental points over $T(S)$ with respect to $S$ which are numbered so that if $(i, j, k)$ is a permutation of $(1, 2, 3)$, then $P_i^*$ corresponds* *to the line $l_i$ which goes through $P_j$ and $P_k$ i.e., $P_i^* = T(l_i - P_j - P_k)$.*

By virtue of the above lemma, we define an operator $\mathfrak{q}$ acting to numerical types with at least three base points, as follows: $\mathfrak{q}(d, m_1, \cdots, m_r) = (d+a, m_1+a, m_2+a, m_3+a, m_4, \cdots, m_r)$ where $a = d - m_1 - m_2 - m_3$. Then lemma 2.3 implies that

LEMMA 2.4. *Two numerical types $(d, m_1, \cdots, m_r)$ and $(d', m_1', \cdots, m_r')$ are similar to each other if and only if there are permutations $\pi_1, \cdots, \pi_s$ of multiplicities such that $(d', m_1', \cdots, m_r') = \pi_s \mathfrak{q} \pi_{s-1} \mathfrak{q} \cdots \pi_1(d, m_1, \cdots, m_r)$.*

(ほ) Given ordinary points $P_1, \cdots, P_r$ on an $S$ are said to be *non-special with respect to Cremona transformations* if for any Cremona transformation $T$ with centers within $P_i$, the corresponding base points $P_1', \cdots, P_r'$ to the $P_i$ under $T$ are ordinary points on $T(S)$ such that no three points among the $P_i'$ are colinear. In the contrary case, we say that the $P_i$ are *special with respect to Cremona transformations.*

LEMMA 2.5. *If $P_1, \cdots, P_r$ are independent generic points of an $S$ over the prime field, then they are non-special with respect to Cremona transformations.*

Proof. If $r \leq 2$, then the assertion is obvious, and we assume that $r \geq 3$. Let $T$ be the quadratic Cremona transformation with centers $P_1, P_2, P_3$, and let $P_1^*, P_2^*, P_3^*$, be the corresponding base points to the $P_1, P_2, P_3$ under $T$. $P_4, \cdots, P_r$ are independent generic points over the smallest field of definition $K$ of $T$. Since $P_1^*, P_2^*, P_3^*$ are ordinary points which are not colinear and since they are rational over $K$, we can choose coordinates of $T(S)$ so that $P_1^*, P_2^*, P_3^*, P_4, \cdots, P_r$ are independent generic points over the prime field. Therefore the same is applied to any Cremona transformation with centers within $P_i$, and we prove the assertion.

PROPOSITION 3. *Given mutually distinct ordinary points $P_1, \cdots, P_r$ on an $S$ are special with respect to Cremona transformations if and only if there is a numerical type $(d, m_1, \cdots, m_r)$ which is similar to $(1, 1, 1, 1, 0, \cdots, 0)$ (with $r-3$ zeros) and such that $\mathcal{L}^{\oplus}(d; \Sigma m_i P_i)$ is not empty.*

The proof is immediate from Lemma 2.6 below and from the definition.

(へ) The following two lemmas are obvious:

LEMMA 2.6. *Assume that $P_1, \cdots, P_r$ are non-special points with* *respect to Cremona transformations on an S. A numerical type* $(d, m_1, \cdots, m_r)$ *is similar to* $(d', m'_1, \cdots, m'_r)$ *if and only if there is a Cremona transformations T with centers within* $P_i$ *such that* $T(\mathcal{L}(d\,;\,\Sigma\, m_iP_i))=\mathcal{L}(d'\,;\,\Sigma\, m'_iP'_i)$ *with a suitable numering of the corresponding base points* $P'_i$.

LEMMA 2.7. *When these numerical types as above are similar to each other, the following two assertions hold good*: (1) *If we can choose* $P_i$ *(non-special) so that* $\mathcal{L}^{\oplus}(d\,;\,\Sigma\, m_iP_i)$ *is not empty, then either* $d'>0$ *or* $d'=0$, $m'_i\leq 0$; (2) *if one can choose* $P_i$ *so that* $\mathcal{L}(d\,;\,\Sigma\, m_iP_i)$ *is irreducible, then either* $d'>0$, $m'_i\geq 0$ *or* $d'=0$, $m'_i$ *are zero except for at most one of the* $m'_i$ *which is equal to* $-1$.

(と) For a given numerical type $(d, m_1, \cdots, m_r)$, the set of all numerical types which are similar to $(d, m_1, \cdots, m_r)$ is denoted by $\mathfrak{C}(d, m_1, \cdots, m_r)$. Elements of $\mathfrak{C}(1, 1, 1, 0, \cdots, 0)$ (with $r-2$ zeros) and elements of $\mathfrak{C}(1, 0, \cdots, 0)$ (with $r$ zeros) are called *pre-exceptional types* and *Cremona types* (with $r$ base points) respectively.

$I(\mathcal{L}(d\,;\,\Sigma\, m_iP_i))$ and $vg(\mathcal{L}(d\,;\,\Sigma\, m_iP_i))$ do not depend on the particular choice of the points $P_i$ but depend only on the numerical type $(d, m_1, \cdots, m_r)$ and are invariant under Cremona transformations. These numbers are defined to be the *grade* and the *virtual genus* of the numerical type $(d, m_1, \cdots, m_r)$ and are denoted by $I(d, m_1, \cdots, m_r)$ and $vg(d, m_1, \cdots, m_r)$: Since they are invariants of the similarity class $\mathfrak{C}(d, m_1, \cdots, m_r)$, and since $3d-\Sigma\, m_i=I(d, m_1, \cdots, m_r)-2\cdot vg(d, m_1, \cdots, m_r)+2$, we have

LEMMA 2.8. *The grade, virtual genus and* $3d-\Sigma\, m_i$ *are invariants of* $\mathfrak{C}(d, m_1, \cdots, m_r)$.

## 3. Exceptional curves of the first kind.

THEOREM 2a. *Assume that* $P_1, \cdots, P_r$ *are points on an S which are non-special with respect to Cremona transformations. Then there is a one-one correspondence between all of pre-exceptional types* $(d, m_1, \cdots, m_r)$ *with r base points and all of irreducible exceptional curves c of the first kind on* $\mathrm{dil}_{(P_1,\cdots,P_r)}(S)$ *in such a way that* $c\in \mathrm{dil}_{(P_1,\cdots,P_r)}(\mathcal{L}(d\,;\,\Sigma\, m_iP_i))$.

Proof. Assume that $(d, m_1, \cdots, m_r)$ is a pre-exceptional type. By Lemma 2.6, there is a Cremona transformation $C$ with centers within $P_i$ such that $C(\mathcal{L}(d\,;\,\Sigma\, m_iP_i))$ is of the form $\mathcal{L}(0\,;\,-P'_1)$ on $C(S)$, whence $c=\mathrm{dil}_{(P'_1,\cdots,P'_r)}(P'_1)$, where the $P'_i$ are corresponding base points to the $P_i$. Since the $P'_i$ are ordinary points on $C(S)$, we see that $c$ is an irreducible exceptional curve of the first kind. Conversely, assume that $c$ is an irreducible exceptional curve of the first kind on $\mathrm{dil}_{(P_1,\cdots,P_r)}(S)$. Set $c^*=(\mathrm{dil}_{(P_1,\cdots,P_r)})^{-1}(c)$, $L=\mathcal{L}(c^*)$ and let $(d, m_1, \cdots, m_r)$ be such that $L=\mathcal{L}(d;\Sigma m_iP_i)$. We are to prove that $(d, m_1, \cdots, m_r)$ is a pre-exceptional type by induction on $d$. If $d=0$, then $c^*$ must be a point and the assertion is obvious. We assume that $d>0$. Since $c$ is irreducible, $c$ is the proper transform of the divisorial part $c'$ of $c^*$, hence $m_i=m(P_i;c')\geq 0$. We may assume that $m_1\geq m_2\geq\cdots\geq m_r$. Since $I(L)=I(c)=-1$ and since $vg(L)=p_a(c)=0$, we have $m_1+m_2+m_3>d$ by virtue of Proposition 4 in Part I[2]. Let $T$ be the quadratic Cremona transformation with centers $P_1, P_2, P_3$. Then $T(L)$ on $T(S)$ is of smaller degree than $d$, whence, by our induction assumption, we see that $\mathfrak{q}(d, m_1, \cdots, m_r)$ is a pre-exceptional type, which implies that $(d, m_1, \cdots, m_r)$ is also a pre-exceptional type, and the proof is completed.

PROPOSITION 4. *Assume that, for a given numerical type* $\mathfrak{t}=(d, m_1, \cdots, m_r)$ *with* $r\geq 3$, *every member of the class* $\mathfrak{C}(\mathfrak{t})$ *has positive degree and non-negative multiplicities except for those of degree zero. If furthermore* $vg(\mathfrak{t})\leq 1$ *and if* $3d-\Sigma m_i\geq 0$, *then it holds one of the following six cases*: (1) *there is a member of* $\mathfrak{C}(\mathfrak{t})$ *of degree zero, say* $(0, n_1, \cdots, n_r)$ *such that the sum of any three of the* $n_i$ *is non-positive*; (2) $vg(\mathfrak{t})\leq 0$ *and* $(n, n, 0, \cdots, 0)\in\mathfrak{C}(\mathfrak{t})$ *with a natural number* $n$; (3) $vg(\mathfrak{t})=0$ *and* $(n, n-1, 0, \cdots, 0)\in\mathfrak{C}(\mathfrak{t})$ *with a natural number* $n$; (4) $vg(\mathfrak{t})=0$, $(2, 0, \cdots, 0)\in\mathfrak{C}(\mathfrak{t})$; (5) $vg(\mathfrak{t})=1$, $(3, 1, \cdots, 1, 0, \cdots, 0)\in\mathfrak{C}(t)$ *with at most* 9 *ones* (*may have no one*); (6) $(3n, n, \cdots, n, 0, \cdots, 0)\in\mathfrak{C}(\mathfrak{t})$ *with a natural number* $n$ *and with exactly* 9 *n's*.

Proof. By virtue of our assumption, we can adapt the last half of the proof of Theorem 2a, and we yield the result by Proposition 4 in Part I.

PROPOSITION 5. *If* $L=\mathcal{L}(d;\Sigma m_iP_i)$ *is virtually connected* ($P_i$ *being mutually distinct points over an* $S$) *and if either* $\mathrm{vdim}\, L\geq 0$ *or the* $P_i$ *are non-special with respect to Cremona transformations then* $(d, m_1, \cdots, m_r)$ *satisfies the first assumption in Proposition* 4 *above.*

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2) $(2i\leq t<r)$ in the second line of Proposition 4 in Part I should be $(2\leq t<r)$.

Proof. If vdim $L \geq 0$, then we may change the $P_i$ to independent generic points and we may assume that the $P_i$ are non-special points with respect to Cremona transformations, whence the assertion is obvious by virtue of Lemma 2.7 and by the fact that if some $m_i$, say $m_1$, is negative, then a curve $c$ in $\mathrm{dil}_{(P_1,\cdots,P_r)}(L)$ contains $(-m_1)p$ with $p = \mathrm{dil}_{(P_1,\cdots,P_r)}(P_1)$, and $(c-(-m_1)p, -m_1p)=0$.

By virtue of Proposition 5, we have the following result as a particular case of Proposition 4.

THEOREM 3a. *Assume that $L = \mathcal{L}(d; \Sigma m_iP_i)$ is virtually connected ($P_i$ being mutually distinct points over an S) and that $I(L) = -1$. If $vg(L)=0$, then $(d, m_1, \cdots, m_r)$ is a pre-exceptional type*[3].

REMARK. Two of the three conditions $I(L) = -1$, $vg(L)=0$ and vdim $L=0$ imply the remaining by virtue of Lemma 2.1.

Since the irreducibility of a curve implies the virtual connectedness of the curve, we have the following corollary to Theorem 3a.

COROLLARY. *If $\mathcal{L}(d; \Sigma m_iP_i)$ is exceptional, then $(d, m_1, \cdots, m_r)$ is a pre-exceptional type.*

The above corollary and Theorem 2a imply the following

PROPOSITION 6a. *If $P_1, \cdots, P_r$ are non-special points of an S with respect to Cremona transformations, then every exceptional curve of the first kind on $\mathrm{dil}_{(P_1,\cdots,P_r)}(S)$ is irreducible.*

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3) It seems to the writer that Franchetta [4] is asserting that if a virtually connected curve $c$ is such that $I(c) = -1$ and $p_a(c)=0$, then either $c$ is an exceptional curve of the first kind or the transform of a curve on a ruled surface. But this statement is not true. A counter example is given as follows: Let $l$ be a line on an $S$ and let $P$ and $Q$ be mutually distinct ordinary points on $l$. Let $P'$ and $Q'$ be infinitely near points of $P$ and $Q$ such that they lie on $l$. (The assumption that $P'$ and $Q'$ lie on $l$ is not important. But the treatment becomes different in the other cases.) Set $D = \mathrm{dil}_{(P,Q,P',Q')}$, $l' = D(l-P-Q-P'-Q')$, $p = D(P-P')$, $q = D(Q-Q')$, $p' = D(P')$, $q' = D(Q')$ and $c = D(l-P'-Q')$. Then $l'$, $p$, $p'$, $q$, $q'$ are irreducible and $c$ is the sum of them. Let $c'$ be a minimal curve of $c$ which is dominated by $c$. If $c \neq c'$, then there is an irreducible exceptional curve $e$ of the first kind on $D(S)$ such that $e$ is a component of $c$ and such that $\mathrm{cont}_e(c)$ is a curve. Exceptional curves of the first kind among $l'$, $p$, $q$, $p'$ and $q'$ are only $p'$ and $q'$. If $e = p'$, then the contracted point $\mathrm{cont}_{p'}(p')$ is a double point of $\mathrm{cont}_{p'}(c)$, whence $2p'$ must be contained in $c$, which is a contradiction, and $e \neq p'$. Similarly, $e \neq q'$ and $e$ does not exist, which shows that $c$ itself is a minimal curve. There is no ruled rational surface which carries $c$ because a rational ruled surface has at most one irreducible curve of negative grade (Proposition 2 in Part I). On the other hand, $c$ is obviously virtually connected. That $p_a(c)=0$ and that $I(c) = -1$ are obvious. $c$ is not a transform of any curve on a ruled surface by virtue of Theorem 1.

Above treatment can be adapted to the case of Cremona transformations, and we have:

THEOREM 2b. *If $P_1, \cdots, P_r$ are non-special points of an $S$ with respect to Cremona transformations, then there is a one-one correspondence between all of Cremona transformations $C$ with centers within $P_i$ and all of Cremona types $(d, m_1, \cdots, m_r)$ with $r$ base points in such a way that $C$ is defined by the linear system $\mathcal{L}^{\oplus}(d\,;\,\Sigma\, m_iP_i)$.*

THEOREM 3b. *Assume that $L=\mathcal{L}(d\,;\,\Sigma\, m_iP_i)$ is virtually connected and that $I(L)=1$, $vg(L)=0$, then $(d, m_1, \cdots, m_r)$ is a Cremona type.*

COROLLARY. *Let $P_1, \cdots, P_r$ be points over an $S$ such that $D=\mathrm{dil}_{(P_1,\cdots,P_r)}$ is well defined on the $S$. Then every $D(S)$-admissible Cremona transformation defined on the $S$ is defined by the linear system $\mathcal{L}^{\oplus}(d\,;\,\Sigma\, m_iP_i)$ with a suitable Cremona type $(d, m_1, \cdots, m_r)$.*

PROPOSITION 6b. *If $P_1, \cdots, P_r$ are non-special points on an $S$ with respect to Cremona transformations, then every $\mathrm{dil}_{(P_1,\cdots,P_r)}(S)$-admissible Cremona transformation defined on the $S$ is a Cremona transformation with centers within $P_i$.*

THEOREM 4a. *Let $P_1, \cdots, P_r$ be points over an $S$ such that $D=\mathrm{dil}_{(P_1,\cdots,P_r)}$ is well defined on the $S$. If $D(S)$ carries infinitely many exceptional curves of the first kind, then $r\geq 9$. Conversely, if $r\geq 9$ and if the $P_i$ are non-special with respect to Cremona transformations, then $D(S)$ carries infinitely many exceptional curves of the first kind*[4].

Proof. Theorem 2a and Proposition 6a show that we have only to prove that there are infinitely many pre-exceptional types with $r$ base points if and only if $r\geq 9$. Assume first that $r\geq 9$, and let $(d, m_1, \cdots, m_r)$ be a pre-exceptional type such that $m_1\leq m_2 \leq \cdots \leq m_r$. Since $3d-\Sigma\, m_i$ is an invariant of the similarity class by Lemma 2. 8, we have $3d-\Sigma\, m_i=1$, whence $m_1+m_2+m_3<d$, and therefore the operator $\mathfrak{q}$ yields a new member of higher degree of the class of pre-exceptional types. Hence there are infinitely many pre-exceptional types with $r$ base points. Assume next that

4) The existence of a surface carrying infinitely many exceptional curves of the first kind was claimed by Franchetta [3]. But his proof was not complete; he proved the existence of infinitely many Cremona types with 9 base points, which was already known (see Coble [2]). The writer was told the existence of such a surface by K. Kodaira.

$r=8$. Then the following table gives all pre-exceptional types up to permutations of the multiplicities:

| | $d$ | $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ | $m_6$ | $m_7$ | $m_8$ |
|---|---|---|---|---|---|---|---|---|---|
| (1) | 0 | $-1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| (2) | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| (3) | 2 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| (4) | 3 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| (5) | 4 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 |
| (6) | 5 | 2 | 2 | 2 | 2 | 2 | 2 | 1 | 1 |
| (7) | 6 | 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |

For, as is easily seen, the following shows all possible change of types by the operator $\mathfrak{q}$ up to permutations:

$$\begin{array}{c} (4) \\ \swarrow\!\!\nearrow \qquad \nwarrow\!\!\searrow \\ (1)\leftrightarrow(2)\leftrightarrow(3)\longleftrightarrow(5)\leftrightarrow(6)\leftrightarrow(7) \end{array}$$

Therefore, by a stronger reason, we see that if $r\leq 8$, then there are only a finite number of pre-exceptional types with $r$ base points. Thus we complete the proof.

THEOREM 4b. *Let $P_i$ and $D$ be as above. If there are infinitely many $D(S)$-admissible Cremona transformations defined on the $S$, then $r\geq 9$. Conversely, if $r\geq 9$ and if the $P_i$ are non-special with respect to Cremona transformations, then there are infinitely many Cremona transformations with centers within $P_i$.*

Proof. If $r<9$, then $D(S)$ carries only a finite number of exceptional curves of the first kind by Theorem 4a, hence $D(S)$ dominates only a finite number of members of $B$ and the first assertion is proved. The last assertion is proved by a similar way as in Theorem 4a.

We add here a modification of Proposition 2 in the case of a projective plane $S$.

PROPOSITION 7. *Let $P_1, \cdots, P_r$ be mutually distinct points over an $S$. Then $L=\mathcal{L}(d\,;\,\Sigma\, m_iP_i)$ is exceptional if (and only if) $L$ is irreducible, $I(L)<0$ and* vdim $L\geq 0$.

Proof. Since $L$ is irreducible, $vg(L)\geq 0$, whence $0\leq \mathrm{vdim}\, L+vg(L)=I(L)+1\leq 0$. It follows that $\mathrm{vdim}\, L=vg(L)=I(L)+1=0$, hence $L$ is exceptional by Proposition 2.

## 4. Abnormal curves.

Let $P_1, \cdots, P_r$ be given mutually distinct points over an $S$.

A curve $c$ is called an *abnormal curve* with respect to the points $P_i$ if $\deg c/\Sigma\, m(P_i\,;\, c) < 1/\sqrt{r}$. From this definition, it follows that

LEMMA 4.1. *If $c$ is an abnormal curve with respect to the $P_i$, then there exists an irreducible component $c'$ of $c$ which is abnormal with respect to the $P_i$.*

Let $G$ be the set of permutations $\sigma$ of the $P_i$ such that some linear translation of $(P_1, \cdots, P_r)$ can be specialized to $\sigma(P_1, \cdots, P_r)$ over the prime field. This $G$ is called the *geometric permutation group* of the $P_i$. If $G$ is transitive, we say that the $P_i$ are *transitive*; if $G$ is symmetric, we say that the $P_i$ are *symmetric*. If $c$ is a curve and if $\sigma \in G$, then $\sigma(c)$ denotes a specialization of a translation of $c$ compatible with $(P_1, \cdots, P_r) \to \sigma(P_1, \cdots, P_r)$ as above. ($\sigma(c)$ may not be unique.)

We say that a curve $c$ is *uniform* with respect to the $P_i$ if all $m(P_i\,;\, c)$ are equal.

THEOREM 5. *If $c$ is a uniform abnormal curve and if $c'$ is an irreducible abnormal curve with respect to the points $P_i$ then $c'$ is a component of $c$.*

Proof. Let $m_i = m(P_i\,;\, c')$, $m = m(P_i\,;\, c)$, $d = \deg c$ and $d' = \deg c'$. Then $d/mr < 1/\sqrt{r}$, $d'/\Sigma\, m_i < 1/\sqrt{r}$ and therefore $dd' < \Sigma\, mm_i$ and the assertion is proved.

COROLLARY. *If $c$ is an irreducible uniform abnormal curve with respect to the $P_i$, then $c$ is the unique irreducible abnormal curve, hence every abnormal curve must contain $c$ as a component.*

Assume that the $P_i$ are transitive. For a curve $c$, $\Sigma\, \sigma(c)$ is called a *uniformization* of $c$ if $\sigma$ runs over a complete set of representatives of the geometric permutation group $G$ of the $P_i$ modulo its subgroup $H$ which consists of $\sigma'$ such that $m(P_i\,;\, c) = m(\sigma'(P_i)\,;\, c)$ for all $i$ ($G = \Sigma\, \sigma H$).

THEOREM 6a. *Assume that $c$ is an irreducible abnormal curve with respect to the $P_i$ and that, setting $d = \deg c$, $m_i = m(P_i\,;\, c)$, for any permutation $\sigma$ of the $i$, there exists an irreducible abnormal curve $c_\sigma$ of degree $d$ such that $m(P_i\,;\, c_\sigma) = m_{\sigma(i)}$. Then,* (1) *for any $\sigma$, $c_\sigma$ is unique,* (2) *any irreducible abnormal curve with respect to* *the $P_i$ coincides with one of the $c_\sigma$ and* (3) *setting* $H=\{\sigma ; m_i=m_{\sigma(i)}$ *for all* $i\}$, *let* $\sigma_1, \cdots, \sigma_s$ *be a complete set of representatives of the symmetric group* $\mathfrak{S}_r$ *of the* $\{i\}$ *modulo* $H$ $(\mathfrak{S}_r=\Sigma\, \sigma H)$; *then any uniform abnormal curve with respect to the* $P_i$ *contains* $\Sigma\, c_{\sigma_i}$ *as a component.*

Proof. Let $c'$ be any irreducible abnormal curve with respect to the $P_i$. Then $c'$ must be a component of $\Sigma\, c_{\sigma_i}$, which proves (1) and (2). If $h$ is a uniform abnormal curve, then every $c_\sigma$ must be a component of $h$, which proves (3).

THEOREM 6b. *Assume that* $c$ *is an irreducible abnormal curve with respect to the* $P_i$ *and that the* $P_i$ *are transitive. Then for any element* $\sigma$ *of the geometric permutation group* $G$ *of the* $P_i$, $\sigma(c)$ *is unique* (*and irreducible*) *and any irreducible abnormal curve coincide with one of the* $\sigma(c)$. *Furthermore, uniformization of* $c$ *is unique.*

THEOREM 7. *If* $L=\mathcal{L}(d ; \Sigma\, m_iP_i)$ *is irreducible, if* $\operatorname{supab} L=0$ *and if* $d/\Sigma\, m_i<1/\sqrt{r}$ (*hence* $L^{\oplus}$ *consists of an irreducible abnormal curve*), *then, assuming that* $m_1\geq m_2\geq \cdots \geq m_r$,

(1) $r=2,\ m_1=m_2=1,\ d=1,$

*or* (2) $r=3,\ m_1=m_2=1,\ m_3=0,\ d=1,$

*or* (3) $r=5,\ m_1=m_2=m_3=m_4=m_5=1,\ d=2,$

*or* (4) $r=6,\ m_1=m_2=m_3=m_4=m_5=1,\ m_6=0,\ d=2,$

*or* (5) $r=7,\ m_1=2,\ m_2=m_3=m_4=m_5=m_6=m_7=1,\ d=3,$

*or* (6) $r=8,\ m_1=3,\ m_2=m_3=m_4=m_5=m_6=m_7=m_8=2,\ d=6.$

*Conversely, if the* $P_i$ *are non-special with respect to Cremona transformations, then each of the above conditions gives an irreducible abnormal curve.*

Observe that these curves are of pre-exceptional type.

Proof. The converse part is obvious by virtue of Theorem 2a. Assume that $L$ satisfies the conditions. Since $\operatorname{supab} L=0$, even if the $P_i$ are independent generic points, $L^{\oplus}$ consists of an irreducible abnormal curve. Hence we may assume that the $P_i$ are independent generic points. Let $c$ be the unique member of $L^{\oplus}$ and let $\sigma_1, \cdots, \sigma_t$ be elements of the geometric permutation group $G$ of the $P_i$ (which is symmetric) such that $\Sigma\, \sigma_i(c)$ is the uniformization of $c$, let $L_i=\mathcal{L}(d-\Sigma_j\, m(P_j, \sigma_i(c))P_j)$ and let $L'$ be the complete sum of $L_1, \cdots, L_t$. First of all, $L_i$ is exceptional by virtue of Proposition 7. On the other hand, Theorem 6b (or Theorem 6a) shows that effdim $L'=0$. effdim $L'=0$ shows that effdim $[L_i+L_j]=0$ for any $i\neq j$, hence vdim $[L_i+L_j]\leq 0$. Therefore, Lemma 2.1 implies that $(L_i, L_j)=0$ if $i\neq j$, and the above inequality is really an equality. Then, by successive application of the same, we see that vdim $L'=0$ and supab $L'=0$. Therefore, denoting by $m'$ the multiplicity of the $P_i$ on the uniformization of $c$, we have $(td)^2+3dt-r(m'^2+m')=0$. By our assumption that $c$ is abnormal, we have $td/rm'<1/\sqrt{r}$, hence $td<m'\sqrt{r}$. Therefore $rm'^2+3r\cdot m'-r(m'^2+m')>0$, which shows that $3>\sqrt{r}$. It follows that $r\leq 8$. Non-existence of abnormal curves in the case where $r=1$ or 4 is obvious. Therefore $r=2$ or 3 or 5 or 6 or 7 or 8. In each of these cases, the condition described in the theorem gives an irreducible curve by Theorem 2a. Therefore, by virtue of Theorem 6a, we prove the theorem.

We note here that Proposition in Nagata [5], § 3 implies by virtue of the above results and Theorem 9 below that *if $P_1, \cdots, P_r$ are independent generic points of $S$ over the prime field and if $r$ is the square of a natural number, then there is no abnormal curve with respect to the $P_i$.*

## 5. Some remarks on superabundance.

Let $P_1, \cdots, P_r$ be mutually distinct points over an $S$ such that $D=\mathrm{dil}_{(P_1,\cdots,P_r)}$ is well defined on the $S$.

THEOREM 8. *Let $d^*$ be a given number or infinity. Assume that if $L=\mathcal{L}(d; \Sigma m_iP_i)$ is irreducible and if $d$ is less than $d^*$, then* supab $L=0$. *Then, for an arbitrary $L=\mathcal{L}(d; \Sigma m_iP_i)$ with $d<d^*$ and such that $\mathcal{L}^{\oplus}(d; \Sigma m_iP_i)$ is not empty, the superabundance of $L$ is given as follows: Let $L_1, \cdots, L_t$ be mutually distinct irreducible complete virtual linear systems on $S$ such that $(D(L))^{\oplus}$ is the minimal sum $\Sigma e_i(D(L_i))^{\oplus}$. Then* supab $L=\Sigma_f f(f-1)/2$, *where $f$ runs over all $e_i$ such that $L_i$ is exceptional. We have, in the above case, that $(L_i, L_j)=0$ if $i\neq j$ and that if $e_i\geq 2$ then $I(L_i)$ is either $-1$ or $0$ accordingly to whether $L_i$ is exceptional or not.*

Proof. Since supab $(L_i)=0$, we have effdim $L_i=$ vdim $L_i\geq 0$. Hence if $(L_i, L_j)>0$ ($i\neq j$ or $i=j$ and $e_i>2$), then effdim $[L_i+L_j]>$ effdim $(L_i+L_j)$ by Lemma 2.1, which is a contradiction, whence the last assertion is true by virtue of Proposition 7. Since $(L_i, L_j)=0$, it follows from Lemma 2.1 that supab $(L)=\Sigma$ supab $(e_iL_i)$, and we may assume that $L=e_1L_1$. If $e_1=1$, then the assertion is obvious, and we assume that $e_1\geq 2$. If $I(L_1)=0$, then the assertion is proved by Lemma 2.1 and we assume that $I(L_1)<0$. Then $L_1$ is exceptional by Proposition 7, hence $\mathrm{supab}(e_1L_1)=\mathrm{supab}(e_1\mathcal{L}(0\,;\,-P_1))$ $=-e_1(-e_1+1)/2=e_1(e_1-1)/2$, and the assertion is proved completely.

THEOREM 9. *Assume that the points $P_i$ are ordinary points.*

(1) *If $r\leq 8$, then the condition in Theorem 8 is satisfied for $d^*$ = infinity if and only if the $P_i$ are non-special with respect to Cremona transformations.*

(2) *If $r=9$, then the condition in Theorem 8 is satisfied for $d^*$ = infinity if and only if the following two conditions are satisfied:*
(i) *The $P_i$ are non-special with respect to Cremona transformations.*
(ii) *For any natural number $n$, the system $\mathcal{L}(3n\,;\,\Sigma\,nP_i)$ is of dimension zero.*

Proof. Assume that the condition in Theorem 8 is satisfied for $d^*$ = infinity. Then Proposition 3 shows that the $P_i$ are non-special. Therefore the "only if" part (in each of (1) and (2)) is proved.

In order to prove the "if" part, we shall use the following fact, which will be proved later in a more general form:

LEMMA 5.1. *If* $\mathrm{supab}\,\mathcal{L}(d\,;\,\Sigma m_iP_i)=0$, *$P_i$ being ordinary points, if no $P_j$ ($j\geq 3$) is on the line $P_1P_2$, and if either* (i) $a_1=a_2=0$, *or* (ii) $a_1=1$, $a_2=0$ *or* (iii) $a_1=a_2=1$, $m_1+m_2\leq d$, *then* $\mathrm{supab}\,\mathcal{L}(d+1\,;$ $(m_1+a_1)P_1+(m_2+a_2)P_2+\Sigma_3^r\,m_iP_i)=0$.

By virtue of Lemma 2.1, we have the following

COROLLARY. *If, besides the conditions in Lemma* 5.1, $\mathcal{L}(d\,;\,\Sigma\,m_iP_i)$ *is irreducible, then $\mathcal{L}(d+1\,;\,(m_1+a_1)P_1+(m_2+a_2)P_2+\Sigma_3^r\,m_iP_i)$ is irreducible except for the cases where $d=m_1=a_1=1$ and where $a_1=a_2=1$, $m_1+m_2=d$.*

Now we consider the if part of Theorem 9. Assume that $L=\mathcal{L}(d\,;\,\Sigma\,m_iP_i)$ is irreducible. If there are three multiplicities of $L$ whose sum is greater than $d$, then we can reduce the degree of $L$ by a quadratic Cremona transformation with centers within $P_i$, or to a system of degree zero; this last case has to be the case of pre-exceptional type and such case is known. Observing that the condition (ii) in Theorem 9, (2) is invariant under Cremona transformations with centers within $P_i$, we see that it is sufficient to prove the following

PROPOSITION 8. (1) *If $r=8$ and if the $P_i$ are non-special with respect to Cremona transformations, then any system $L=\mathcal{L}(d\,;\,\Sigma\, m_iP_i)$, with $d\geq m_1+m_2+m_3$, $m_1\geq m_2\geq\cdots\geq m_8\geq 0$, has superabundance 0; $L$ is irreducible except for the case where $\mathcal{L}=L(d\,;\,dP_1)$, $d>1$.*

(2) *Assume that $r=9$ and that* (i) *the $P_i$ are non-special with respect to Cremona transformations and that* (ii) *$L_n=\mathcal{L}(3n\,;\,\Sigma\, nP_i)$ is of dimension zero for any natural number $n$. If $d\geq m_1+m_2+m_3$, $m_1\geq m_2\geq\cdots\geq m_9\geq 0$, then $L=\mathcal{L}(d\,;\,\Sigma\, m_iP_i)$ has superabundance 0; $L$ is irreducible except for the cases where $L=L_n$ for some $n\neq 1$ and where $L=\mathcal{L}(n\,;\,nP_1)$ for some $n\neq 1$.*

Proof of (1). Set $L_t=\mathcal{L}(3t\,;\,\Sigma\, tP_i)$ for $t=0,1,\cdots$. If we see the irreducibility of $L_t$ for $t>0$ and that $\operatorname{supab} L_t=0$, then, using $L_{m_3}$ we see the assertion easily by virtue of Lemma 5.1 and its corollary. Therefore we shall prove them by induction on $t$ (for $t>0$). Assume that $L_1$ is reducible, or rather that $\mathcal{L}^{\oplus}(3\,;\,\Sigma\, P_i)$ has a reducible number, then either there exists a conic which goes through 6 of the $P_i$ or there is a line going through 3 of the $P_i$, which contradicts to the non-speciality of the $P_i$. If $\operatorname{supab} L_1>0$, then there are 7 of the $P_i$, say $P_1,\cdots,P_7$ such that $P_8$ is a base point of $L'=\mathcal{L}^{\oplus}(3\,;\,\Sigma_1^7 P_i)$, whence $L'=L_1^{\oplus}$. Since $L'=|\mathcal{L}^{\oplus}(1\,;\,P_1+P_2)+\mathcal{L}^{\oplus}(2\,;\,\Sigma_3^7 P_i)|$, $L_1^{\oplus}$ has a reducible member, which is a contradiction as was proved above. Thus the case where $t=1$ is proved. We assume now that $t>1$. $\operatorname{vdim} L_t=t(t+1)/2\geq t+1$ and therefore $\mathcal{L}^{\oplus}(3t\,;\,(t+1)P_1+\Sigma_2^8\, tP_i)$ is not empty. Since $L_t$ does not change its type under Cremona transformations with centers within $P_i$ and since $(0,-1,0,0,0,0,0,0,0)$ is similar to $(-6,-3,-2,-2,-2,-2,-2,-2,-2)$ by virtue of the table in the proof of Theorem 4a, there is a Cremona transformation $T$ with centers within $P_i$ such that $T(\mathcal{L}(3t\,;\,(t+1)P_1+\Sigma_2^8\, tP_i))$ is of the form $[L_{t-2}+\mathcal{L}(0\,;\,-P_1)]$ (of course, the points must be changed to the corresponding base points under $T$). Since the non-speciality is preserved by the corresponding base points, we can use the induction assumption and we see the irreducibility and the vanishing of superabundance of the system $T(\mathcal{L}(3t\,;\,(t+1)P_1+\Sigma_2^8\, tP_i))$. Therefore the same assertions are true for $\mathcal{L}(3t\,;\,(t+1)P_1+\Sigma_2^8\, tP_i))$, whence, by a stronger reason, then hold also for $L_t$, which completes our proof.

Proof of (2). That $\operatorname{supab} L=0$ is proved by the same way as in the first part of the proof of (1), because we assumed that supab $L_n=0$. For the irreducibility of $L$, it is sufficient to prove the irreducibility of $L'_n=[L_n-\mathcal{L}(0\,;P_9)]$. Since $L'_n=[L_1+L'_{n-1}]$, we prove the irreducibility by induction on $n$ by virtue of Lemma 2.1.

The generalization of Lemma 5.1 which we want to prove is

THEOREM 10. *Assume that $P_1, \cdots, P_r$ are ordinary points. Let $c$ be a curve and let $L^*=\mathcal{L}(d^*\,;\Sigma m_i^*P_i)$ be such that $d^*=\deg c$, $m_i^*=m(P_i\,;c)$. Assume that $m_i^*=0$ if and only if $i>s$. If* supab $\mathcal{L}(d\,;\Sigma m_iP_i)=0$ *and if* supab $\mathcal{L}(d+d^*\,;\Sigma_1^s(m_i+m_i^*)P_i)=0$, *then* supab $\mathcal{L}(d+d^*\,;\Sigma_1^r(m_i+m_i^*)P_i)=0$.

In order to prove Theorem 10, we introduce some notations. Let $\boldsymbol{k}$ be a ground field over which the $P_i$ are rational and let $\mathfrak{H}=\boldsymbol{k}[x, y, z]$ be the homogeneous coordinate ring of the projective plane $S$. Let $\mathfrak{p}_i$ be the homogeneous prime ideal of the point $P_i$ for each $i$ and let $\mathfrak{p}(m_1, \cdots, m_r)$ be the intersection of $\mathfrak{p}_i^{m_i}$. When $\mathfrak{a}$ is a homogeneous ideal of $\mathfrak{H}$, $\chi(\mathfrak{a}\,;d)$ denotes the Hilbert characteristic function of $\mathfrak{a}$. Then it is obvious that if $d>0$, $m_i\geq 0$, then supab $\mathcal{L}(d\,;\Sigma m_iP_i)=(\Sigma m_i(m_i+1)/2)-\chi(\mathfrak{p}(m_1, \cdots, m_r)\,;d)$. Note that this last equality implies that if $d'\geq d$, then supab $\mathcal{L}(d'\,;\Sigma m_iP_i)\leq$ supab $\mathcal{L}(d\,;\Sigma m_iP_i)$, hence that if $d'\geq d\geq 0$, $0\leq m_i'\leq m_i$, then supab $\mathcal{L}(d'\,;\Sigma m_i'P_i)\leq$ supab $\mathcal{L}(d\,;\Sigma m_iP_i)$.

LEMMA 5.2. *Set* $\mathfrak{a}=\mathfrak{p}(m_1, \cdots, m_s, 0, \cdots, 0)$ *and* $\mathfrak{b}=\mathfrak{p}(0, \cdots, 0, m_{s+1}, \cdots, m_r)$. *Then* supab $L=$ supab $\mathcal{L}(d\,;\Sigma_1^s m_iP_i))$ + subab $(\mathcal{L}(d\,;\Sigma_{s+1}^r m_iP_i)+\chi(\mathfrak{a}+\mathfrak{b}\,;d)$, *where* $L=\mathcal{L}(d\,;\Sigma_1^r m_iP_i)$ *with arbitrary* $d>0$, $m_i\geq 0$.

Proof. This follows from the known formula $\chi(\mathfrak{a}\cap\mathfrak{b}\,;d)=\chi(\mathfrak{a}\,;d)+\chi(\mathfrak{b}\,;d)-\chi(\mathfrak{a}+\mathfrak{b}\,;d)$.

LEMMA 5.3. *Let $\mathfrak{a}$ be a homogeneous ideal and let $f$ be a homogeneous form of degree $d^*$ (in a homogeneous polynomial ring). Assume that $\mathfrak{a}:f=\mathfrak{a}$. Then, for any $n\geq d^*$ and for any homogeneous ideal $\mathfrak{c}$,* $\chi(\mathfrak{a}\,;n)-\chi(\mathfrak{a}\,;n-d^*)=\chi(\mathfrak{a}+\mathfrak{c}f\,;n)-\chi(\mathfrak{a}+\mathfrak{c}\,;n-d^*)$.

Proof. $\chi(\mathfrak{a}\,;n)-\chi(\mathfrak{a}\,;n-d^*)=\chi(\mathfrak{a}+(f)\,;n)=\chi(\mathfrak{a}+\mathfrak{c}f+(f)\,;n)=\chi(\mathfrak{a}+\mathfrak{c}f\,;n)-\chi((\mathfrak{a}+\mathfrak{c}f):f\,;n-d^*)$, which proves the assertion.

Now we shall prove Theorem 10. We apply Lemma 5.3 to the case where $\mathfrak{c}=\mathfrak{p}(m_1, \cdots, m_s, 0, \cdots, 0)$, $\mathfrak{a}=\mathfrak{p}(0, \cdots, 0, m_{s+1}, \cdots, m_r)$, $f$ is a homogeneous form of degree $d^*$ which defines $c$ ($\mathfrak{a}:f=\mathfrak{a}$ is obvious), and $n=d+d^*$. That supab $(\mathcal{L}(d\,;\Sigma_1^r m_iP_i)=0$ implies that (i) supab $(\mathcal{L}(d\,;\Sigma_{s+1}^r m_iP_i))=0$ and (ii) $\chi(\mathfrak{a}+\mathfrak{c}\,;d)=0$ (by Lemma

5.2). (i) implies that $\chi(\mathfrak{a}; d+d^*)-\chi(\mathfrak{a}; d)=0$. Therefore the equality in Lemma 5.3 implies that $\chi(\mathfrak{a}+\mathfrak{c}f; d+d^*)=0$. Set $\mathfrak{b}=\mathfrak{p}(m_1+m_1^*, \cdots, m_s+m_s^*, 0, \cdots, 0)$. Then $\mathfrak{b}$ contains $\mathfrak{c}f$ and we have $\chi(\mathfrak{a}+\mathfrak{b}; d+d^*)=0$. Therefore Theorem 10 follows from Lemma 5.2 (applying it to $d+d^*$ instead of $d$).

It should be remarked here that the following nice results were given by Castelnuovo [1].

Let $c$ be an irreducible curve of degree $d$ on an $S$ and let $P_1, \cdots, P_r$ be mutually distinct points over $S$.

(1) If $0 \leq m_i \leq m(P_i; c)$ for every $i$, then $\operatorname{supab}(\mathcal{L}(d-3; \Sigma(m_i-1)P_i))=0$.

(2) If furthermore $L=\mathcal{L}(d; \Sigma m_i P_i)$ is irreducible, if $\operatorname{effdim} L \geq 1$ and if every point $P$ over $S$ such that $m(P; c) \geq 2$ is some of the $P_i$, then the superabundance of $\mathcal{L}(d; \Sigma m_i P_i)$ coincides with the index of speciality of the trace of $\mathcal{L}(d; \Sigma m_i P_i)$ on $c$ with respect to the points $P_i$.

## 6. Suplementary remarks.

Proposition 9. *Set* $\mathfrak{a}_1=(1, 1, 1, 1, 0, 0, 0, 0, 0, 0)$, $\mathfrak{a}_2=(2, 1, 1, 1, 1, 1, 0, 0, 0, 0)$, $\mathfrak{a}_3=(3, 2, 1, 1, 1, 1, 1, 1, 1, 0)$ *and* $\mathfrak{b}(n)=(3n, n, n, n, n, n, n, n, n, n)$. *Set* $\mathfrak{c}(n, i)=\mathfrak{b}(n)+\mathfrak{a}_i$ *(for* $n \geq 0$, $i=1, 2, 3$*). Then 9 ordinary points* $P_1, \cdots, P_9$ *are special with respect to Cremona transformations if and only if there exists a non-empty system* $\mathcal{L}^{\oplus}(d; \Sigma m_i P_i)$ *such that* $\sigma(d, m_1, \cdots, m_9)=\mathfrak{c}(n, i)$ *with a permutation* $\sigma$ *of the* $m_i$ *and with some* $n$ *and* $i$.

Proof. We shall show first that if we allow $n$ to be negative, then the set of all $\mathfrak{c}(n, i)$ is the class $\mathfrak{C}(\mathfrak{a}_1)$ up to permutations of multiplicities. It is obvious that $\mathfrak{a}_i$ $(=\mathfrak{c}(0, i)) \in \mathfrak{C}(\mathfrak{a}_1)$. Furthermore, it is easy to see that $\mathfrak{c}(1, 1) \in \mathfrak{C}(\mathfrak{a}_1)$. Since $\mathfrak{b}(n)$ is the unique member of $\mathfrak{C}(\mathfrak{b}(n))$, we see that all $\mathfrak{c}(n, i)$ are in $\mathfrak{C}(\mathfrak{a}_1)$ for $n \geq 0$, inductively on $n$. Similarly, we see that $\mathfrak{c}(n, i) \in \mathfrak{C}(\mathfrak{a}_1)$ even if $n$ is negative. Now, considering $\mathfrak{q}\pi\mathfrak{c}(n, i)$ ($\pi$ is a permutation of multiplicity and $\mathfrak{q}$ is the operator which corresponds to quadratic Cremona transformation), we see easily that $\mathfrak{c}(n, i)$ exhaust all elements of $\mathfrak{C}(\mathfrak{a}_1)$ up to permutations of multiplicities. Therefore Proposition 3 implies our assertion.

Corollary. *If* $P_1, \cdots, P_8$ *are ordinary points, then they are special with respect to Cremona transformations if and only if one* *of the following conditions holds*:

(1) *There are three points among them which are colinear.*

(2) *There are six points among them which lie on a conic.*

(3) *There is a cubic curve which goes through all of them and has a double point at one of them.*

PROPOSITION 10. *If* $P_1, \cdots, P_r$ *are ordinary points on an* $S$ *and if* $r \leq 9$, *then the* $P_i$ *are non-special with respect to Cremona transformations if and only if the surface* $F = \mathrm{dil}_{(P_1,\cdots,P_r)}(S)$ *carries no non-singular rational curve of grade at most* $-2$.

Proof. If the $P_i$ are special, then the existence of such a curve is obvious. Assume, conversely, that the $P_i$ are non-special and that $c'$ is a non-singular rational curve on $F$ such that $I(c') \leq -2$. Since the $P_i$ are ordinary points, $c'$ is the proper transform of a curve, say $c$. Set $d = \deg c$, $m_i = m(P_i; c)$, and $L = \mathcal{L}(d; \Sigma m_i P_i)$. Then, by virtue of Cremona transformations with centers within $P_i$, we may assume that $d \geq m_1 + m_2 + m_3$, $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$. Then $\Sigma m_i \leq 3d$, and Proposition 4 in Part I yields a contradiction.

LEMMA 6.1. *Let* $P_i$ *be points over an* $S$ *such that* $D = \mathrm{dil}_{(P_1,\cdots,P_9)}$ *is well defined on the* $S$. *Then* $D(S)$ *carries only a finite number of non-singular rational curve* $c$ *of grade* $-2$.

Proof. If $c$ is such a curve as above, then $c$ is the unique member of $(\mathcal{L}(c))^+$. Now, with the notations as in Proposition 9, let $\mathfrak{a}_{ij}$ be the set of $\pi \cdot \mathfrak{a}_i$ with permutations $\pi$ of multiplicities. Each $c$ corresponds to a unique $\mathfrak{b}(n) + \mathfrak{a}_{ij}$ ($n$ may be negative) so that $D^{-1}(\mathcal{L}(c)) = \mathcal{L}(d; \Sigma m_i P_i)$ with $(d, m_1, \cdots, m_r) = \mathfrak{b}(n) + \mathfrak{a}_{ij}$. Assume that another $c$, say $c'$, corresponds to $\mathfrak{b}(n') + \mathfrak{a}_{ij}$ and that $n' > n$. Then $\mathcal{L}(c')$ must contain $c + (n'-n) \cdot D(c^* - \Sigma P_i)$ with a cubic curve $c^*$ going through all the $P_i$, and $(\mathcal{L}(c'))^+$ contains at least two members, which is a contradiction. This means that the number of all the $c$ does not exceed the number of all the $\mathfrak{a}_{ij}$, and the assertion is proved.

PROPOSITION 11. *Assume that* $\mathcal{L}^{\oplus}(3n; \Sigma nP_i)$ *is not empty for a natural number* $n$. *If* $c$ *is an irreducible curve such that* $(L, L) + 2 < 2 \cdot vg(L)$ *with* $L = \mathcal{L}(c - \Sigma m(P_i; c)P_i)$, *then* $c$ *is a fixed component of* $\mathcal{L}^{\oplus}(3n; \Sigma nP_i)$.

Proof. By the equality $3d - \Sigma m_i = (L, L) - 2 \cdot vg(L) + 2$, we see that $(L, \mathcal{L}(3n, \Sigma nP_i)) < 0$, which proves our assertion.

COROLLARY. *If* $r \leq 9$ *and if* $c$ *is an irreducible curve such that*

$(L, L)+2<2\cdot vg(L)$ *with* $L=\mathcal{L}(c-\Sigma\, m(P_i\,;\, c)P_i)$, *then either* $c$ *is a line going through at least* 4 *of the* $P_i$ *or* $c$ *is conic going through at least* 7 *of the* $P_i$ *or* $c$ *is a cubic curve going through all the* $P_i$ *and having one double point at one of the* $P_i$ *and the number* $r$ *is equal to* 9.

The above corollary and Lemma 6.1 yield the following result.

PROPOSITION 12. *If* $r\leq 9$, *then* $\mathrm{dil}_{(P_1,\cdots,P_r)}(S)$ *carries only a finite number of irreducible curves* $c'$ *such that* (i) $c'$ *is not an exceptional curve of the first kind and that* (ii) *the grade of* $c'$ *is negative. The existence of such* $c'$ *is equivalent to that the* $P_i$ *are special with respect to Cremona transformations.*

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