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A note on the pseudo-compactness of the product of two spaces

By

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As is well known, the Stone-Čech compactification of a product space is not generally identical (more precisely, homeomorphic) with the product of the Stone-Čech compactifications of coordinate spaces. M. Henriksen and J. R. Isbell [5] pointed out that the relation $\beta(X\times Y)=\beta X\times\beta Y$[1] implies the pseudo-compactness of the product $X\times Y$[2,3]. Recently, the converse has been established by I. Glicksberg [4]. He proved more generally that the relation $\beta(\Pi X_\alpha)=\Pi\beta X_\alpha$ holds true if and only if ΠX_α[4] is pseudo-compact.

In this note, we shall restrict ourselves to consider the product of two spaces, and give some conditions equivalent to that the relation $\beta(X\times Y)=\beta X\times\beta Y$ hold. We shall show that $\beta(X\times Y)=\beta X\times\beta Y$ if and only if the tensor product $C^*(X)\otimes C^*(Y)$ is dense in $C^*(X\times Y)$.

The pseudo-compactness of the product $X\times Y$ implies the pseudo-compactness of each coordinate space. However, it is not true that the product of pseudo-compact spaces must be pseudo-compact[5]. Several additional conditions sufficient to insure the pseudo-compactness of the product of pseudo-compact spaces are given and discussed in [1], [4] and [5]. We shall generalize those results in somewhat unific form.

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1) Throughout, we shall consider X as a subspace of βX.

2) The trivial case that X or Y is a finite set will be excluded throughout. If X is a finite set, then $\beta(X\times Y)=\beta X\times\beta Y$ for any space Y.

3) T. Ishiwata [7] has proved that if $\beta(X\times X)=\beta X\times\beta X$, then X is totally bounded for any uniform structure of X. (X is pseudo-compact if and only if it is totally bounded for any uniform structure of X. C. f. *T. Ishiwata*: *On uniform spaces, Sugaku Kenkyuroku,* Vol. 2 (1953) (*in Japanese*).)

4) ΠX_α denotes the product of X_α.

5) C.f. [9], [10].

All spaces mentioned here will be assumed to be infinite completely regular T_1-spaces, and all functions to be real-valued.

A compactification of X is a compact Hausdorff space containing X as a dense subspace. The Stone-Čech compactification βX is characterized among compactifications of X by the property that every bounded continuous function on X has a continuous extension over βX[6].

Let $C^*(X)$ denote the Banach space of all bounded continuous functions on X with the usual nom $\|f\|=\sup_{x\in X}|f(x)|$. We shall denote by $Z(F)$ the set of zero points of $F\in C^*(X\times Y)$, that is, $Z(F)=\{(x, y)\in X\times Y;\ F(x, y)=0\}$.

THEOREM 1. *The following conditions on the product* $X\times Y$ *are equivalent.*

(a) *Both* X *and* Y *are pseudo-compact and* $pr_X[Z(F)]$[7] *is closed in* X *for each* $F\in C^*(X\times Y)$.

(b) *Both* X *and* Y *are pseudo-compact and* $pr_Y[Z(F)]$ *is closed in* Y *for each* $F\in C^*(X\times Y)$.

(c) *The tensor product* $C^*(X)\otimes C^*(Y)$ *is dense in* $C^*(X\times Y)$.

(d) $\beta(X\times Y)=\beta X+BY$.

The pattern of proof is (a) ↘ (c), (b) ↗ (c); (c) → (d); (d) ↗ (a), (d) ↘ (b).

Proof of (a)→(c): Let F be an element of $C^*(X\times Y)$ and let F_x denote the restriction of F on $x\times Y$. Then F_x defines a continuous function on Y. By assigning F_x to $x\in X$, we have a map $\hat{F}$ of X into $C^*(Y)$. The map $\hat{F}$ is continuous as we now verify: Put $H_\varepsilon(x, y)=\varepsilon-\min(\varepsilon, |F(x, y)-1\otimes F_x(y)|)$, then $H_\varepsilon(x, y)=\varepsilon$ on $x\times Y$ and $H_\varepsilon(x, y)\neq 0$ implies that $|F(x, y)-1\otimes F_x(y)|<2\varepsilon$. Since $pr_X[Z(H_\varepsilon)]$ is closed in X by (a), there is a neighborhood $U_\varepsilon(x)$ of x such that $U_\varepsilon(x)\times Y\cap Z(H_\varepsilon)=\phi$. If $x'\in U_\varepsilon(x)$, then $|F_x(y)-F_{x'}(y)|<\varepsilon$ for each $y\in Y$, and consequently $\|F_x-F_{x'}\|<\varepsilon$ for each $x'\in U(x)$. Therefore $\hat{F}$ is continuous.

It follows that the image $\hat{F}(X)\subset C^*(Y)$ of X is compact, since the continuous image of a pseudo-compact space is pseudo-compact and since pseudo-compact metrizable space is compact[8]. Therefore

6) See [3], P. 831.

7) $pr_X[Z(F)]$ denotes the projection of $Z(F)$ into X.

8) Note that every metrizable space is paracompact (c.f. [8], P. 160) and that pseudo-compact paracompact space is compact (c.f. [6]).

we have a finite number of functions, say $F_1, \cdots, F_m$ in $\hat{F}(X) \subset C^*(Y)$ such that $\bigcup_{i=1}^{m} V_n(F_i)$ covers $\hat{F}(X)$, where $V_n(F_i) = \{f \in C^*(Y); \|f - F_i\| < 1/n\}$. Put $f_i(x) = \max[0, 1/n - \|F_i - F_x\|]$, then $0 \leqslant f_i(x) \leqslant 1/n$ and $\sum_{i=1}^{m} f_i(x) > 0$ for each $x \in X$. Letting $\varphi_i(x) = f_i(x) / \sum_{i=1}^{m} f_i(x)$, we have a finite partition of unity $\sum_{i=1}^{m} \varphi_i(x) = 1$. Now, let us consider the function $F_n(x, y) = \sum_{i=1}^{m} \varphi_i(x) \otimes F_i(y)$ which is evidently an element of $C^*(X) \otimes C^*(Y)$. Obviously $\varphi_i(x) \neq 0$ implies that $|F(x, y) - 1 \otimes F_i(y)| < 1/n$ for each $y \in Y$, and therefore we have $\|F(x, y) - F_n(x, y)\| = \|(\sum_{i=1}^{m} \varphi_i(x) \otimes 1) \cdot F(x, y) - \sum_{i=1}^{m} (\varphi_i(x) \otimes 1) \cdot (1 \otimes F_i(y))\| = \sum_{i=1}^{m} \|\varphi_i(x) \otimes 1\| \cdot \|F(x, y) - 1 \otimes F_i(y)\| \leqslant 1/n \|\sum_{i=1}^{m} \varphi_i(x)\| = 1/n$. It follows that $C^*(X) \otimes C^*(Y)$ is dense in $C^*(X \times Y)$.

Proof of (c)$\rightarrow$(d): To prove (d), we have only to show that each $F \in C^*(X \times Y)$ has a continuous extension over $\beta X \times \beta Y$. Let $F(x, y)$ be any element of $C^*(X \times Y)$. Then there is, regarding our hypothesis, a sequence $\{F_n(x, y)\}$ of elements of $C^*(X) \otimes C^*(Y)$ which converges to $F(x, y)$. It is clear that each element of $C^*(X) \otimes C^*(Y)$ has a continuous extension over $\beta X \times \beta Y$, and we shall denote by $F_n^*(x, y)$ the extension of $F_n(x, y)$ over $\beta X \times \beta Y$. Then $\{F_n^*(x, y)\}$ forms a Cauchy sequence of $C^*(\beta X \times \beta Y)$[9], and since $C^*(\beta X \times \beta Y)$ in complete $\{F_n^*(x, y)\}$ converges to a function $F^* \in C^*(\beta X \times \beta Y)$, which is the desired extension of F over $\beta X \times \beta Y$.

Proof of (d)$\rightarrow$(a): The first statement of (a) is an easy consequence of Stone-Čech's theorem (see [8], P. 153) which states that if h is a continuous map of X to a compact Hausdorff space Y, then h has a continuous extension h^* which carries βX to Y. Let R^* denote the one point compactification of real space R (i.e. $R^* = R \cup \infty$), then each continuous function $f \in C(X)$ (where $C(X)$ denotes the set of all real-valued continuous functions on X) has a continuous extension f^* (R^*-valued function) over βX, and f is unbounded if and only if $f^*(p) = \infty$ for some $p \in \beta X$. If Y is not pseudo-compact, then there is an unbounded continuous function $g(y) \in C(Y)$. Since X is assumed to be infinite, we can

9) See [2], P. 17, Proposition 5.

see that there is a bounded function $h\in C(X)$ such that $Z(h^*)$ is not open in βX[10], where h^* denotes the extension of h over βX. Consider the function $G(x, y)=h(x)\otimes g(y)$, then it is easy to see that $G(x, y)$ has no (R^*-valued) extension over $\beta X\times\beta Y$. But this contradicts the assumption that $\beta(X\times Y)=\beta X\times\beta Y$. It follows that both X and Y are pseudo-compact. We now prove that $pr_X[Z(F)]$ is closed in X for each $F\in C^*(X\times Y)$. To this end, we first observe that $pr_{\beta X}[Z(F^*)\cap(X\times Y)]=pr_{\beta X}[Z(F^*)\cap(X\times\beta Y)]$, where F^* denotes the extension of F over $\beta X\times\beta Y$. Suppose not, then there is a point $x_0\in X$ such that $F(x_0, y)\neq 0$ for each $y\in Y$ and $F(x_0, q)=0$ for some $q\in\beta Y$. Let F_0 be the restriction of F^* on $x_0\times Y$, then $F_0(y)\neq 0$ for each $y\in Y$ and $F_0^*(q)=0$ for some $q\in\beta Y$. Evidently, $(1/F_0)^2$ is an unbounded continuous function on Y, and hence Y can not be pseudo-compact. This is contradictory, therefore we have $pr_{\beta X}[Z(F^*)\cap(X\times Y)]=pr_{\beta X}[Z(F^*)\cap(X\times\beta Y)]$. On the other hand, it is clear that $Z(F)=Z(F^*)\cap(X\times Y)$ and it follows that $pr_X[Z(F)]=pr_{\beta X}[Z(F^*)\cap(X\times Y)]=pr_{\beta X}[Z(F^*)\cap(X\times\beta Y)]=pr_{\beta X}[Z(F^*)]\cap X$. Since $Z(F^*)$ is compact $pr_{\beta X}[Z(F^*)]$ is compact and consequently $pr_{\beta X}[Z(F^*)]\cap X=pr_X[Z(F)]$ is closed in X. The proof is completed.

The proof of (b)→(c) ((d)→(b)) is entirely similar to that of (a)→(c) ((d)→(a)).

We now discuss the pseudo-compactness of the product $X\times Y$. Throughout the sequel, both X and Y are assumed to be pseudo-compact. By virtue of the theorem due to I. Glicksberg ([4], Theorem 1), the pseudo-compactness of the product $X\times Y$ is equivalent to that the relation $\beta(X\times Y)=\beta X\times\beta Y$ hold true. It follows from Theorem 1 that $X\times Y$ is pseudo-compact if and only

10) Suppose that $Z(h^*)$ is open for each $h\in C^*(X)$ then $Z(f)$ is open for each $f\in C^*(\beta X)$. It follows that every continuous function on βX assumes only finitely many values, since $\{x\in\beta X;\ f(x)=a,\ a\in R,\ f\in C^*(\beta X)\}$ is open (and closed) in βX. Take two points x, y of βX and let f be a continuous function on βX such that $f(x)=0$ and $f(y)=1$. Then both $\{x\in\beta X;\ f(x)=0\}$ and $\{x\in\beta X;\ f(x)\neq 0\}$ are open and closed, and at least one of them must be infinite because βX is infinite. Consequently, there is an open and closed subset A_1 containing infinitely many points such that $\beta X-A_1\neq\phi$. Similarly, A_1 contains an open and closed A_2 containing infinitely points such that $A_1-A_2\neq\phi$.$\cdots$It follows that there is a sequence $\{A_n\}$ of open and closed subset of βX such that $A_n\supset A_{n+1}$ and $A_n-A_{n+1}\neq\phi$ for each n. Let g_n be a characteristic function of A_n, then $g=\sum_{n=1}^{\infty}g_n/2^n$ is a continuous function of βX assuming infinitely many values. But this is a contradiction.

if $pr_X[Z(F)]$ is closed in X for each $F\in C^*(X\times Y)$ (or, equivalently, if and only if $pr_Y[Z(F)]$ is closed in Y for each $F\in C^*(X\times Y)$).

We first give a simple proof of the following proposition.

PROPOSITION 1. *If X is compact, then $X\times Y$ is pseudo-compact for any pseudo-compact space Y.*

Proof. We shall show that $pr_Y[Z(F)]$ is closed for each $F\in C^*(X\times Y)$, which will complete the proof. If $y\notin pr_Y[Z(F)]$, then $F(x, y)\neq 0$ for each $x\in X$. There is, for each point $(x, y)\in X\times y$, an open neighborhood $U(x)\times V(y)$ on which $F(x, y)\neq 0$. Since $X\times y$ is compact, $X\times y$ can be covered by a finite number of such neighborhoods, say $U_1(x_1)\times V_1(y), \cdots, U_m(x_m)\times V_m(y)$. Put $W(y)=\bigcap_{i=1}^{m} V_i(y)$; then $W(y)$ is open and $W(y)\cap pr_Y[Z(F)]=\phi$. It follows that $pr_Y[Z(F)]$ is closed in Y.

The next proposition shows that $X\times Y$ is pseudo-compact for any pseudo-compact space Y if X has a "rich" supply of compact sets, even if it is not compact. Recall that X is a k-space[11] provided every subset of X intersects every compact subset of X in a closed set is itself closed. Every locally compact space, and every space satisfying the first axiom of countability is a k-space.

PROPOSITION 2. *If X is a pseudo-compact k-space, then $X\times Y$ is pseudo-compact for any pseudo-compact space Y.*

Proof. Suppose that $X\times Y$ is not pseudo-compact, then there is a function $F\in C^*(X\times Y)$ such that $pr_X[Z(F)]$ is not closed in X. Since X is assumed to be a k-space, there is a compact set C such that $C\cap pr_X[Z(F)]$ is not closed. Let F' be the restriction of F on $C\times Y$, then $F'\in C^*(C\times Y)$. Evidently $Z(F')=Z(F)\cap(C\times Y)$ and we can conclude without difficulty that $pr_C[Z(F')]=pr_C[Z(F)\cap(C\times Y)]$ $=pr_X[Z(F)\cap(C\times Y)]=pr_X[Z(F)]\cap C$. Therefore $pr_C[Z(F')]$ is not closed in C. On the other hand, it follows from Proposition 1 and Theorem 1 that $pr_C[Z(F')]$ is closed, since C is compact. This is contradictory, and hence $X\times Y$ is pseudo-compact.

The preceding proposition can be generalized, by utilizing the notion of P-point[12], and Glicksberg's technique on the equicontinuity

11) See [8], P. 231.

12) $x\in X$ is said to be a P-point if every countable intersection of neighborhoods of x contains a neighborhood of x. C.f. *L. Gillman and M. Henriksen: Concerning rings of continuous functions, Trans. Amer. Math. Soc.* 77 (1954) 340–362.

of $\{F_y(x)\}_{y\in Y}$, where $F_y(x)$ denotes the restriction of $F\in C^*(X\times Y)$ on $X\times y$.

Now, let us agree to call $x\in X$ as a k-point of X if x satisfies the following condition: If x is an accumulation point of a subset H of X, then there is a compact set C in X such that x is also an accumulation point of $C\cap H$. Every discrete point of X is a k-point, and X is a k-space if and only if every point of X is a k-point.

THEOREM 2. *If X is pseudo-compact and if every non-P-point of X is a k-point, then $X\times Y$ is pseudo-compact for any pseudo-compact space Y.*

Proof. Reviewing the proof of Prop. 2, we can see that $x\notin \overline{Pr_X[Z(F)]}-Pr_X[Z(F)]$ for each $F\in C^*(X\times Y)$ if x is a k-point of X. Consequently, $\{F_y(x)\}_{y\in Y}$ is equicontinuous at each k-point of X, because $\{F_y(x)\}_{y\in Y}$ is equicontinuous at x if and only if $x\notin \overline{A(\varepsilon)}-A(\varepsilon)$ for any $\varepsilon>0$, where $A(\varepsilon)=Pr_X[Z(\varepsilon-\min(\varepsilon, |F(x,y)-1\otimes F_x(y)|))]$. On the other hand, equicontinuity of $\{F_y(x)\}_{y\in Y}$ is equivalent to the equicontinuity of each countable subset by virtue of the fact that Ascoli's theorem holds in a pseudo-compact space (c.f. [4], P. 370). Each countable subset of $\{F_y(x)\}_{y\in Y}$ is obviously equicontinuous at each P-point, and consequently each countable subset of $\{F_y(x)\}_{y\in Y}$ is equicontinuous on X. It follows that $\{F_y(x)\}_{y\in Y}$ is equicontinuous on X, and hence $Pr_X[Z(F)]$ is closed for each $F\in C^*(X\times Y)$. Therefore $X\times Y$ is pseudo-compact.

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