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On the harmonic boundary of an open Riemann surface, II

By

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Introduction

The present paper contains some generalizations and supplements of our previous results [6]. Furthermore the relation between harmonic boundary points and the minimal functions (in $\underline{HD}$) will be studied.

We shall denote by R an open Riemann surface and by R^* the Royden compactification of R. In §1 the structure of R^* and some definitions are stated. §2 is concerned with the harmonic measure with respect to any compact subset of harmonic boundary Δ of R. In particular, the harmonic measure with respect to a single point becomes a minimal function in class $\underline{HD}$ studied by Constantinescu-Cornea [1] and one-to-one correspondence between minimal functions (in $\underline{HD}$) and some points in Δ will be established. These results are the contents of §3. Finally in §4 from our point of view we shall study the properties of non-compact subregions G on R, by which some theorems in previous paper [6] will be made more clear and complete. Particularly, Theorem 6 (or 6′) gives the characterization of $G \notin SO_{HD}$, which has some remarkable applications.

§1. Structure of R^*

1. Let R denotes an arbitrary open Riemann surface. For the sake of definiteness and convenience we shall state briefly the structure of the compactification R^* of R (for some proofs see Gelfand-Silov [2] and Royden [11]. cf. Nakai [10] for another approach). Let BD denotes the class of bounded continuous func-

tions which are piecewise smooth on R and have finite Dirichlet integrals. The class of BD-functions with compact carriers forms an ideal K in the ring BD. By $\bar{K}$ we denote the closure of K in BD-topology[1]. Introducing the BD a norm given by

$$(1) \qquad \|f\| = \sup_R |f| + \sqrt{D_R(f)},$$

where $D_R(f)=\iint_R |\text{grad}\, f|^2 dxdy$, we get a normed ring A. $K, \bar{K}$ are ideals still in A. Let $\mathfrak{M}$ be the set of all maximal ideals in A. Now the topology of $\mathfrak{M}$ is defined as follows: a maximal ideal M is called a limit point of a subset $\mathfrak{A}\subset\mathfrak{M}$, if M contains an ideal $\bigcap_{N\in\mathfrak{A}} N$. The totality of limit points of $\mathfrak{A}$ constitutes the closure $\bar{\mathfrak{A}}$. For any two sets $\mathfrak{A}$ and $\mathfrak{B}$ we have

$$(2) \qquad \bar{\mathfrak{A}}\cup\bar{\mathfrak{B}} = (\overline{\mathfrak{A}\cup\mathfrak{B}}).$$

It is proved that $\mathfrak{M}$ with the topology induced by this closure operation becomes a compact Hausdorff space, which will be written as R^*. For every point $a\in R$ the set

$$M_a = \{x\,;\, x\in A,\ x(a) = 0\}$$

makes obviously a maximal ideal. With each $x\in A$ a real number $x(M_a)$ is associated (A/M_a is isomorphic to the real number field). Then it is proved that $x(M_a)=x(a)$ and the mapping $a\to M_a$ gives a homeomorphism of R into R^*. Since $\bigcap_{a\in R} M_a$ consists of a function $x\equiv 0$, every point of R^* is a limit point of $\bigcup_{a\in R} M_a$, i.e. the image of R is a subset dense in R^*, which will be denoted again by R. Every function in A becomes continuous on R^*.

2. We call a non dense closed set

$$\Gamma = R^*-R, \qquad R = \bigcup_{a\in R} M_a$$

the ideal boundary of R. It is easily seen that every point $M_a\in R$ does not include an ideal K, while every point M (maximal ideal) of Γ necessarily contains K. Now a point M in the ideal boundary Γ is called a *harmonic boundary point* of R, if maximal ideal M contains not only K, but also an ideal $\bar{K}$. The set of harmonic boundary points of R constitutes a closed set Δ, the harmonic boundary of R, which plays an important role in our studies.

1) $f_n\to 0$ in BD-topology if $|f_n|$ are uniformly bounded, $f_n\to 0$ uniformly on every compact set of R and $D_R(f_n)\to 0$.

In the sequel, we shall assume that Riemann surface R *does not belong to* O_{HD}, unless otherwise stated.

§ 2. Harmonic measures with respect to harmonic boundary points.

3. LEMMA 1. *Let* u *be a non-constant BD-function which is subharmonic (resp. superharmonic) on* R, *then* u *attains its maximum (resp. minimum) on* Δ.

Proof. Let

$$u = U+\varphi, \qquad U\in HBD,\ \varphi\in\bar{K} \tag{3}$$

be the orthogonal decomposition on R. Take an exhaustion $\{R_n\}$ of R and consider a sequence of harmonic functions u_n $(n=1,2,\cdots)$ which have, on ∂R_n, the same boundary values as u. Then u_n (subsequence) converge uniformly to U on every compact set in R. Since $u_n\geqq u$ on R_n, we have for $n\to\infty$ $U\geqq u$ on R, therefore by means of maximum principle (Mori-Ôta [9])

$$\sup_R u\leqq\sup_R U = U(q^*), \qquad q^*\in\Delta$$

While $\varphi(q^*)=0$, hence we have $\sup_R u=u(q^*)$, q.e.d.

Let u, v be any two harmonic functions on R, then the notations

$$u\wedge v \quad\text{and}\quad u\vee v$$

mean respectively the greatest harmonic minorant, and the least harmonic majorant of u and v. Now we have

LEMMA 2. *Let* u_1, u_2 *be HBD-functions on* R, *then for* $p^*\in\Delta$

$$\begin{aligned}(u_1\vee u_2)(p^*) &= \max[u_1(p^*), u_2(p^*)]\\ (u_1\wedge u_2)(p^*) &= \min[u_1(p^*), u_2(p^*)]\end{aligned} \tag{4}$$

Proof. Since the function

$$u(p) = \max[u_1(p), u_2(p)], \qquad p\in R$$

belongs to the class BD, we have the orthogonal decomposition (3) of u. Since

$$u(q) = U(q) \qquad \text{for } q\in\Delta \tag{5}$$

and u is subharmonic, we have, by Lemma 1, $u(p)\leqq U(p)$ on R, hence $u_1\vee u_2\leqq U$. While, $u\leqq u_1\vee u_2$ and $U(q)=u(q)$, therefore

$U \leqq u_1 \vee u_2$ by Lemma 1. Thus we have

$$(6) \qquad U = u_1 \vee u_2 .$$

(5) and (6) show the first equality in (4). As for the second one, the proof is quite analogous.

4. After Constantinescu-Cornea [1] we consider a class $\underline{HD}$ of harmonic functions which are limits of monotone decreasing HD-functions. Evidently $HD \subset \underline{HD}$.

THEOREM 1. *Let α be a compact subset $(\neq \phi)$ of Δ and β its complementary set $(\neq \phi)$ in Δ. Then there exists a function Ω_α defined on R^* such that*

(i) *Ω_α is upper semi-continuous on R^* and $\Omega_\alpha \in \underline{HD}$ in R*

(ii) *$\Omega_\alpha = 1$ on α, $=0$ on β and $0 \leqq \Omega_\alpha \leqq 1$ on R^*.*

We call Ω_α the harmonic measure with respect to α.

Proof. Consider the set of functions;

$$(7) \qquad \mathfrak{F}_\alpha = \{v \in HBD\,;\ v = 1 \text{ on } \alpha \text{ and} \geqq 0 \text{ on } R\} .$$

First we note that $\mathfrak{F}_\alpha$ is non empty, because $\mathfrak{F}_\alpha$ contains a non-negative HBD-function $u_{\alpha,q}$ with the property

$$(8) \qquad u_{\alpha,q} = 1 \text{ on } \alpha \text{ and } = 0 \text{ at a point } q \in \beta$$

(cf. Lemma 2, [6]). Now the function

$$(9) \qquad \Omega_\alpha(p) = \inf_{v \in \mathfrak{F}_\alpha} v(p)\,, \qquad p \in R^*$$

has the required properties. This is proved by the Perron's method for Dirichlet problem. Take any point p_0 on R and the sequence of functions $\{v_n\}$ such that $v_n(p_0) \to \Omega_\alpha(p_0)$ $(n \to \infty)$, $v_n \in \mathfrak{F}_\alpha$. Then we see that by Lemma 2 the functions

$$(10) \qquad u_n = v_1 \wedge v_2 \wedge \cdots \wedge v_n$$

belong to $\mathfrak{F}_\alpha$, moreover the monotone decreasing sequence $\{u_n\}$ converges to a harmonic function Ω and $\Omega(p_0) = \Omega_\alpha(p_0)$. Next take any point p_1 $(\neq p_0)$ on R, then we obtain analogously the monotone decreasing sequence $\{u_n'\}$, which converges to a harmonic function Ω' and $\Omega'(p_1) = \Omega_\alpha(p_1)$. Now the functions $w_n = u_n \wedge u_n'$ belong also to $\mathfrak{F}_\alpha$ and we find that the limit function $\Omega'' = \lim_{n \to \infty} w_n$ are majorized by Ω and Ω', hence $\Omega \equiv \Omega' \equiv \Omega''$ by the minimum principle, because $\Omega''(p_0) = \Omega(p_0) = \Omega_\alpha(p_0)$, $\Omega''(p_1) = \Omega'(p_1) = \Omega_\alpha(p_1)$. Since p_1 is arbitrary, it follows that

(11) $$\Omega_\alpha \equiv \Omega = \lim_{n\to\infty} u_n \in \underline{HD}\,, \qquad u_n \in \mathfrak{F}_\alpha$$

Since $u_n \in HBD$ are continuous on R^* and $u_{n+1} \leqq u_n$ $(n=1, 2, \cdots)$, we see that Ω_α is upper semi-continuous on R^*, and $=1$ on α. It suffices now to show that Ω_α vanishes on β. Indeed, for any point $q \in \beta$ we have by (8) and (9)

$$0 \leqq \Omega_\alpha(q) \leqq u_{\alpha,q}(q) = 0\,,$$

which implies that $\Omega_\alpha(q)=0$ and Ω_α is continuous at q, q.e.d.

THEOREM 2. *$\Omega_\alpha \in HBD$ if and only if $\alpha \cap \bar{\beta}$ is empty.*

Proof. If $\alpha \cap \bar{\beta} = \phi$, β becomes compact and $\alpha \cup \beta = \Delta$. Hence we can construct a non-negative function $u_{\alpha,\beta} \in HBD$ such that $u_{\alpha,\beta}=1$ on α and $=0$ on β (cf. Lemma 3, [6]). Since $u_{\alpha,\beta} \in \mathfrak{F}_\alpha$

$$\Omega_\alpha \leqq u_{\alpha,\beta}\,.$$

While, for any $v \in \mathfrak{F}_\alpha$ we have $u_{\alpha,\beta} \leqq v$ by the maximum principle, hence

$$u_{\alpha,\beta} \leqq \inf_{v \in \mathfrak{F}_\alpha} v = \Omega_\alpha\,.$$

Thus $\Omega_\alpha \equiv u_{\alpha,\beta} \in HBD$. Next assume that $\Omega_\alpha \in HBD$, but $\alpha \cap \bar{\beta} \neq \phi$. Take a point $M \in \alpha \cap \bar{\beta}$, then M contains an ideal $\bigcap_{N \in \beta} N$. Since HBD-function Ω_α vanishes at every point N of β, $\Omega_\alpha \in N$, hence $\Omega_\alpha \in \bigcap_{N\in\beta} N \subset M$, which shows that $\Omega_\alpha(M)=0$. While, $M \in \alpha$ therefore $\Omega_\alpha(M)=1$ by (ii). This is a contradiction.

5. A compact subset e of $\Delta(R)$ is said to be of harmonic measure zero if $\Omega_e \equiv 0$ in R. We state here some properties on sets of harmonic measure zero, but not prove as they are not used in the sequel.

THEOREM 3. 1°. *If $\Omega_e=0$, then $e \subset \overline{\Delta - e}$, moreover $\Omega_{e'}=0$ for every compact subset $e' \subset e$.*

2°. *If $\Omega_{e_i}=0$ for $i=1, \cdots, n$ $(<\infty)$, then $\Omega_{\bigcup_i e_i}=0$.*

3°. (*Generalized maximum principle*). *If f is a BD-function which is subharmonic on R and $\sup_{\Delta - e} f \leqq m$, where $\Omega_e=0$, then $f \leqq m$ throughout R.*

§ 3. Minimal functions in $\underline{HD}$.

6. A non-negative function $u(\not\equiv 0)$ is said to be minimal in certain class L of real-valued functions if for any $\omega \in L$ satisfying

$0 \leqq \omega \leqq u$ we have $u \equiv \text{const} \cdot \omega$. (cf. Martin [7], Kjellberg [4], Heins [3], Kuramochi [5] and Constantinescu-Cornea [1]).

THEOREM 4. *Let Ω_q be the harmonic measure with respect to a single point $q \in \Delta$, then Ω_q is minimal in the class $\underline{HD}$, provided that $\Omega_q \not\equiv 0$ in R.*

Proof. From the assumption

$$0 \leqq \omega \leqq \Omega_q \leqq 1, \qquad \omega \in \underline{HD} \tag{12}$$

we shall prove that $\Omega_q = \text{const} \cdot \omega$. Since $\omega \in \underline{HD}$, there exists a sequence $\{\omega_n\}$ such that $\omega_n \downarrow \omega$, $\omega_n \in HD$ ($\downarrow$ means monotone decreasing). We may assume that ω_n are *bounded* $\leqq 1$ (since $\omega_n \wedge 1 \downarrow \omega$). Let $c_n = \omega_n(q)$, then $c_n \downarrow c$ ($\geqq 0$). Suppose $c > 0$. Since $\omega_n / c_n \in \mathfrak{F}_q$, it follows that $\omega_n \geqq c_n \Omega_q$ ($n = 1, 2, \cdots$), hence $\omega \geqq c\Omega_q$ for $n \to \infty$. Then it is proved that the equality holds in R. Suppose the contrary:

$$\omega(p_0) - c\Omega_q(p_0) = \delta_0 > 0$$

at $p_0 \in R$. We recall that $u_n \downarrow \Omega_q$, $u_n \in \mathfrak{F}_q$ (cf. (11)) and u_n are bounded (since $1 \wedge u_n \downarrow \Omega_q$). Therefore $\omega(p_0) - cu_n(p_0) \geqq \delta_0/2$ for $n \geqq n_0$. We fix a number n ($\geqq n_0$) and consider the set

$$G = \{p \in R;\ \omega(p) - cu_n(p) \geqq \delta_0/4\}. \tag{13}$$

G is non-compact, moreover the double $\hat{G}$ of G with respect to ∂G is of hyperbolic type, because the anti-symmetric extension of harmonic function $\omega - cu_n - \delta_0/4$ is a non-constant HB-function on $\hat{G}$. Therefore $\bar{G}$ contains some harmonic boundary points by Proposition 1, [6], but this is impossible by the following reasons. First, suppose $q \in \bar{G}$. Take a positive number $\varepsilon < \delta_0/2(1+c)$. Since ω is upper semi-continuous and u_n is continuous on R^*, we can find a neighborhood V_q (of q) such that for $p \in V_q$

$$\omega(p) < c + \varepsilon/2, \quad |u_n(p) - u_n(q)| < \varepsilon/2.$$

Since $u_n(q) = 1$, we have in V_q (hence $V_q \cap G$)

$$\omega(p) - cu_n(p) < \varepsilon(1+c)/2 < \delta_0/4$$

which contradicts with (13). Next suppose that $\bar{G}$ contains a harmonic boundary point q' distinct from q. Since Ω_q is upper semi-continuous on R^*, we have for $p \in V_{q'} \cap G$

$$\omega(p) \leqq \Omega_q(p) \leqq \Omega_q(q') + \varepsilon/2 = \varepsilon/2 \qquad \text{(see (12))}$$

whence

$$\omega(p) - cu_n(p) \leqq \omega(p) \leqq \varepsilon/2 < \delta_0/4$$

which is also absurd. In case of $c=0$ we would have more easily $\omega \equiv c\Omega_q = 0$.

7. Now we call temporally a harmonic boundary point q a *HD-singular point*, if $\Omega_q \not\equiv 0$ in R, i.e. the harmonic measure of q is positive, and split Δ into two parts

$$(14) \qquad \Delta = \Delta_0 + \tilde{\Delta}_0$$

where Δ_0 denotes the set of all HD-singular points. As an example of HD-singular point, every isolated point q in Δ belongs to Δ_0, because $\Omega_q \equiv u_{\alpha,\beta} \not\equiv 0$ by Theorem 2. A criterion for HD-singular points will be given later. As the converse of Theorem 4 we shall prove now the following

THEOREM 5. *For any minimal function* ω $(\sup \omega = 1)$ *in* $\underline{HD}$ *there exists a HD-singular point* $P_\omega \in \Delta_0$ *such that the harmonic measure* Ω_{P_ω} *is identical with* ω. *Moreover the mapping* $\omega \leftrightarrow P_\omega$ *between minimal functions (in* $\underline{HD}$*) and points of* Δ_0 *is one-to-one.*

Proof. According to [1], to every minimal function ω $(\sup \omega = 1)$ in $\underline{HD}$ there corresponds a maximal HD-indivisible set M on $|z|=1$ and ω is equal to the harmonic measure with respect to M, where $\{|z|<1\}$ is the conformal image of the universal covering surface of R. Moreover, let

$$(15) \quad \tilde{\mathfrak{F}} = \{v \in HBD \text{ on } R;\ \lim_{r\to 1} v(re^{i\theta}) = 1 \text{, a.e. on } M \text{ and } v \geqq 0\}$$

then we have (pp. 213-215, [1])

$$(16) \qquad \omega(p) = \inf_{v \in \tilde{\mathfrak{F}}} v(p)\,, \qquad p \in R$$

Now consider the sets

$$E_n = \{p \in R;\ \omega(p) > 1 - 1/n\} \qquad (n = 2, 3, \cdots)$$

then E_n are non-compact and $E_n \supset E_{n+1}$. Let $\bar{E}_n$ be closures (in R^*) of E_n and

$$\Delta_n = \bar{E}_n \frown \Delta\,,$$

then Δ_n are non empty by Proposition 1, [6], because the doubles $\hat{E}_n$ are of hyperbolic type. Furthermore, Δ_n are compact in the compact Hausdorff space R^* and $\Delta_n \supset \Delta_{n+1}$ for every n, hence by a well-known theorem

$$\Delta' = \bigcap_{n=2}^{\infty} \Delta_n$$

is not void. Now we show that Δ' consists of a single point. To see this it suffices to prove that any $v \in HBD$ has a constant value on Δ'. Since v is bounded, we may assume that v is positive. From the Poisson integral formula we have

$$v(z) = \frac{1}{2\pi}\int_{M^c} v(e^{i\theta})\, Re\frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta + \gamma\,\omega(z) \tag{17}$$

where M^c is the complementary set of M and γ the radial limit of v on M. From (17) we get $\gamma\omega \leqq v \leqq K(1-\omega)+\gamma\omega$, $(K=\sup v)$, hence for $z \in E_n$

$$\gamma(1-1/n) \leqq v(z) \leqq \gamma + K/n\,.$$

These inequalities $(n=2, 3, \cdots)$ are also valid on Δ', hence we have

$$v = \gamma \quad \text{on } \Delta'. \tag{18}$$

Thus we know that Δ' consists of a single point, P_ω say. Now let u be any element of $\mathfrak{F}_{P_\omega}$, then u has a constant radial limit γ (a.e.) on HD-indivisible set M, moreover we know that $\gamma=1$ from above. Therefore $\mathfrak{F}_{P_\omega} \subset \tilde{\mathfrak{F}}$. While, $\tilde{\mathfrak{F}} \subset \mathfrak{F}_{P_\omega}$ (see (18)), hence we have $\mathfrak{F}_{P_\omega} = \tilde{\mathfrak{F}}$, i.e.

$$\Omega_{P_\omega} \equiv \omega \not\equiv 0\,.$$

Finally, the mapping $\omega \leftrightarrow P_\omega$ is one-to-one. Let P_1, P_2 be two distinct points in Δ_0, and $\omega_1 = \Omega_{P_1}$, $\omega_2 = \Omega_{P_2}$ be corresponding minimal functions, then $\omega_1 \neq \omega_2$, because ω_1, ω_2 have respectively radial limits 1 and 0 (a.e.) on the indivisible set for ω_1, q.e.d.

COROLLARY 1. *If a minimal function* ω $(\sup \omega = 1)$ *in* $\underline{HD}$ *belongs to the HBD, then* ω *is identical with the harmonic measure* Ω_{q^*} *of an isolated point* q^* *in* Δ.

Proof. By Theorem 5 we know that $\omega = \Omega_{q^*}$ for some point $q^* \in \Delta_0$. Since $\Omega_{q^*} = \omega \in HBD$, q^* must be an isolated point by Theorem 2.

COROLLARY 2 (Proposition 3.2, [6]). $R \in O_{HD_n} - O_{HD_{n-1}}$ *if and only if* $\sigma(R) = n$.

Proof. Here we shall give a direct proof for the equivalence (i)$\leftrightarrow$(ii) in Proposition 3.2, [6]. Suppose $R \in O_{HD_n} - O_{HD_{n-1}}$, then by Theorem 5 there are n HD-singular points $q_i \in \Delta_0$. Hence it suffices to show that $\tilde{\Delta}_0 = \phi$. Since any $v \in HBD$ can be expressible as

$$v = \sum_{i=1}^{n} \gamma_i \omega_i(z, M_i) = \sum_{i=1}^{n} \gamma_i \Omega_{q_i}(p) \qquad (\gamma_i = v(q_i)),$$

where M_i are maximal HD-indivisible sets, hence if $\tilde{\Delta}_0 \neq \phi$, $v(q)=0$ at $q \in \tilde{\Delta}_0$ which is absurd, because there exists a $v \in HBD$ with $v(q) \neq 0$ (cf. Lemma 2, [6]). Conversely if $\sigma(R)=n$, these points q_i $(i=1, \cdots, n)$ are isolated in Δ, hence $q_i \in \Delta_0$, which implies $R \in O_{HD_n} - O_{HD_{n-1}}$ by Theorem 5.

REMARK. From the above proof we see also that if R has an infinite number of maximal HD-indivisible sets, $\sigma(R)=\infty$. But the converse is not true. For example, the harmonic boundary of the unit circle $R=\{|z|<1\}$ contains an uncountable number of harmonic boundary points, but there is no HD-indivisible set. We note that this example shows also the non-validity of Proposition 4, [6] in case of $\sigma(R)=\infty$.

§4. Non-compact subregions.

8. SO_{HD} *and* O_{HD_n}.

By a non-compact subregion G we mean a non-compact domain on R whose boundary ∂G consists of an at most countable number of disjoint analytic Jordan curves not clustering to any point of R.

THEOREM 6. *Let G be a non-compact subregion on R, then $G \notin SO_{HD}$ if and only if $\bar{G} - \overline{\partial G}$ contains harmonic boundary points, i.e.* $(\bar{G} - \overline{\partial G}) \cap \Delta \neq \phi$.

Proof. If $G \notin SO_{HD}$, then $(\bar{G} - \overline{\partial G}) \cap \Delta \neq \phi$ by Proposition 2, [6]. To prove the converse, take a point $q^* \in (\bar{G} - \overline{\partial G}) \cap \Delta$. Since q^* is disjoint with $\overline{\partial G}(R^*)$, there is a BD-function f such that $f \in \bigcap_{N \in \overline{\partial G}} N$, but $f \notin q^*$, that is, $f(q^*) \neq 0$ and $f=0$ on $\overline{\partial G}$. Let f^* be a BD-function defined so that $f^*=f$ on G and $=0$ on $R-G$, then we have

$$f^*(q^*) = f(q^*) \neq 0. \tag{19}$$

Let $\hat{f}^*$ be the anti-symmetric extension of f^* onto the double $\hat{G}$ of G (with respect to ∂G), then $\hat{f}^* \in BD(\hat{G})$, hence we have the orthogonal decomposition

$$\hat{f}^* = \hat{u} + \hat{\phi}, \qquad \hat{u} \in HBD(\hat{G}), \quad \hat{\phi} \in \bar{K}(\hat{G}).$$

From the construction it is easily verified that $\hat{u}$ and $\hat{\phi}$ are anti-symmetric with respect to ∂G, hence the conclusion follows if we could show that $\hat{u} \not\equiv 0$. Assume the contrary, then $\hat{f}^* = \phi \in \bar{K}(\hat{G})$ is a limit (in BD-topology) of functions $\phi_n \in K(\hat{G})$. Here ϕ_n may be considered as anti-symmetric. Indeed, if we decompose ϕ_n into anti-symmetric parts $\phi_n^{(-)}$ and symmetric parts $\phi_n^{(+)}$:

$$
(20) \qquad \begin{aligned} \phi_n &= \phi_n^{(-)} + \phi_n^{(+)} \\ \phi_n^{(-)} &= (\phi_n(p) - \phi_n(\tilde{p}))/2\,, \quad \phi_n^{(+)} = (\phi_n(p) + \phi_n(\tilde{p}))/2 \end{aligned}
$$

where $\tilde{p}$ denotes the symmetric point of p, then

$$
\begin{aligned} 4D_{\hat{G}}(\phi_n^{(-)} - \hat{f}^*) &= D_{\hat{G}}[(\phi_n(p) - \hat{f}^*(p)) - (\phi_n(\tilde{p}) - \hat{f}^*(\tilde{p}))] \\ &\leqq 4D_{\hat{G}}[\phi_n(p) - \hat{f}^*(p)] \to 0 \quad (n \to \infty)\,. \end{aligned}
$$

Now we define functions ψ_n on R which are $= \phi_n$ on G and $= 0$ on $R - G$, then $\psi_n \in K(R)$ and $\psi_n \to f^*$ in BD-topology on R. Hence $f^* \in \bar{K}(R)$, that is, $f^*(q^*) = 0$, which contradicts with (19), q.e.d.

REMARK. From above proof we see that if $(\bar{G} - \overline{\partial G}) \frown \Delta \neq \phi$, then $G \notin SO_{HB}$ (without use of the general inclusion relation: $SO_{HB} \subset SO_{HD}$), because $\hat{u}$ is bounded. The character of a domain $G \notin SO_{HD}$ will be made more clear by the following theorem.

THEOREM 6′. *Let G be a non-compact subregion on R and $G^c = R - (G \cup \partial G)$. Then $G \notin SO_{HD}$ if and only if $(\bar{G} - \bar{G}^c) \frown \Delta \neq \phi$, that is, $\bar{G}$ contains a point $\in \Delta$ not belonging to the closure of G^c. Furthermore $(\bar{G} - \overline{\partial G}) \frown \bar{G}^c$ does not contain any points $\in \Delta$, provided $G \notin SO_{HD}$.*

Proof. Suppose $G \notin SO_{HD}$ and $(\bar{G} - \bar{G}^c) \frown \Delta = \phi$. Then according to Theorem 6, there is a point $q \in \Delta$ which belongs to $\bar{G} \frown \bar{G}^c$ but not to $\overline{\partial G} = \overline{\partial G^c}$. Let $\{E_j\}$ $(j = 1, 2, \cdots)$ be the components of G^c, then $q \in \overline{\bigcup_j E_j}$. We note here that q must belong to some $\bar{E}_j$. Indeed, if

$$
q \notin \bar{E}_j \qquad (j = 1, 2, \cdots)\,,
$$

then any neighborhood $U(q)$ of q must contain two points belonging respectively to distinct domains E_i, E_j, that is, $U(q)$ contains a point $\in \partial E_i \subset \partial G$. Hence $q \in \overline{\partial G}$, which is absurd. Now let $q \in \overline{E}_j$ $(q \notin \overline{\partial E}_j)$. We construct as above a BD-function u on G (resp. u' on E_j) such that $u = u' = 0$ on ∂G resp. ∂E_j and $u(q) = 1$ (resp. $u'(q) = 1/2$). Let u^* be a BD function on R such that $u^* = u$ on $G \cup \partial G$, $= u'$ on E_j and $= 0$ elsewhere. Then u^* should be continuous on R^*, while $u^*(p) \to 1$ for $p \to q$ $(p \in G)$ and $u^*(p') \to 1/2$

for $p' \to q$ $(p' \in E_j)$. This is a contradiction. The converse is concluded by Theorem 6.

COROLLARY 1. *Let G_0 be a compact set such that $G - G_0$ is a non-compact subregion on R, then $G \notin SO_{HD}$ if and only if $G - G_0 \notin SO_{HD}$.*

By Theorem 6′ and Proposition 1, [6] we get the following result which contains a criterion due to Matsumoto [8].

COROLLARY 2. *If there exist $(n+1)$ non-compact subregions $G_0, G_1, \cdots, G_n$ on R such that $G_i \notin SO_{HD}$ $(i=1, \cdots, n)$ and the double $\hat{G}_0$ (of G_0) $\notin O_G$, then $R \notin O_{HD_n}$. The converse is also true.*

COROLLARY 3 (Proposition 5, [6]). *$R \notin O_{HD_n}$ if and only if there exist $(n+1)$ non-compact subregions G_i such that $G_i \notin SO_{HD}$.*

9. *Symmetric harmonic boundary points. NO_{HD} and O_{HD}.*

Let G be a non-compact subregion on R and $\hat{G}^*$ be the compactification of the double $\hat{G}$ obtained from G. It is assumed $\hat{G} \notin O_G$. Among harmonic boundary points of $\hat{G}^*$ we consider any point $q \notin \bar{G} \cap \bar{\tilde{G}}$ $(\tilde{G} = \hat{G} - (G \cup \partial G)$ and bar means the closure taken on $\hat{G}^*$). Let $\{V_\alpha\}_\alpha$ be the neighborhoods of q, then obviously $q = \bigcap_\alpha \bar{V}_\alpha$. Denoting by $\tilde{V}_\alpha$ the symmetric sets of V_α' (restriction of V_α to $\hat{G}$) with respect to ∂G, it is proved that $\bigcap_\alpha \bar{\tilde{V}}_\alpha$ determines a harmonic boundary point of $\hat{G}^*$, $\tilde{q}$ say, which will be called the *symmetric harmonic boundary point* of q. To prove this, take any $\varphi \in \bar{K}(\hat{G})$, then two functions $\varphi^{(-)}$, $\varphi^{(+)}$ (cf. (20)) belong to $\bar{K}(\hat{G})$, hence, vanish at q and

$$\inf_\alpha M(\bar{V}_\alpha) = 0$$

where $M(\bar{V}_\alpha) = \max(\sup|\varphi^{(-)}(p)|, \sup|\varphi^{(+)}(p)|)$ (sup is taken on $\bar{V}_\alpha$). Since $\varphi^{(-)}(\tilde{p}) = -\varphi^{(-)}(p)$, $\varphi^{(+)}(\tilde{p}) = \varphi^{(+)}(p)$ we find that

$$\inf_\alpha M(\bar{\tilde{V}}_\alpha) = 0\,,$$

and thus, $\bigcap \bar{\tilde{V}}_\alpha = e$ consists of points $\in \Delta(\hat{G}^*)$. That e consists of a single point, is seen from the fact that any $u \in HBD(\hat{G})$ takes on e a constant value. The symmetric point of $\tilde{q}$ returns q itself.

Under these considerations we say that harmonic boundary points of G appear symmetrically and *q, $\tilde{q}$ are mutually symmetric.*

THEOREM 7. *Let G be a non-compact subregion on R, then the harmonic boundary of the double $\hat{G}$ of G consists of symmetric two* *points if and only if* $G \notin SO_{HD}$ *and* $G \in NO_{HD}$.[1)]

Proof. From the proof to Proposition 6, [6] we know that the condition is sufficient. Next, let u be any HBD-function on G whose normal derivative vanishes on ∂G, then the symmetric extension $\tilde{u}$ becomes an HBD-function on $\hat{G}$ and we find $\tilde{u}(p_0)=\tilde{u}(\tilde{p}_0)=c$, where p_0 and $\tilde{p}_0$ are mutually symmetric two points $\in \Delta(\hat{G}^*)$, hence $\tilde{u}(p)\equiv c$ by maximum principle, i.e. $G \in NO_{HD}$. Evidently $G \notin SO_{HD}$ by Theorem 6′, q.e.d.

THEOREM 8. *If G is a non-compact subregion on R such that $G \notin NO_{HD}$ and SO_{HD}, the closure $\bar{G}$ contains at least two harmonic boundary points of R.*

COROLLARY. *$R \notin O_{HD}$ if and only if there exists a non-compact subregion G such that $G \notin NO_{HD}$ and SO_{HD}. In particular, if the boundary ∂G is compact, the condition $G \notin SO_{HD}$ is unnecessary.*

Indeed, if $R \notin O_{HD}$ it suffices to take as G the complementary domain of a compact set. because $\hat{G}^*$ would contain at least two pairs of symmetric points (cf. Theorem 6′), hence $G \notin NO_{HD}$ by Theorem 7.

The proof of Theorem 8 is contained in Theorem 7 and the following Lemma 3. We say that a non-compact subregion G is *HD-singular* if the closure $\bar{G}$ contains only one point of $\Delta(R)$. For instance, for a minimal function ω ($\sup \omega = 1$) every domain $G=\{p\,;\,\omega(p)>\lambda,\ 0<\lambda<1\}$ is HD-singular, provided $\omega \in HBD(R)$. In fact, ω would become identical with the harmonic measure Ω_{P_ω} of an isolated point of Δ and $P_\omega \in \bar{G}$ (Corollary to Theorem 5).

LEMMA 3. *Let G be a non-compact subregion $\notin SO_{HD}$ and be HD-singular, then $\Delta(\hat{G})$ consists of two symmetric points.*

Proof. Since $G \notin SO_{HD}$ it follows that $\hat{G} \notin O_{HD}$ and $\Delta(\hat{G})$ contains at least two symmetric points. Now suppose that $\Delta(\hat{G})$ contains at least two pairs of symmetric points $(p_i, \tilde{p}_i)$ $(i=1,2)$, which do not belong to $\bar{G} \cap \bar{\tilde{G}}$. Then we can construct two HBD-functions U_i on $\hat{G}$ such that

$$U_i(p_j) = \delta_{ij} \text{ (Kronecker) and } U_i = 0 \text{ on } \partial G\,.$$

For example, construct $u_1 \in HBD(\hat{G})$ such that $u_1(p_1)=1$ and $=0$ at three other points, then $U_1 = 2u_1^{(-)}$ fulfils the condition. Now for

1) $G \in NO_{HD}$ means that there does not exist non-constant HD-functions on G whose normal derivatives vanishes on ∂G. It is known that $NO_{HD}=NO_{HBD}$.

suitable constants λ_i $(0<\lambda_i<1)$ two sets $E_i=\{p\,;\,U_i(p)>\lambda_i\}$ are disjoint with $\overline{\partial G}$ and $E_1\cap E_2=\phi$. Then the images (some components) of E_i on R are two non-compact subregions (in G) such that $E_i\notin SO_{HD}$, hence G must contain at least two points of $\Delta(R)$, which is absurd. Thus we know that $\Delta(\hat{G})$ contains at most two symmetric points $(p_1, \tilde{p}_1)$. Next suppose that $\Delta(\hat{G})$ contains further a point $q\in\bar{G}\cap\bar{\tilde{G}}$, then it would also lead us to a contradiction as follows.

By Theorem 6′ it remains to consider the case where $q\in\overline{\partial G}(\hat{G}^*)$. Let q^* be a point $\in\bar{G}\cap\Delta(R)$. There is an *HBD*-function v such that $v(q^*)=1$ and $v=0$ on ∂G. Then a suitable neighborhood $U=\{v>\lambda,\ 0<\lambda<1\}$ of q^* is disjoint with $\overline{\partial G}$. Write $F=\overline{(G-U)}\cap\Gamma(R^*)$, then we can find a function $\varphi\in\bar{K}(R)$ such that $\varphi>0$ on R and $\inf_F\varphi>0$ (cf. proof to Proposition 1, [6]). Let $\tilde{\varphi}$ be the symmetric extension onto $\hat{G}$ of the restriction of φ to G, then $\tilde{\varphi}\in\bar{K}(\hat{G})$ and $\inf\tilde{\varphi}>0$ on $\overline{\partial G}(\hat{G}^*)$, i.e. $\tilde{\varphi}(q)>0$. This is a contradiction.

10. *SO_{HB} and O_{HD_n}.*

THEOREM 9. *Let $\{\omega_n\}_{n=1,\dots,N}$ $(2\leq N\leq\infty)$ be all minimal functions* ($\sup\omega_n=1$) *on $R\in O_{HD_\infty}$ and*

$$G_n=\{p\,;\,p\in R,\ \omega_n(p)>\lambda,\ 0<\lambda<1\}\,, \tag{21}$$

then any non-compact subregion G (on R) outside $\bigcup_n G_n$ belongs to SO_{HB}.

Proof. According to [1] (pp. 195–196), for the inextremisation I_G (to G) of the harmonic measure for a set M of ideal boundary points of R

$$I_G\,\omega(z, M, R)=\omega(z, I^*M, G)\,. \tag{22}$$

We insist here that

$$I_G\,\omega_n(z, M_n, R)=0$$

where M_n are maximal *HD*-indivisible sets corresponding to ω_n. Otherwise we would have $\sup_G I_G\,\omega_n=1$ by (22). While, G lies outside $\bigcup_n G_n$, hence

$$I_G\,\omega_n\leqq\omega_n\leqq\lambda<1 \qquad \text{on } G$$

which is absurd. Now since $\sum_n\omega_n\equiv1$, we have

$$I_G \cdot 1 = I_G(\sum_n \omega_n) = \sum_n I_G(\omega_n) = 0,$$

which shows that $G \in SO_{HB}$, q.e.d.

Two HD-singular subregions are called disjoint if they determine two distinct points of Δ. With this terminology we shall give another characterisation of O_{HD_n}.

THEOREM 10. *$R \in O_{HD_n}$ $(1 \leqslant n < \infty)$ if and only if there exist at most n mutually disjoint HD-singular subregions G_i, and any subregion G outside $\bigcup_i G_i$ belongs to SO_{HB}.*

Proof. In case of $R \in O_{HD_n}$ $(2 \leqq n < \infty)$, the regions G_i ((21) with suitable λ) become mutually disjoint HD-singular subregions and $G \in SO_{HB}$ by Theorem 9. In case of $R \in O_{HD_1}$ $(= O_{HD} - O_G)$ it suffices to take $G_1 = R - R_0$ (R_0 is compact). Conversely, from the assumption Δ contains at least m $(\leqq n)$ distinct points $q_i \in \bar{G}_i$, thus it suffices to prove that $\Delta = \{q_i\}$. Suppose Δ contains another point $q^* \neq q_i$. Since by (2)

$$\overline{\bigcup_i G_i} = \bigcup_i \bar{G}_i = \bigcup_i \overline{(G_i + \partial G_i)},$$

$q^* \notin \overline{(\bigcup_i G_i)}$ hence $q^* \in \overline{\bigcup_i E_j}$, where E_j are the components of $R - \bigcup_i (G_i + \partial G_i)$. Moreover by the same reasoning as one in proof to Theorem 6′ we know easily that q^* is contained in some $\bar{E}_i$ and $q^* \notin \overline{\partial E_i}$ $(\subset (\bigcup_i \overline{\partial G_i}))$. Then $E_i \notin SO_{HB}$ by Theorem 6 and its remark, which is absurd, q.e.d.

Finally we give a criterion for HD-singular points (sec. 7):

THEOREM 11. *Let G be a HD-singular subregion containing a point $q^* \in \Delta$. If $G \notin SO_{HB}$, then q^* is a HD-singular point, i.e. $q^* \in \Delta_0$.*

Proof. Since $G \notin SO_{HB}$, the relative harmonic measure ω is non-constant. ω vanishes on ∂G, hence for any $u \in \mathfrak{F}_{q^*}$ we have $u(p) \geqq \omega(p)$ for $p \in \partial G$. It is shown that this holds for any point of G. Suppose that at some point $p_0 \in G$

$$u(p_0) - \omega(p_0) = \lambda < 0.$$

Then a subregion

$$D = \{p \in G\,;\ u(p) - \omega(p) < \lambda/2\} \tag{23}$$

becomes non-compact and $\bar{D} \cap \Delta \neq \phi$ by means of Proposition 1, [6]. Since $D \subset G$, q^* must belong to $\bar{D} \cap \Delta$. Now we have

$\sup_D (u-\omega) \geqq 0$, because u is continuous on R^* and $u(q^*)=1$, while $\sup_G \omega = 1$. But this contradicts with (23). Thus

$$\Omega_{q^*} = \inf_{u \in \mathfrak{F}_{q^*}} u \geqq \omega \not\equiv 0 \quad \text{on } G.$$

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