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Additive functionals of the Brownian path

By

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1. INTRODUCTION.

A functional $\mathfrak{f}(t)=\mathfrak{f}(t, w)$ of $t(\geq 0)$ and a several-dimensional Brownian path $w: t\rightarrow x(t)$ is said to be *additive* if

1.1 $\mathfrak{f}(t, w)$ *depends upon* t *and* $x(s): s\leq t$ *alone.*

1.2 $0 = \mathfrak{f}(0)\leq\mathfrak{f}<+\infty$

1.3 $\mathfrak{f}(t\pm) = \mathfrak{f}(t)$

1.4 $\mathfrak{f}(t) = \mathfrak{f}(s)+\mathfrak{f}(t-s, w_s^+) \qquad t\geq s,$

where w_s^+ is the shifted path $w_s^+: t\rightarrow x(t+s)$; for example, $\mathfrak{f}(t)=\int_0^t f(x(s))ds$ is an additive functional if $f\geq 0$ is bounded and Borel.

K. Itô and H. P. McKean, Jr. [13] proved that in the 1-dimensional case such an additive functional is an *integral*

1.5 $\mathfrak{f}(t) = \int\mathfrak{t}(t, b)e(db)$

of the *standard Brownian local times*

1.6 $\mathfrak{t}(t, a) = \lim_{b\downarrow a}\dfrac{\text{measure }(s: a\leq x(s)<b, s\leq t)}{b-a}$

with respect to a non-negative measure e, finite on bounded intervals.

Brownian local times are not available in $d\geq 2$ dimensions, but $\mathfrak{f}$ can still be expressed as a (*formal*) Hellinger *integral*

1.7 $\mathfrak{f}(t) = \int\dfrac{\text{measure }(s: x(s)\in db, s\leq t)e(db)}{db}$

with a non-negative measure e which is *smooth* in the sense that *each bounded open D is the union of an increasing series of sets $B_n (n \geq 1)$, closed in D, such that*

1.8a *the charge distributions $e|B_n$ have bounded potentials $\int_{B_n} G de \leq n (n \geq 1)$, where G is the Green function of D;*

1.8b *for large n, depending upon the path, the Brownian particle lies in B_n until it leaves D, i.e.,*

$$P_{\bullet}[x(t) \in B_n,\ t < \min (s: x(s) \notin D),\ n \uparrow +\infty] = 1,$$

where $P_{\bullet}(B)$ is the probability of the event B as a function of the starting point of the Brownian motion.

The correspondence embodied in 1.7 between the class of smooth measures e and the class of additive functionals $\mathfrak{f}$ is one to one and onto.

1.8a implies that $e(B)=0$ unless B has positive logarithmic capacity in the 2-dimensional case or has positive Newtonian capacity in the $d \geq 3$ dimensional case; thus, a smooth measure cannot attach positive mass to a line in 3 dimensions, nor, in 4 dimensions, to a surface. But it can be singular relative to Lebesgue measure; the simplest example in 3 dimensions is the uniform distribution on the spherical surface $|a|=1$.

Choose $d=3$, $e(db)=|b|^{\alpha} db$, and let G be the Green function of D: $|a|<1$; then

1.9a
$$\begin{aligned} p = \int G de & \\ \leq \text{constant } (2+\alpha)^{-1} & \qquad \alpha > -2 \\ < +\infty \text{ except at } 0 & \qquad -3 < \alpha \leq -2 \\ \equiv +\infty & \qquad \alpha \leq -3, \end{aligned}$$

1.9b
$$\mathfrak{E}(e) \equiv \iint G de de < +\infty \qquad \alpha > -5/2,$$

and it follows that e is smooth for $\alpha > -2$. *e is not smooth for $\alpha \leq -2$*; in fact, choosing $B_n \uparrow D$ as needed for 1.8,

1.10
$$\begin{aligned} E_0\left[\int_0^{\varepsilon} |x(s)|^{\alpha} ds,\ x(t) \in B_n,\ t < \varepsilon\right] & \\ \leq \int_0^{+\infty} ds \int_{B_n} P_0[x(s) \in db, \max_{t \leq s} |x(t)| < 1] |b|^{\alpha} & \\ \leq \int_{B_n} G(0, b) de < +\infty, & \end{aligned}$$

thanks to 1.8a, and

1.11 $\lim_{\varepsilon \downarrow 0} \lim_{n \uparrow +\infty} P_0[x(t) \in B_n, t < \varepsilon] = 1$

thanks to 1.8b, while, as is not hard to prove,

1.12 $P_0\left[\int_0^\varepsilon |x(s)|^\alpha ds \equiv +\infty, \ \varepsilon > 0\right] = 1,$

contradicting 1.10.

V. A. Volkonskii [15, 16] also studied additive functionals, establishing a special case of the above for a wider class of motions; the method used below is similar to his.

Given a 1-dimensional diffusion with the same hitting probabilities as the standard Brownian motion:

1.13 $$P_\xi[\min(t: x(t) = a) < \min(t: x(t) = b)] = \frac{b-\xi}{b-a} \qquad a < \xi < b,$$

W. Feller [8] explained how to express the associated generator $\mathfrak{G}$ as a *differential opearator* based upon a *speed measure e*, positive on open intervals:

1.14 $$\mathfrak{G}u = \frac{u^+(da)}{e(da)} = \lim_{b \downarrow a} \frac{u^+(b) - u^+(a)}{e(a, b)}$$
$$u^+(a) = \lim_{b \downarrow a} \frac{u(b) - u(a)}{b-a},$$

and K. Itô and H. P. McKean, Jr. [13] found that its sample paths could be expressed as standard Brownian sample paths run with the *stochastic clock* $\mathfrak{f}^{-1}$ which is the inverse function of the additive functional (local time integral) $\mathfrak{f} = \int \mathfrak{t}\, de$ associated with the speed measure.

V. A. Volkonskiĭ [15] also studied such time substitutions; his method is less explicit because it does not use local times but has the advantage that it can be applied in higher dimensions.

As will be explained below, a $d \geq 2$ dimensional *diffusion with Brownian hitting probabilities* has as its generator *the closure of a differential operator*

1.15 $\mathfrak{G}u = -\dfrac{e^u(db)}{e(db)}$

based upon a (smooth) *speed measure* e, positive on the neighborhoods of H. Cartan's *fine topology* [2]; moreover, the associated motion is the standard Brownian motion run with the inverse function $\mathfrak{f}^{-1}$ of the additive functional $\mathfrak{f}$ associated with e, and *this correspondence between the class of diffusions with Brownian hitting probabilities and the class of smooth measures e positive on fine neighborhoods is one to one and onto.*

2. BROWNIAN MOTION.

Choose $d \geq 2$, let $E^d = R^d$ it $d=2$, let it be the one-point compactification $R^d + \infty$ if $d \geq 3$, introduce the space of continuous *sample paths* $w: [0, +\infty) \to E^d$ with

2.1 $$\begin{aligned} w(t) &\in R^d & t &< \mathfrak{m}_\infty \\ &= \infty & t &\geq \mathfrak{m}_\infty, \end{aligned}$$

where $\mathfrak{m}_\infty = \mathfrak{m}_\infty(w) \leq +\infty$ and $\mathfrak{m}_\infty \equiv +\infty$ in case $d=2$, let $w(t) = x(t, w) = x(t)$ as need be, note that $x(+\infty) \equiv \infty$ even if $d=2$, and, introducing the corresponding *coordiante fields* $\boldsymbol{B}_t = \boldsymbol{B}[x(s): s \leq t]$ and $\boldsymbol{B} = \boldsymbol{B}_{\infty+}$, let $P_\cdot(B)$ be the *probability* (*Wiener measure*) of the event $B \in \boldsymbol{B}$ as a function of the starting point of the *d-dimentional Brownian motion* with *generator* $\mathfrak{G} = \dfrac{\partial^2}{\partial b_1^2} + \dfrac{\partial^2}{\partial b_2^2} + \cdots + \dfrac{\partial^2}{\partial b_d^2}$.[1]

Brownian motion enthusiasts are familiar with the fact that the Brownian traveller *starts afresh* at a passage time; the full significance of this was explained by E. B. Dynkin [6] and G. Hunt [9] as follows.

An instant of time $0 \leq \mathfrak{m} \leq +\infty$ depending upon the path is said to be a *Markov time* if

2.2 $$(w: \mathfrak{m} < t) \in \boldsymbol{B}_t \qquad t \geq 0;$$

for example, the passage time $\mathfrak{m}_Q = \inf(t: x(t) \in Q)$ to a closed or

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[1] $\mathfrak{G}/2$ is often used as the generator of the Brownian motion, but for our purpose it is simpler to omit the factor 1/2.

open d-dimensional figure Q is a Markov time and so is $\mathfrak{m}=\min\left(t:\int_0^t f(x(s))ds=1\right)$ if $0<f\leq 1$ is a Borel function.

Given such a Markov time $\mathfrak{m}$, if $w_{\mathfrak{m}}^+$ is the *shifted path*

2.3 $\quad w_{\mathfrak{m}}^+ : t \to x(t+\mathfrak{m})$

and if $\boldsymbol{B}_{\mathfrak{m}+}$ is the *field* of events $B\in\boldsymbol{B}$ such that

2.4 $\quad B\cap(w:\mathfrak{m}<t)\in\boldsymbol{B}_t \qquad t\geq 0,$

then the Brownian particle *starts afresh* at time $t=\mathfrak{m}$, i.e.,

2.5 $\quad P_a[w_{\mathfrak{m}}^+\in B\,|\,\boldsymbol{B}_{\mathfrak{m}+}]=P_b(B) \qquad a\in E^d,\ B\in\boldsymbol{B},\ b\equiv x(\mathfrak{m})$[2].

Blumenthal's 01 law [1]:

2.6 $\quad P_\cdot(B)=\begin{matrix}0\\1\end{matrix} \qquad B\in\boldsymbol{B}_{0+}=\bigcap_{\varepsilon>0}\boldsymbol{B}_\varepsilon$

is a special case of 2.5.

A. R. Galmarino[3] has pointed out that a *non-negative* Borel *function* $\mathfrak{m}$ *of the sample path is a Markov time if and only if*

2.7a $\quad \mathfrak{m}(u)<t$
2.7b $\quad x(u,u)=x(s,v) \qquad s\leq t$

imply $\mathfrak{m}(u)=\mathfrak{m}(v)$ and that *an event* $B\in\boldsymbol{B}$ *is a member of* $\boldsymbol{B}_{\mathfrak{m}+}$ *if and only if* 2.7 *coupled with* $u\in B$ *implies* $v\in B$. As a simple application of this test, note that $B_{\mathfrak{m}+}$ measures both $\mathfrak{m}$ and the *past* $x(\theta\wedge\mathfrak{m})$ $(\theta\geq 0)$[4] because 2.7 implies $\theta\wedge\mathfrak{m}(u)=\theta\wedge\mathfrak{m}(v)<t$ and hence $x(\theta\wedge\mathfrak{m}(u),u)=x(\theta\wedge\mathfrak{m}(v),v)$.

Given bounded open $D\subset R^d$ with boundary ∂D, if $\mathfrak{m}_{\partial D}$ is the *exit time* $\min(t: x(t)\in\partial D)$, then the *hitting probability*

2.8 $\quad h_{\partial D}(a,db)=P_a[x(\mathfrak{m}_{\partial D})\in db] \qquad a\in D,\ db\in\partial D$

is the *classical harmonic measure* of db as viewed from the point a, and, if $G_D(a,b)$ is the *classical Green function* of D, then

2.9 $\quad E_a[\text{measure}\,(t: x(t)\in db,\ t<\mathfrak{m}_{\partial D}]=G_D(a,b)db$[5] $\quad a,b\in D;$

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2 $x(\mathfrak{m})\equiv\infty$ in case $\mathfrak{m}=+\infty$; it is understood that $P_\infty[x(t)\equiv\infty,\ t\geq 0]=1$.
3 private communication.
4 $a\wedge b$ is the smaller of a and b.
5 $E_\cdot(f)=\int f dP_\cdot$.

for the proofs, see J. Doob [5] and G. Hunt [9].

G. Hunt [10] has called a non-negative Borel function p *excessive* on D if

2.10 $E_a[p(x(t)), t < \mathfrak{m}_{\partial D}] \uparrow p(a) \qquad t \downarrow 0, \ a \in D.$

An excessive function can be split into its *greatest harmonic minorant* h plus the *potential* $\int G_D de$ of a non-negative (Riesz) measure e, indeed, Hunt's excessive functions are the same as the *superharmonic* functions of F. Riesz [14]. J. Doob [5] proved that *an excessive function is continuous on the Brownian path* ($t < \mathfrak{m}_{\partial D}$) and that *a potential tends to* 0 *along the Brownian path* ($t \uparrow \mathfrak{m}_{\partial D}$).

3. THE ASSOCIATED MEASURE OF AN ADDITIVE FUNCTIONAL.

Consider an additive functional $\mathfrak{f}$ of the Brownian sample path as described in seition 1, interpreting 1.1 to mean

3.1 $\mathfrak{f}(t, w)$ *is measurable* $\boldsymbol{B}_t$ *for each* $t \geq 0$.

The purpose of this section is *to associate with* $\mathfrak{f}$ *a unique non-negative measure* e *such that, for each bounded open* $D \subset R^d$,

3.2 $1 - p_\alpha = \alpha \int G p_\alpha de \qquad p_\alpha = E_\cdot[e^{-\alpha \mathfrak{f}(\mathfrak{m}_{\partial D})}], \ \alpha > 0,$

where G is the Green funetion of D and the integral is extended over D; *it will follow from* 3.2 *that* e *is smooth.*

Ccnsider, for this purpose, the additive functional

3.3 $\mathfrak{f}_\alpha(t) = \int_0^{t \wedge \mathfrak{m}_{\partial D}} p_\alpha(x(s)) \mathfrak{f}(ds) \qquad t \geq 0,$

and let us begin with the following simple lemmas:

a) $1 - p_\alpha$ *is the potential of a non-negative measure* αe_α.

b) $E_\cdot[\mathfrak{f}_\alpha(\mathfrak{m}_{\partial D})] < +\infty$.

c) $\mathfrak{f}_\alpha \uparrow \mathfrak{f}$ *as* $\alpha \downarrow 0$.

d) $1 - p_\alpha = \alpha E_\cdot[\mathfrak{f}(\mathfrak{m}_{\partial D})] = \alpha \int G de_\alpha$.

e) $E_\cdot\left[\int_0^{\mathfrak{m}_{\partial D}} f(x(s)) \mathfrak{f}_\alpha(ds)\right] = \int G f de_\alpha$ *if* $f \geq 0$ *is a Borel function.*

f) *$p_\alpha^{-1}e_\alpha(db) = e(db)$ is independent of α and of D.*
g) *e is unique.*

Because

3.3 $$E.(1-p_\alpha(x(t)), t < \mathfrak{m}_{\partial D}) = E.(1-e^{-\alpha[\mathfrak{f}(\mathfrak{m}_{\partial D})-\mathfrak{f}(t)]}, t < \mathfrak{m}_{\partial D}) \uparrow 1-p_\alpha \qquad t \downarrow 0,$$

$1-p_\alpha$ is excessive; it is, in fact, a potential thanks to

3.4 $$\int_{\partial \dot{D}} h_{\partial \dot{D}}(a, db)[1-p_\alpha] = E_a(1-e^{-\alpha[\mathfrak{f}(\mathfrak{m}_{\partial D})-\mathfrak{f}(\mathfrak{m}_{\partial \dot{D}})]}) \downarrow 0 \qquad \dot{D} \uparrow D,$$

and this completes the proof of a). As to b), p_α is continuous on the Brownian path because of 1, and, since $\mathfrak{f}$ is continuous, b) follows on letting $n \uparrow +\infty$ in

3.5 $$\begin{aligned} 1-p_\alpha &= E.\left[\int_0^{\mathfrak{m}_{\partial D}} e^{-\alpha[\mathfrak{f}(\mathfrak{m}_{\partial D})-\mathfrak{f}(t)]}\mathfrak{f}(dt)\right] \\ &\geq \alpha E.\left[\sum_{k2^{-n}<\mathfrak{m}_{\partial D}} e^{-\alpha\mathfrak{f}(\mathfrak{m}_{\partial D}(w^+_{k2^{-n}}),\, w^+_{k2^{-n}})} e^{-\alpha\mathfrak{f}(I_k)}\,\mathfrak{f}(I_k)\right] \\ I_k &= [(k-1)2^{-n}, k2^{-n}) \\ &= \alpha E.\left[\sum_{k2^{-n}<\mathfrak{m}_{\partial D}} p_\alpha(x(k2^{-n}))e^{-\alpha\mathfrak{f}(I_k)}\,\mathfrak{f}(I_k)\right]. \end{aligned}$$

Because of c), which is obvious,

3.6 $$\alpha^{-1}(1-p_\alpha) = \lim_{\varepsilon\downarrow 0} E.\left[\int_0^{\mathfrak{m}_{\partial D}} e^{-\alpha[\mathfrak{f}(\mathfrak{m}_{\partial D})-\mathfrak{f}(t)]}\mathfrak{f}_\varepsilon(dt)\right],$$

and, using the method of 3.5 and $E.[\mathfrak{f}_\varepsilon(\mathfrak{m}_{\partial D})] < +\infty$, it appears that

3.7 $$\alpha^{-1}(1-p_\alpha) = \lim_{\varepsilon\downarrow 0} E.\left[\int_0^{\mathfrak{m}_{\partial D}} p_\alpha(x(t))\mathfrak{f}_\varepsilon(dt)\right] = E.\left[\int_0^{\mathfrak{m}_{\partial D}} p_\alpha d\mathfrak{f}\right]$$

proving d).

K. Itô (private communication) pointed out the following neat method for proving e). Choose closed $B \subset D$ such that $e_\alpha(\partial B)=0$ and let e_1 and e_2 be the charge distributions of the potentials $p_1 = E.\left[\int_0^{\mathfrak{m}_{\partial D}} f d\mathfrak{f}_\alpha\right]$ and $p_2 = E.\left[\int_0^{\mathfrak{m}_{\partial D}} (1-f) d\mathfrak{f}_\alpha\right]$, in which f is the indicator function of B. Because p_1 is harmonic *outside* B and differs from $\alpha^{-1}(1-p_\alpha) = E.[\mathfrak{f}_\alpha(\mathfrak{m}_{\partial D})]$ by a harmonic function *inside* B, e_1 is not smaller than the restriction of e_α to B, and, for the same reasons, e_2 is not smaller than the restriction of e_α to $D-B$. But $p_1+p_2=\alpha^{-1}(1-p_\alpha)$, whence

$$3.8 \qquad p_1 = E.\left[\int_0^{\mathfrak{m}_{\partial D}} f d\mathfrak{f}_\alpha\right] = \int_B G de_\alpha ,$$

and since such figures B generate the class of Borel subsets of D, e) follows.

As to f), $p_\alpha > 0$ because $\mathfrak{f}(\mathfrak{m}_{\partial D}) < +\infty$, and, choosing $0<\beta<\alpha$, e) implies

$$3.9 \qquad \int G de_\alpha = E.\left[\int_0^{\mathfrak{m}_{\partial D}} p_\alpha / p_\beta d\mathfrak{f}_\beta\right] = \int G p_\alpha / p_\beta de_\beta ,$$

i.e., $de \equiv p_\alpha^{-1} de_\alpha$ is indeqendent of α; it is also independent of D because if $\dot{D} \supset D$ and if $p_\alpha \equiv 0$ outside D, then, with an obvious notation,

$$\begin{aligned} 3.10 \qquad \int G p_\alpha \dot{p}_\beta de &= E.\left[\int_0^{\mathfrak{m}_{\partial D}} p_\alpha d\dot{\mathfrak{f}}_\beta\right] \\ &= E.\left[\int_0^{\mathfrak{m}_{\partial \dot{D}}} p_\alpha d\dot{\mathfrak{f}}_\beta\right] - E.\left[\int_{\mathfrak{m}_{\partial D}}^{\mathfrak{m}_{\partial \dot{D}}} p_\alpha d\dot{\mathfrak{f}}_\beta\right] \\ &= \int \dot{G} p_\alpha \dot{p}_\beta d\dot{e} - \int dh_{\partial D} \int \dot{G} p_\alpha \dot{p}_\beta d\dot{e} \\ &= \int G \dot{p}_\alpha p_\beta d\dot{e} . \end{aligned}$$

g) is immediate from 3.2.

To establish the *smoothness* of e, take bounded open D and put $B_n = D \cap (p_1 \geq n^{-1})$. Because $1-p_1$ is a potential, B_n is closed in D and increases to D as $n \uparrow +\infty$; moreover, according to 3.2,

$$3.11 \qquad \int_{B_n} G de \leq n \int G p_1 de = n(1-p_1) \leq n ,$$

which is 1.8a, and, since, along the Brownian path, $0<p_1$ is continuous and tends to 1 $(t \uparrow \mathfrak{m}_{\partial D})$,

$$\begin{aligned} 3.12 \qquad & P.[\inf (t : x(t) \notin B_n) < \mathfrak{m}_{\partial D}] \\ &= P.[\inf_{t<\mathfrak{m}_{\partial D}} p_1(x(t)) < n^{-1}] \downarrow 0 \qquad n \uparrow +\infty , \end{aligned}$$

which verifies 1.8b.

4. UNIQUENESS.

The following simple lemma is useful in later sections: *two additive functionals* $\mathfrak{f}_1$ and $\mathfrak{f}_2$ *with the same bounded mean*

4.1 $$p = E.[\mathfrak{f}(\mathfrak{m}_{\partial D})] = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}(1-p_\varepsilon) = \int Gde \qquad \mathfrak{f} = \mathfrak{f}_1, \mathfrak{f}_2$$

are the same for $t \leq \mathfrak{m}_{\partial D}$.

An argument similar to 3.5 implies

4.2 $$E.\left(\int_0^{\mathfrak{m}_{\partial D}} [\mathfrak{f}_i(\mathfrak{m}_{\partial D}) - \mathfrak{f}_i(t)]\mathfrak{f}_k(dt)\right)$$
$$= E.\left(\int_0^{\mathfrak{m}_{\partial D}} p d\mathfrak{f}_k\right)$$
$$= \int Gpde \leq \|p\|_\infty^2 < +\infty ;$$

thus, putting $\mathfrak{f} = \mathfrak{f}_2 - \mathfrak{f}_1$,

4.3 $$E.[\mathfrak{f}(\mathfrak{m}_{\partial D})^2]$$
$$= 2E.\left[\int_0^{\mathfrak{m}_{\partial D}} [\mathfrak{f}(\mathfrak{m}_{\partial D}) - \mathfrak{f}(t)]\mathfrak{f}(dt)\right]$$
$$= 0,$$

and hence

4.4 $$0 = E.[\mathfrak{f}(\mathfrak{m}_{\partial D}) \mid \boldsymbol{B}_{t \wedge \mathfrak{m}_{\partial D}+}] = E.[\mathfrak{f}(\mathfrak{m}_{\partial D}) - \mathfrak{f}(t) \mid \boldsymbol{B}_{t \wedge \mathfrak{m}_{\partial D}+}] + \mathfrak{f}(t)$$
$$= \mathfrak{f}(t) \qquad t \leq \mathfrak{m}_{\partial D}$$

as desired.

It is a simple matter to deduce from this that *two additive functionals with the same associated measure are the same*; indeed, the difference p of two solutions of 3.2 satisfies $-p = \alpha \int Gpde$, which implies

4.5 $$0 \geq -\int p^2 de = \mathfrak{E}(pde) = \int Gpdepde,$$

and it follows that two additive functionals with the same associated measure have the same p_α, hence the same $\alpha^{-1}(1-p_\alpha) = E.[\mathfrak{f}_\alpha(\mathfrak{m}_{\partial D})]$ ($< \alpha^{-1} < +\infty$), and hence the same $\mathfrak{f}_\alpha$. But this means that the two functionals are the same up to time $\mathfrak{m}_{\partial D}$, and, to finish the proof, it is enough to make D swell out to R^d.

5. CONSTRUCTION OF AN ADDITIVE FUNCTIONAL FROM ITS ASSOCIATED MEASURE.

Given a *smooth* non-negative measure e, *our task is to find an additive functional* $\mathfrak{f}$ *with* e *as its associated measure.*

Consider, for this purpose, a non-negative measure e on a bounded open figure D with *bounded potential* $p=\int Gde$ and *finite energy* $\mathfrak{E}(e)=\int Gdede$, let p_n be the potential $\int Gf_n db$ of

5.1 $$f_n = n(p - E_\bullet[p(x(n^{-1})),\ n^{-1} < \mathfrak{m}_{\partial D}]),$$
let

5.2 $$\mathfrak{f}_n(t) = \int_0^{t \wedge \mathfrak{m}_{\partial D}} f_n(x(s))ds,$$

and let us construct a functional $\mathfrak{f}$ associated with e as a limit of $\mathfrak{f}_n$ with the aid of the following simple lemmas:

a) *p is a Brownian excessive function*; *esp.* $0 \leq f_n$.

b) $E_\bullet[p(x(t)),\ t < \mathfrak{m}_{\partial D}] \downarrow 0$ *inside* D *as* $t \uparrow +\infty$.

c) $p_n = n\int_0^{n^{-1}} E_\bullet[p(x(t)),\ t < \mathfrak{m}_{\partial D}]dt$ *increases to* p *inside* D *as* $n \uparrow +\infty$.

d) $\lim_{n\uparrow+\infty} \mathfrak{E}(e - f_n db) = 0$, *where* $\mathfrak{E}$ *is the energy* $\mathfrak{E}(e) \equiv \int Gdede$.

e) $$\begin{aligned} E_\bullet[\mathfrak{f}_n(+\infty) \mid \boldsymbol{B}_{t\wedge \mathfrak{m}_{\partial D}+}] &\equiv \mathfrak{l}_n(t) \\ &= p_n(x(t)) + \mathfrak{f}_n(t) \qquad \mathfrak{m}_{\partial D} > t \geq 0 \\ &= \mathfrak{f}_n(\mathfrak{m}_{\partial D}) \qquad\qquad \mathfrak{m}_{\partial D} \leq t, \end{aligned}$$
i.e., $\mathfrak{l}_n$ *is a martingate with respect to the fields* $\boldsymbol{B}_{t\wedge\mathfrak{m}_{\partial D}+}$; *moreover*, $\mathfrak{l}_n$ *is continuous in* t.

f) $$P_\bullet[\max_{t\geq 0} |\mathfrak{l}_n(t, w_s^+) - \mathfrak{l}_m(t, w_s^+)| > \varepsilon] \leq \textit{constant} \times \varepsilon^{-2} s^{-d/2} \sqrt{\mathfrak{E}(f_n db - f_m db)}.$$

g) $P_\bullet[\lim_{n\uparrow+\infty} \mathfrak{f}_n(t, w_s^+) = \mathfrak{f}(t, w_s^+),\ t \geq s > 0] = 1$.
where the limit is taken as $n \uparrow +\infty$ *via suitable* $n_1 < n_2 <$ *etc.*, $\mathfrak{f}(t)$ *is continuous*, $\mathfrak{f}(0) \equiv \mathfrak{f}(0+) = 0$, *and* $\mathfrak{f}(t) = \mathfrak{f}(s) + \mathfrak{f}(t-s, w_s^+)$ $(\mathfrak{m}_{\partial D} \geq t \geq s)$.

h) $E_\bullet[\mathfrak{f}(\mathfrak{m}_{\partial D})] = p$.

Because p is a potential, it is excessive; a) is obvious from this, b) is obvious from the bound

5.3 $$E_{\bullet}[p(x(t)), t<\mathfrak{m}_{\partial D}]\leq \|p\|_{\infty}P_{\bullet}(t<\mathfrak{m}_{\partial D})\downarrow 0 \qquad t\uparrow +\infty,$$

and c) follows from b):

5.4 $$\begin{aligned} p_n &= E_{\bullet}\left[\int_0^{\mathfrak{m}_{\partial D}} f_n(x(s))ds\right] \\ &= n\int_0^{+\infty} dsE_{\bullet}[p(x(s))-E_{x(s)}[p(x(n^{-1})), n^{-1}<m_{\partial D}], s<\mathfrak{m}_{\partial D}] \\ &= n\int_0^{+\infty} ds[E_{\bullet}[p(x(s)), s<\mathfrak{m}_{\partial D}]-E_{\bullet}[p(x(s+n^{-1})), s+n^{-1} \\ &\quad <\mathfrak{m}_{\partial D}]] \\ &= n\int_0^{n^{-1}} E_{\bullet}[p(x(s)), s<\mathfrak{m}_{\partial D}]ds\uparrow p \qquad n\uparrow +\infty . \end{aligned}$$

An application of c) establishes

5.5 $$\mathfrak{E}(e) = \int pde \geq \int p_n de = \int pf_n db \geq \mathfrak{E}(f_n db)\uparrow \mathfrak{E}(e) \qquad n\uparrow +\infty,$$

and this implies d):

5.6 $$\lim_{n\uparrow +\infty} \mathfrak{E}(e-f_n db) = \lim_{n\uparrow +\infty} [\mathfrak{E}(e)-2\int p_n de+\mathfrak{E}(f_n db)] = 0 .$$

Because $t\wedge\mathfrak{m}_{\partial D}$ is a Markov time,

5.7 $$\begin{aligned} E_{\bullet}[\mathfrak{f}_n(\mathfrak{m}_{\partial D})\,|\,B_{t\wedge\mathfrak{m}_{\partial D}+}] &= E_{\bullet}\left[\int_{t\wedge\mathfrak{m}_{\partial D}}^{\mathfrak{m}_{\partial D}} f_n(x(s))ds\,|\,B_{t\wedge\mathfrak{m}_{\partial D}+}\right]+\mathfrak{f}_n(t) \\ &= E_{x(t\wedge\mathfrak{m}_{\partial D})}\left[\int_0^{\mathfrak{m}_{\partial D}} f_n ds\right]+\mathfrak{f}_n(t) = p_n(x(t))+\mathfrak{f}_n(t)\equiv\mathfrak{l}_n(t) \quad t<\mathfrak{m}_{\partial D}, \end{aligned}$$

i.e., $\mathfrak{l}_n$ is martingale, and since p_n is continuous and tends to 0 along the Brownian path ($t\uparrow\mathfrak{m}_{\partial D}$), $\mathfrak{l}_n$ is continuous. e) is now established, and f) follows from Doob's submartingale extension of Kolmogorov's inequality [4], the Schwarz inequality

5.8 $$\left(\int Gde_1de_2\right)^2\leq \mathfrak{E}(e_1)\mathfrak{E}(e_2)$$

(see H. Cartan [2]), and the resulting

5.9 $$\begin{aligned} &E_{\bullet}[\,|\mathfrak{l}_n(+\infty, w_s^+)-\mathfrak{l}_m(+\infty, w_s^+)|^2, s<\mathfrak{m}_{\partial D}] \\ &\leq E_{\bullet}[E_{x(s)}[\,|\mathfrak{f}_n(\mathfrak{m}_{\partial D})-\mathfrak{f}_m(\mathfrak{m}_{\partial D})|^2], s<\mathfrak{m}_{\partial D}] \\ &\leq E_{\bullet}\left[\int G(x(s), b)(f_n-f_m)(p_n-p_m)db, s<\mathfrak{m}_{\partial D}\right] \end{aligned}$$

$$= \iint G(a, b) e_{nm}(da) E_{\bullet}[G(x(s), b), s < \mathfrak{m}_{\partial D}] e_{nm}(db) \qquad e_{nm}(db) = (f_n - f_m)db$$
$$\leq \mathfrak{E}(e_{nm})^{1/2} \mathfrak{E}(E_{\bullet}[G(x(s), b), s < \mathfrak{m}_{\partial D}] de_{nm})^{1/2}$$
$$\leq \mathfrak{E}(e_{nm})^{1/2} \text{ constant } \times s^{-d/2} \mathfrak{E}(e_{nm})^{1/2}.$$

Choose $n_1 < n_2 <$ etc. so as to make $P_{\bullet}[\max_{t \geq 0} | \mathfrak{l}_n(t, w_s^+) - \mathfrak{l}_s(t) | \downarrow 0, s > 0] = 1$, where $\mathfrak{l}_s$ is a continuous function of t. Because $p = \lim_{n \uparrow \infty +} p_n$ is continuous and tends to 0 along the Brownian path $(t \uparrow \mathfrak{m}_{\partial D})$, it follows that $\mathfrak{f}_s(t) = \lim_{n \uparrow +\infty} \mathfrak{f}_n(t, w_s^+)$ is continuous $(t \geq 0)$ and additive $(t \leq \mathfrak{m}_{\partial D}(w_s^+))$; moreover

5.10 $$E_{\bullet}[\mathfrak{f}_n(+\infty)^2] = 2 \int G p_n f_n db \leq 2 \| p \|_\infty^2 < +\infty$$

implies

5.11 $$E_{\bullet}[\mathfrak{f}_s(t-s)] = \lim_{n \uparrow +\infty} E_{\bullet}[\mathfrak{f}_n(+\infty, w_s^+) - \mathfrak{f}_n(+\infty, w_t^+)]$$
$$= \lim_{n \uparrow +\infty} [E_{\bullet}(p_n(x(s)), s < \mathfrak{m}_{\partial D}) - E_{\bullet}(p_n(x(t)), t < \mathfrak{m}_{\partial D})]$$
$$= E_{\bullet}(p(x(s)), s < \mathfrak{m}_{\partial D}) - E_{\bullet}(p(x(t)), t < \mathfrak{m}_{\partial D}),$$

and, to finish the proof of g) and h), it is enough to define

5.12 $$\mathfrak{f}(t) = \lim_{s \downarrow 0} \mathfrak{f}_s(t)$$

and to make $s \downarrow 0$ and $t \uparrow +\infty$ in 5.11.

Now take a smooth measure e, choose $B_n \uparrow D$ as needed for 1.8 with the additional property that $e | B_n$ has *finite energy* $\mathfrak{E}(e | B_n) = \int_{B_n \times B_n} G de de$[6], let $\mathfrak{f}_n$ be the additive functional associated with $e | B_n$ as in 5.12 above, and note that

5.13 $$E_{\bullet}\left[\int_0^{\mathfrak{m}_{\partial D}} f d\mathfrak{f}_n\right] = \int_{B_n} G f de \qquad f \geq 0$$

as in e) of section 3. It follows that if f is the indicator function

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[6] $\mathfrak{E}(e | B_n) < +\infty$ is achieved as follows: take $\dot{D} \supset D$ with $\partial \dot{D}$ at a positive distance from ∂D, choose $\dot{B}_n \uparrow \dot{D}$ as needed for 1.8, and let $B_n = \dot{B}_n \cap D$; then $\mathfrak{E}(e | B_n) \leq \int_{B_n} de \int_{B_n} \dot{G} de \leq n e(B_n) \leq n (\inf_{D \times D} \dot{G})^{-1} \sup_D \int_{B_n} \dot{G} de \leq n^2 (\inf_{D \times D} \dot{G})^{-1} < +\infty$.

of B_m, then the functionals $\int_0^t f d\mathfrak{f}_n$ $(n>m)$ and $\int_0^t f d\mathfrak{f}_m$ have the same (bounded) mean:

$$5.14 \qquad E.\left[\int_0^{\mathfrak{m}_{\partial D}} f d\mathfrak{f}_n\right] = E.\left[\int_0^{\mathfrak{m}_{\partial D}} f d\mathfrak{f}_m\right] = \int_{B_m} G de$$

and are therefore identical up to time $\mathfrak{m}_{\partial D}$ according to the first uniqueness lemma for additive functionals of section 4. But this means that $\mathfrak{f}_n = \mathfrak{f}_m$ up to the exit time from B_m, and since this exit time $= \mathfrak{m}_{\partial D}$ for large m, it is legitimate to define a functional $\mathfrak{f}$ for $t \leq \mathfrak{m}_{\partial D}$ by means of

$$5.15 \qquad \mathfrak{f}(t) = \mathfrak{f}_n(t) \qquad t \leq \inf(t: x(t) \notin B_n),\ n \geq 1 .$$

Introducing $p_\alpha = E.[e^{-\alpha \mathfrak{f}(\mathfrak{m}_{\partial D})}]$ and using the method of 3.5, 5.13 implies

$$5.16 \qquad 1-p_\alpha = \lim_{n \uparrow +\infty} \alpha E.\left[\int_0^{\mathfrak{m}_{\partial D}} e^{-\alpha[\mathfrak{f}(\mathfrak{m}_{\partial D}) - \mathfrak{f}(t)]} \mathfrak{f}_n(dt)\right]$$

$$= \lim_{n \uparrow +\infty} \alpha E.\left[\int_0^{\mathfrak{m}_{\partial D}} p_\alpha d\mathfrak{f}_n\right] = \lim_{n \uparrow +\infty} \alpha \int_{B_n} G p_\alpha de = \alpha \int G p_\alpha de ;$$

in brief, e is the measure associated with $\mathfrak{f}$ as in 3.2.

Because two additive functionals with same associated measure are the same, it follows that if e is smooth and if $D_1 \subset D_2 \subset$ *etc.* swell out to R^d, then the functionals $\mathfrak{f}_n$ associated with $e_n = e | D_n$ as in 5.15 above determine an additive functional

$$5.17 \qquad \mathfrak{f}(t) = \mathfrak{f}_n(t) \qquad t < \mathfrak{m}_{\partial D_n},\ n \geq 1$$

depending upon e alone, for which 3.2 holds for each D.

$\mathfrak{f}$ is additive for *almost all* Brownian paths, but of course it can be modified on a negligeable class of paths so as to be Borel in the pair (t, w) and to satisfy 1.1, 1.2, 1.3, 1.4, as *identities*; with *this modification*, $\mathfrak{f}$ *is the additive functional associated with e.*

6. DIFFUSIONS WITH BROWNIAN HITTING PROBABILITIES.

To avoid confusion, let w, x, $\mathfrak{m}$, B, $\boldsymbol{B}$, $\boldsymbol{B}_{\mathfrak{m}+}$, *etc.* be used to describe the Brownian motion, introduce the same paths, times, events, and fields with the new names $\dot{w}$, $\dot{x}$, $\dot{\mathfrak{m}}$, $\dot{B}$, $\dot{\boldsymbol{B}}$, $\dot{\boldsymbol{B}}_{\dot{\mathfrak{m}}+}$, *etc.*, and take a *new motion* $\dot{\boldsymbol{D}}$ with probabilities $\dot{P}_a(\dot{B})$.

$\dot{D}$ is said to be a *diffusion* if it *starts afresh* at each Markov time $\mathfrak{m}$:

6.1 $\dot{P}_a[\dot{w}^+_{\mathfrak{m}} \in \dot{B} \mid \dot{\boldsymbol{B}}_{\mathfrak{m}+}] = \dot{P}_b(\dot{B}) \qquad a \in E^d,\ \dot{B} \in \dot{\boldsymbol{B}},\ b \equiv \dot{x}(\mathfrak{m})$;

it is said to have Brownian *hitting probabilities* if, for each bounded open D,

6.2 $\dot{P}_a[\dot{x}(\dot{\mathfrak{m}}_{\partial D}) \in db] = P_a[x(\mathfrak{m}_{\partial D}) \in db] = h_{\partial D}(a, db)$
$a \in D,\ db \subset \partial D$.

Given such a diffusion with Brownian hitting probabilities

6.3a $\dot{p}_\alpha \equiv \dot{E}_\bullet[e^{-\alpha \dot{\mathfrak{m}}_{\partial D}}]$[7]

solves

6.3b $1 - \dot{p}_\alpha = \alpha \int G\dot{p}_\alpha de$,

where $e \geq 0$ is independent of α and of D. e is the so-called *speed measure* of the diffusion; the speed measure of the Brownian motion is the Lebesgue measure db. Because of 6.2, $\dot{P}_\bullet[\dot{\mathfrak{m}}_{\partial D} < +\infty] = 1$ and $\dot{p}_\alpha > 0$, which implies that *e is smooth, i.e., that it is the associated measure of some additive functional $\mathfrak{f}$* of the Brownian path; moreover, *e is positive on the neighborhoods of H. Cartan's fine topology, and this implies that its associated additive functional satisfies*

6.4 $P_\bullet[\mathfrak{f}(s) < \mathfrak{f}(t),\ 0 \leq s < t] = 1$.

Introducing the *inverse function* $\mathfrak{f}^{-1}$ of $\mathfrak{f}$, it turns out that

6.5 $\dot{P}_\bullet(\dot{B}) = P_\bullet[x(\mathfrak{f}^{-1}) \in \dot{B}] \qquad \dot{B} \in \dot{\boldsymbol{B}}$;

in brief, $\dot{D}$ is identical in law to the Brownian motion run with the stochastic clock $\mathfrak{f}^{-1}$; moreover, *each smooth measure e, positive on fine neighborhoods, is the speed measure of a diffusion with Brownian hits.*

The proofs are carried out in the next few sections.

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[7] $\dot{E}_\bullet(f) = \int f d\dot{P}_\bullet$.

7. SPEED MEASURES

Beginning with the speed measure e of 6.3b, if a $a\in R^d$ and if $\mathfrak{m}_\varepsilon=\min(t: |\dot{x}(t)-a|\geq\varepsilon)$, then $\mathfrak{m}_{0+}=\lim_{\varepsilon\downarrow 0}\mathfrak{m}_\varepsilon$ satisfies

$$7.1 \qquad \dot{E}_a(e^{-\mathfrak{m}_{0+}}) = \dot{E}_a[e^{-\mathfrak{m}-\mathfrak{m}(w^{+}_{\mathfrak{m}})}] = \dot{E}_a(e^{-\mathfrak{m}_{0+}})^2, \qquad \mathfrak{m}=\mathfrak{m}_{0+}$$

and since $\dot{E}_a(e^{-\mathfrak{m}_{0+}})=0$ implies $\dot{P}_a[\mathfrak{m}_{0+}=+\infty]=1$, violating the fact that the dot motion has Brownian hitting probabilities, it follows that $\dot{P}_a[\mathfrak{m}_{0+}=0]=1$.

Because

$$7.2 \qquad 1-\dot{p}_\alpha = \alpha\dot{E}_\cdot\left[\int_0^{\mathfrak{m}_{\partial D}} e^{-\alpha(\mathfrak{m}_{\partial D}-t)}\,dt\right] = \alpha\dot{E}_\cdot\left[\int_0^{\mathfrak{m}_{\partial D}} \dot{p}_\alpha(x(t))\,dt\right],$$

$1-\dot{p}_\alpha$ satisfies

$$7.3 \qquad 1-\dot{p}_\alpha(a)\geq\alpha\dot{E}_a\left[\int_{\mathfrak{m}_{\partial\dot{D}}}^{\mathfrak{m}_{\partial D}}\dot{p}_\alpha\,dt\right] = \int h_{\partial\dot{D}}(a,db)(1-\dot{p}_\alpha) \qquad a\in\dot{D}\subset D,$$

and, using 7.3 and $\dot{P}_a[\mathfrak{m}_{0+}=0]=1$ to establish

$$7.4\text{a} \qquad \int h_{\partial\dot{D}}(a,db)(1-\dot{p}_\alpha)\downarrow 0 \qquad \dot{D}\uparrow D$$

$$7.4\text{b} \qquad \int h_{|b-a|=\varepsilon}(a,db)(1-\dot{p}_\alpha)\uparrow 1-\dot{p}_\alpha \qquad \varepsilon\downarrow 0,$$

it appears that $1-\dot{p}_\alpha$ is the potential $\alpha\int Gde_\alpha$ of a non-negative charge distribution αe_α

It remains to verify that $\dot{p}_\alpha^{-1}de_\alpha\equiv de$ is independent of α and of D, which is done with the aid of the additive functional $\dot{\mathfrak{f}}_\alpha(t) =\int_0^t\dot{p}_\alpha(\dot{x}(s))ds$ $(t<\mathfrak{m}_{\partial D})$ and the method of section 3; in outline,

$$7.5 \qquad 1-\dot{p}_\alpha = \alpha\dot{E}_\alpha[\dot{\mathfrak{f}}_\alpha(\mathfrak{m}_{\partial D})] = \alpha\int Gde_\alpha$$

implies

$$7.6 \qquad \dot{E}\left[\int_0^{\mathfrak{m}_{\partial D}} f(\dot{x}(s))\dot{\mathfrak{f}}_\alpha(ds)\right] = \int Gfde_\alpha$$

as in e) of section 3, and 3.9 follows as before, *etc.*.

8. TWO DIFFUSIONS WITH BROWNIAN HITS AND THE SAME SPEED MEASURE ARE THE SAME.

Consider a pair of diffusions with Brownian hitting probabilities and the *same* speed measure.

Because 6.3b=3.2 has unique solutions as noted in section 4, both diffusions have the same $\dot{p}_\alpha = \dot{E}_\cdot[e^{-\alpha \mathfrak{m}_{\partial D}}]$ and hence the same

8.1 $$\dot{G}_{0+} : f \to \dot{E}_\cdot\left[\int_0^{\mathfrak{m}_{\partial D}} f(\dot{x})\dot{\mathfrak{f}}_\varepsilon(dt)\right] = \int G f \dot{p}_\varepsilon de$$

$$\dot{\mathfrak{f}}_\varepsilon(t) \equiv \int_0^t \dot{p}_\varepsilon(\dot{x}(s))ds \qquad t \leq \mathfrak{m}_{\partial D}.$$

Choose $\varepsilon > 0$ and introduce the Green operators

8.2 $$\dot{G}_\alpha : f \to \dot{E}_\cdot\left[\int_0^{\mathfrak{m}_{\partial D}} e^{-\alpha \dot{\mathfrak{f}}_\varepsilon(t)} f(\dot{x})\dot{\mathfrak{f}}_\varepsilon(dt)\right] \qquad \alpha > 0;$$

then

8.3 $$\alpha \dot{G}_{0+}\dot{G}_\alpha f = \alpha \dot{E}_\cdot\left[\int_0^{\mathfrak{m}_{\partial D}} \dot{\mathfrak{f}}_\varepsilon(dt)(\dot{G}_\alpha f)(\dot{x})\right]$$

$$= \alpha \dot{E}_\cdot\left[\int_0^{\mathfrak{m}_{\partial D}} \dot{\mathfrak{f}}_\varepsilon(dt)\int_t^{\mathfrak{m}_{\partial D}} e^{-\alpha[\dot{\mathfrak{f}}_\varepsilon(s)-\dot{\mathfrak{f}}_\varepsilon(t)]} f(\dot{x}(s))\dot{\mathfrak{f}}_\varepsilon(ds)\right]$$

$$= \alpha \dot{E}_\cdot\left[\int_0^{\mathfrak{m}_{\partial D}} e^{-\alpha \dot{\mathfrak{f}}_\varepsilon(s)} f(\dot{x})\dot{\mathfrak{f}}_\varepsilon(ds)\int_0^s e^{\alpha \dot{\mathfrak{f}}_\varepsilon} d\dot{\mathfrak{f}}_\varepsilon\right]$$

$$= \dot{E}_\cdot\left[\int_0^{\mathfrak{m}_{\partial D}} (1-e^{-\alpha \dot{\mathfrak{f}}_\varepsilon}) f d\dot{\mathfrak{f}}_\varepsilon\right] = \dot{G}_{0+}f - \dot{G}_\alpha f,$$

i.e.,

8.4 $$\dot{G}_\alpha = \dot{G}_{0+} - \alpha \dot{G}_{0+}\dot{G}_\alpha,$$

and, using the bound

8.5 $$\dot{G}_{0+}1 = \int G\dot{p}_\varepsilon de = \varepsilon^{-1}(1-\dot{p}_\varepsilon) \leq \varepsilon^{-1} < +\infty,$$

the obvious iteration of 8.4 implies

8.6 $$\dot{G}_\alpha = \sum_{n\geq 0}(-)^n \alpha^n \dot{G}_{0+}^{n+1} \qquad \alpha < \varepsilon.$$

Because $\dot{G}_\alpha$ is a Laplace transform in its dependence on α, 8.6 implies that both diffusions have the same Green operators ($\alpha \geq 0$) and hence the same

8.7 $$\dot{E}_\cdot\left[\int_0^{\mathfrak{m}_\infty} e^{-\alpha t} f(\dot{x})dt\right] = \lim_{D\uparrow R^d}\lim_{\varepsilon\downarrow 0} \dot{G}_\alpha f.$$

But this implies that both diffusions have the same

8.8 $$\dot{P}_a[\dot{x}(t) \in db, t < \mathfrak{m}_\infty] \qquad (t, a, b) \in (0, +\infty)\times R^{2d}$$

and hence are the same in all respects as stated in the section title.

9. SPEED MEASURES ARE POSITIVE ON FINE NEIGHBORHOODS.

Given a point a of bounded open D and a (Brownian) excessive function p on D, the set of points $b \in D$ at which $|p(b)-p(a)|<n^{-1}$ is said to be a *fine neighborhood* of a; the corresponding topology is called the *fine topology of H. Cartan* [2]. E. B. Dynkin [7] has pointed out that a point a is in the fine interior of $B \subset R^d$ if and only if almost all Brownian paths starting at a remain in B for some positive time. Because an excessive function in 1-dimension is concave and hence continuous, the 1-dimensional *fine* topology is the same as the usual one; in higher dimensions it is different.

Given a diffusion with Brownian hitting probabilities, *its speed measure e has to be positive on fine neighborhoods*; for the proof, it is enough to verify that if $Z \subset R^d$ is bounded and Borel and if $e(Z)=0$, then Z has no fine interior.

Choose an open ball D and a decreasing series of open figures D_n so as to have

9.1a $\quad D \supset D_1 \supset D_2 \supset etc. \supset Z$

9.1b $\quad \int_{D_n} G_D \dot{p}_1 de \downarrow 0 \qquad n \uparrow +\infty, \ \dot{p}_1 = \dot{E}_\cdot[e^{-\mathfrak{m}_{\partial D}}],$

and let

9.2 $\quad \dot{\mathfrak{f}}_1(t) = \int_0^t \dot{p}_1(\dot{x}(s))ds \qquad t \leq \mathfrak{m}_{\partial D}.$

Because of

9.3a $\quad 1-\dot{E}_a[e^{-\dot{\mathfrak{f}}_1(\partial \mathfrak{m}_{D_n})}] = \int_{D_n} G_{D_n} \dot{p}_1 de \leq \int_{D_n} G_D \dot{p}_1 de \downarrow 0$

$$n \uparrow +\infty, \ a \in Z$$

9.3b $\quad \dot{p}_1 > 0,$

it is apparent that

9.4a
$$\begin{aligned} 1 &= \dot{P}_a[\mathfrak{m}_{\partial D_n} \downarrow 0, \ n \uparrow +\infty] \\ &= \dot{P}_a[\lim_{n\uparrow+\infty} \dot{x}(\mathfrak{m}_{\partial D_n}) = a] \\ &= \lim_{n\uparrow+\infty} \int h_{\partial D_n}(a, db) e^{-|b-a|} \qquad a \in Z. \end{aligned}$$

But this final expression depends upon the (Brownian) hitting probabilities $h_{\partial D}$ alone; thus,

9.4b $\quad P_a[\lim_{n\uparrow+\infty} x(\mathfrak{m}_{\partial D}) = a] = 1 \qquad a \in Z$

for the Brownian motion also, and since almost no Brownian path meets its starting point at a *positive* time, it follows that

9.5 $\quad P_a[\mathfrak{m}_{\partial D_n} \downarrow 0, n \uparrow +\infty] = 1 \qquad a \in Z;$

an application of Dynkin's description of fine neighborhoods completes the proof.

10. SPEED MEASURES GIVE RISE TO INCREASING ADDITIVE FUNCTIONALS.

Because the speed measure of a diffusion with Brownian hits is smooth, it has associated with it an additive functional $\mathfrak{f}$ of the Brownian path; moreover, e is positive on fine neighborhoods, and this is reflected in the fact that $\mathfrak{f}$ *is increasing as in* 6.4: $\mathfrak{f}(s) < \mathfrak{f}(t) (s < t)$.

It will be enough to verify that the set A of points a at which

10.1 $\quad P_a(\mathfrak{m} > 0) = P_a[\mathfrak{f}(t) = 0 \text{ for some } t > 0] = 1$
$\qquad \mathfrak{m} = \inf(t: f(t) > 0)$

is vaccuous; note that A is Borel and that $P_{\bullet}(\mathfrak{m} > 0) = 0$ or 1 according to Blumenthal's 01 law.

Because A is either *void* or *fine open* and e is positive on fine neighborhoods, it is enough to show that $e(A) = 0$, and, for this, it is enough to show that for each bounded open D, the points of D at which $P_{\bullet}[\mathfrak{f}(\mathfrak{m}_{\partial D}) > 0] < 1$ have e-mass 0. But this is immediate on letting $\alpha \uparrow +\infty$ in

10.2 $\quad \alpha^{-1}(1-p_\alpha) = \int G p_\alpha de \qquad p_\alpha = E_{\bullet}[e^{-\alpha \mathfrak{f}(\mathfrak{m}_{\partial D})}].$

11. WHEN IS $\mathfrak{f}(+\infty) = +\infty$?

In making up the stochastic clock $\mathfrak{f}^{-1}$ for use in 6.5, two cases arise according as $\mathfrak{f}(+\infty) = +\infty$ or not, and it is desirable to have a test for this.

Because $p \equiv P_{\bullet}[\mathfrak{f}(+\infty) < +\infty]$ satisfies

11.1 $$p(a) = P_a[\mathfrak{f}(\mathfrak{m}_{\partial D}) < +\infty, \mathfrak{f}(+\infty, w^+_{\mathfrak{m}_{\partial D}}) < +\infty]$$
$$= E_a[p(x(\mathfrak{m}_{\partial D}))] = \int h_{\partial D}(a, db)p(b) \qquad a \in D$$

for bounded open D, it is harmonic and since $p \geq 0$, it must be constant; moreover, letting first t and then $n \uparrow +\infty$ in

11.2 $$P.[\mathfrak{f}(+\infty) < n] \leq E.[\mathfrak{f}(t) < n, p(x(t))] \leq P.[\mathfrak{f}(t) < n]p,$$

it appears that $p \leq p^2$ and hence that $p = 0$ or 1.

Our test states that $p = 1$ *if and only if one of the following conditions is met*:

a) $d \geq 3$ *and* $1 - p_1 = \int G p_1 de$ *admits a solution* $0 < p_1 \leq 1$ *on the whole of* R^d, *where G is the function of* R^d.

b) $\mathfrak{G} p_1 = p_1$ *admits a solution* $0 < p_1 \leq 1$ *on the whole of* R^d, *where* $\mathfrak{G} p_1$ *is the negative of the Radon-Nikodym derivative of the Riesz measure of* p_1 *with respect to e* (see section 13 for the meaning of $\mathfrak{G}$).

c) $d \geq 3$ *and* $R^d = A \cup B$, *where A is thin at* ∞ *in the sense that* $P.[x(t) \in A$ *for some* $t > n] \downarrow 0$ $(n \uparrow \infty)$, *and* $\int_B G de < +\infty$.[8]

Beginning with $p = 1$, $p_1 \equiv E.[e^{-\mathfrak{f}(+\infty)}]$ is positive and ≤ 1, and, since $G_D \uparrow G$ as $D \uparrow R^d$,

11.3 $$1 - p_1 = \lim_{D \uparrow R^d} E.[e^{-\mathfrak{f}(+\infty, w^+_{\mathfrak{m}_{\partial D}})}] - p_1$$
$$= \lim_{D \uparrow R^d} \int G_D p_1 de = \int G p_1 de.$$

Because $G \equiv +\infty$ in case $d = 2$, it follows that $p = 1$ implies a), that a) implies b) is clear, that c) implies $p = 1$ is evident from

11.4 $$E.\left[\int_0^{+\infty} f d\mathfrak{f}\right] = \int_B G de < +\infty,$$

where f is the indicator function of B, and now it remains to verify that b) implies c).

But, if $0 < p_1 \leq 1$ is a solution of $\mathfrak{G} p_1 = p_1$ and if h_1 is its

[8] $A \subset R^d (d \geq 3)$ is thin at ∞ if and only if (Wiener's test) $\sum_{n \geq 1} 2^{-n(d-2)} C(A_n) < +\infty$, where A_n is the meet of A with the spherical $2^{n-1} \leq |b| < 2^n$ and C is the d-dimensional Newtonian capacity; for the proof in the case of the d-dimensional random walk, see K. Itô and H. P. McKean, Jr. [12].

harmonic part $E_{\bullet}[p_1(x(\mathfrak{m}_{\partial D}))]$ inside D, then $h_1 - p_1 = \int G_D p_1 de$ inside D, and, as $D \uparrow R^d$, h_1 decreases to a non-negative (and hence constant) harmonic function $p_1(\infty)$ such that $p_1(\infty) - p_1 = \int G p_1 de$. R^d is now split into $A = (p_1 < \frac{1}{2} p_1(\infty))$ and $B = (p_1 \geq \frac{1}{2} p_1(\infty))$ and the fact that $p_1(\infty) - p_1$ excessive is used to ensure that p_1 has a limit along the Brownian path as $t \uparrow +\infty$, permitting us to conclude from

11.4a $p_1 \leq p_1(\infty)$

11.4b $p_1(\infty) = \lim_{D \uparrow R^d} E_{\bullet}[p_1(x(\mathfrak{m}_{\partial D}))] = E_{\bullet}[\lim_{t \uparrow +\infty} p_1(x(t))]$

that A is thin at ∞. Because $0 < p_1(\infty)$,

11.5 $\int_B G de \leq 2 p_1(\infty)^{-1} \int G p_1 de < +\infty$,

and this completes the verification of c).

12. PERFORMING THE TIME SUBSTITUTION.

Coming to the actual time substitution $t \to \mathfrak{f}^{-1}$ which is supposed to send the Brownian motion into the diffusion $\dot{D}$, let $\mathfrak{f}$ be modified on a negligeable class of Brownian paths so as to have

12.1 $0 = \mathfrak{f}(0) \leq \mathfrak{f}(t \pm) = \mathfrak{f}(t) < +\infty$

12.2 $\mathfrak{f}(t) = \mathfrak{f}(s) + \mathfrak{f}(t-s, w_s^+) \qquad t \geq s$

12.3 $\mathfrak{f}(t) > \mathfrak{f}(s) \qquad t > s$

as *identities*, let x^{-1} denote the sample path

12.4 $w^{-1}: t \to x^{-1}(t) \equiv x^{-1}(t, w^{-1}) \equiv x[\mathfrak{f}^{-1}(t, w), w]$,

note that this path is continuous even if $\mathfrak{f}(+\infty) < +\infty (d \geq 3)$, and let us check that *the motion* $\dot{D}$ *with sample paths*

12.5a $\dot{w}: t \to \dot{x}(t)$

and probabilities

12.5b $\dot{P}_a(\dot{B}) \equiv P_a(w^{-1} \in \dot{B})$

is the diffusion with Brownian hitting probabilities and speed measure e.

Beginning with the proof of the *diffusive character* 6.1 of this motion, the problem is to check that if $\dot{\mathfrak{m}}$ is a Markov time and if $\dot{B} \in \dot{\boldsymbol{B}}$, then

12.6 $\dot{P}_a[\dot{w}^+_{\dot{\mathfrak{m}}} \in \dot{B} \mid \dot{\boldsymbol{B}}_{\dot{\mathfrak{m}}+}] = \dot{P}_b(\dot{B}) \qquad b \equiv \dot{x}(\dot{\mathfrak{m}})$.

Given such a Markov time $\dot{\mathfrak{m}}$,

12.7 $\mathfrak{m}(w) \equiv \mathfrak{f}^{-1}(\dot{\mathfrak{m}}(w^{-1}), w)$

is a Markov time for the Brownian path; indeed, using Galmarino's test, if

12.8a $\mathfrak{m}(u) < t$

12.8b $x(s, u) = x(s, v) \qquad s \leq t$,

then

12.9a $\mathfrak{f}(s, u) = \mathfrak{f}(s, v) \qquad s \leq t$

12.9b $\mathfrak{f}^{-1}(s, u) = \mathfrak{f}^{-1}(s, v) \leq t \qquad s \leq \dot{t} = \mathfrak{f}(t, u) = \mathfrak{f}(t, v)$,

and it follows that

12.10a $\dot{\mathfrak{m}}(u^{-1}) = \mathfrak{f}(\mathfrak{m}(u), u) < \mathfrak{f}(t, u) = \dot{t}$

12.10b $x^{-1}(s, u^{-1}) = x[\mathfrak{f}^{-1}(s, u), u] = x[\mathfrak{f}^{-1}(s, v), v]$
$= x^{-1}(s, v^{-1}) \qquad s \leq \dot{t}$.

Because $\dot{\mathfrak{m}}$ was a Markov time for the dot motion, 12.10 implies

12.11a $\dot{\mathfrak{m}}(u^{-1}) = \dot{\mathfrak{m}}(v^{-1}) < \dot{t}$,

and an application of 12.9b implies

12.11b $\dot{\mathfrak{m}}(u) = \mathfrak{f}^{-1}(\dot{\mathfrak{m}}(u^{-1}), u) = \mathfrak{f}^{-1}(\mathfrak{m}(v^{-1}), v) = \mathfrak{m}(v)$;

in brief, 12.8 *implies*, 12.11b, as needed to conclude by Galmarino's test that $\mathfrak{m}$ is a Markov time.

Given $\dot{A} \in \dot{\boldsymbol{B}}_{\dot{\mathfrak{m}}+}$, if $A = (w : w^{-1} \in \dot{A})$ and if

12.12a $\mathfrak{m}(u) < t$

12.12b $x(s, u) = x(s, v) \qquad s \leq t$

12.12c $u \in A$,

then, using 12.10,

12.13a $\dot{\mathfrak{m}}(u^{-1}) < \dot{t}$

12.13b $x^{-1}(s, u^{-1}) = x^{-1}(x, v^{-1}) \qquad s \leq \dot{t}$

12.13c $u^{-1} \in A$,

and it follows that $v^{-1}\in\dot{A}$, or, what is the same, that $v\in A$; thus, by Galmarino's test, $A\in\boldsymbol{B}_{\mathfrak{m}+}$, and now it appears that

12.14 $$\begin{aligned}&\dot{P}_a[\dot{A},\ \dot{w}^+_{\dot{\mathfrak{m}}}\in\dot{B}]\\&=P_a[w^{-1}\in\dot{A},\ (w^+_{\mathfrak{m}})^{-1}\in\dot{B}]\\&=P_a[w\in A,\ (w^+_{\mathfrak{m}})^{-1}\in\dot{B}]\\&=E_a[A,\ P_b(w^{-1}\in\dot{B})]\qquad b\equiv x(\mathfrak{m})=x^{-1}(\dot{\mathfrak{m}}(w^{-1}),\ w^{-1})\\&=E_a[w^{-1}\in\dot{A},\ \dot{P}_b(\dot{B})]\\&=\dot{E}_a[\dot{A},\ \dot{P}_b(\dot{B})]\qquad b=\dot{x}(\dot{\mathfrak{m}}),\end{aligned}$$

completing the proof of 12.6.

$\dot{\boldsymbol{D}}$ is now identified as a *diffusion*; that it has *Brownian hitting probabilities* is clear, and to complete the discussion, it suffices to verify that *it has e as its speed measure.* But this is clear because

12.15 $$\mathfrak{m}_{\partial D}^{-1}\equiv\min\,(t:x^{-1}(t)\in\partial D)=\mathfrak{f}(\mathfrak{m}_{\partial D}),$$

and $\mathfrak{f}$ has e as its associated measure.

13. GENERATORS.

Given a diffusion with Brownian hitting probabilities, the *Green operators*

13.1 $$\dot{G}_\alpha:\ f\to\dot{E}.\left[\int_0^{+\infty}e^{-\alpha t}f(\dot{x})dt\right]\qquad\alpha>0$$

map into itself the space $\dot{C}(E^d)$ of real, bounded, fine-continuous functions having ordinary limits at ∞ in case $d\geq 3$; in fact, if $d\geq 3$, then $\dot{P}.[\min_{t\geq 0}|\dot{x}(t)|\geq n]=P.[\min_{t\geq 0}|x(t)|\geq n]$ tends to 1 at ∞, so that $\dot{G}_\alpha f$ tends to $\alpha^{-1}f(\infty)$, and, if $a\in R^d$ $(d\geq 2)$ and if the ball $D\ni a$ is so small that $1-\dot{p}_\alpha(a)=1-\dot{E}.[e^{-\alpha\dot{\mathfrak{m}}_{\partial D}}]<n^{-1}$, then, inside the *fine neighborhood* $B=D\cap(\dot{p}_\alpha>1-n^{-1})$, the difference between

13.2a $$\begin{aligned}u&=\dot{G}_\alpha f\\&=\dot{E}.\left[\int_0^{\dot{\mathfrak{m}}_{\partial D}}e^{-\alpha t}f(\dot{x})dt+e^{-\alpha\dot{\mathfrak{m}}_{\partial D}}\int_0^{+\infty}e^{-\alpha t}f(\dot{x}(t,\ \dot{w}^+_{\dot{\mathfrak{m}}_{\partial D}}))dt\right]\\&=\dot{E}.\left[\int_0^{\dot{\mathfrak{m}}_{\partial D}}e^{-\alpha t}f(\dot{x})\,dt+e^{-\alpha\dot{\mathfrak{m}}_{\partial D}}u(\dot{x}(\dot{\mathfrak{m}}_{\partial D}))\right]\end{aligned}$$

and the harmonic (and hence continuous) function

13.2b $h = \dot{E}_\cdot[u(\dot{x}(\mathfrak{m}_{\partial D}))]$

is not greater than

13.3 $constant \times (1-\dot{p}_\alpha) < constant \times n^{-1}$.

Because

13.4 $\dot{G}_\alpha - \dot{G}_\beta + (\alpha-\beta)\dot{G}_\alpha \dot{G}_\beta = 0 \qquad \alpha, \beta > 0$,

it is evident that $\dot{G}_\alpha$ maps our space of fine continuous functions onto some subspace $D(\dot{\mathfrak{G}})$ independent of α and that its null-space $\dot{G}_\alpha^{-1}(0)$ is likewise independent of α. But, for fine-continuous $f \in \dot{G}_\alpha^{-1}(0)$,

13.5 $0 = \lim_{\beta\uparrow+\infty} \beta \dot{G}_\beta f = \dot{E}_\cdot\left[\lim_{\beta\uparrow+\infty} \beta \int_0^{+\infty} e^{-\beta t} f(\dot{x})\, dt\right] = f$

according to E. B. Dynkin's description of fine neighborhoods; thus the null-space is trivial, $\dot{G}_\alpha$ is invertable, and another application of 13.4 verifies that $\dot{\mathfrak{G}} \equiv \alpha - \dot{G}_\alpha^{-1}$ acting on $D(\dot{\mathfrak{G}})$ is independent of α.

$\dot{\mathfrak{G}}$ is the so-called generator; it is *closed* in the sense that if $u_n \in D(\dot{\mathfrak{G}})$ and $\dot{\mathfrak{G}} u_n = f_n$ converge pointwise under fixed bounds to u and $f \in \dot{C}(E^d)$, then $u \in D(\dot{\mathfrak{G}})$ and $\dot{\mathfrak{G}} u = f$.

Consider the *differential operator*

13.6 $$\mathfrak{D}u = \frac{-e^u(db)}{e(db)} \qquad |b| < +\infty$$
$$= 0 \qquad b = \infty, \ d \geq 3$$

acting on the class $D(\mathfrak{D})$ of functions $u \in \dot{C}(E^d)$ such that

a) *inside each* D, u *is the sum of the harmonic function* $h = \int h_{\partial D}(a, db) u(b)$ *and the potential* $\int G_D de^u$ *of its Riesz measure* e^u.

b) $\int G_D |de^u|$ *is bounded.*

c) $\mathfrak{D}u$, *as described in* 13.6, *exists and belongs to* $\dot{C}(E^d)$.

$\dot{\mathfrak{G}}$ *is the closure* $\overline{\mathfrak{D}}$ *of* $\mathfrak{D}$ *in the topology of bounded pointwise convergence* as will now be explained.

Choose a fine-continuous function $0 < p_n \leq 1$ tending to 0 at ∞ in the ordinary topology such that $\int G_D p_n de$ is bounded for each bounded open D and $p_n \uparrow 1$ as $n \uparrow +\infty$, and introduce the additive functional $\dot{\mathfrak{f}}_n = \int_0^t p_n(\dot{x}) ds \ (t \geq 0)$.

$\dot{x}(\mathfrak{f}_n^{-1})$ *is a diffusion with Brownian hitting probabilities and speed measure* $de_n = p_n \times de$,

13.7 $\dot{E}.[\min(t: \dot{x}(\mathfrak{f}_n^{-1}) \in \partial D)] = \dot{E}.[\mathfrak{f}_n(\mathfrak{m}_{\partial D})] = \int G_D p_n de < +\infty$,

and it follows from 13.2a that if u is in the domain of its generator $\dot{\mathfrak{G}}_n$, then

13.8 $$u = -\dot{E}.\left[\int_0^{\mathfrak{m}_{\partial D}} (\dot{\mathfrak{G}}_n u)(\dot{x}(t))\mathfrak{f}_n(dt)\right] + E.[u(\dot{x}\mathfrak{m}_{\partial D}))]$$
$$= -\int G_D(\dot{\mathfrak{G}}_n u)p_n de + \textit{a harmonic function},$$

which is a special case of a formula of E. B. Dynkin [6]; *in brief*,

13.9 $$-e^u(db) = (\dot{\mathfrak{G}}_n u)p_n de \qquad u \in D(\dot{\mathfrak{G}}_n).$$

Choose $u = \dot{G}_1 f \in D(\dot{\mathfrak{G}})$, then

13.10 $$u_n \equiv \dot{E}.\left[\int_0^{+\infty} e^{-t} f(\dot{x}(\mathfrak{f}_n^{-1}))dt\right]$$

tends pointwise under the bound $\|f\|_\infty$ to u. Because $u_n \in D(\dot{\mathfrak{G}}_n)$,

13.11 $$\mathfrak{Q}u_n = \frac{-e^u(db)}{e(db)} = p_n \dot{\mathfrak{G}}_n u_n = p_n(u_n - f)$$

satisfies all the conditions for u_n to belong to $D(\mathfrak{Q})$, and, what is more, $\mathfrak{Q}u_n$ converges pointwise under the bound $2\|f\|_\infty$ to $u - f = \dot{\mathfrak{G}}u$; thus, $u \in D(\bar{\mathfrak{Q}})$ and $\bar{\mathfrak{Q}}u = \dot{\mathfrak{G}}u$, i.e.,

13.12 $$\bar{\mathfrak{Q}} \supset \dot{\mathfrak{G}}.$$

As to the proof of $\bar{\mathfrak{Q}} \subset \dot{\mathfrak{G}}$, it is enough to show that if $u \in D(\bar{\mathfrak{Q}})$, then $\dot{G}_1(1 - \bar{\mathfrak{Q}})u = u$, and, for this, it is enough to deduce from $u \in D(\bar{\mathfrak{Q}})$ and $\bar{\mathfrak{Q}}u = u$ that $u \equiv 0$.

Given such a $u \in D(\bar{\mathfrak{Q}})$ with $\bar{\mathfrak{Q}}u = u$ and choosing $u_n \in D(\mathfrak{Q})$ so as to make u_n and $\mathfrak{Q}u_n$ converge pointwise and boundedly to u and $\bar{\mathfrak{Q}}u = u$,

13.13 $$u_n - h_n = -\int G_D \mathfrak{Q}u de = -\dot{E}.\left[\int_0^{\mathfrak{m}_{\partial D}} (\mathfrak{Q}u_n)(\dot{x})ds\right]$$
$$h_n = \int h_{\partial D}(0, db)u_n(b)$$

implies

13.14 $$\dot{E}_\bullet[(u_n-h_n)(\dot{x}(t)),\, t<\dot{\mathfrak{m}}_{\partial D}]-(u_n-h_n) = \dot{E}_\bullet\left[\int_0^{t\wedge\dot{\mathfrak{m}}_{\partial D}}(\mathfrak{D}u_n)(\dot{x})ds\right] \qquad t\geq 0\,,$$

which, in turn, implies

13.15 $$\dot{E}_\bullet[(u-h)(\dot{x}(t)),\, t<\dot{\mathfrak{m}}_{\partial D}]-(u-h) = \dot{E}_\bullet\left[\int_0^{t\wedge\dot{\mathfrak{m}}_{\partial D}}(\bar{\mathfrak{D}}u)(\dot{x})ds\right] \qquad t\geq 0$$

$$h = \int h_{\partial D}(\cdot,\, db)u(b)\,,$$

and, letting $D\uparrow R^d$ so as to make h tend to a bounded (and hence constant) harmonic function $h(\infty)$,

13.16a $\dot{P}_\bullet[\dot{\mathfrak{m}}_\infty<+\infty] = P_\bullet[\mathfrak{f}(\mathfrak{m}_\infty)<+\infty]=0 \qquad d=2$

13.16b $u(\infty)=h(\infty)=0 \qquad d\geq 3$

implies

13.17 $$\dot{E}_\bullet[u(\dot{x}(t)]-u = \dot{E}_\bullet\left[\int_0^t u(\dot{x})ds\right] \qquad t\geq 0\,,$$

and the desired $u\equiv 0$ follows.

The Green operators leave invariant the space $C(E^d)$ of bounded functions continuous in the ordinary topology of E^d if and only if, for each D, the mean exit time

13.18 $$\dot{p} = \dot{E}_\bullet[\dot{\mathfrak{m}}_{\partial D}] = \int G_D de$$

is continuous inside D and tends to 0 *on* ∂D; *in this case, the generator* $\dot{\mathfrak{G}}$ *coincides with the differential operator* $\mathfrak{D}$ *acting on the class of functions* $u\in C(E^d)$ *such that* $\mathfrak{D}u\in C(E^d)$; the reader will easily supply the details of the proof.

Here is an example in which the Green operators *do not map* $C(E^d)$ *into itself.*

Choose $d=3$ and $e=f\times db$, where $f=1+\sum_{n\geq 1}f_n$ and the f_n are the indicators of little non-overlapping open balls D_n converging to as $n\uparrow+\infty$ but not covering 0 itself and so small that

13.19 $P_0[\mathfrak{m}_{\partial D} < -\infty] < 2^{-n} \qquad n \geq 1$.

Because of the first Borel-Cantelli lemma, $p \equiv P_\cdot[\mathfrak{m}_{\partial D_n} < +\infty, \, i.o.] = 0$ at the origin, and it is also clear that $p \equiv 0$ on the rest of R^3. But then $\mathfrak{f} = \int_0^t f(x(s))ds$ is a continuous additive functional, $\mathfrak{f}(s) < \mathfrak{f}(t)$ $(s < t)$, and $x(\mathfrak{f}^{-1})$ is a diffusion with Brownian hitting probabilities and speed measure e.

Given a neighborhood D of 0,

13.20
$$\begin{aligned} &E_0[\min(t: x(\mathfrak{f}^{-1}) \in \partial D)] \\ &= E_0[\mathfrak{f}(\mathfrak{m}_{\partial D})] \\ &\geq \sum_{n \geq 1} E_0\left[\int_0^{\mathfrak{m}_{\partial D}} f_n(x(s))ds\right] \\ &= \sum_{n \geq 1} \int_{D_n} G_D(0, b)db/\text{volume}\,(D_n) \\ &= +\infty. \end{aligned}$$

But, as E. B. Dynkin [6] has pointed out, this cannot happen for all small neighborhoods if the Green operators map $C(E^d)$ into itself.

14. DISCONTINUOUS ADDITIVE FUNCTIONALS.

V. A. Volkonskii [16] has studied *discontinuous* additive functionals; in the present Brownian case their structure is very simple.

A functional $\mathfrak{t}$ *of the Brownian path which satisfies*

14.1 $\mathfrak{t}(t, w)$ is measurable $\boldsymbol{B}_t$ for each $t \geq 0$.

14.2 $0 \leq \mathfrak{t} < +\infty$

14.3 $\mathfrak{t}(t-) = \mathfrak{t}(t)$

14.4 $\mathfrak{t}(t) = \mathfrak{t}(s) + \mathfrak{t}(t-s, w_s^+) \qquad t \geq s$

is the sum of a continuous additive functional $\mathfrak{f}$ *and a discontinuous additive functional* $\mathfrak{j}$ *with*

14.5 $P_\cdot[\mathfrak{j}(t) = \mathfrak{j}(0+), t > 0] = 1$

14.6a $P_\cdot[\mathfrak{j}(0+) > 0] = 0$ *or* 1

14.6b $C(E) = 0$, *where* E *is the set of points at which*

$P.[\mathfrak{j}(0+)>0]=1$ *and* C *is the Newtonian (logarithmic) capacity in* $d\geq 3$ $(=2)$ *dimensions.*[9]

Consider the (discontinuous) additive functional $\mathfrak{j}_n(t)=$ *the sum of the jumps of* $\mathfrak{t}$ *of magnitude* $\geq n^{-1}$ *taking place before time* t, note that $\mathfrak{j}_n(0+)>0)\in \boldsymbol{B}_{0+}$ so that $P.[\mathfrak{j}_n(0+)>0]=0$ or 1 according to Blumenthal's 01 law, let E_n be the (Borel) set on which $P.[\mathfrak{j}_n(0+)>0]=1$, and introduce the least *positive* jumping time $\mathfrak{m}$ of $\mathfrak{j}_n$.

If $P.(\mathfrak{m}<+\infty)>0$ at some point, then

14.7 $$0<P.[0<\mathfrak{m}<+\infty,\ \mathfrak{j}_n(\mathfrak{m})<\mathfrak{j}_n(\mathfrak{m}+)] = P.[0<\mathfrak{m}<+\infty,\ x(\mathfrak{m})\in E_n];$$

this implies $C(E_n)>0$[10], and it follows that E_n contains a subcompact A of positive capacity, having a (regular) point at which $P.[x(t)\in A$, i.o., $t\downarrow 0]=1.$[11] But then $P.[\mathfrak{j}_n(t)\equiv+\infty,\ t>0]=1$ at that point, contradicting $\mathfrak{j}_n\leq\mathfrak{t}<+\infty$, and it follows that

14.8a $$P.[\mathfrak{j}_n(t)\equiv\mathfrak{j}_n(0+),\ t>0]\equiv 1$$

14.8b $$C(E_n)=0\,.$$

The rest is clear: $\mathfrak{j}=\lim_{n\uparrow+\infty}\mathfrak{j}_n$ satisfies 14.5, the remainder $\mathfrak{f}=\mathfrak{t}-\mathfrak{j}$ is a continuous additive functional, 14.6a is immediate from Blumenthal's 01 law, and $C(E)=\lim_{n\uparrow+\infty}C(E_n)=0.$

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9 G. Choquet [3] found that if $E\subset R^d$ is Borel, then
$$\inf C(B):B \text{ open},\ B\supset E$$
$$=\sup C(A):A \text{ compact},\ A\subset E;$$
this common value is the *capacity* of E.

10 $P.[x(t)\in E$ *at some positive time*$]$ is positive or $\equiv 0$ according as $C(E)>0$ or not; see, for example, G. Hunt [9].

11 See O. D. Kellogg [11] for the classical significance of regular points and J. Doob [5] for the probabilistic interpretation.

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Note added in proof. The authors came to know that A. D. Ventsel' [19] obtained almost the same result as in sect. 3-5 of the present paper and that A. Meyer [18] studied the same problem for a more general class of Markov processes. As for signed additive functionals, E. B. Dynkin [17] constructed them using stochastic integrals in case of a Brownian motion.

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