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Supplements and corrections to my former papers

By

Yukio Kusunoki

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I. Corrections

("*nt*"("*nb*") means respectively the n-th line from top (bottom))

1. "Contributions to Riemann-Roch's theorem" This memoirs, vol. 31 (1958) pp. 161-180

The following slight modifications would make our argument correct and more neat.

12b, p. 162 Delete the sentence "We normalize $\cdots$ at Q_1" and insert, after the definition of the space S, "In case of the non-integral divisor $\delta=\delta_{(P)}/\delta_{(Q)}$, we identify two integrals or functions in $M(\supset S)$ if they are identical except a constant. Therefore every element of S is then an equivalent class (in M) containing a single-valued function which is a multiple of $\delta_{(Q)}/\delta_{(P)}$. Anyway, $\dim S$ is equal to the number of linearly independent functions which are single-valued and multiples of $1/\delta$".

3t, p. 163 Omit "and vanish at Q_1"

8t, p. 166 Replace "Now if $\cdots$ absurd" by "If we choose $\varphi=\phi_{Q_1 Q_t}\in E$, then we have $c=\Omega(Q_t)=\Omega(Q_1)$ $(t=2, \cdots, s)$".

10t. p. 166 Replace Ω by $\Omega-c$.

Corresponding modifications should be made for the spaces M' (15t, p. 167), M(W) (17t, p. 169), and also the space M (6t, p. 249) in my paper;

[*] "Theory of Abelian integrals and its applications to conformal mappings" This memoirs, vol. 32 (1959) pp. 235-258

One should treat suitably the integral constants also at some other places in above two papers, e.g. 3b. p. 255 [*].

2. In my paper

[**] "On the harmonic boundary of an open Riemann surface". Jap. Journ. of Math. vol. 29 (1960) pp. 52-56

	Read	for
4t p. 53	$D_{\hat{G}}[\phi-\phi_n]$	$D_G[\phi-\phi]$
3t p. 54	$\bar{G}$	$\bar{K}$
6t p. 54	$\bar{F}$	F
3t p. 55	$\overline{\partial G_i}(R^*)$	$\partial G_i(R^*)$
16b	$(R_n \cap G)^\wedge$	$(R_n \cap G)$
7t p. 56	U_{HD}	O_{HD}

II. Supplements

Here we shall give some short notes supplementary to papers [*] [**].

1. We shall use the same notations as in [*] without any repetition. Let R be an arbitrary open Riemann surface of genus $g\,(<\infty)$, then we can always find a pair of g points $P_1^0, \cdots, P_g^0$ on R for which

$$B(P_1^0 P_2^0 \cdots P_g^0) = 0^{1)} \tag{1}$$

Let $K_j\,(j=1, 2, \cdots, g)$ be the neighborhoods of P_j^0 and $z_j = x_j + iy_j$ the local parameters at P_j^0 such that $z_j(P_j^0)=0$. For any pair of g points $P_j(z_j) \in K_j$ consider the functions

$$\varphi_j(z_1, \cdots, z_g) = \sum_{i=1}^{g} \int_0^{z_i} du_{A_j},\ \varphi_{g+j}(z_1, \cdots, z_g) = \sum_{i=1}^{g} \int_0^{z_i} du_{B_j}\ (j=1, \cdots, g)$$

where $du_{A_j} = Re\,\varphi_{A_j}^*$, $du_{B_j} = Re\,\varphi_{B_j}^*$ and the integrations are taken on respective K_j. Then the vector-valued function

$$F(z_1, \cdots, z_g) = (\varphi_1(z_1, \cdots, z_g), \cdots, \varphi_{2g}(z_1, \cdots, z_g))$$

gives a mapping of the polycylinder $K_1 \times K_2 \times \cdots \times K_g$ into $2g$-

1) 13b p. 255.

dimensional euclidean space R^{2g}. Since the Jacobian of this mapping

$$\frac{\partial(\varphi_1, \varphi_2, \cdots\cdots, \varphi_{2g})}{\partial(x_1, y_1, \cdots, x_g, y_g)} = \frac{\partial(u_{A_1}, \cdots, u_{A_g}, u_{B_1}, \cdots, u_{B_g})}{\partial(x_1, y_1, \cdots\cdots\cdots\cdots, x_g, y_g)}$$

$$= (-1)^g \begin{vmatrix} Re\,\check{\phi}_{A_1}(z_1) & Im\,\check{\phi}_{A_1}(z_1) \cdots Re\,\check{\phi}_{A_1}(z_g) & Im\,\check{\phi}_{A_1}(z_g) \\ \cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ Re\,\check{\phi}_{B_g}(z_1) & Im\,\check{\phi}_{B_g}(z_1) \cdots Re\,\check{\phi}_{B_g}(z_g) & Im\,\check{\phi}_{B_g}(z_g) \end{vmatrix}$$

does not vanish at the origin $(0, \cdots, 0)$ on account of (1), F gives a topological mapping of a suitable polycylinder $U_1 \times U_2 \times \cdots \times U_g$ ($P_i(0) \in U_i \subset K_i$) onto a neighborhood U of the origin in R^{2g}. That is, for sufficiently small real numbers $c_1, \cdots, c_{2g}$ for which $(c_1, \cdots, c_{2g}) \in U$ there exists a pair of g points $P_j(\in U_j \subset K_j)$ such that

$$(2) \qquad \sum_{i=1}^{g} \int_{P_i^0}^{P_i} du_{A_j} = c_j, \quad \sum_{i=1}^{g} \int_{P_i^0}^{P_i} du_{B_j} = c_{g+j} \quad (j = 1, 2, \cdots, g)$$

This is the solution of a restricted *Jacobi's inversion problem* on open Riemann surface R. We shall apply this to a canonical conformal mapping of R whose boundary consists of a finite number of disjoint Jordan curves. Let P_0^0, P_1^0 be any two points on R. If $g=0$ the function $\varphi(P) = \exp. \int^P \phi^*_{P_0^0 P_1^0}$ is single-valued and its pole (zero point) is located at $P_0^0(P_1^0)$. Next suppose $g \geqq 1$, then we can take $(g-1)$ points $P_2^0, \cdots, P_g^0$ on R for which the relation (1) is fulfilled. Consider the function

$$f(P) = \int^P \phi^*_{Q_0 P_0^0} + \sum_{i=1}^{g} \int^P \phi^*_{P_i^0 Q_i} \in \mathfrak{K}$$

where the points Q_0, Q_i will be determined in the following. If some Q_i ($1 \leqq i \leqq g$) coincide with P_i^0, the corresponding terms on the right hand side should be removed. The function f is not single-valued in general, and has periods along canonical cycles (A_μ, B_μ);

$$\int_{\substack{A_\mu \\ B_\mu}} df = 2\pi i \left(\int_{Q_0}^{P_0^0} du_{\substack{A_\mu \\ B_\mu}} + \sum_{i=1}^{g} \int_{P_i^0}^{Q_i} du_{\substack{A_\mu \\ B_\mu}} \right) \quad (\mu = 1, \cdots, g).$$

Therefore if the conditions

$$(3)\quad \sum_{i=1}^{g}\int_{P_i^0}^{Q_i} du_{A_\mu}=\int_{P_0^0}^{Q_0} du_{A_\mu},\quad \sum_{i=1}^{g}\int_{P_i^0}^{Q_i} du_{B_\mu}=\int_{P_0^0}^{Q_0} du_{B_\mu}\quad (\mu=1,\cdots,g)$$

are satisfied, f will have no periods along A_μ, B_μ and boundary contours of R. Now we choose a point $Q_0\,(\neq P_0^0)$ so near P_0^0 that $\left(\int_{P_0^0}^{Q_0} du_{A_1},\cdots,\int_{P_0^0}^{Q_0} du_{B_g}\right)\in U(\subset R^{2g})$. Then by (2) we can find g points $Q_i\,(i=1,\cdots,g)$ on R for which relation (3) holds. For such points $Q_0, Q_1,\cdots,Q_g$ $\varphi(P)=\exp f(P)$ becomes a single-valued function on R whose poles (zero points) are located at $Q_0, P_1^0,\cdots,P_g^0$ ($P_0^0, Q_1,\cdots,Q_g$), moreover $|\varphi|$ is constant on each boundary component of R. Hence for $g\,(<\infty)$ the function φ gives a solution of the following mapping problem;

Let R be an open Riemann surface of genus $g\,(<\infty)$ whose boundary consists of a finite number of disjoint Jordan curves. Then R can be mapped conformally onto an at most $(g+1)$-sheeted covering surface bounded by slits along circular arcs centered at origin. We can moreover prescribe the location of one pole and one zero of the mapping function. The same statement holds if the circular slits are replaced by radial slits.

We note that by a suitable choice of Q_0, R can be mapped in the same way onto an *exactly* $(g+1)$*-sheeted* surface and that if $g\geqq 1$, the location of two poles (or zero points) can be prescribed, more generally, two points can be *prescribed exactly* as poles or zero points, while the other $(g-1)$ points *almost exactly prescribed* in the sense that each point can be taken in any neighborhood of a given point, because $P_2^0,\cdots,P_g^0$ can be chosen so.

2. For [**], we note that Lemma 1 (maximum principle due to S. Mori and Ota) can be easily derived from Proposition 1. Indeed, suppose that an HBD-function u does not attain on Δ its maximum M on R^*. Since u is continuous on R^*, harmonic in R, moreover Δ is compact, we find that $\max_{\Delta} u=M'<M$, and every component E_α of the set

$$E=\{p\,;u(p)>M'',\ M'<M''<M\}$$

becomes a non-compact subregion on R whose closure $\bar{E}_\alpha$ (on R^*)

is disjoint with Δ. While as is easily seen, the double $\hat{E}_\alpha$ of E_α does not belong O_G. Therefore by Proposition 1 $\bar{E}_\alpha$ must contain some points of Δ, which is a contradiction.

Reflecting our paper it seems therefore that Proposition 1 is a most fundamental lemma.

3. Finally we remark that any single-valued canonical potential u associated with an elementary domain B([*] p. 241) is characterized by the simple way on the Royden compactification of R. Let $B^c = \bigcup_{i=1}^{k} G_i$, $\bar{G}_i \cap \partial B = \gamma_i$ and $u = u_i + c_i$ $(i=1, \cdots, k)$, where u_i denote normalized potentials on G_i. Consider the harmonic functions u_{in} on $R_n \cap G_i$ ($\{R_n\}$ denotes an exhaustion of R) whose boundary values are $=u$ on γ_i, $=c_i$ on $\partial R_n \cap G_i$, then u_{in} converge to u (for $n \to \infty$) on every compact subset of $G_i \cup \gamma_i$. We define BD-functions v_n on R so that $v_n = u$ on B, $= u_{in}$ on $R_n \cap G_i$ and $v_n = c_i$ on $G_i - R_n$. Then it is proved by usual computations on Dirichlet integrals that $v_n \to u$ in BD-topology. Let

$$(4) \qquad v_n = U_n + \varphi_n, \quad U_n \in HBD, \quad \varphi_n \in \bar{K}$$

be the orthogonal decomposition of v_n. Now dividing curves γ_i separate the harmonic boundary Δ into disjoint pieces Δ_i $(i=1, \cdots, k)$. Therefore we can construct an HBD-function U such that $U = c_i$ on Δ_i (Lemma 3 [**]). While, $U_n = v_n = c_i$ on Δ_i, it follows that $U_n \equiv U$ by the maximum principle. Hence letting $n \to \infty$ in (4), we have $u = U + \varphi$, where $\varphi_n \to \varphi$ in BD-topology, that is $\varphi \in \bar{K}$. While $\varphi = u - U \in HBD$, hence $\varphi \equiv 0$. Therefore $u \equiv U$, which implies that u is an HBD-function which takes a constant value c_i on each Δ_i. We find also that u can be written as a linear combination of harmonic measures.

KYOTO UNIVERSITY