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# On holomorphic curves in algebraic torus

By

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**Abstract**

We study entire holomorphic curves in the algebraic torus, and show that they can be characterized by the "growth rate" of their derivatives.

## 1. Introduction

Let $z = x + y\sqrt{-1}$ be the natural coordinate in the complex plane $\mathbb{C}$, and let $f(z)$ be an entire holomorphic function in the complex plane. Suppose that there are a non-negative integer $m$ and a positive constant $C$ such that

$$|f(z)| \leq C|z|^m \quad (|z| \geq 1).$$

Then $f(z)$ becomes a polynomial with $\deg f(z) \leq m$. This is a well-known fact in the complex analysis in one variable. In this paper, we prove an analogous result for entire holomorphic curves in the algebraic torus $(\mathbb{C}^*)^n := (\mathbb{C} \setminus \{0\})^n$.

Let $[z_0 : z_1 : \cdots : z_n]$ be the homogeneous coordinate in the complex projective space $\mathbb{C}P^n$. We define the complex manifold $X \subset \mathbb{C}P^n$ by

$$X := \{[1 : z_1 : \cdots : z_n] \in \mathbb{C}P^n |\, z_i \neq 0\ (1 \leq i \leq n)\} \cong (\mathbb{C}^*)^n.$$

$X$ is a natural projective embedding of $(\mathbb{C}^*)^n$. We use the restriction of the Fubini-Study metric as the metric on $X$. (Note that this metric is different from the natural flat metric on $(\mathbb{C}^*)^n$ induced by the universal covering $\mathbb{C}^n \to (\mathbb{C}^*)^n$.)

For a holomorphic map $f : \mathbb{C} \to X$, we define the pointwise norm $|df|(z)$ by setting

$$|df|(z) := \sqrt{2}\,|df(\partial/\partial z)| \quad \text{for all } z \in \mathbb{C}. \tag{1.1}$$

Here $\partial/\partial z = \frac{1}{2}(\partial/\partial x - \sqrt{-1}\partial/\partial y)$, and the normalization factor $\sqrt{2}$ comes from $|\partial/\partial z| = 1/\sqrt{2}$.

The first result of this paper is the following.

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**Theorem 1.1.** *Let $f : \mathbb{C} \to X$ be a holomorphic map. Suppose there are a non-negative integer $m$ and a positive constant $C$ such that*

$$|df|(z) \le C|z|^m \quad (|z| \ge 1). \tag{1.2}$$

*Then there are polynomials $g_1(z)$, $g_2(z)$, $\cdots$, $g_n(z)$ with $\deg g_i(z) \le m+1$ $(1 \le i \le n)$ such that*

$$f(z) = [1 : e^{g_1(z)} : e^{g_2(z)} : \cdots : e^{g_n(z)}]. \tag{1.3}$$

*Conversely, if a holomorphic map $f(z)$ is expressed by* (1.3) *with polynomials $g_i(z)$ of degree $\le m+1$, then $f(z)$ satisfies the "polynomial growth condition"* (1.2)*.*

The direction (1.3) $\Rightarrow$ (1.2) is easier, and the substantial part of the argument is the direction (1.2) $\Rightarrow$ (1.3).

If we set $m = 0$ in the above, then we get the following corollary.

**Corollary 1.1.** *Let $f : \mathbb{C} \to X$ be a holomorphic map with bounded derivative, i.e., $|df|(z) \le C$ for some positive constant $C$. Then there are complex numbers $a_i$ and $b_i$ $(1 \le i \le n)$ such that*

$$f(z) = [\,1 : e^{a_1 z + b_1} : e^{a_2 z + b_2} : \cdots : e^{a_n z + b_n}\,].$$

This is the theorem of F. Berteloot and J. Duval in [2, Appendice]. (I also gave a proof of this result in [4, Section 6].) Holomorphic curves with bounded derivative are usually called "Brody curves" (cf. Brody [3]). Hence the condition (1.2) is an extension of the Brody condition.

**Remark 1.** Let $T(r, f)$ be the Shimizu-Ahlfors characteristic function of a holomorphic curve $f : \mathbb{C} \to X$:

$$T(r, f) := \int_1^r \frac{dt}{t} \int_{|z| \le t} |df|^2(z)\, dxdy.$$

It is easy to see that $f(z)$ can be expressed by (1.3) with polynomials $g_i(z)$ of degree $\le m+1$ if and only if

$$T(r, f) \le \mathrm{const} \cdot r^{m+1} \quad (r \ge 1). \tag{1.4}$$

(See Section 4.) Hence we have to prove (1.4) under the condition (1.2). But the direct consequence of (1.2) is

$$T(r, f) \le \mathrm{const} \cdot r^{2m+2} \quad (r \ge 1).$$

From this estimate, we can only prove that $f(z)$ can be expressed by (1.3) with polynomials $g_i(z)$ of degree $\le 2m+2$. The proof of (1.2) $\Rightarrow$ (1.4) needs a precise analysis on the behavior of $|df|$, and this is the main task of the paper. The pointwise norm $|df|(z)$ is actually very complicated object (see the beginning of Section 2), and one of the purposes of this paper is to develop techniques to handle it.

**Remark 2.** If the metric on $(\mathbb{C}^*)^n$ is the natural flat metric induced by the universal covering $\mathbb{C}^n \to (\mathbb{C}^*)^n$, then the statement of Theorem 1.1 is trivial. Our theorem tells us that the same conclusion holds even if we use the Fubini-Study metric.

Theorem 1.1 states that holomorphic curves in $X$ can be characterized by the growth rate of their derivatives. We can formulate this fact more clearly as follows;

Let $g_1(z)$, $g_2(z)$, $\cdots$, $g_n(z)$ be polynomials, and define $f : \mathbb{C} \to X$ by (1.3). We define the integer $m \geq -1$ by setting

$$m := -1 + \max(\deg g_1(z), \deg g_2(z), \cdots, \deg g_n(z)). \tag{1.5}$$

We have $m = -1$ if and only if $f$ is a constant map. This integer $m$ can be obtained as the growth rate of $|df|$:

**Theorem 1.2.** *If $m \geq 0$, then we have*

$$\limsup_{r\to\infty} \frac{\max_{|z|=r} \log |df|(z)}{\log r} = m.$$

The order $\rho_f$ of a holomorphic curve $f$ is usually defined by

$$\rho_f := \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

It is easy to see that $\rho_f = m + 1$ under the condition of this theorem. Hence this theorem tells us that we can prove a similar result for the pointwise norm $|df|$.

**Corollary 1.2.** *Let $\lambda$ be a non-negative real number, and let $[\lambda]$ be the maximum integer not greater than $\lambda$. Let $f : \mathbb{C} \to X$ be a holomorphic map, and suppose that there is a positive constant $C$ such that*

$$|df|(z) \leq C|z|^{\lambda} \quad (|z| \geq 1). \tag{1.6}$$

*Then we have a positive constant $C'$ such that*

$$|df|(z) \leq C'|z|^{[\lambda]} \quad (|z| \geq 1).$$

*Proof.* If $f$ is a constant map, then the statement is trivial. Hence we can suppose that $f$ is not constant. From Theorem 1.1, $f$ can be expressed by (1.3) with polynomials $g_i(z)$ of degree $\leq [\lambda]+2$. Since $f$ satisfies (1.6), we have

$$\limsup_{r\to\infty} \frac{\max_{|z|=r} \log |df|(z)}{\log r} \leq \lambda.$$

From Theorem 1.2, this shows $\deg g_i(z) \leq [\lambda] + 1$ for all $g_i(z)$. Then, Theorem 1.1 gives the conclusion. ∎

## 2. Proof of (1.3) ⇒ (1.2)

Let $f : \mathbb{C} \to X$ be a holomorphic map. From the definition of $X$, we have holomorphic maps $f_i : \mathbb{C} \to \mathbb{C}^*$ $(1 \le i \le n)$ such that $f(z) = [1 : f_1(z) : \cdots : f_n(z)]$. The norm $|df|(z)$ in (1.1) is given by

(2.1)

$$|df|^2(z) = \frac{1}{4\pi}\Delta \log\left(1 + \sum_{i=1}^{n} |f_i(z)|^2\right) \quad \left(\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}\right).$$

Suppose that $f$ is expressed by (1.3) with polynomials $g_i(z)$ of degree $\le m+1$. Then $f_i(z) = e^{g_i(z)}$ $(1 \le i \le n)$, and we have

$$\begin{aligned}
|df|^2 &= \frac{1}{\pi}\left[\frac{\sum_i |f_i'|^2}{(1+\sum_i |f_i|^2)^2} + \frac{\sum_{i<j} |g_i' - g_j'|^2 |f_i|^2 |f_j|^2}{(1+\sum_i |f_i|^2)^2}\right], \\
&\le \frac{1}{\pi}\left[\sum_i \frac{|f_i'|^2}{(1+|f_i|^2)^2} + \sum_{i<j} \frac{|g_i' - g_j'|^2 |f_i|^2 |f_j|^2}{(|f_i|^2 + |f_j|^2)^2}\right], \\
&= \frac{1}{\pi}\left[\sum_i \frac{|f_i'|^2}{(1+|f_i|^2)^2} + \sum_{i<j} \frac{|(f_i/f_j)'|^2}{(1+|f_i/f_j|^2)^2}\right], \\
&= \sum_i |df_i|^2 + \sum_{i<j} |d(f_i/f_j)|^2.
\end{aligned} \tag{2.2}$$

Here we have set

$$|df_i| := \frac{1}{\sqrt{\pi}}\frac{|f_i'|}{1+|f_i|^2} \quad \text{and} \quad |d(f_i/f_j)| := \frac{1}{\sqrt{\pi}}\frac{|(f_i/f_j)'|}{1+|f_i/f_j|^2}.$$

These are the norms of the differentials of the maps $f_i$, $f_i/f_j : \mathbb{C} \to \mathbb{C}P^1$. (We will repeatedly use the above (2.2) in this paper.)

We have $f_i(z) = \exp(g_i(z))$ and $f_i(z)/f_j(z) = \exp(g_i(z) - g_j(z))$, and the degrees of the polynomials $g_i(z)$ and $g_i(z) - g_j(z)$ are at most $m+1$. Then, the next lemma gives the desired conclusion:

$$|df|(z) \le C|z|^m \quad (|z| \ge 1)$$

for some positive constant $C$. □

**Lemma 2.1.** *Let $g(z)$ be a polynomial of degree $\le m+1$, and set $h(z) := e^{g(z)}$. Then there is a positive constant $C$ such that*

$$|dh|(z) = \frac{1}{\sqrt{\pi}}\frac{|h'(z)|}{1+|h(z)|^2} \le C|z|^m \quad (|z| \ge 1).$$

*Proof.* We have

$$\sqrt{\pi}\,|dh| = \frac{|g'|}{|h| + |h|^{-1}} \le |g'| \min(|h|, |h|^{-1}) \le |g'|.$$

Since the degree of $g'(z)$ is at most $m$, we easily get the conclusion. $\square$

## 3. Preliminary estimates

In this section, $k$ is a fixed positive integer.

The following is a standard fact in the Nevanlinna theory.

**Lemma 3.1.** *Let $g(z)$ be a polynomial of degree $k$, and set $h(z) = e^{g(z)}$. Then there is a positive constant $C$ such that*

$$\int_1^r \frac{dt}{t} \int_{|z|\le t} |dh|^2(z)\, dxdy \le Cr^k \quad (r \ge 1).$$

*Proof.* Since $|dh|^2 = \frac{1}{4\pi}\Delta \log(1+|h|^2)$, Jensen's formula gives

$$\int_1^r \frac{dt}{t} \int_{|z|\le t} |dh|^2\, dxdy = \frac{1}{4\pi}\int_{|z|=r} \log(1+|h|^2)\, d\theta - \frac{1}{4\pi}\int_{|z|=1} \log(1+|h|^2)\, d\theta.$$

Here $(r,\theta)$ is the polar coordinate in the complex plane. We have

$$\log(1+|h|^2) \le 2\,|\mathrm{Re}\, g(z)| + \log 2 \le Cr^k \quad (r := |z| \ge 1).$$

Thus we get the conclusion. $\square$

Let $I$ be a closed interval in $\mathbb{R}$, and let $u(x)$ be a real valued function defined on $I$. We define its $\mathcal{C}^1$-norm $\|u\|_{\mathcal{C}^1(I)}$ by setting

$$\|u\|_{\mathcal{C}^1(I)} := \sup_{x\in I} |u(x)| + \sup_{x\in I} |u'(x)|.$$

For a Lebesgue measurable set $E$ in $\mathbb{R}$, we denote its Lebesgue measure by $|E|$.

**Lemma 3.2.** *There is a positive number $\varepsilon$ satisfying the following: If a real valued function $u(x) \in \mathcal{C}^1[0,\pi]$ satisfies*

$$\|u(x) - \cos x\|_{\mathcal{C}^1[0,\,\pi]} \le \varepsilon, \tag{3.1}$$

*then we have*

$$|u^{-1}([-t,t])| \le 4t \quad \textit{for any } t \in [0,\varepsilon].$$

*Proof.* The proof is just an elementary calculus. We choose $\delta > 0$ so that $\sin x \ge 3/4$ for all $x \in [\pi/2-\delta, \pi/2+\delta]$. (Note that $\sin(\pi/2) = 1$.) We choose a positive number $\varepsilon < 1/4$ sufficiently small so that for any $u \in \mathcal{C}^1[0,\pi]$ satisfying (3.1) and for any $t \in [0,\varepsilon]$

$$u^{-1}([-t,t]) \subset [\pi/2-\delta, \pi/2+\delta].$$

Let $x_1$ and $x_2$ be any two elements in $u^{-1}([-t,t])$. From the mean value theorem, there exists $y \in [\pi/2-\delta, \pi/2+\delta]$ such that

$$u(x_1) - u(x_2) = u'(y)\,(x_1 - x_2).$$

We have $|u'(y)| \geq |\sin y| - |\sin y + u'(y)| \geq 3/4 - \varepsilon \geq 1/2$. Hence

$$|x_1 - x_2| \leq 2\,|u(x_1) - u(x_2)| \leq 4t.$$

Thus we get

$$|u^{-1}([-t, t])| \leq 4t.$$

□

Using a scale change of the coordinate, we get the following.

**Lemma 3.3.** *There is a positive number $\varepsilon$ satisfying the following: If a real valued function $u(x) \in \mathcal{C}^1[0, 2\pi]$ satisfies*

$$\|u(x) - \cos kx\|_{\mathcal{C}^1[0,\,2\pi]} \leq \varepsilon,$$

*then we have*

$$|u^{-1}([-t, t])| \leq 8t \quad \textit{for any } t \in [0, \varepsilon].$$

*Proof.*

$$u^{-1}([-t, t]) = \bigcup_{j=0}^{2k-1} u^{-1}([-t, t]) \cap [j\pi/k,\, (j+1)\pi/k].$$

Applying Lemma 3.2 to $u(x/k)$, we have

$$|u^{-1}([-t, t]) \cap [0, \pi/k]\,| \leq 4t/k.$$

In a similar way,

$$|u^{-1}([-t, t]) \cap [j\pi/k,\, (j+1)\pi/k]\,| \leq 4t/k \quad (j = 0, 1, \cdots, 2k-1).$$

Thus we get the conclusion. □

Let $E$ be a subset of $\mathbb{C}$. For a positive number $r$, we set

$$E(r) := \{\theta \in [0, 2\pi] |\, re^{i\theta} \in E\}.$$

In the rest of this section, we always assume $k \geq 2$.

**Lemma 3.4.** *Let $C$ be a positive constant, and let $g(z) = z^k + a_1 z^{k-1} + \cdots + a_k$ be a monic polynomial of degree $k$. Set*

$$E := \{z \in \mathbb{C} |\, |\mathrm{Re}\, g(z)| \leq C|z|\}.$$

*Then there exists a positive number $r_0$ such that*

$$|E(r)| \leq 8C/r^{k-1} \quad (r \geq r_0).$$

*Proof.* Set $v(z) := \mathrm{Re}\,(a_1 z^{k-1} + a_2 z^{k-2} + \cdots + a_k)$. Then we have

$$|\mathrm{Re}\, g(re^{i\theta})| \le Cr \quad \Longleftrightarrow \quad |\cos k\theta + v(re^{i\theta})/r^k| \le C/r^{k-1}.$$

Set $u(\theta) := \cos k\theta + v(re^{i\theta})/r^k$. It is easy to see that

$$\|u(\theta) - \cos k\theta\|_{\mathcal{C}^1[0,2\pi]} \le \mathrm{const}/r \quad (r \ge 1).$$

Then we can apply Lemma 3.3 to this $u(\theta)$, and we get

$$|E(r)| = |u^{-1}([-C/r^{k-1},\, C/r^{k-1}])| \le 8C/r^{k-1} \quad (r \gg 1).$$

Here we have used $k \ge 2$. □

The following is the key lemma.

**Lemma 3.5.** *Let $g(z) = a_0 z^k + a_1 z^{k-1} + \cdots + a_k$ be a polynomial of degree $k$ $(a_0 \ne 0)$. Set*

$$E := \{z \in \mathbb{C} |\, |\mathrm{Re}\, g(z)| \le |z|\}.$$

*Then there exists a positive number $r_0$ such that*

$$|E(r)| \le \frac{8}{|a_0| r^{k-1}} \quad (r \ge r_0).$$

*Proof.* Let $\arg a_0$ be the argument of $a_0$, and set $\alpha := \arg a_0/k$. We define the monic polynomial $g_1(z)$ by

$$g_1(z) := \frac{1}{|a_0|} g(e^{-i\alpha} z) = z^k + \cdots .$$

Then we have

$$|\mathrm{Re}\, g(re^{i\theta})| \le r \Longleftrightarrow |\mathrm{Re}\, g_1(re^{i(\theta+\alpha)})| \le r/|a_0|.$$

Hence the conclusion follows from Lemma 3.4. □

**Lemma 3.6.** *Let $g(z)$ be a polynomial of degree $k$, and we define $E$ as in Lemma 3.5. Set $h(z) := e^{g(z)}$. Then we have*

$$\int_{\mathbb{C}\setminus E} |dh|^2(z)\, dxdy < \infty.$$

*Proof.* Since $|h| = e^{\mathrm{Re}\, g}$, the argument in the proof of Lemma 2.1 gives

$$\sqrt{\pi}\, |dh| \le |g'| \min(|h|, |h|^{-1}) = |g'|\, e^{-|\mathrm{Re}\, g|}.$$

Note that $g'(z)$ is a polynomial of degree $k-1$ and that we have $|\mathrm{Re}\, g| > |z|$ for $z \in \mathbb{C} \setminus E$. Hence we have a positive constant $C$ such that

$$|dh|(z) \le C|z|^{k-1} e^{-|z|} \quad \text{if } z \in \mathbb{C} \setminus E \text{ and } |z| \ge 1.$$

The conclusion follows from this estimate. □

## 4. Proof of (1.2) ⇒ (1.3)

Let $f = [1 : f_1 : f_2 : \cdots : f_n] : \mathbb{C} \to X$ be a holomorphic map with $|df|(z) \le C|z|^m$ ($|z| \ge 1$). Since $\exp : \mathbb{C} \to \mathbb{C}^*$ is the universal covering, there exist entire holomorphic functions $g_i(z)$ such that $f_i(z) = e^{g_i(z)}$. We will prove that all $g_i(z)$ are polynomials of degree $\le m+1$. The proof falls into two steps. In the first step, we prove all $g_i(z)$ are polynomials. In the second step, we show $\deg g_i(z) \le m+1$. The second step is the harder part of the proof.

Schwarz's formula (see Ahlfors [1, p. 168]) gives[*1]

$$g_i(\zeta) = \frac{1}{\pi}\int_{|z|=r} \frac{z}{z-\zeta}\mathrm{Re}\,(g_i(z))d\theta + \mathrm{const}, \quad \text{where } |\zeta| < r \text{ and } \theta = \arg z.$$

Differentiating this equation, we get ($k \ge 1$)

$$\pi r^k g_i^{(k)}(0) = k!\int_{|z|=r} \mathrm{Re}\,(g_i(z))\, e^{-k\sqrt{-1}\theta}d\theta = k!\int_{|z|=r} \log|f_i(z)|\, e^{-k\sqrt{-1}\theta}d\theta.$$

We have

$$|\log|f_i|| \le \log(|f_i|+|f_i|^{-1}) = \log(1+|f_i|^2)-\log|f_i| \le \log(1+\sum|f_j|^2)-\log|f_i|.$$

Hence

$$\pi r^k|g_i^{(k)}(0)| \le k!\int_{|z|=r} \log(1+\sum|f_j|^2)\,d\theta - k!\int_{|z|=r}\log|f_i|\,d\theta.$$

Since $\log|f_i| = \mathrm{Re}\,g_i(z)$ is a harmonic function, the second term in the right-hand side is equal to the constant $-2\pi k!\,\mathrm{Re}\,g_i(0)$. Since $|df|^2 = \frac{1}{4\pi}\Delta\log(1+\sum|f_j|^2)$, Jensen's formula gives

$$\begin{aligned}\frac{1}{4\pi}\int_{|z|=r}\log(1+\sum|f_j|^2)\,d\theta - \frac{1}{4\pi}\int_{|z|=1}\log(1+\sum|f_j|^2)\,d\theta,\\ = \int_1^r \frac{dt}{t}\int_{|z|\le t}|df|^2(z)\,dxdy.\end{aligned}$$

Thus we get

$$\frac{r^k}{4k!}|g_i^{(k)}(0)| \le \int_1^r \frac{dt}{t}\int_{|z|\le t}|df|^2(z)\,dxdy + \mathrm{const}. \tag{4.1}$$

Since $|df|(z) \le C|z|^m$ ($|z| \ge 1$), the right-hand side is $O(r^{2m+2})$. Hence $g_i^{(k)}(0) = 0$ for $k \ge 2m+3$, and all $g_i(z)$ are polynomials (cf. Remark 1).

[*1] I learned the idea of using Schwarz's formula from Berteloot-Duval [2, Appendice]. I gave a different approach in [4, Section 6].

Next we will prove $\deg g_i(z) \leq m+1$. We define $E_i$, $E_{ij} \subset \mathbb{C}$ $(1 \leq i \leq n, 1 \leq i < j \leq n)$ by setting

$$\deg g_i(z) \leq m+1 \Longrightarrow E_i := \emptyset,$$
$$\deg g_i(z) \geq m+2 \Longrightarrow E_i := \{z \in \mathbb{C} \,|\, |\mathrm{Re}\, g_i(z)| \leq |z|\},$$
$$\deg(g_i(z) - g_j(z)) \leq m+1 \Longrightarrow E_{ij} := \emptyset,$$
$$\deg(g_i(z) - g_j(z)) \geq m+2 \Longrightarrow E_{ij} := \{z \in \mathbb{C} \,|\, |\mathrm{Re}\,(g_i(z) - g_j(z))| \leq |z|\}.$$

We set $E := \bigcup_i E_i \cup \bigcup_{i<j} E_{ij}$. Then we have $E(r) = \bigcup_i E_i(r) \cup \bigcup_{i<j} E_{ij}(r)$ for $r > 0$. From Lemma 3.5, we have positive constants $r_0$ and $C'$ such that

$$|E(r)| \leq C'/r^{m+1} \quad (r \geq r_0). \tag{4.2}$$

We have

(4.3)
$$\int_1^r \frac{dt}{t} \int_{|z| \leq t} |df|^2(z)\, dxdy = \int_1^r \frac{dt}{t} \int_{E \cap \{|z| \leq t\}} |df|^2(z)\, dxdy + \int_1^r \frac{dt}{t} \int_{E^c \cap \{|z| \leq t\}} |df|^2(z)\, dxdy.$$

Using (4.2) and $|df|(z) \leq C|z|^m$ $(|z| \geq 1)$, we can estimate the first term in the right-hand side of (4.3) as follows:

$$\int_{E \cap \{1 \leq |z| \leq t\}} |df|^2(z)\, dxdy \leq C^2 \int_{E \cap \{1 \leq |z| \leq t\}} r^{2m+1}\, drd\theta, \\ = C^2 \int_1^t r^{2m+1} |E(r)| dr.$$

If $t \geq r_0$, then

$$\int_{r_0}^t r^{2m+1} |E(r)| dr \leq C' \int_{r_0}^t r^m dr = \frac{C'}{m+1} t^{m+1} - \frac{C'}{m+1} {r_0}^{m+1}.$$

Thus

$$\int_1^r \frac{dt}{t} \int_{E \cap \{|z| \leq t\}} |df|^2(z)\, dxdy \leq \mathrm{const} \cdot r^{m+1} \quad (r \geq 1). \tag{4.4}$$

Next we will estimate the second term in the right-hand side of (4.3) by using the inequality (2.2) given in Section 2:

$$|df|^2 \leq \sum_i |df_i|^2 + \sum_{i<j} |d(f_i/f_j)|^2.$$

If $\deg g_i(z) \leq m+1$, then Lemma 3.1 gives

$$\int_1^r \frac{dt}{t} \int_{E^c \cap \{|z| \leq t\}} |df_i|^2(z)\, dxdy \leq \int_1^r \frac{dt}{t} \int_{|z| \leq t} |df_i|^2(z)\, dxdy \leq \mathrm{const} \cdot r^{m+1}.$$

If $\deg g_i(z) \geq m+2$, then Lemma 3.6 gives

$$\int_{E^c\cap\{|z|\leq t\}} |df_i|^2(z)\,dxdy \leq \int_{E_i^c\cap\{|z|\leq t\}} |df_i|^2(z)\,dxdy \leq \text{const.}$$

The terms for $|d(f_i/f_j)|$ can be estimated in the same way, and we get

$$\int_1^r \frac{dt}{t}\int_{E^c\cap\{|z|\leq t\}} |df|^2(z)\,dxdy \leq \text{const}\cdot r^{m+1} \quad (r\geq 1). \tag{4.5}$$

From (4.3), (4.4), (4.5), we get

$$\int_1^r \frac{dt}{t}\int_{|z|\leq t} |df|^2(z)\,dxdy \leq \text{const}\cdot r^{m+1} \quad (r\geq 1).$$

From (4.1), this shows $g_i^{(k)}(0) = 0$ for $k \geq m+2$. Thus $g_i(z)$ are polynomials with $\deg g_i(z) \leq m+1$. This concludes the proof of Theorem 1.1. □

## 5. Proof of Theorem 1.2

The proof of Theorem 1.2 needs the following lemma.

**Lemma 5.1.** *Let $k \geq 1$ be an integer, and let $\delta$ be a real number satisfying $0 < \delta < 1$. Let $g(z) = a_0z^k + a_1z^{k-1} + \cdots + a_k$ be a polynomial of degree $k$ ($a_0 \neq 0$). We set $h(z) := e^{g(z)}$ and define $E \subset \mathbb{C}$ by*

$$E := \{z \in \mathbb{C} |\, |\operatorname{Re} g(z)| \leq |z|^\delta\}.$$

*Then we have*

$$\int_{\mathbb{C}\setminus E} |dh|^2(z)\,dxdy < \infty,$$

*and there is a positive number $r_0$ such that*

$$|E(r)| \leq \frac{8}{|a_0|r^{k-\delta}} \quad (r \geq r_0).$$

*Proof.* This can be proved by the methods in Section 3. We omit the detail. □

Let $g_1(z), g_2(z), \cdots, g_n(z)$ be polynomials, and define the holomorphic map $f : \mathbb{C} \to X$ and the integer $m \geq -1$ by (1.3) and (1.5). Here we suppose $m \geq 0$, i.e., $f$ is not a constant map. We will prove Theorem 1.2.

From Theorem 1.1, we have

$$|df|(z) \leq \text{const}\cdot |z|^m \quad (|z| \geq 1).$$

It follows

$$\limsup_{r\to\infty} \frac{\max_{|z|=r}\log|df|(z)}{\log r} \leq m.$$

We want to prove that this is actually an equality. Suppose

$$\limsup_{r\to\infty} \frac{\max_{|z|=r} \log |df|(z)}{\log r} \lneqq m.$$

Then, if we take $\varepsilon > 0$ sufficiently small, there exists a positive number $r_0$ such that

$$|df|(z) \le |z|^{m-\varepsilon} \quad (|z| \ge r_0). \tag{5.1}$$

Schwarz's formula gives the inequality (4.1):

$$\frac{r^k}{4k!}|g_i^{(k)}(0)| \le \int_1^r \frac{dt}{t} \int_{|z|\le t} |df|^2(z)\, dxdy + \text{const} \quad (k \ge 0). \tag{5.2}$$

Let $\delta$ be a positive number such that $0 < \delta < 2\varepsilon$. We define $E_i$ and $E_{ij}$ $(1 \le i \le n,\ 1 \le i < j \le n)$ by setting

$$\begin{aligned}
&\deg g_i(z) \le m \Longrightarrow E_i := \emptyset,\\
&\deg g_i(z) = m+1 \Longrightarrow E_i := \{z \in \mathbb{C} |\, |\mathrm{Re}\, g_i(z)| \le |z|^\delta\},\\
&\deg (g_i(z) - g_j(z)) \le m \Longrightarrow E_{ij} := \emptyset,\\
&\deg (g_i(z) - g_j(z)) = m+1 \Longrightarrow E_{ij} := \{z \in \mathbb{C} |\, |\mathrm{Re}\,(g_i(z) - g_j(z))| \le |z|^\delta\}.
\end{aligned}$$

We set $E := \bigcup_i E_i \cup \bigcup_{i<j} E_{ij}$. Then, if we take $r_0$ sufficiently large, we have (from Lemma 5.1)

$$|E(r)| \le \text{const}/r^{m+1-\delta} \quad (r \ge r_0). \tag{5.3}$$

We have

$$\begin{aligned}
&\int_1^r \frac{dt}{t} \int_{|z|\le t} |df|^2(z)\, dxdy\\
&\qquad = \int_1^r \frac{dt}{t} \int_{E\cap\{|z|\le t\}} |df|^2(z)\, dxdy + \int_1^r \frac{dt}{t} \int_{E^c\cap\{|z|\le t\}} |df|^2(z)\, dxdy.
\end{aligned}$$

From (5.1) and (5.3), the first term in the right-hand side can be estimated as in Section 4:

$$\int_1^r \frac{dt}{t} \int_{E\cap\{|z|\le t\}} |df|^2(z)\, dxdy \le \text{const} \cdot r^{m+1-(2\varepsilon-\delta)} \quad (r \ge 1).$$

Using Lemma 5.1 and the inequality $|df|^2 \le \sum_i |df_i|^2 + \sum_{i<j} |d(f_i/f_j)|^2$, we can estimate the second term:

$$\int_1^r \frac{dt}{t} \int_{E^c\cap\{|z|\le t\}} |df|^2(z)\, dxdy \le \text{const} \cdot \log r + \text{const} \cdot r^m \quad (r \ge 1).$$

Thus we get

$$\int_1^r \frac{dt}{t} \int_{|z|\le t} |df|^2(z)\, dxdy \le \text{const} \cdot r^{m+1-(2\varepsilon-\delta)} \quad (r \ge 1).$$

Note that $2\varepsilon - \delta$ is a positive number. Using this estimate in (5.2), we get

$$g_i^{(k)}(0) = 0 \quad (k \geq m+1).$$

This shows $\deg g_i(z) \leq m$. This contradicts the definition of $m$. □

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