J. Math. Kyoto Univ. (JMKYAZ)
47-4 (2007), 827–836

Synge type theorems for positively curved Finsler manifolds

By

Chang-Wan Kim

Abstract

We study Synge type theorems for positively curved Finsler manifolds.

Introduction

Finsler geometry is actually the geometry of a simple integral and hence is differentiable metric geometry. Since the notion of Finsler manifolds is a generalization of Riemannian manifolds, it seems natural to consider the problem: How to distinguish Finsler manifolds from Riemannian manifolds? In this paper we will obtain Synge type and symmetry rank theorems for positively curved Finsler manifolds.

The reader recall Weinstein's theorem in Finsler geometry that any isometry f of an n-dimensional oriented Finsler manifold of positive flag curvature has a fixed point if either n is even and f preserves orientation or n is odd and f reverses orientation ([6]). Since Synge's theorem for Finsler manifolds ([2], [3]) is an easy corollary of this, it should not be surprising that with a little more work was we can prove the following theorem.

Theorem 1. *Assume that M be an oriented Finsler manifold with k-th Ricci curvature* $\mathrm{Ric}_k \geq k$. *Let f be an isometry satisfied* $\mathrm{dist}(x, f(x)) > \pi\sqrt{(k-1)/k}$ *for all $x \in M$.*

(1) *If M is even dimensional, then f reverses the orientation.*

(2) *If M is odd dimensional, then f is orientation preserving.*

The extra condition on the displacement function is not only necessary, as already observed, it is also the optimal extra conditions pointed out in [8]. This generalizes Weinstein's theorem for Finsler manifolds [6] because in case $k = 1$ the pinching constant is zero. Theorem 1 remains valid when the condition of the displacement function is replaced by the first systole of a given manifold, i.e., the length of the shortest closed noncontractible curve.

2000 *Mathematics Subject Classification*(*s*). 53C60, 53B40
Received January 9, 2007

Even before Weinstein's [7] observations Berger in 1965 proved another result along these lines for Riemannian manifolds. The following is only a reformulation of the some of the Synge type results as in the previous discussions.

Theorem 2. *Let M be an n-dimensional Finsler manifold of positive flag curvature and admit an effective isometric k-dimensional torus T^k-action.*

(1) *If n is even, then the fixed point set is not empty.*

(2) *If n is odd, then there is a circle orbit, i.e., not all isotropy subgroups can be finite.*

We can actually say something about Killing fields in positively curved Finsler manifolds. A vector field is Killing if and only if it is an infinitesimal isometry of manifolds. Recall that every vector field on a two-dimensional sphere has zero, since the Euler characteristic is two not zero. At some point Hopf conjectures that in fact even-dimensional Riemannian manifold with positive sectional curvature has positive Euler characteristic. From Theorem 2(1) we may derive the following partial justification for the Hopf conjecture in Finsler geometry:

Corollary. *If M is an even-dimensional compact Finsler manifold with positive flag curvature, then every Killing field X has a zero.*

Proof. If X has no zeros, then $\exp(tX)$ has no fixed points for a small t. Since X is an infinitesimal isometry of M, this is a contradiction by Theorem 2(1). □

Note that any isometry of a torus action must preserve orientation. We define the *symmetry rank* of a Finsler manifold M to be the rank of the isometry group of M. Clearly, M has symmetry rank $\geq k$ if and only if M admits an isometric k-dimensional torus T^k-action.

Theorem 3 (symmetry rank)**.** *Assume that M is an n-dimensional Finsler manifold with positive flag curvature. If M admits an isometric T^k-action, then $k \leq [(n+1)/2]$ (the integer part).*

Note that a sphere and a complex projective space with the quotient Riemannian metric from the Hopf fibration, $S^1 \to S^{2n+1} \to S^{2n+1}/S^1 \cong \mathrm{CP}^n$, have the maximal symmetric rank.

The author would like to thank the referee for useful comments and remarks.

1. Preliminaries

In this section, we shall recall some well-known facts about Finsler geometry. See [3], for more details. Let M be an n-dimensional smooth manifold and TM denote its tangent bundle. A *Finsler structure* on a manifold M is a map $F : TM \to [0, \infty)$ which has the following properties:

- F is smooth on $\widetilde{TM} := TM \setminus \{0\}$;

- $F(tv) = tF(v)$, for all $t > 0,\ v \in T_xM$;
- F^2 is strongly convex, i.e.,

$$g_{ij}(x,y) := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x,y)$$

is positive definite for all $(x,y) \in \widetilde{TM}$.

The Finsler structure F induces a distance dist on $M \times M$ by

$$\mathrm{dist}(p,q) := \inf_\gamma \int_0^1 F\big(\dot\gamma(t)\big)dt,$$

where the infimum is taken over all Lipschitz continuous curves $\gamma : [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. It is easy to verify that for all $p, q, r \in M$

$$\mathrm{dist}(p,r) \le \mathrm{dist}(p,q) + \mathrm{dist}(q,r).$$

At any point $p \in M$, there are an open neighborhood U of p, a constant $C \ge 1$ and a diffeomorphism $\psi : U \to \mathrm{B} \subset \mathrm{R}^n$ such that

$$|u - v|_{\mathrm{R}^n}/C \le d_F(\psi^{-1}(u), \psi^{-1}(v)) \le C \cdot |u - v|_{\mathrm{R}^n},\ u, v \in \mathrm{B}.$$

Thus $\mathrm{dist}(p,q) = 0$ if and only if $p = q$. We conclude that (M, dist) is a metric space and the Finsler manifold topology coincides with metric topology. A diffeomorphism is an isometry on a Finsler manifold M if it preserves this metric. By the classical van Dantzing and van der Waerden Theorem and Montgomery-Zippin Theorem, the group of isometries on a Finsler manifold form a Lie group (see [5, Chapter 1, Theorem 4.6]).

Let π^*TM denote the pull-back of the tangent bundle TM by $\pi : \widetilde{TM} \to M$. Denote vectors in π^*TM by $(v; w), v \in \widetilde{TM}, w \in T_{\pi(v)}M$. For the sake of simplicity, we denote by $\partial_i|_v = (v; \frac{\partial}{\partial x^i}|_x), v \in T_xM$, the natural local basis for π^*TM. The Finsler metric F defines two tensors g and A in π^*TM by

$$\begin{aligned} g(\partial_i|v, \partial_j|v) &:= g^v_{ij} := g_{ij}(x,y), \\ A(\partial_i|v, \partial_j|v, \partial_k|v) &:= \frac{1}{2}F(x,y)\frac{\partial g_{ij}}{\partial y^k}(x,y) := A_{ijk}(v), \end{aligned}$$

where $v = y^i \frac{\partial}{\partial x^i}|_x$. The tensors g and A are called the *fundamental* and *Cartan* tensors, respectively.

Unlike the Levi-Civita connection in Riemannian geometry, there is no unique natural connection in Finsler geometry. Among the various possible connections on π^*TM, we use the Chern connection which is defined by the unique set of local 1-forms $\{\omega_j{}^i\}_{1\le i,j\le n}$ on $\widetilde{TM}$ such that

$$\begin{aligned} d\omega^i &= \omega^j \wedge \omega_j{}^i, \\ dg_{ij} &= g_{kj}\omega_i{}^k + g_{ik}\omega_j{}^k + 2A_{ijk}\omega_n{}^k. \end{aligned}$$

We call $\{\omega_j{}^i\}$ the set of *local connection forms*. It defines a linear connection ∇ in π^*TM by

$$\nabla_{\hat{X}} Y := \{\hat{X}Y^i + Y^j \omega_j{}^i(\hat{X})\}E_i, \hat{X} \in T(\widetilde{TM}),\ Y = Y^i E_i \in C^\infty(\pi^*TM).$$

Define the set of local curvature forms $\Omega_j{}^i$ by

$$\Omega_j{}^i := d\omega_j{}^i - \omega_j{}^k \wedge \omega_k{}^i.$$

Then one can write

$$\Omega_j{}^i = \frac{1}{2} R_j{}^i{}_{kl} \omega^k \wedge \omega^l + P_j{}^i{}_{kl} \omega^k \wedge \omega^{n+l}.$$

Define the curvature tensors R, P in π^*TM by

$$R(U,V)W = u^k v^l w^j R_j{}^i{}_{kl} E_i, \qquad P(U,V)W = u^k v^l w^j P_j{}^i{}_{kl} E_i,$$

where $U = u^i E_i, V = v^i E_i, W = w^i E_i \in \pi^*TM$. The Chern connection ∇ in π^*TM defines the covariant derivative $D_v u$ of a vector field u on M in the direction $v \in T_xM$ as follows. Let c be a curve in M with $\dot{c}(0) = v$. Let $\hat{c} = \frac{dc}{dt}$ be the canonical lift of c in $\widetilde{TM}$. Let $u(t) = u|_{c(t)}$ and $U(t) := \big(\hat{c}; u(t)\big) \in \pi^*TM$. Define $D_v u$ by $(v; D_v u) := \nabla_{\frac{d\hat{c}}{dt}} U(0)$. A vector field $u = u(t)$ along c is called *parallel* if $D_{\frac{dc}{dt}} u = 0$. A curve $\gamma : [0,a] \to M$ is a geodesic if and only if $\dot{\gamma}$ is parallel along γ, i.e., $D_{\dot{\gamma}}\dot{\gamma} = 0$. In this case, γ must be parametrized proportional to the arc length. For the sake of simplicity, we shall denote $D_t = D_{\dot{\gamma}}$.

For a fixed $v \in T_xM$ let γ_v be the geodesic from x with $\dot{\gamma}_v(0) = v$. Along γ_v, we have a family of inner products $g^t = g^{\dot{\gamma}_v(t)}$ in $T_{\gamma_v(t)}M$. Define the *Riemann curvature* $R^t : T_{\gamma_v(t)}M \to T_{\gamma_v(t)}M$ by

$$R^t\big(u(t)\big) := R^{\dot{\gamma}_v(t)}\big(u(t)\big) := R\big(U(t), V(t)\big)V(t),$$

where $U(t) = \big(\hat{\gamma}_v(t); u(t)\big), V(t) = \big(\hat{\gamma}_v(t); \gamma_v(t)\big) \in \pi^*TM$. The Riemann curvature is independent of connections, that is, the term appears in the second variation formula of arc length, thus is of particular interest to us. We remark that if the Finsler metric F is Riemannian, then the Riemann curvature is the curvature tensor. Suppose that a two parameter variation $H(s,t) : (-\epsilon, \epsilon) \times [0,l] \to M$ is given. Further assume that it is the unit speed geodesic $H(0,t) = \gamma(t)$. The variation of arc length is the function

$$\mathcal{L}(s) = \int_0^l F\Big(\frac{\partial H}{\partial t}\Big)\, dt,$$

where l is the length of the curve. Since $\gamma(t)$ is a geodesic we know that the first variation formula vanishes. The second variation formula is

$$\begin{aligned}\frac{d^2\mathcal{L}}{ds^2}(0) = \int_0^l \Big\{ g^t\big(D_tX, D_tX\big) - g^t\big(D_tX(t), \dot{\gamma}_v(t)\big)^2 - g^t\big(R^t(X(t)), X(t)\big)\Big\}\, dt \\ + g^{\dot{\gamma}(l)}\big(\kappa_l(0), \dot{\gamma}(l)\big) - g^{\dot{\gamma}(0)}\big(\kappa_0(0), \dot{\gamma}(0)\big) + T_{\dot{\gamma}(0)}\big(X(0)\big) - T_{\dot{\gamma}(l)}\big(X(l)\big),\end{aligned}$$

where $X(t) = \frac{\partial H}{\partial s}|_{s=0}(s,t)$ is the variational vector field, $\kappa_l(0)$ and $\kappa_0(0)$ are the geodesic curvatures of the transversal curves $H(s,t)$ for $s=0$, and T is the tangent curvature. In the special case where the variation is transversal we have $g^{\dot\gamma(l)}\big(\kappa_l(0),\dot\gamma(l)\big) = g^{\dot\gamma(0)}\big(\kappa_0(0),\dot\gamma(0)\big) = 0$. We can also assume that the variation is g^t-perpendicular to the geodesic and the drop the term $g^t\big(D_tX(t),\dot\gamma_v(t)\big)^2$. In that case we arrive at the following simple formula,

$$\frac{d^2\mathcal{L}}{ds^2}(0)$$
$$= \int_0^l \Big\{g^t\big(D_tX, D_tX\big) - g^t\big(R^t(X(t)), X(t)\big)\Big\}\,dt + T_{\dot\gamma(0)}\big(X(0)\big) - T_{\dot\gamma(l)}\big(X(l)\big).$$

A vector field $J = J(t)$ along γ_v is called a *Jacobi field* if it satisfies

$$D_tD_tJ(t) + R^t(J(t)) = 0.$$

The *exponential map* $\exp_x : T_xM \to M$ is defined as usual, that is, $\exp_x(v) = \gamma_v(1)$. A vector field J_u along γ_v with $J_u(0) = 0$ and $D_tJ_u(0) = u$ is a Jacobi field if and only if $J_u(t) = d\exp_x|_{tv}tu$. For every Jacobi field J_u, $J_u(t)$ is perpendicular to $\gamma_v(t)$ for all t with respect to g^t if and only if u is perpendicular to v with respect to g^v (the Gauss Lemma). We can see that $\exp_x$ is singular at $rv \in T_xM$ if and only if there is $u \in T_xM\setminus\{0\}$, such that the Jacobi field J_u satisfies $J_u(r) = 0$. In this case we call $\gamma_v(r)$ a *conjugate point* with respect to x. For a unit vector $v \in T_xM$, we define c_v to be the first number $r > 0$ such that there is a Jacobi field $J(t)$ along $\gamma(t) = exp_x(tv)$, $0 \le t \le r$, satisfying

$$J(0) = 0 = J(r).$$

c_v is called the *conjugate value* of v.

To study the conjugate value of a tangent vector, we introduce the notion of index form. Let $\gamma(t) : [0,l] \to M$ be a unit speed geodesic. For vector fields $U = U(t)$ and $V = V(t)$ along $\gamma(t)$, define

$$\mathcal{I}_\gamma(U,V) := \int_0^l \Big\{g^t\big(D_tU(t), D_tV(t)\big) - g^t\big(R^t(U(t)), V(t)\big)\Big\}\,dt.$$

$\mathcal{I}_\gamma(U,V)$ is called the *index form* along γ. We first need the following lemma.

Lemma 1.1. *Let $0 \le r \le c_v$ and $\gamma_v : [0,r] \to M$ be the unit speed geodesic with $\dot\gamma(0) = v$. For any piecewise $\mathcal{C}^\infty$ vector field $V(t) \ne 0$ along $\gamma_v(t)$ with $V(0) = 0 = V(r)$, we have*

$$\mathcal{I}_\gamma(V,V) \ge 0$$

and the equality holds if and only if $r = c_v$ and V is Jacobi field along γ_v.

For a $(k+1)$-dimensional subspace $\mathcal{V} \subset T_xM$, define the Ricci curvature $\mathrm{Ric}_\mathcal{V}$ on $\mathcal{V}$ to be the trace of the Riemann curvature restricted to $\mathcal{V}$. $\mathrm{Ric}_\mathcal{V}$ is given by

$$\mathrm{Ric}_{\mathcal{V}}(v) := \sum_{i=1}^{k+1} g^v\big(R^v(e_i), e_i)\big), v \in \mathcal{V},$$

where $\{e_i\}_{i=1}^{k+1}$ is an orthonormal basis for $(\mathcal{V}, g^v)$. Put, k-th Ricci curvature,

$$\mathrm{Ric}_k := \inf_{\dim \mathcal{V}=k+1} \inf_{v\in\mathcal{V}} \frac{\mathrm{Ric}_{\mathcal{V}}(v)}{F^2(v)},$$

where the infimum is taken over all $(k+1)$-dimensional subspace $\mathcal{V} \subset T_xM$ and $v \in \mathcal{V} \setminus \{0\}$. $\mathrm{Ric}_1 \geq h$ is the same as flag curvature $\geq h$, and $\mathrm{Ric}_{\dim M-1} \geq h$ is the same as the usual $\mathrm{Ric} \geq h$.

A submersion $\rho : M \to N$ between Finsler manifolds is called an *isometric submersion* if for any point $x \in M$, the differential $d\rho_x : T_xM \to T_{\rho(x)}N$ maps the closed unit ball of T_xM onto the closed unit ball of $T_{\rho(x)}N$. In [1], Álvarez and Duran proved that

Lemma 1.2. *For any isometric submersion $\rho : M \to N$, the flag curvatures of M and N satisfy*

$$K_N(\Pi, w_{\rho(x)}) \geq K_M(\Xi, v_x),$$

where the flag (Ξ, v_x) of T_xM is the horizontal lift of the flag $(\Pi, w_{\rho(x)})$ of $T_{\rho(x)}N$.

As a consequence of the theory of Riemannian submersions, Finsler submersions do not decrease the flag curvature.

A large class of example of isometric submersions can be constructed by taking a principal fiber bundle $\rho : M \to N$ with structure group G, together with a G-invariant Finsler metric F on M. By the invariance of F, the image under the differential of ρ of the unit ball in any tangent space of M depends only on the fiber. It is possible then to define a Finsler metric $\hat{F}$, the subduced metric, on N such that the map ρ is an isometric submersion (see [1]).

2. Proof of Theorem 1

In this section we prove Theorem 1. Again the ideology of our proof is the same as in the original [8]; it is another illustration of the work of our second variation formula in Finsler manifolds. Before proving Theorem 1, we need a simple but useful lemma. The hypotheses on orientability and dimension parity are used to appeal to the following standard lemma of linear algebra.

Lemma 2.1. *Let B be an orthogonal linear transformation of* R^{n-1} *and suppose* $\det B = (-1)^n$. *Then B fixes some nonzero vector of* R^{n-1}.

Now we are ready to prove Theorem 1.

Proof. Part (1) follows if we show that the orientable isometry on even dimensional Finsler manifolds with $\mathrm{Ric}_k \geq k$ has minimal displacement $\leq \pi\sqrt{(k-1)/k}$. On the other hand, part (2) would follow if non-orientable isometry on odd dimensional Finsler manifolds with $\mathrm{Ric}_k \geq k$ has minimal displacement $\leq \pi\sqrt{(k-1)/k}$. The two part are proven together.

Since the Finsler manifold M is compact, the displacement function attains its minimum at a point $p \in M$. Let $\gamma : [0,l] \to M$ be a unit speed geodesic joining p and $f(p)$. To prove Theorem 1 it suffices to show that $l \leq \pi\sqrt{(k-1)/k}$. Consider the curve $f \circ \gamma$ which joins $f(p)$ and $f \circ f(p)$, and a point $q = \gamma(t), t \in (0,l)$. By the triangle inequality, it follows that

$$\begin{aligned}\mathrm{dist}(q, f(q)) &\leq \mathrm{dist}(q, f(p)) + \mathrm{dist}(f(p), f(q))\\ &= \mathrm{dist}(q, f(p)) + \mathrm{dist}(p, q)\\ &= \mathrm{dist}(p, f(p)).\end{aligned}$$

Since p is a minimum for the displacement function $\mathrm{dist}(x, f(x))$, we have

$$\mathrm{dist}(q, f(q)) = \mathrm{dist}(q, f(p)) + \mathrm{dist}(f(p), f(q)), \tag{2.2}$$

so this implies that it is smooth, that is $(f \dot{\circ} \gamma)(0) = \dot\gamma(l)$.

Denote by P_γ the parallel transport induced by the Chern connection along the minimal geodesic γ between the tangent spaces T_pM and $T_{f(p)}M$. Consider the map $P_\gamma^{-1} \circ df_p : T_pM \to T_pM$. We have the following relations

$$\begin{aligned}\big(P_\gamma^{-1} \circ df_p\big)\big(\dot\gamma(0)\big) &= P_\gamma^{-1}\big((f \dot{\circ} \gamma)(0)\big)\\ &= P_\gamma^{-1}\big(\dot\gamma(l)\big) = \dot\gamma(0),\end{aligned}$$

i.e., $P_\gamma^{-1} \circ df_p$ leaves $\dot\gamma(0)$ fixed. Let B be the restriction of $P_\gamma^{-1} \circ df_p$ to the $g^{\dot\gamma(0)}$-orthogonal complement of $\dot\gamma(0)$ in T_pM. Our hypotheses on f combined with Lemma 2.1 implies that the map $P_\gamma^{-1} \circ df_p : T_pM \to T_pM$ given by parallel transport around γ fixes a nonzero vector $g^{\dot\gamma(0)}$-perpendicular to $\dot\gamma(0)$. We can therefore find a unit parallel vector field $E_1(t)$ along $\gamma(t)$. Let $H : (-\epsilon, \epsilon) \times [0,l] \to M$ be the variation

$$H(s,t) := \exp_{\gamma(t)}\big(s \cdot E_1(t)\big),$$

and $\mathcal{L}(s)$ denote the length of the curve $t \mapsto H(s,t)$. Then from the second variation formula and the fact γ is a unit speed minimal geodesic we get

$$\begin{aligned}0 \leq \frac{d^2\mathcal{L}}{ds^2}(0) = &\int_0^l \Big\{g^t\big(D_tE_1, D_tE_1\big) - g^t\big(R^t(E_1(t)), E_1(t)\big)\Big\}\, dt\\ &+ T_{\dot\gamma(0)}\big(E_1(0)\big) - T_{\dot\gamma(l)}\big(E_1(l)\big).\end{aligned}$$

Since E_1 is parallel vector field along γ, we have the first term D_tE_1 is zero along γ. Finally by assumption of f and $df_p\big(E_1(0)\big) = E_1(l)$, we obtain $T_{\dot\gamma(0)}(E_1(0)) = T_{\dot\gamma(l)}(E_1(l))$. Hence the second variation formula reduces to

$$0 \leq \frac{d^2\mathcal{L}}{ds^2}(0) = -\int_0^l \Big\{g^t\big(R^t(E_1(t)), E_1(t)\big)\Big\}\, dt.$$

Let $\{E_i(t)\}_{i=1}^k$ be a parallel g^t-orthonormal k-frame along γ that is g^t-perpendicular to $\dot\gamma(t)$. Since $\mathrm{Ric}_k \geq k$ it follows from reduced second variation formula,

$$\frac{1}{l}\int_0^l \sum_{i=2}^k \Big\{ g^t\big(R^t(E_i(t)), E_i(t)\big)\Big\}\, dt \geq k. \tag{2.3}$$

For $2 \leq i \leq k$, let $W_i = \sin\left(\frac{\pi t}{l}\right) E_i(t)$, $0 \leq t \leq l$. By Lemma 1.1 we have

$$\begin{aligned} 0 &\leq \sum_{i=2}^k \mathcal{I}_\gamma(W_i, W_i) \\ &= \sum_{i=2}^k \int_0^l \sin^2\left(\frac{\pi t}{l}\right)\Big\{\frac{\pi^2}{l^2}\cdot g^t\big(E_i(t), E_i(t)\big) - g^t\big(R^t(E_i(t)), E_i(t)\big)\Big\}\, dt. \end{aligned}$$

Consider now the curve $(\gamma \cup f\circ\gamma) : [l/2, 3l/2] \to M$ given by

$$(\gamma \cup f\circ\gamma)(t) = \begin{cases} \gamma(t), & l/2 \leq t \leq l, \\ (f\circ\gamma)(t-l), & l \leq t \leq 3l/2, \end{cases}$$

and $\{E_i(t)\}_{i=1}^k$ parallelly extend along $(\gamma\cup f\circ\gamma)(t)$. By equality (2.2), $(\gamma \cup f\circ \gamma)|_{[l/2,3l/2]}$ is also minimal, so arguing as above we find

$$0 \leq \sum_{i=2}^k \int_{l/2}^{3l/2} \cos^2\left(\frac{\pi t}{l}\right)\Big\{\frac{\pi^2}{l^2}\cdot g^t\big(E_i(t), E_i(t)\big) - g^t\big(R^t(E_i(t)), E_i(t)\big)\Big\}\, dt.$$

We show below that if we make an appropriate choice of parametrization of γ and an appropriate choice of $\{E_i\}_{i=2}^k$, then the right hand side of the above term is

$$\sum_{i=2}^k \int_0^l \cos^2\left(\frac{\pi t}{l}\right)\Big\{\frac{\pi^2}{l^2}\cdot g^t\big(E_i(t), E_i(t)\big) - g^t\big(R^t(E_i(t)), E_i(t)\big)\Big\}\, dt.$$

Adding the resulting inequality given us

$$\sum_{i=2}^k \int_0^l \Big\{\frac{\pi^2}{l^2}\cdot g^t\big(E_i(t), E_i(t)\big) - g^t\big(R^t(E_i(t)), E_i(t)\big)\Big\}\, dt \geq 0.$$

Combining this with inequality (2.3) we have

$$\begin{aligned} \frac{\pi^2}{l}(k-1) &= \sum_{i=2}^k \int_0^l \Big\{\frac{\pi^2}{l^2}\cdot g^t\big(E_i(t), E_i(t)\big)\Big\}\, dt \\ &\geq \sum_{i=2}^k \int_0^l \Big\{ g^t\big(R^t(E_i(t)), E_i(t)\big)\Big\}\, dt \geq kl, \end{aligned}$$

or

$$\pi\sqrt{\frac{k-1}{k}} \geq l$$

as desired and this contradicts the minimality of γ, the curved joins p and $f(p)$. □

3. Proof of Theorems 2, 3

In this section we shall prove Theorems 2, 3. First we show Theorem 2.

Proof. Recall that if a Finsler manifold is not orientable, then there is a double covering space, $\hat{M} \to M$, which is orientable ($\hat{M}$ is obtained by the universal covering space quotient off the index two subgroup of orientation preserving isometries). Clearly Theorem 2 holds for the T^k-action if and only if it holds for the lifting $\hat{T}^k$-action on $\hat{M}$. Hence without loss of the generality, we would assume that M is orientable.

Part (1): We first prove for $k = 1$. If $T^1 = S^1$ has no fixed point set, then we may assume the isometry $f \in S^1$ that f has no fixed point on M. Note that any isometry of a torus action is orientation preserving, and in particular f is an orientation preserving isometry without fixed point, it is a contradiction to Theorem 1(1). For $k > 1$, if the T^k-action has no fixed point set, then there is a circle subgroup $S^1 \subset T^k$ without fixed point sets, it is also a contradiction to the above.

Part (2): We may assume that $k \geq 2$. If all isotropy subgroups of the T^k-action are finite. Since the action of a compact group on a topological space has finitely many orbit types, there are finitely many isotropy subgroups up to conjugacy. Since T^k is abelian, this implies that there are finitely many subgroups which are isotropy groups, all of which are finite. Hence, we can find circle subgroup $S^1 \subset T^k$ and $T^{k-1} = T^k/S^1$ such that $S^1 \cap T^{k-1} = \{1\}$ and S^1, T^{k-1} intersect every isotropy subgroup trivially. By construction S^1 and T^{k-1} both acts freely and isometrically on M. By Lemma 1.2, $T^{k-1} = T^k/S^1$ acts freely and isometrically on M/S^1 which is an even dimensional Finsler manifold with positive flag curvature. This contradicts Theorem 2(1). □

As an immediate consequence of this theorem, we shall prove Theorem 3.

Proof. We first assume that n is even . By Theorem 2(1) and isotropy representation of a T^k-fixed point, we may assume that T^k acts linearly and isometrically on T_xM. Hence T^k is the subgroup of orthogonal group $\mathrm{O}(n)$, and therefore $k \leq n/2$.

We then assume that n is odd. Similarly, by Theorem 2(2), we can consider the isotropy representation of a T^{k-1} in the normal space to a circle orbit, and conclude that $k - 1 \leq (n-1)/2$. □

Remark. In the case when the manifold M is of Riemannian type with maximal symmetry rank, M is equivariantly diffeomorphic to either a sphere, a real or complex projective space or a lens space (see [4]). Grove and Searle used the convexity of distance functions to prove this result in [4]. However a Finsler manifold with positive flag curvature does not have the convexity of distance, we cannot directly generalize the classification of Finsler manifolds as in Riemannian case.

Korea Institute for Advanced Study
207-43 CheongNyangNi 2-Dong
DongDaeMun-Gu, Seoul 130–722
Republic of Korea
e-mail: cwkimgrf@kias.re.kr

References

[1] J. C. Álvarez Paiva and C. E. Duran, *Isometric submersions of Finsler manifolds*, Proc. Amer. Math. Soc. **129** (2001), 2409–2417, MR1823926 (2002g:53128), Zbl0992.53055.

[2] L. Auslander, *On curvature in Finsler geometry*, Trans. Amer. Math. Soc. **79** (1955), 378–388, MR0071833 (17,190d), Zbl0066.16202.

[3] D. Bao, S. S. Chern and Z. Shen, *An introduction to Riemann-Finsler geometry*, Springer-Verlag, Grad. Texts in Math. **200** (2000), xx+431 pp., MR1747675 (2001g:53130), Zbl0954.53001.

[4] K. Grove and C. Seale, *Positively curved manifolds with maximal symmetry-rank*, J. Pure Appl. Algebra **91** (1994), 137–142, MR1255926 (95i:53040), Zbl0793.53040.

[5] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol. I, Interscience Publishers, a division of John Wiley & Sons, New York-Lond (1963), xi+329 pp., MR0152974 (27#2945), Zbl0508.53002.

[6] L. Kozma and I. R. Peter, *Weinstein's theorem for Finsler manifolds*, J. Math. Kyoto Univ. **46** (2006), 377–382, MR2284350 (2007k:53122), Zbl-pre05128994.

[7] A. Weinstein, *A fixed point theorem for positively curved manifolds*, J. Math. Mech. **18** (1968/1969), 149–153, MR0227894 (37#3478), Zbl0159.51401.

[8] F. Wilhelm, *On intermediate Ricci curvature and fundamental groups*, Illinois J. Math. **41** (1997), 488–494, MR1458186 (98h:53065), Zbl0883.53042.