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Strong unique continuation property for some second order elliptic systems with two independent variables

By

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Abstract

We show the strong unique continuation property for certain elliptic systems of second order with two independent variables.

1. Introduction

In this paper we prove the strong unique continuation property for some second order systems with two independent variables. As far as we know, there are few results for second order systems. On the other hand, there are many results for first order systems (for example [1], [3], [4] and [5]). In [3], Hile and Protter obtained an interesting result. They considered a system of the form

$$|\partial_x u + N(x,y)\partial_y u| \leq M|u| \quad \text{for all} \quad (x,y) \in \Omega \tag{1.1}$$

where Ω is a nonempty open connected subset of $\mathbf{R}^2$ containing the origin and $N(x,y)$ is an $n \times n$ matrix with complex entries of the class $C^1(\Omega)$. They proved, roughly speaking, that if N is a normal elliptic matrix, any u satisfying (1.1) and

$$\lim_{r\to 0} \exp{(x^2+y^2)^{-\beta/2}} u(x,y) = 0 \quad \text{for all} \quad \beta \geq 0 \tag{1.2}$$

vanishes in Ω where $r = \sqrt{x^2+y^2}$.

Okaji improved (1.2) in [5]. He proved that: Suppose that all the eigenvalues of $N(0,0)$ are ζ or $\bar{\zeta}$ with a non-real complex number ζ. Then there is a positive constant M_0 such that if $u \in C^1$ satisfies the inequality

$$|\partial_x u + N(x,y)\partial_y u| \leq M|u|/r \quad \text{for all} \quad (x,y) \in \Omega \tag{1.3}$$

with $M < M_0$ and vanishes of infinite order at the origin, then u is identically zero.

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Assuming that u verifies (1.3) and vanishes of infinite order at the origin he derives a stronger vanishing of u at the origin. Therefore he could use a stronger weight function than the usual weight $r^{-\beta}$.

We study the strong unique continuation property of solutions to some second order elliptic systems verifying (1.2) or vanishing of infinite order at the origin. In both cases we reduce our system to a first order system. In particular, in the case that u vanishes of infinite order at the origin we use a similar method as [5]. Then we shall apply Grammatico's result in [2].

We emphasize that there is no regularity assumptions on the eigenvalues of N as well as in [3] and [5].

2. Main results

Let Ω be a nonempty open connected subset of $\mathbf{R}^2$ containing the origin. We define $\dot{\Omega} = \Omega\backslash\{0\}$ and $B(\rho) = \{(x,y); x^2+y^2 \le \rho^2\}$. We denote by r the distance between (x,y) and the origin. The letter C stands for a generic constant whose value may vary from line to line. $X^1(\Omega)$ denotes the class of functions f defined on Ω satisfying the following properties (2.1) and (2.2):

$$f(x,y) \in C^0(\bar{\Omega}) \cap C^1(\dot{\Omega}) \tag{2.1}$$

where $\bar{\Omega}$ is the closure of Ω and

$$|\nabla f(x,y)| = O(r^{-1}) \tag{2.2}$$

where we shall use the notation $g(x,y) = O(h(x,y))$ if

$$\lim_{\rho\to 0} \sup_{0\le r\le\rho} |g(x,y)/h(x,y)| < \infty.$$

$X^{1,\kappa}(\Omega)$ denotes the class of functions $f \in X^1(\Omega)$ satisfying the following properties (2.3) and (2.4): $f(x,y)$ is Hölder continuous of order κ, that is, there exists a positive C such that

$$|f(x,y) - f(x',y')| \le C|(x,y)-(x',y')|^\kappa \tag{2.3}$$

for all $(x,y),(x',y') \in \Omega$ and

$$|\nabla f(x,y)| = o(r^{-1}) \tag{2.4}$$

where we shall use the notation $g(x,y) = o(h(x,y))$ if

$$\lim_{\rho\to 0} \sup_{0\le r\le\rho} |g(x,y)/h(x,y)| = 0.$$

Put $L^{(k)} = \partial_x + N_k(x,y)\partial_y, k = 1,2$ where $N_k(x,y)$ is an $n\times n$ normal matrix with complex entries of the class $X^1(\Omega)$. Moreover we shall assume that there exists a positive number δ such that

$$|\mathrm{Im}\lambda_j^{(k)}(x,y)| \ge \delta \quad j = 1,2,\dots,n$$

for all $(x,y) \in \Omega$ where $\lambda_j^{(k)}(x,y), j = 1,2,\dots,n$, are the eigenvalues of $N_k(x,y)$.

Theorem 2.1. *Let $L = L^{(1)}L^{(2)}$. Let $u \in H^2_{loc}(\Omega; \mathbf{C}^n)$ satisfy*

$$|Lu| \leq C_0 r^{-\beta_0}|u| + C_1 r^{-1}|\nabla u| \tag{2.5}$$

with some $C_0, C_1 \geq 0$ and $\beta_0 \in \mathbf{R}$. If u satisfies

$$\lim_{r\to 0} \exp(r^{-\beta}) \int_{B(r)} (|u|^2 + |\nabla u|^2)dxdy = 0 \quad \textit{for all } \beta > 0, \tag{2.6}$$

then u is identically zero in Ω.

Corollary 2.1. *Let $L = \partial_x^2 + A(x,y)\partial_y^2$ where $A(x,y)$ is an $n\times n$ normal matrix with complex entries of the class $X^1(\Omega)$. Let $\mu_j(x,y), j = 1,2,\dots,n$, be the eigenvalues of $A(x,y)$ and suppose that there exists a positive number δ such that*

$$\mathrm{dist}(\mu_j(x,y), (-\infty, 0]) \geq \delta \qquad j = 1,2,\dots,n$$

for all $(x,y) \in \Omega$. Let $u \in H^2_{loc}(\Omega; \mathbf{C}^n)$ satisfy (2.5) *with some $C_0, C_1 \geq 0$ and $\beta_0 \in \mathbf{R}$. If u satisfies* (2.6)*, then u is identically zero in Ω.*

Corollary 2.2. *Let $L = \partial_x^2 + 2B(x,y)\partial_{xy}^2 + A(x,y)\partial_y^2$ where $A(x,y)$ and $B(x,y)$ are $n \times n$ Hermitian matrices with complex entries of the class $X^1(\Omega)$ and satisfy $AB = BA$. Suppose that L is elliptic, that is, there exists a positive δ such that*

$$((\xi^2 + 2B(x,y)\xi\eta + A(x,y)\eta^2)v, v) \geq \delta(\xi^2+\eta^2)^{1/2}|v|^2 \tag{2.7}$$

for any $(\xi,\eta) \in \mathbf{R}^2 \setminus \{(0,0)\}$ and any $v \in \mathbf{C}^n$. Let $u \in H^2_{loc}(\Omega; \mathbf{C}^n)$ satisfy (2.5) *with some $C_0, C_1 \geq 0$ and $\beta_0 \in \mathbf{R}$. If u satisfies* (2.6)*, then u is identically zero in Ω.*

Remark 1. For $L = L^{(1)}L^{(2)}\cdots L^{(m)}$, we obtain a similar result as Theorem 2.1 if $N_k(x,y), k = 1,2,\dots,m$, belong to the class $C^m(\Omega)$.

Next we relax the assumption (2.6). In this case, we consider the system of differential operators $L = \partial_x^2 + N(x,y)^2\partial_y^2$ where $N(x,y)$ is an $n \times n$ normal matrix with complex entries of the class $X^{1,\kappa}(\Omega)$. Let $\lambda_j(x,y), j = 1,2,\dots,n$, be eigenvalues of $N(x,y)$. We suppose that there exists a positive number δ such that

$$|\mathrm{Re}\lambda_j(x,y)| \geq \delta \quad j = 1,2,\dots,n$$

for all $(x,y) \in \Omega$. Moreover we suppose that there exists $a \in \mathbf{R}$ such that

$$\lambda_j(0,0) = a \ \text{ or } \ -a \qquad j = 1,2,\dots,n.$$

Theorem 2.2. *Let $u \in H^2_{loc}(\Omega; \mathbf{C}^n)$ satisfy*

$$|Lu| \leq C_0 r^{-2}|u| + C_1 r^{-1}|\nabla u| \tag{2.8}$$

with $C_0 \geq 0$ and $0 \leq C_1 < \min\{1, |a|\}/\sqrt{2}$. If u satisfies

$$\lim_{r\to 0} r^{-\beta} \int_{B(r)} (|u|^2 + |\nabla u|^2)dxdy = 0 \quad \textit{for all } \beta > 0, \tag{2.9}$$

then u is identically zero in Ω.

Remark 2. In [6], he proved a similar result as Theorem 2.2 with $n = 1$ in $\mathbf{R}^d$.

Corollary 2.3. *Let $L = \partial_x^2 + A(x,y)\partial_y^2$, where $A(x,y)$ is an $n\times n$ normal matrix with complex entries of the class $X^{1,\kappa}(\Omega)$. Let $\mu_j(x,y)$ be eigenvalues of $A(x,y)$ and suppose that there exists a positive number δ such that*

$$\mathrm{dist}(\mu_j(x,y), (-\infty, 0]) \geq \delta \qquad j = 1, 2, \ldots, n$$

for all $(x,y) \in \Omega$. Moreover we suppose that there exists a positive number a such that

$$\mu_j(0,0) = a \qquad j = 1, 2, \ldots, n.$$

Let $u \in H^2_{loc}(\Omega; \mathbf{C}^n)$ satisfy (2.8) with $C_0 \geq 0$ and $0 \leq C_1 < \min\{1, \sqrt{a}\}/\sqrt{2}$. If u satisfies (2.9), then u is identically zero in Ω.

3. Proof of Theorem 2.1

Let $P = \partial_x + M(x,y)\partial_y$ where $M(x,y)$ is an $n \times n$ normal matrix with complex entries of the class $C^0(\bar{\Omega}) \cap C^1(\Omega)$ and

$$|\mathrm{Im}(\text{eigenvalues of } M(x,y))| \geq \delta.$$

Then in [3], they proved the following estimate.

Proposition 3.1 (Hile and Protter [3]). *There exists a positive C such that*

$$C \iint_\Omega e^{2\varphi} |Pu|^2 dxdy \geq \beta^2 \iint_\Omega e^{2\varphi} r^{-\beta-2} |u|^2 dxdy \tag{3.1}$$

for any $u \in C_0^1(\Omega)$ and any large β where $\varphi = r^{-\beta}$.

Remark 3. In [3], they assume $M(x,y) \in C^0(\bar{\Omega}) \cap C^1(\Omega)$. We obtain the same result if $M(x,y) \in X^1(\Omega)$.

We may assume $\Omega \subset B(1)$. By Proposition 3.1 we have the following Carleman estimate.

Lemma 3.1. *There exists a positive C such that*

$$C \iint e^{2\varphi} r^{-2\gamma} |L^{(2)} u|^2 dxdy \geq \beta^2 \iint e^{2\varphi} r^{-\beta-2-2\gamma} |u|^2 dxdy$$

for any large β and any $u \in C_0^1(\dot{\Omega})$ where $\varphi = r^{-\beta}$ and γ is a linear function of β.

Proof. Applying (3.1) with $P = L^{(2)}$ and $u = r^{-\gamma} u$, we have

$$C \iint e^{2\varphi} |L^{(2)}(r^{-\gamma} u)|^2 dxdy \geq \beta^2 \iint e^{2\varphi} r^{-\beta-2-2\gamma} |u|^2 dxdy.$$

Since $L^{(2)}$ is a first order operator

$$|L^{(2)}(r^{-\gamma}u)|^2 \leq Cr^{-2\gamma}|L^{(2)}u|^2 + C\gamma^2 r^{-2\gamma-2}|u|^2.$$

Therefore we obtain the desired estimate in Lemma 3.1 if β is large enough. □

We require the following elliptic estimate.

Lemma 3.2. *There exists a positive C such that*

$$\iint |\nabla u|^2 dxdy \leq C \iint (|L^{(2)}u|^2 + |u|^2)dxdy$$

for any $u \in C_0^1(\dot{\Omega})$.

Proof. Using a partition of unity, we reduce the problem to the case of finite number of constant matrices $\{N_2(x_j, y_j)\}_{j=1}^N$. Then the assertion can be easily verified in the standard manner. □

Applying Lemma 3.2 with $u = e^{\varphi}r^{-\gamma}u$, we have the following elliptic estimate with weight function.

Lemma 3.3. *There exists a positive C such that*

$$\iint e^{2\varphi}r^{-2\gamma}|\nabla u|^2 dxdy \leq C \iint e^{2\varphi}r^{-2\gamma}(|L^{(2)}u|^2 + \beta^2 r^{-2\beta-2}|u|^2)dxdy$$

for any $u \in C_0^1(\dot{\Omega})$ and any large β where $\gamma = \gamma_0\beta + \gamma_1$ with $\gamma_0, \gamma_1 \in \mathbf{R}$.

In order to prove Theorem 2.1, it suffices to prove the following Carleman estimate.

Proposition 3.2. *There exists a positive number C such that*

$$\int_\Omega e^{2\varphi}|Lu|^2 dxdy \geq C\beta^2 \int_\Omega e^{2\varphi}r^{-2}|\nabla u|^2 dxdy + C\beta^4 \int_\Omega e^{2\varphi}r^{-2\beta-4}|u|^2 dxdy$$

for all large $\beta \geq 0$ and any $u \in C_0^2(\dot{\Omega})$.

Proof. Applying (3.1) with $P = L^{(1)}$ and $u = L^{(2)}u$, we have

$$C \iint e^{2\varphi}|Lu|^2 dxdy \geq \beta^2 \iint e^{2\varphi}r^{-\beta-2}|L^{(2)}u|^2 dxdy. \tag{3.2}$$

By Lemma 3.1 we have

$$\beta^2 \iint e^{2\varphi}r^{-\beta-2}|L^{(2)}u|^2 dxdy \geq \beta^4 \iint e^{2\varphi}r^{-2\beta-4}|u|^2 dxdy. \tag{3.3}$$

On the other hand, by Lemma 3.3 we have

$$\begin{aligned}\beta^2 \iint e^{2\varphi}r^{-\beta-2}|L^{(2)}u|^2 dxdy &\geq \beta^2 \iint e^{2\varphi}r^{-2}|L^{(2)}u|^2 dxdy \\ &\geq C\beta^2 \iint e^{2\varphi}r^{-2}|\nabla u|^2 dxdy - C\beta^4 \iint e^{2\varphi}r^{-2\beta-4}|u|^2 dxdy.\end{aligned} \tag{3.4}$$

Combining (3.2), (3.3) and (3.4), we obtain the desired estimate in Proposition 3.2. □

Next we shall prove Corollary 2.1.

Proof. We define

$$B(x,y) = (2\pi i)^{-1} \oint_\Gamma \sqrt{\zeta}\ (\zeta - A(x,y))^{-1} d\zeta$$

where Γ is a closed curve in $\mathbf{C}\backslash(-\infty, 0]$ enclosing $\mu_j(x,y)$ $(j = 1,2,\dots,n)$, symmetric with respect to the real axis and $\sqrt{\zeta}$ means $r^{1/2}e^{\theta/2}$ when $\zeta = re^{\theta}$. Then applying Theorem 2.1 with $N_1(x,y) = iB(x,y)$ and $N_2(x,y) = -iB(x,y)$, we can prove Corollary 2.1. In fact, from the first resolvent equation

$$(z - A)^{-1}(\zeta - A)^{-1} = \{(z - A)^{-1} - (\zeta - A)^{-1}\}/(\zeta - z),$$

we have

$$\begin{aligned} B^2 &= (2\pi i)^{-2} \oint_\Gamma \sqrt{z}\ (z - A)^{-1} dz \oint_{\Gamma^-} \sqrt{\zeta}\ (\zeta - A)^{-1} d\zeta \\ &= (2\pi i)^{-2} \oint_\Gamma \sqrt{z}\ (z - A)^{-1}\{\oint_{\Gamma^-} \sqrt{\zeta}\ /(\zeta - z) d\zeta\} dz \\ &\quad + (2\pi i)^{-2} \oint_{\Gamma^-} \sqrt{\zeta}\ (\zeta - A)^{-1}\{\oint_\Gamma \sqrt{z}\ /(z - \zeta) dz\} d\zeta \end{aligned}$$

where Γ^- is a closed curve inside Γ and satisfies the same conditions as Γ. From

$$\oint_{\Gamma^-} \sqrt{\zeta}\ /(\zeta - z) d\zeta = 0 \quad \text{and} \quad (2\pi i)^{-1} \oint_\Gamma \sqrt{z}\ /(z - \zeta) dz = \sqrt{\zeta},$$

it verifies

$$B(x,y)^2 = (2\pi i)^{-1} \oint_{\Gamma^-} \zeta(\zeta - A)^{-1} d\zeta = A(x,y).$$

In what follows, we denote this $B(x,y)$ by $\sqrt{A(x,y)}$. Since Γ is symmetric, we have

$$\sqrt{A(x,y)}^{\,*} = (2\pi i)^{-1} \oint_\Gamma \sqrt{\zeta}\ (\zeta - A(x,y)^*)^{-1} d\zeta.$$

Hence it easily follows that $\sqrt{A(x,y)}$ is a normal matrix. Moreover it is easy to see that the eigenvalues of $\sqrt{A(x,y)}$ are $\sqrt{\mu_j}$ and entries of $\sqrt{A(x,y)}$ belong to $X^1(\Omega)$. □

In the rest of this section, we shall prove Corollary 2.2.

Proof. From our hypothesis, there exists a unitary matrix $U(x,y)$ such that

$$U^*AU = \operatorname{diag}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \quad \text{and} \quad U^*BU = \operatorname{diag}\begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix}.$$

Hence

$$U^*(A-B^2)U = \mathrm{diag}\begin{pmatrix} \lambda_1-\mu_1^2 & & \\ & \ddots & \\ & & \lambda_n-\mu_n^2 \end{pmatrix}.$$

By (2.7) we see that $\lambda_j(x,y)-\mu_j(x,y)^2 \geq \delta$ $(j=1,2,\ldots,n)$ for any $(x,y)\in\Omega$. Repeating the same arguments as the proof of Corollary 2.1 we can define $\sqrt{A(x,y)-B(x,y)^2}$. Since eigenvalues of $\sqrt{A-B^2}$ are $\sqrt{\lambda_j-\mu_j^2}$ and $\mu_j\in\mathbf{R}$, we have

$$\begin{aligned}|\mathrm{Im}(\text{eigenvalues of } B\pm i\sqrt{A-B^2})| &= |\mathrm{Im}(\mu_j \pm i\sqrt{\lambda_j-\mu_j^2})| \\ &= \sqrt{\lambda_j-\mu_j^2} \geq \delta.\end{aligned}$$

By Theorem 2.1 with $N_1(x,y)=B(x,y)+i\sqrt{A(x,y)-B(x,y)^2}$ and $N_2(x,y)=B(x,y)-i\sqrt{A(x,y)-B(x,y)^2}$, we obtain the desired conclusion of Corollary 2.2. □

4. Proof of Theorem 2.2

First we shall give the proof of Theorem 2.2 with $a=1$. We consider

$$L_0 = \partial_x^2 u + N(0,0)^2\partial_y^2 u.$$

Then the first result we will show is the Carleman estimate of L_0.

Proposition 4.1. *For an arbitrary positive $B<1$, there exists a positive number $\beta_0(B)$ such that if $\beta\geq\beta_0$ with $\beta\in\mathbf{N}+1/2$ then*

$$\begin{aligned}(1+\epsilon)\iint r^{-2\beta+2}|L_0u|^2dxdy \\ \geq B/2\iint r^{-2\beta}|\nabla u|^2dxdy + \epsilon\beta^2/4\iint r^{-2\beta-2}|u|^2dxdy.\end{aligned}$$

for any $u\in C_0^2(\dot{\Omega})$ and any positive ϵ.

Proof. By our hypothesis there exists a unitary matrix U_0 such that $U_0^{-1}L_0U_0=(\partial_x^2+\partial_y^2)I$. Introduce the polar coordinates $(x,y)=(r\cos\theta, r\sin\theta)$ and making the change of variables $z=\log r$ we see the following.

Lemma 4.1. *For arbitrary $B<1$ and $B'<1$, there exists a positive $\beta_0=\beta_0(B,B')$ such that if $\beta\geq\beta_0$ with $\beta\in\mathbf{N}+1/2$ then*

$$\begin{aligned}(1+\epsilon)\iint e^{-2\beta z}|(\partial_z^2+\partial_\theta^2)u|^2dzd\theta \geq \alpha B\iint e^{-2\beta z}|\partial_\theta u|^2dzd\theta \\ +(1-\alpha)B'\iint e^{-2\beta z}|\partial_z u|^2dzd\theta + \epsilon\beta^2/4\iint e^{-2\beta z}|u|^2dzd\theta\end{aligned}$$

for any positive $\epsilon>0$, any $\alpha\in[0,1]$ and $u\in C_0^2(\dot{\Omega})$.

Proof. We use the same method as [2]. We show it briefly (see [2] in detail). Putting $u = e^{\beta z} v$, we have

$$\iint e^{-2\beta z} |(\partial_z^2 + \partial_\theta^2) u|^2 dz d\theta = \iint |\partial_z^2 v + 2\beta \partial_z v + \beta^2 v + \partial_\theta^2 v|^2 dz d\theta.$$

By integration by parts, it follows that

$$\begin{aligned} 2\mathrm{Re}(\partial_z^2 v, \partial_z v) &= 2\mathrm{Re}(\partial_z v, v) = 2\mathrm{Re}(\partial_z v, \partial_\theta^2 v) = 0, \\ 2\mathrm{Re}(\partial_z^2 v, v) &= -2\|\partial_z v\|^2, \\ 2\mathrm{Re}(\partial_z^2 v, \partial_\theta^2 v) &= 2\|\partial_{z,\theta}^2 v\|^2, \\ 2\mathrm{Re}(v, \partial_\theta^2 v) &= -2\|\partial_\theta v\|^2, \end{aligned}$$

where $(\cdot, \cdot)$ is the L^2 inner product, and $\|\cdot\|$ is the L^2 norm. Therefore, we have

$$\iint e^{-2\beta z} |(\partial_z^2 + \partial_\theta^2) u|^2 dz d\theta \geq 2\beta^2 \|\partial_z v\|^2 + \|\partial_\theta^2 v\|^2 - 2\beta^2 \|\partial_\theta v\|^2 + \beta^4 \|v\|^2.$$

We use Fourier series expansion of $v(z, \cdot) \in L^2(\mathbf{S}^1)$:

$$v(z, \theta) = \sum_{k \in \mathbf{Z}} v_k(z) e^{ik\theta}, \quad \int_0^{2\pi} |v|^2 d\theta = \sum_{k \in \mathbf{Z}} |v_k(z)|^2.$$

Note that

$$\begin{aligned} \partial_\theta v(z, \theta) &= \sum_{k \in \mathbf{Z}} ik v_k(z) e^{ik\theta}, \quad \int_0^{2\pi} |\partial_\theta v|^2 d\theta = \sum_{k \in \mathbf{Z}} k^2 |v_k(z)|^2, \\ \partial_\theta^2 v(z, \theta) &= \sum_{k \in \mathbf{Z}} (-k^2) v_k(z) e^{ik\theta}, \quad \int_0^{2\pi} |\partial_\theta^2 v|^2 d\theta = \sum_{k \in \mathbf{Z}} k^4 |v_k(z)|^2. \end{aligned}$$

Thus, we have

$$\iint e^{-2\beta z} |(\partial_z^2 + \partial_\theta^2) u|^2 dz d\theta \geq 2\beta^2 \|\partial_z v\|^2 + \sum_{k \in \mathbf{Z}} (\beta^2 - k^2)^2 \int |v_k|^2 dz.$$

For any positive $B < 1$, there exists $\beta_0(B)$ such that if $\beta \geq \beta_0(B)$ with $\beta \in \mathbf{N} + 1/2$, we have

$$\sum_{k \in \mathbf{Z}} (\beta^2 - k^2)^2 \int |v_k|^2 dz \geq B \sum_{k \in \mathbf{Z}} k^2 \int |v_k|^2 dz = B \|\partial_\theta v\|^2.$$

Hence, we have

$$\iint e^{-2\beta z} |(\partial_z^2 + \partial_\theta^2) u|^2 dz d\theta \geq B \iint e^{-2\beta z} |\partial_\theta u|^2 dz d\theta. \tag{4.1}$$

On the other hand, for any positive B', there exists $\beta_1(B')$ such that if $\beta \geq \beta_1(B')$ with $\beta \in \mathbf{N} + 1/2$, we have

$$\sum_{k\in\mathbf{Z}}(\beta^2 - k^2)^2 \int |v_k|^2 dz \geq B'\beta^2 \sum_{k\in\mathbf{Z}} \int |v_k|^2 dz = B'\beta^2\|v\|^2.$$

Hence, we have

$$\begin{aligned} \iint e^{-2\beta z}|(\partial_z^2 + \partial_\theta^2)u|^2 dzd\theta &\geq B'(\beta^2\|v\|^2 + \|\partial_z v\|^2) \\ &\geq B'\|\partial_z v + \beta v\|^2 \\ &= B' \iint e^{-2\beta z}|\partial_z u|^2 dzd\theta. \end{aligned} \tag{4.2}$$

Combining (4.1) and (4.2), for any positive $B > 1$ and $B' > 1$ there exists $\beta_0(B, B')$ such that if $\beta \geq \beta_0(B, B')$ with $\beta \in \mathbf{N} + 1/2$, we have

$$\begin{aligned} &\iint e^{-2\beta z}|(\partial_z^2 + \partial_\theta^2)u|^2 dzd\theta \\ &\quad\geq \alpha B \iint e^{-2\beta z}|\partial_\theta u|^2 dzd\theta + (1-\alpha)B' \iint e^{-2\beta z}|\partial_z u|^2 dzd\theta \end{aligned} \tag{4.3}$$

for any $\alpha \in [0, 1]$. We recall that the inequality

$$\iint e^{-2\beta z}|(\partial_z^2 + \partial_\theta^2)u|^2 dzd\theta \geq \beta^2/4 \iint e^{-2\beta z}|u|^2 dzd\theta \tag{4.4}$$

holds (see the appendix of [2]). (4.3) and (4.4) show the desired conclusion of Lemma 4.1. □

Now, we proceed to the proof of Proposition 4.1. From Lemma 4.1 with $B = B'$ and $\alpha = 1/2$, it follows

$$\begin{aligned} &(1+\epsilon) \iint r^{-2\beta+2}|(\partial_x^2 + \partial_y^2)u|^2 dxdy \\ &\qquad\qquad \geq B/2 \iint r^{-2\beta}|\nabla u|^2 dxdy + \epsilon\beta^2/4 \iint r^{-2\beta-2}|u|^2 dxdy, \end{aligned}$$

which proves the desired result. □

Proposition 4.1 and (2.3) give the following Carleman inequality with a remainder term.

Proposition 4.2. *For arbitrary $B < 1$, there exists a positive $\beta_0 = \beta_0(B)$ such that if $\beta \geq \beta_0$ with $\beta \in \mathbf{N} + 1/2$ then*

(4.5)

$$\begin{aligned} &(1+\epsilon)(1+\delta) \iint r^{-2\beta+2}|Lu|^2 dxdy + C(1+\epsilon)(1+\delta^{-1}) \iint r^{-2\beta+2+2\kappa}|\partial_y^2 u|^2 dxdy \\ &\qquad\qquad \geq B/2 \iint r^{-2\beta}|\nabla u|^2 dxdy + \epsilon\beta^2/4 \iint r^{-2\beta-2}|u|^2 dxdy \end{aligned}$$

for any positive ϵ, δ and any $u \in C_0^2(\dot{\Omega})$.

Proof. We can write

$$Lu = L_0 u + (N(x,y)^2 - N(0,0)^2)\partial_y^2 u,$$

and

$$|N(x,y)^2 - N(0,0)^2| \le Cr^{\kappa}$$

because of their Hölder continuity. Using

$$\begin{aligned} &|Lu - (N(x,y)^2 - N(0,0)^2)\partial_y^2 u|^2 \\ &\qquad \le (1+\delta)|Lu|^2 + C(1+\delta^{-1})|(N(x,y)^2 - N(0,0)^2)\partial_y^2 u|^2, \end{aligned}$$

the proof is clear. □

We require the following elliptic estimate.

Lemma 4.2. *There exists a positive constant C such that*

$$\iint_{\Omega} (|\partial_x^2 u|^2 + |\partial_y^2 u|^2)dxdy \le C \iint_{\Omega} (|Lu|^2 + |\nabla u|^2 + |u|^2)dxdy$$

for any $u \in C_0^2(\Omega)$.

Applying Lemma 4.2 with $u = r^{-\beta}u$, we have

Lemma 4.3. *There exists a positive constant C such that*

$$\begin{aligned} &\iint_{\Omega} r^{-2\beta}(|\partial_x^2 u|^2 + |\partial_y^2 u|^2)dxdy \\ &\qquad \le C \iint_{\Omega} r^{-2\beta}(|Lu|^2 + \beta^2 r^{-2}|\nabla u|^2 + \beta^4 r^{-4}|u|^2)dxdy \end{aligned}$$

for any $u \in C_0^2(\dot{\Omega})$.

Proposition 4.3. *Under the assumption of Theorem 2.2, there exist positive constants C_2 and C_3 such that*

$$\iint_{0 \le R(x,y) \le \rho} (|u|^2 + |\nabla u|^2 + |\partial_x^2 u|^2 + |\partial_y^2 u|^2)dxdy \le C_2 \exp(-C_3 \rho^{-\kappa})$$

for any small positive ρ.

Proof. Let $\chi(r)$ be a nonnegative function belonging to $C_0^1([0,2))$ such that $\chi(r) = 1$ when $0 \le r < 1$. We shall consider $\tilde{u}(x,y) = \chi(M\beta^{1/\kappa}r)u(x,y)$. Here, M is a large positive parameter, which will be determined later. By

Proposition 4.2 and Lemma 4.3, we have

(4.6)
$$\begin{aligned}(B/2 - C/K) \iint r^{-2\beta} |\nabla \tilde{u}|^2 dxdy + (\epsilon/4 - C/K)\beta^2 \iint r^{-2\beta-2} |\tilde{u}|^2 dxdy \\ + (K\beta^2)^{-1} \iint r^{-2\beta+2} (|\partial_x^2 \tilde{u}|^2 + |\partial_y^2 \tilde{u}|^2) dxdy \\ \leq \{(1+\epsilon)(1+\delta) + C(K\beta^2)^{-1}\} \iint r^{-2\beta+2} |L\tilde{u}|^2 dxdy \\ + C(1+\epsilon)(1+\delta^{-1}) \iint r^{-2\beta+2+2\kappa} |\partial_y^2 \tilde{u}|^2 dxdy\end{aligned}$$

where K is a large parameter which will be determined later. On the other hand, for all positive ϵ_1 we have

(4.7)
$$\begin{aligned}\iint r^{-2\beta+2} |L\tilde{u}|^2 dxdy \leq (1+\epsilon_1) \iint r^{-2\beta+2} |\chi L u|^2 dxdy + C(1+\epsilon_1^{-1}) \times \\ \times \iint_{B(2M^{-1}\beta^{-1/\kappa}) \setminus B(M^{-1}\beta^{-1/\kappa})} r^{-2\beta+2} (M^2 \beta^{2/\kappa} |\nabla u|^2 + M^4 \beta^{4/\kappa} |u|^2) dxdy.\end{aligned}$$

because of

$$1 \leq M^2 \beta^{2/\kappa} r^2 \leq 4 \quad \text{if } (x,y) \in B(2M^{-1}\beta^{-1/\kappa}) \setminus B(M^{-1}\beta^{-1/\kappa}).$$

From (2.8), we have

(4.8)
$$\begin{aligned}\iint r^{-2\beta+2} |\chi L u|^2 dxdy \leq (1+\epsilon_2)(1+\epsilon_3) C_1^2 \iint r^{-2\beta} |\nabla \tilde{u}|^2 dxdy \\ + C(1+\epsilon_2)(1+\epsilon_3^{-1}) C_1^2 M^2 \beta^{2/\kappa} \iint_{B(2M^{-1}\beta^{-1/\kappa}) \setminus B(M^{-1}\beta^{-1/\kappa})} r^{-2\beta} |u|^2 dxdy \\ + (1+\epsilon_2^{-1}) C_0^2 \iint r^{-2\beta-2} |\tilde{u}|^2 dxdy\end{aligned}$$

for all positive ϵ_2 and ϵ_3. Thus, combining (4.6), (4.7) and (4.8), we see that

$$\begin{aligned}T_1 \iint r^{-2\beta} |\nabla \tilde{u}|^2 dxdy + T_2 \beta^2 \iint r^{-2\beta-2} |\tilde{u}|^2 dxdy \\ + (K\beta^2)^{-1} \iint r^{-2\beta+2} (|\partial_x^2 \tilde{u}|^2 + |\partial_y^2 \tilde{u}|^2) dxdy \\ \leq T_3 \iint r^{-2\beta+2+2\kappa} |\partial_y^2 \tilde{u}|^2 dxdy \\ + T_4 \iint_{B(2M^{-1}\beta^{-1/\kappa}) \setminus B(M^{-1}\beta^{-1/\kappa})} r^{-2\beta+2} (M^2 \beta^{2/\kappa} |\nabla u|^2 + M^4 \beta^{4/\kappa} |u|^2) dxdy \\ + T_5 M^2 \beta^{2/\kappa} \iint_{B(2M^{-1}\beta^{-1/\kappa}) \setminus B(M^{-1}\beta^{-1/\kappa})} r^{-2\beta} |u|^2 dxdy,\end{aligned}$$

where

$$T_1 = B/2 - C/K - (1+\epsilon_1)(1+\epsilon_2)(1+\epsilon_3)\{(1+\epsilon)(1+\delta) + C(K\beta^2)^{-1}\}C_1^2,$$
$$T_2 = (\epsilon/4 - C/K) - (1+\epsilon_1)(1+\epsilon_2^{-1})\{(1+\epsilon)(1+\delta) + C(K\beta^2)^{-1}\}C_0^2\beta^{-2}$$

and T_3, T_4, T_5 are positive constants depending only on δ, ϵ_1 and ϵ_3. Take $\epsilon, \delta, \epsilon_1, \epsilon_2$ and ϵ_3 to be small enough. Moreover taking K to be large enough, by our assumption, T_1 and T_2 are positive if β is large enough. Choose M such that $T_3 M^{-2\kappa} < 1/(8K)$. Then it holds that

$$T_3 r^{2\kappa} \le 1/(2K\beta^2)$$

if $(x,y) \in B(2M^{-1}\beta^{-1/\kappa})$. Then it follows that

$$\begin{aligned} T_1 &\iint_{B(1/2M^{-1}\beta^{-1/\kappa})} r^{-2\beta}|\nabla u|^2 dxdy \\ &+ T_2\beta^2 \iint_{B(1/2M^{-1}\beta^{-1/\kappa})} r^{-2\beta-2}|u|^2 dxdy \\ &+ (2K\beta^2)^{-1} \iint_{B(1/2M^{-1}\beta^{-1/\kappa})} r^{-2\beta+2}(|\partial_x^2 u|^2 + |\partial_y^2 u|^2) dxdy \\ \le C &\iint_{B(2M^{-1}\beta^{-1/\kappa})\setminus B(M^{-1}\beta^{-1/\kappa})} r^{-2\beta}|\nabla u|^2 dxdy \\ &+ C \iint_{B(2M^{-1}\beta^{-1/\kappa})\setminus B(M^{-1}\beta^{-1/\kappa})} r^{-2\beta-2}|u|^2 dxdy. \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} &\iint_{B(1/2M^{-1}\beta^{-1/\kappa})} (|\nabla u|^2 + |u|^2 + |\partial_x^2 u|^2 + |\partial_y^2 u|^2) dxdy \\ &\le C 2^{-2\beta+2}(M\beta^{1/\kappa})^4 \beta^2 \times \\ &\times \iint_{B(2M^{-1}\beta^{-1/\kappa})\setminus B(M^{-1}\beta^{-1/\kappa})} (|\nabla u|^2 + |u|^2 + |\partial_x^2 u|^2 + |\partial_y^2 u|^2) dxdy \end{aligned}$$

for any large $\beta \in \mathbf{N} + 1/2$. This gives the conclusion of Proposition 4.3. □

Now we recall an estimate in the case of a first order system. Let $P = \partial_x + M(x,y)\partial_y$ where $M(x,y)$ is an $n \times n$ normal matrix with complex entries of the class $X^{1,\kappa}(\Omega)$ and

$$|\mathrm{Im}(\text{eigenvalues of } M(x,y))| \ge \delta.$$

Moreover suppose that all the eigenvalues of $M(0,0)$ are ζ or $\bar{\zeta}$ with a non-real complex number ζ. Then in [5], he proved the following estimate.

Proposition 4.4 (Okaji [5]). *For a sufficiently small $\tilde{\Omega}$, there exists a positive C independent of $\tilde{\Omega}$ such that*

$$C \iint_{\tilde{\Omega}} e^{\beta(\log r)^2}|Pu|^2 r^{-1} dxdy \ge \beta \iint_{\tilde{\Omega}} r^{-2}|\log r| e^{\beta(\log r)^2}|u|^2 r^{-1} dxdy$$

for any $u \in C_0^1(\dot{\Omega})$ and any large β.

By Proposition 4.4 with $u = r^{-1}|\log r|^{1/2}u$ we have the following estimate.

Lemma 4.4. *For a sufficiently small $\tilde{\Omega}$, there exists a positive C independent of $\tilde{\Omega}$ such that*

$$\iint_{\tilde{\Omega}} r^{-2}|\log r|e^{\beta(\log r)^2}|Pu|^2 r^{-1}dxdy \geq C\beta \iint_{\tilde{\Omega}} r^{-4}|\log r|^2 e^{\beta(\log r)^2}|u|^2 r^{-1}dxdy$$

for any $u \in C_0^1(\dot{\Omega})$ and any large β.

Thus, we have the following Carleman estimate with a stronger weight function.

Proposition 4.5. *For a sufficiently small $\tilde{\Omega}$, there exists a positive C independent of $\tilde{\Omega}$ such that*

$$\int_{\tilde{\Omega}} e^{\beta(\log r)^2}|Lu|^2 r^{-1}dxdy \geq C\beta \iint_{\tilde{\Omega}} r^{-2}|\log r|e^{\beta(\log r)^2}|\nabla u|^2 r^{-1}dxdy$$
$$+ C\beta^2 \iint_{\tilde{\Omega}} r^{-4}|\log r|^2 e^{\beta(\log r)^2}|u|^2 r^{-1}dxdy$$

for any $u \in C_0^2(\dot{\Omega})$ and any large β.

Proof. Putting

$$\tilde{L} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \partial_x + \begin{pmatrix} N & 0 \\ 0 & -N \end{pmatrix} \partial_y \quad \text{and} \quad U = \begin{pmatrix} u \\ u \end{pmatrix},$$

it follows that

$$\tilde{L}U = \begin{pmatrix} \partial_x u + N(x,y)\partial_y u \\ \partial_x u - N(x,y)\partial_y u \end{pmatrix}, \tilde{L}^2 U = \begin{pmatrix} Lu + A(x,y)u \\ Lu + B(x,y)u \end{pmatrix}$$

where $A(x,y)u = -N_x\partial_y u + NN_y\partial_y u$ and $B(x,y)u = N_x\partial_y u + NN_y\partial_y u$. Since $|A(x,y)u|, |B(x,y)u| \leq Cr^{-1}|\nabla u|$ we have

$$\int e^{\beta(\log r)^2}|Lu|^2 r^{-1}dxdy \geq C\int e^{\beta(\log r)^2}|\tilde{L}(\tilde{L}U)|^2 r^{-1}dxdy$$
$$- C\int e^{\beta(\log r)^2} r^{-2}|\nabla u|^2 r^{-1}dxdy.$$

By Proposition 4.4 with $P = \tilde{L}$ and $u = \tilde{L}U$ we have

$$C\iint_{\tilde{\Omega}} e^{\beta(\log r)^2}|\tilde{L}(\tilde{L}U)|^2 r^{-1}dxdy \geq \beta \iint_{\tilde{\Omega}} r^{-2}|\log r|e^{\beta(\log r)^2}|\tilde{L}U|^2 r^{-1}dxdy$$

for a sufficiently small $\tilde{\Omega}$. Moreover applying Lemma 4.4 with $P = \tilde{L}$ and $u = U$ we have

$$\iint_{\tilde{\Omega}} r^{-2}|\log r|e^{\beta(\log r)^2}|\tilde{L}U|^2 r^{-1}dxdy$$
$$\geq C\beta \iint_{\tilde{\Omega}} r^{-4}|\log r|^2 e^{\beta(\log r)^2}|U|^2 r^{-1}dxdy.$$

On the other hand, we have

$$\iint_{\tilde{\Omega}} r^{-2}|\log r|e^{\beta(\log r)^2}|\tilde{L}U|^2 r^{-1}dxdy \geq C \iint_{\tilde{\Omega}} r^{-2}|\log r|e^{\beta(\log r)^2}|\nabla u|^2 r^{-1}dxdy$$

from

$$\begin{aligned}|\tilde{L}U|^2 &= |\partial_x u + N(x,y)\partial_y u|^2 + |\partial_x u - N(x,y)\partial_y u|^2 \\ &= 2|\partial_x u|^2 + 2|N(x,y)\partial_y u|^2 \\ &\geq 2\min\{1,\delta^2\}|\nabla u|^2.\end{aligned}$$

Thus we obtain the desired estimate in Proposition 4.5. □

Theorem 2.2 with $a = 1$ follows from Proposition 4.3 and 4.5.

Proof. Suppose that R_0 is sufficiently small so that Proposition 4.5 holds for $\tilde{\Omega} = B(R_0)$. Fix $0 < R_1 < R_0$ and take $\delta > 0$ and a smooth function $\chi_\delta \in C_0^\infty(0, R_0)$ such that

$$\chi_\delta(r) = \begin{cases} 1 & \text{if} \quad \delta \leq r \leq R_1 \\ 0 & \text{if} \quad r \leq \delta/2 \end{cases}, \quad |\chi'_\delta(r)| = \begin{cases} C\delta^{-1} & \text{if} \quad \delta/2 \leq r \leq \delta \\ C & \text{if} \quad R_1 \leq r \leq R_0 \end{cases}$$

and

$$|\chi''_\delta(r)| = \begin{cases} C\delta^{-2} & \text{if} \quad \delta/2 \leq r \leq \delta \\ C & \text{if} \quad R_1 \leq r \leq R_0 \end{cases}$$

for a positive constant C. By Proposition 4.5 it follows that

$$\begin{aligned}C\beta &\iint_{B(R_1)\setminus B(\delta)} r^{-2}|\log r|e^{\beta(\log r)^2}|\nabla u|^2 r^{-1}dxdy \\ &\quad + C\beta^2 \iint_{B(R_1)\setminus B(\delta)} r^{-4}|\log r|^2 e^{\beta(\log r)^2}|u|^2 r^{-1}dxdy \\ &\leq C\beta \iint_{B(R_0)} r^{-2}|\log r|e^{\beta(\log r)^2}|\nabla(\chi_\delta u)|^2 r^{-1}dxdy \\ &\quad + C\beta^2 \iint_{B(R_0)} r^{-4}|\log r|^2 e^{\beta(\log r)^2}|\chi_\delta u|^2 r^{-1}dxdy \\ &\leq \iint_{B(R_0)} e^{\beta(\log r)^2}|L(\chi_\delta u)|^2 r^{-1}dxdy.\end{aligned}$$

From (2.8) we have

$$\begin{aligned}
&\iint_{B(R_0)} e^{\beta(\log r)^2}|L(\chi_\delta u)|^2 r^{-1}dxdy \\
&\quad\le C\iint_{B(R_0)\setminus B(R_1)} e^{\beta(\log r)^2}(|Lu|^2+|\nabla u|^2+|u|^2)r^{-1}dxdy \\
&\qquad+\iint_{B(R_1)\setminus B(\delta)} e^{\beta(\log r)^2}|Lu|^2 r^{-1}dxdy \\
&\qquad+C\iint_{B(\delta)\setminus B(\delta/2)} e^{\beta(\log r)^2}(|Lu|^2+\delta^{-2}|\nabla u|^2+\delta^{-4}|u|^2)r^{-1}dxdy \\
&\quad\le C\iint_{B(R_0)\setminus B(R_1)} e^{\beta(\log r)^2}(r^{-2}|\nabla u|^2+r^{-4}|u|^2)r^{-1}dxdy \\
&\qquad+C\iint_{B(R_1)\setminus B(\delta)} e^{\beta(\log r)^2}(r^{-2}|\nabla u|^2+r^{-4}|u|^2)r^{-1}dxdy \\
&\qquad+C\iint_{B(\delta)\setminus B(\delta/2)} e^{\beta(\log r)^2}(\delta^{-2}r^{-2}|\nabla u|^2+\delta^{-4}r^{-4}|u|^2)r^{-1}dxdy.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&(C\beta-C)\iint_{B(R_1)\setminus B(\delta)} r^{-2}|\log r|e^{\beta(\log r)^2}|\nabla u|^2 r^{-1}dxdy \\
&\qquad+(C\beta^2-C)\iint_{B(R_1)\setminus B(\delta)} r^{-4}|\log r|^2 e^{\beta(\log r)^2}|u|^2 r^{-1}dxdy \\
&\quad\le C\iint_{B(R_0)\setminus B(R_1)} e^{\beta(\log r)^2}(r^{-2}|\nabla u|^2+r^{-4}|u|^2)r^{-1}dxdy \\
&\qquad+C\iint_{B(\delta)\setminus B(\delta/2)} e^{\beta(\log r)^2}(\delta^{-2}r^{-2}|\nabla u|^2+\delta^{-4}r^{-4}|u|^2)r^{-1}dxdy.
\end{aligned}$$

Since

$$\begin{aligned}
&C\iint_{B(\delta)\setminus B(\delta/2)} e^{\beta(\log r)^2}(\delta^{-2}r^{-2}|\nabla u|^2+\delta^{-4}r^{-4}|u|^2)r^{-1}dxdy \\
&\qquad\le Ce^{\beta(\log\delta/2)^2}\delta^{-4}\iint_{B(\delta)}|\nabla u|^2dxdy+Ce^{\beta(\log\delta/2)^2}\delta^{-8}\iint_{B(\delta)}|u|^2dxdy,
\end{aligned}$$

this integral tend to zero if $\delta\to 0$ by Proposition 4.3. Hence letting δ tend to zero it follows that

$$\begin{aligned}
&(C\beta-C)\iint_{B(R_1)} r^{-2}|\log r|e^{\beta(\log r)^2}|\nabla u|^2 r^{-1}dxdy \\
&\qquad+(C\beta^2-C)\iint_{B(R_1)} r^{-4}|\log r|^2 e^{\beta(\log r)^2}|u|^2 r^{-1}dxdy \\
&\quad\le C\iint_{B(R_0)\setminus B(R_1)} e^{\beta(\log r)^2}(r^{-2}|\nabla u|^2+r^{-4}|u|^2)r^{-1}dxdy.
\end{aligned}$$

Thus we have

$$(C\beta - C)R_1^2|\log R_1| \iint_{B(R_1)} (|\nabla u|^2 + |u|^2)dxdy$$
$$\le C \iint_{B(R_0)\setminus B(R_1)} (|\nabla u|^2 + |u|^2)dxdy < \infty.$$

Letting β large enough, we have that u is identically zero in $B(R_1)$. By Theorem 2.1 with $N_1(x, y) = iN(x, y)$ and $N_2(x, y) = -iN(x, y)$ we have that u is identically zero in Ω. □

Next we prove Theorem 2.2 with $a \in \mathbf{R}$.

Proof. Setting $v(x, y) = u(x, ay)$ it follows that

$$\begin{aligned}|(\partial_x^2 + a^{-2}N(x, ay)^2\partial_y^2)v(x, y)| &= |L(u(x, ay))| \\ &\le C_0 r^{-2}|u(x, ay)| + C_1 r^{-1}|(\nabla u)(x, ay)| \\ &\le C_0 r^{-2}|v(x, y)| + C_1 r^{-1}\max\{1, |a|^{-1}\}|\nabla v(x, y)|.\end{aligned}$$

By Theorem 2.2 with $a = 1$, v is identically zero in Ω if $C_1 < \min\{1, |a|\}/\sqrt{2}$. Therefore u is identically zero in Ω. □

Finally we shall prove Corollary 2.3.

Proof. We define $\sqrt{A(x, y)}$ in the same way as the proof of Corollary 2.1. Then $\sqrt{A(x, y)}$ satisfies the assumptions of Theorem 2.2 because the eigenvalues of $\sqrt{A(x, y)}$ are $\sqrt{\mu_j(x, y)}$. Hence, by Theorem 2.2 with $N(x, y) = \sqrt{A(x, y)}$ the proof is complete. □

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