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On entire curves tangent to a foliation

By

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1. Introduction

Let X be a complex projective manifold, equipped with a holomorphic foliation by curves $\mathcal{F}$, with a possibly nonempty singular set $Sing(\mathcal{F})$ (analytic of codimension at least 2). In this paper we are interested in the structure of the set of (nonconstant) entire curves $f : \mathbb{C} \to X$ tangent to $\mathcal{F}$. That is, $f'(z) \in T_{f(z)}\mathcal{F}$ whenever $f(z) \notin Sing(\mathcal{F})$.

A particular aspect of this problem was already treated in [Br3]. In that paper we defined the leaf L_p of $\mathcal{F}$ through a nonsingular point $p \in X^0 = X \setminus Sing(\mathcal{F})$ (this is not straightforward due to the presence of singularities), and we proved that the set

$$\mathcal{P} = \{p \in X^0 | L_p \text{ is parabolic } \}$$

is either the full X^0 or a *complete pluripolar* subset of X^0, i.e. locally given by the poles of a plurisubharmonic function. Here "L_p parabolic" means that its universal covering is $\mathbb{C}$ or $\mathbb{C}P^1$, equivalently there exists a nonconstant map from $\mathbb{C}$ to L_p. Because the leaf L_p has a natural map to X, we see that if p belongs to $\mathcal{P}$ then there exists an entire curve in X tangent to $\mathcal{F}$ and passing through p. However, the converse statement is generally speaking false: the problem is that an entire curve tangent to $\mathcal{F}$ may pass through singular points of $\mathcal{F}$, and these singular points may not belong to the corresponding leaf of $\mathcal{F}$, which therefore may be hyperbolic (according to the definition of [Br3], in some special cases a leaf may pass through singular points, but this is a very exceptional situation, which does not affect substantially the previous discrepancy). In other words, setting

$$\mathcal{E} = \{p \in X^0 | \text{ there exists a nonconstant } f : \mathbb{C} \to X,\ f(0) = p,\ \text{tangent to } \mathcal{F} \}$$

then $\mathcal{P} \subset \mathcal{E}$, but the inclusion may be strict. It can even happen that $\mathcal{P} = \emptyset$ and $\mathcal{E} = X^0$ (example: take the radial foliation of $\mathbb{C}P^n$ and transform it by a "generic" birational map of $\mathbb{C}P^n$, so that all the leaves become hyperbolic).

Remark that $\mathcal{E}$ is $\mathcal{F}$-invariant, so that in order to understand its (local) structure it is sufficient to look at its trace on $\mathcal{F}$-transverse discs. Let us

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introduce some notation and terminology. Given an embedded $(n-1)$-disc $T \subset X^0$ transverse to the foliation, set $\mathcal{E}(T) = \mathcal{E} \cap T$. Given an embedded k-disc $S \subset T$, $1 \le k \le n-1$, set $\mathcal{E}(S) = \mathcal{E} \cap S$. Here "embedded" means, more precisely, "properly embedded in a neighbourhood of it". Finally, we shall say that a subset of a disc is *thin* if it is contained in a countable union of locally analytic subsets (of positive codimension), where a subset is locally analytic if it is analytic in a neighbourhood of it.

Theorem 1.1. *Let X be a complex projective manifold of dimension n, equipped with a holomorphic foliation $\mathcal{F}$ of dimension 1, with singular set $Sing(\mathcal{F})$. Let $T \subset X^0$ be an embedded $(n-1)$-disc transverse to $\mathcal{F}$, and let $S \subset T$ be an embedded k-disc, $1 \le k \le n-1$. Then there exists a (canonical) splitting*

$$\mathcal{E}(S) = \mathcal{P}(S) \cup \mathcal{Z}(S)$$

where $\mathcal{Z}(S)$ is thin in S and $\mathcal{P}(S)$ is either complete pluripolar in S or full. Moreover, in the latter case ($\mathcal{P}(S) = S$) there exists a meromorphic map

$$F : S \times \mathbb{C} \dashrightarrow X$$

such that for every $s \in S$ the restriction $F(s, \cdot) : \mathbb{C} \to X$ is (after removal of indeterminacies) an entire curve tangent to $\mathcal{F}$ and sending 0 to $s \in S \subset X$.

Warning: $\mathcal{P}(S)$ is *not* the trace on S of parabolic leaves, as defined in [Br3]. The set $\mathcal{P}(S)$ will be defined below, in Section 4, as the set of points through which the "leaf relative to S" is parabolic. Even for $S = T$ we may have that $\mathcal{P}(T)$ is different (larger) than $\mathcal{P} \cap T$, because the leaves defined here are slightly different from the ones defined in [Br3], see remark (a) in Section 3.

The meaning of this Theorem is that $\mathcal{E}(S)$ splits into two parts: the "good" part $\mathcal{P}(S)$, corresponding to entire curves which can be lifted to the covering tube U_S (Section 3), therefore glueing together in a "holomorphic" family; and the "bad" part $\mathcal{Z}(S)$, corresponding to entire curves which, due to holonomy or other accidents, cannot be lifted to U_S. But this bad part is relatively negligible, because it is thin.

An immediate corollary to Theorem 1.1 is that either $\mathcal{E}(S) = S$ or $\mathcal{E}(S)$ is pluripolar in S. However, in the latter case we don't know if $\mathcal{E}(S)$ is actually *complete* pluripolar, i.e. *equal* to the poles of a plurisubharmonic function and not simply *contained* in such poles: the problem is that a thin subset is always pluripolar but it may fail to be complete. In particular, by a connectivity argument $\mathcal{E}$ is either the full X^0 or a pluripolar subset of X^0, but in the latter case we don't know if it is complete pluripolar (we suspect that the answer could be negative, in general).

Anyway, given a transverse disc T we have the splitting $\mathcal{E}(T) = \mathcal{P}(T) \cup \mathcal{Z}(T)$, and we can apply again Theorem 1.1 to a countable collection of subdiscs of T, of positive codimensions, whose union covers $\mathcal{Z}(T)$. By iterating this process we finally obtain

$$\mathcal{E}(T) = \left[\cup_{j=0}^{\infty} \mathcal{P}(S_j) \right] \cup \mathcal{R}$$

where $\mathcal{R}$ is countable, each S_j is an embedded subdisc of T (with $S_0 = T$), and each $\mathcal{P}(S_j) \subset S_j$ is either complete pluripolar in S_j or the full S_j (we could include $\mathcal{R}$ inside $[\dots]$, by allowing 0-dimensional subdiscs). This is perhaps rather noncanonical, but we shall see in the Appendix a potential application.

Theorem 1.1 will be proved in the next three sections, using some ideas from [Br3] and [Br4] developed in a new "relative" context in order to take into account the restriction to S. The first section contains a Levi-type Extension Lemma, independent on foliations but indispensable for further constructions. In fact, such a Lemma was already implicitely used in [Br2, page 124] and consequently in [Br3, page 146]: the direct use of Ivashkovich's theorem [Iv1] was slightly unjustified there [C-I], and should be replaced by our "unparametrized" Levi-type Extension Lemma. The second section introduces holonomy and covering tubes, in the relative context, and states their basic properties, in particular concerning their parabolic fibres. We shall refer to [Br5] for some properties related to the holomorphic convexity of these tubes. The third section is the key one: we prove that most entire curves (i.e. outside a thin subset) tangent to the foliation can be lifted to holonomy and covering tubes, giving the proof of Theorem 1.1. The motivation of Theorem 1.1, including its cumbersome relative statement, comes from a still virtual application to Lang's conjecture for entire curves on surfaces of general type. This will be explained in the Appendix.

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2. An Extension Lemma

Let X be a compact connected Kähler manifold and let A_r be the semi-closed annulus $\{r < |w| \leq 1\}$, $r \in (0,1)$, with boundary $\partial A_r = \{|w| = 1\}$.

Given a holomorphic immersion $f : A_r \to X$, we shall say that $f(A_r)$ **extends to a disc** if there exists a holomorphic map $g : \overline{\mathbb{D}} \to X$ (not necessarily immersive) such that f factorizes as $g \circ j$ for some embedding $j : A_r \to \overline{\mathbb{D}}$ sending ∂A_r to $\partial \mathbb{D}$. That is, f itself does not need to extend to $\{|w| \leq 1\}$, but it extends "after a reparametrization".

Remark that if f is generically injective, i.e. injective outside a discrete subset, and $f(A_r)$ extends to a disc, then we can find an extending map $g : \overline{\mathbb{D}} \to X$ which is also generically injective. Indeed, given any extension g, its image $g(\overline{\mathbb{D}})$ is either a (singular) disc in X with boundary $f(\partial A_r)$, so that g is already generically injective, or a (singular) rational curve in X, over which $f(\partial A_r)$ bounds a (singular) disc. Such a generically injective extension g is then uniquely defined up to a Moëbius reparametrization of $\overline{\mathbb{D}}$. We shall say that g, or $g(\overline{\mathbb{D}})$, is a **simple** extension of $f(A_r)$.

Given a holomorphic immersion $f : \mathbb{D}^k \times A_r \to X$, we shall say that $f(\mathbb{D}^k \times A_r)$ **extends to a meromorphic family of discs** if there exists a complex $(k+1)$-manifold with boundary W, a holomorphic submersion $W \to \mathbb{D}^k$ all of whose fibres are isomorphic to $\overline{\mathbb{D}}$, a meromorphic map $g : W \dashrightarrow X$, such that f factorizes as $g \circ j$ for some embedding $j : \mathbb{D}^k \times A_r \to W$ sending

$\mathbb{D}^k \times \partial A_r$ to ∂W and $\{z\} \times A_r$ to the fibre of W over z, for every z. In particular, for every $z \in \mathbb{D}^k$ the restriction of g to the fibre of W over z gives, after removal of indeterminacies, a disc which extends $f(z, A_r)$. The manifold W is differentiably the product of $\mathbb{D}^k$ with $\overline{\mathbb{D}}$, but in general this does not hold holomorphically. However, it is a standard fact that for any $z \in \mathbb{D}^k$ and any compact $K \subset W \setminus \partial W$ in the fibre over z, there exists a neighbourhood V of K, projecting to a neighbourhood U of z, such that the restriction $V \to U$ is a holomorphically trivial disc-bundle, i.e. $V \simeq U \times \mathbb{D}$.

The following result is a sort of "unparametrized" Levi continuity principle [Siu], [Iv2], and its proof is largely inspired by Ivashkovich's approach to the classical "parametrized" Levi continuity principle [Iv1], [Iv2]. The new difficulty is that we have to construct not only a map but also the space where it is defined.

Proposition 2.1. *Let X be a compact connected Kähler manifold and let $f : \mathbb{D}^k \times A_r \to X$ be a holomorphic immersion, such that $f(z, \cdot)$ is an embedding for every z outside a proper analytic subset $I \subset \mathbb{D}^k$. Suppose that there exists a subset $N \subset \mathbb{D}^k$ such that*:

(i) *for every $z \in N$, $f(z, A_r)$ extends to a disc*;

(ii) *N is* not *thin in $\mathbb{D}^k$.*

Then $f(\mathbb{D}^k \times A_r)$ extends to a meromorphic family of discs.

Proof. The proof will be done in two steps. Firstly, we shall extend f to a *holomorphic* family of discs over some open subset $V \subset \mathbb{D}^k$ with thin complement. Secondly, we shall meromorphically extend over $\mathbb{D}^k \setminus V$ by a Riemann type argument and the Thullen type extension theorem of Siu [Siu], [Iv1].

As a preliminary remark, let us observe that we may assume that the annuli $f(z, A_r)$, $z \in \mathbb{D}^k$, are pairwise disjoint, up to replacing $f : \mathbb{D}^k \times A_r \to X$ with $(f, \jmath) : \mathbb{D}^k \times A_r \to X \times \mathbb{C}P^k$ where $\jmath$ is the composition of the projection to $\mathbb{D}^k$ and any embedding $\mathbb{D}^k \hookrightarrow \mathbb{C}P^k$. Such a replacement allows also to assume that f is an embedding on the full $(\mathbb{D}^k \setminus I) \times A_r$.

Set

$$Z = \{z \in \mathbb{D}^k \setminus I \mid f(z, A_r) \text{ extends to a disc}\}.$$

We give to Z the following metrizable topology: if $z, z' \in Z$ then their distance is the Hausdorff distance in X between $g(\overline{\mathbb{D}})$ and $g'(\overline{\mathbb{D}})$, where $g, g' : \overline{\mathbb{D}} \to X$ are simple extensions of $f(z, A_r)$, $f(z', A_r)$. Remark that this topology may be *finer* than the topology induced by the inclusion $Z \subset \mathbb{D}^k$: two maps g and g' may be very close on $\partial\mathbb{D}$ but very distant inside $\mathbb{D}$ (think to blowing-up).

Next, we give to Z a natural structure of an analytic space. This is "well known", but let us give anyway some details for reader's convenience and for lack of an appropriate precise reference (see, however, [Iv2] and references therein).

Take $z \in Z$ and $g : \overline{\mathbb{D}} \to X$ a simple extension of $f(z, A_r)$. Recall that $f(z, \cdot)$ is an embedding but g is only a generically injective map which may have singular points and selfintersections (if g also is an embedding everything below becomes much more simpler). Take a Stein neighbourhood U of $g(\overline{\mathbb{D}})$ in X. Then we properly embed U in some Euclidean space $\mathbb{C}^N$. In this way, the annuli $f(z', A_r)$, $z' \in \mathbb{D}^k$ close to z, are transferred to embedded semiclosed annuli $A_{z'} \subset \mathbb{C}^N$. If $A_{z'}$ extends to a disc in $\mathbb{C}^N$, then such a disc is necessarily in U, and it is as close to $g(\overline{\mathbb{D}})$ as z' is close to z, by the maximum principle. Therefore, a neighbourhood of z in Z, in the Z-topology, can be identified with a standard neighbourhood of z in the set $\mathcal{D} = \{z' \mid A_{z'}$ extends to a disc in $\mathbb{C}^N\}$. We need to prove that $\mathcal{D}$ is an analytic subset.

Consider the possibly larger set $\mathcal{R} = \{z' \mid A_{z'}$ extends to a compact complex curve with boundary in $\mathbb{C}^N\}$. This is an analytic subset. Indeed, given an embedded circle $\gamma \subset \mathbb{C}^N$, the condition "$\gamma$ bounds a compact curve Γ" is equivalent, by a theorem of Wermer (or Harvey–Lawson) [A-W, Ch. 19], to the moment condition: $\int_\gamma \eta = 0$ for every polynomial 1-form η. If γ depends holomorphically on a parameter z' (like our $\partial A_{z'}$), then $\int_\gamma \eta$ is a holomorphic function of z'. By Noetherianity, we obtain the analyticity of $\mathcal{R}$.

Over $\mathcal{R}$ we have a tautological analytic space (with boundary) $\mathcal{M} \xrightarrow{\pi} \mathcal{R}$, equipped with a holomorphic map to $\mathbb{C}^N$ which sends $\pi^{-1}(z')$ to a curve bounded by $\partial A_{z'}$. More explicitely, $\mathcal{M}$ is the subset of $\mathcal{R} \times \mathbb{C}^N$ which cuts $\{z'\} \times \mathbb{C}^N$ along the curve bounded by $\partial A_{z'}$. The fact that $\mathcal{M}$ is an analytic space follows from inspection of the proof of Wermer theorem recalled above (basically, the holomorphic dependence on parameters of the Cauchy transform, see [A-W, Ch. 19]). The subset $\mathcal{D} \subset \mathcal{R}$ coincides with those z' such that the fibre $\pi^{-1}(z')$ has (geometric) genus zero. It is not difficult to see that this is a closed analytic subset of $\mathcal{R}$; more generally, $\{z' \mid \mathrm{genus}(\pi^{-1}(z')) \leq k\}$ is a closed analytic subset of $\mathcal{R}$. To see this, we replace $\mathcal{M}$ by its normalisation, so that the fibration over $\mathcal{R}$ becomes smooth over a Zariski open dense subset $\mathcal{R}_0$. Thus the genus is constant on $\mathcal{R}_0$, and by an easy topological argument it can only decrease on $\mathcal{R} \setminus \mathcal{R}_0$. Then we restrict over this last subset, and we repeat the argument, obtaining in this way the Zariski lower continuity of the geometric genus. In particular, $\mathcal{D}$ is an analytic subset of $\mathcal{R}$.

Thus, Z is an analytic space, and by construction the inclusion $Z \subset \mathbb{D}^k$ is a holomorphic injection. Moreover, over Z we have the tautological fibration $\pi : Y \to Z$, equipped with a holomorphic map $h : Y \to X$, such that for every $z \in Z$ the fiber $\pi^{-1}(z)$ is a closed disc (after normalisation) and $h|_{\pi^{-1}(z)} : \pi^{-1}(z) \to X$ extends $f(z, A_r)$.

We want now to prove that Z has a *countable* number of irreducible components. To this end, let us firstly recall the following standard fact: there exists a constant $c > 0$ (which we may and will assume equal to 2) such that every rational curve in X has area at least c.

Consider the area function

$$a : Z \longrightarrow \mathbb{R}^+$$

$a(z)$ = area of the disc extending $f(z, A_r)$. It is obviously continuous (in the Z-topology). We can cover Z with the open subsets $Z_n = \{n-1 < a < n+1\}$, $n \in \mathbb{N}$. In each Z_n we may choose a countable subset $L_n \subset Z_n$ which is dense in Z_n with the $\mathbb{D}^k$-topology. We claim that L_n is dense in Z_n also in the Z-topology. Indeed, take $z_\infty \in Z_n$ and let $\{z_j\} \subset L_n$ be a sequence which $\mathbb{D}^k$-converge to z_∞. We thus have, in X, a sequence of discs $\{D_j\}$, extending $\{f(z_j, A_r)\}$, with areas in the interval $(n-1, n+1)$. By Bishop compactness theorem [Iv1, Prop. 3.1], this sequence (or a subsequence of it) converges to a disc with bubbles, necessarily of the form $D_\infty \cup R$ where D_∞ is a disc extending $f(z_\infty, A_r)$ and R is a union of rational curves (the bubbles). Moreover, $area(D_j)$ converges, as $j \to \infty$, to $area(D_\infty) + area(R)$, and from $area(D_\infty) \in (n-1, n+1)$ it follows $area(R) < 2$, hence $R = \emptyset$. This means that D_j converges uniformly to D_∞, i.e. z_j converges to z_∞ also in the Z-topology.

Hence the countable subset $\cup_n L_n$ is dense in Z, and consequently Z has a countable number of irreducible components. Each component is either a thin subset of $\mathbb{D}^k$ or an open subset. By hypothesis (ii), *there exists at least one open component*, which we shall call V.

For every compact $K \subset \mathbb{D}^k$, the area function a is bounded on $V \cap K$. To see this, let $z_0, z_1 \in V$ and join them by a continuous path $\{z_t\}_{t\in[0,1]} \subset V$. We thus have in X a continuous family of discs D_t, extending $f(z_t, A_r)$, and by Stokes theorem we find $a(z_1) - a(z_0) = \int_C \omega$, where C is the real surface $\cup_{t\in[0,1]} \partial D_t$ (and ω the Kähler form). On $\mathbb{D}^k \times A_r$ the pull-back $f^*(\omega)$ has a primitive λ, so that $\int_C \omega = \int_{\cup\{z_t\}\times\partial A_r} f^*(\omega) = \int_{\{z_1\}\times\partial A_r} \lambda - \int_{\{z_0\}\times\partial A_r} \lambda$. When z_0, z_1 belong to a compact $K \subset \mathbb{D}^k$, this last quantity is uniformly bounded.

Look now at the boundary of V in $\mathbb{D}^k \setminus I$. If $z_\infty \in \partial V \setminus I$ is approximated by a sequence $z_j \in V$, then from the boundedness of $a(z_j)$ and Bishop theorem again we obtain a disc in X with boundary $f(z_\infty, A_r)$, that is $z_\infty \in Z$. Obviously, the component of Z which contains z_∞ is a thin one. Thus ∂V is contained in $I \cup \{$ thin components of $Z\}$, i.e. ∂V is thin. From the disconnectedness properties of thin subsets we deduce that *the complement* $\mathbb{D}^k \setminus V$ *is a thin subset of* $\mathbb{D}^k$.

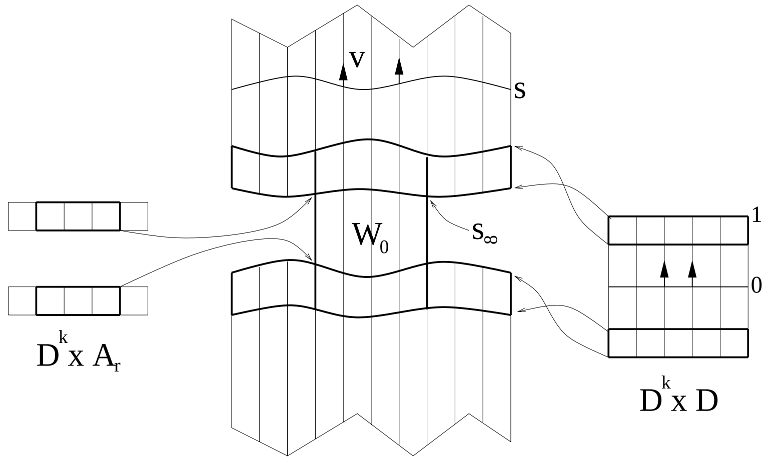
Let $\pi : W_0 \to V$ be the (normalised) tautological fibration over V, equipped with the holomorphic map $h : W_0 \to X$. By construction, ∂W_0 has a neighbourhood isomorphic to $V \times A_r$, sent by h to $f(V \times A_r)$. Hence we can glue $\mathbb{D}^k \times A_r$ to W_0 (using $h^{-1} \circ f$) obtaining a new space W_1 with a fibration $\pi : W_1 \to \mathbb{D}^k$ and a holomorphic map $h : W_1 \to X$ such that:

(1) $\pi^{-1}(z) \simeq \overline{\mathbb{D}}$ for $z \in V$, $\pi^{-1}(z) \simeq A_r$ for $z \in \mathbb{D}^k \setminus V$;

(2) f factorizes through h.

Lemma 2.1. *W_1 can be embedded (respecting the fibration over $\mathbb{D}^k$) into $\mathbb{D}^k \times \mathbb{C}P^1$.*

Proof. We use a "rationalisation trick". By construction, ∂W_1 has a neighbourhood isomorphic to $\mathbb{D}^k \times A_r$ and hence we can glue to W_1 an "exterior polydisc" $\mathbb{D}^k \times \mathbb{D}$ (in other words, we complete $A_r = \{r < |w| \leq 1\}$ to $\{r < |w| \leq \infty\}$). We obtain a new space W_2 with a fibration $\pi : W_2 \to \mathbb{D}^k$ such that $\pi^{-1}(z) \simeq \mathbb{C}P^1$ for $z \in V$, $\pi^{-1}(z) \simeq \mathbb{D}$ for $z \in \mathbb{D}^k \setminus V$. Of course, in this way we lost the map to X.



Let us denote by $s : \mathbb{D}^k \to W_2$ the section which arises from the zero section $\mathbb{D}^k \times \{0\}$ of the exterior polydisc, and by $v : s(\mathbb{D}^k) \to TW_2$ the vertical vector field along the graph of s which arises from the vertical unitary vector field along $\mathbb{D}^k \times \{0\}$. Also, the "boundary section" $\mathbb{D}^k \times \{1\}$ of the exterior polydisc defines a section $s_\infty : V \to W_2$. For every $z \in V$, there exists a *unique* isomorphism $\varphi_z : \pi^{-1}(z) \to \mathbb{C}P^1$ which sends $s(z)$ to 0, $s_\infty(z)$ to ∞, and whose derivative at $s(z)$, computed on $v(z)$, is equal to 1. As is well known, φ_z depends holomorphically on z (any $\mathbb{C}P^1$-fibration is locally trivial,...). Thus, by glueing together these φ_z, we obtain a fibered biholomorphism $\Phi : \pi^{-1}(V) \to V \times \mathbb{C}P^1$. We want to prove that Φ extends to W_2.

Let us look at φ_z in more detail. Using the coordinates given by the exterior polydisc we have a family of univalent maps

$$\varphi_z : \mathbb{D} \longrightarrow \mathbb{C}, \qquad z \in V,$$

with $\varphi_z(0) = 0$, $\varphi_z'(0) = 1$ (and the boundary point 1 is sent to ∞). By Koëbe Theorem, on every compact $K \subset \mathbb{D}$ the distorsion of φ_z is bounded by a constant, independent on z. In particular, for every $\lambda \in \mathbb{D}$ the (holomorphic) function $V \ni z \mapsto \varphi_z(\lambda)$ is bounded. From the thinness of $\mathbb{D}^k \setminus V$ we deduce that this function can be holomorphically prolonged to the full $\mathbb{D}^k$. Hence, even for $z \in \mathbb{D}^k \setminus V$ we obtain a map $\varphi_z : \mathbb{D} \to \mathbb{C}$, which is still holomorphic and univalent by the bounded distorsion, and the assignement $z \mapsto \varphi_z$ is still holomorphic.

This shows that the above Φ extends to an embedding $W_2 \to \mathbb{D}^k \times \mathbb{C}P^1$, and by restriction we obtain the embedding of W_1. □

We can now conclude the proof of Proposition 2.1. By the previous Lemma 2.1, we may fill in the holes of W_1, so that we obtain a manifold W with a $\overline{\mathbb{D}}$-fibration over $\mathbb{D}^k$. It remains to prove that $h : W_1 \to X$ can be meromorphically extended to W. But this is an immediate consequence of the Thullen type extension theorem of Siu [Siu] (by transfinite induction), or of the Hartogs type extension theorem of Ivashkovich [Iv1]. ☐

Remark that the "generic embedding" hypothesis of this Proposition is really needed: without it, we may easily construct an immersion $\mathbb{D}\times A_r \xrightarrow{f} \mathbb{C}P^1$ such that $f(z_0,\cdot)$ is an embedding (so that $f(z,A_r)$ extends to a disc for every z close to z_0), whereas $f(z_1,\cdot)$ is an immersion which bounds no disc. The problem, in the above proof, is that when the boundaries are not embedded the Bishop theorem becomes more complicated, because the discs may break into several components (think to an embedded circle in $\mathbb{C}$ which is deformed to acquire a pinching at some point). A somewhat related problem is that also Wermer theorem becomes more complicated, for immersed nonembedded circles: we need to consider holomorphic chains instead of simple complex curves.

3. Holonomy tubes and covering tubes

Let X be compact, connected, Kähler, of dimension n, equipped with a foliation $\mathcal{F}$ of dimension 1, $X^0 = X\setminus Sing(\mathcal{F})$, $\mathcal{F}^0 = \mathcal{F}|_{X^0}$. Take an embedded disc $T\subset X^0$, $T\simeq \mathbb{D}^{n-1}$, transverse to $\mathcal{F}^0$, and an embedded subdisc $S\subset T$, $S\simeq\mathbb{D}^k$, $1\le k\le n-1$. We need to develop here some ideas of [Br3] in a "relative" context. Roughly speaking, the main problem is concerned with the holonomy of the leaves through points of T: for a generic point this is trivial, but the subdisc S could be contained (when $k<n-1$) in the nongeneric complementary subset.

To start with, let L_s^0 be the leaf of $\mathcal{F}^0$ through $s\in S$, and let

$$Hol : \pi_1(L_s^0, s) \to Diff(T,s)$$

be the corresponding holonomy representation [God], $Diff(T,s)$ being the group of germs of holomorphic diffeomorphisms of (T,s). Let

$$G_{s,S} = \{\gamma\in\pi_1(L_s^0,s) \mid Hol(\gamma)|_{(S,s)} = id_{(S,s)}\}.$$

It is a subgroup of $\pi_1(L_s^0,s)$, but be aware that generally speaking it is *not* a normal subgroup, because an element $Hol(\gamma)$ with $\gamma\notin G_{s,S}$ does not need to preserve $(S,s)\subset(T,s)$. Anyway, we may take the universal covering of L_s^0 and quotient it by $G_{s,S}$, obtaining in this way a complex connected curve $\widehat{L^0_{s,S}}$, called the S**-holonomy covering** of L_s^0. Note that $\widehat{L^0_{s,S}} = L_s^0$ if and only if the holonomy is trivial on S (but possibly not outside). It is useful to think at a point of $\widehat{L^0_{s,S}}$ as an equivalence class of paths in L_s^0 starting at s, where two paths γ_1 and γ_2 are equivalent if they have the same endpoint and if their composition $\gamma_1 * \gamma_2^{-1}$ is a loop based at s with no holonomy on S (i.e. $\gamma_1*\gamma_2^{-1}\in G_{s,S}$). The natural map $\widehat{L^0_{s,S}}\to L_s^0$ is then the endpoint map. It is a covering map which may fail to be galoisian, because $G_{s,S}$ may fail to be normal. On $\widehat{L^0_{s,S}}$ we still have a distinguished point, still denoted by s, corresponding to the constant path.

Set now

$$V_S^0 = \bigcup_{s\in S} \widehat{L_{s,S}^0}$$

and observe that, as in [Br3], proof of Lemma 2.1 (see also [Ily]), this is in a natural way a complex manifold of dimension $k+1$, fibered over S, equipped with an immersion

$$\pi_S^0 : V_S^0 \longrightarrow X^0$$

sending fibres to leaves of $\mathcal{F}^0$.

Then we add to V_S^0 some points according to the following rule. Given any fibre $\widehat{L_{s,S}^0}$ and a parabolic end $E \subset \widehat{L_{s,S}^0}$ (i.e., a closed subset isomorphic to the closed punctured disc $\overline{\mathbb{D}}^*$), we say that E is a **S-vanishing end** if there exists an embedding $f : \mathbb{D}^k \times A_r \to V_S^0$ such that:

(i) $f(0, \partial A_r) = \partial E$, orientation preserving;

(ii) f sends fibres of $\mathbb{D}^k \times A_r \to \mathbb{D}^k$ to fibres of $V_S^0 \to S$;

(iii) the composition $\pi_S^0 \circ f : \mathbb{D}^k \times A_r \to X$ extends to a meromorphic family of discs.

We add to $\widehat{L_{s,S}^0}$ a point $0 \in \overline{\mathbb{D}}$ to each S-vanishing end $E \simeq \overline{\mathbb{D}}^*$ (i.e., we compactify the end). The result, denoted by $\widehat{L_{s,S}}$, will be called **completed S-holonomy covering** of L_s^0, and the union

$$V_S = \bigcup_{s\in S} \widehat{L_{s,S}}$$

will be called **S-holonomy tube**.

The complex structure of V_S^0 extends to V_S in a natural way. Indeed, if $f : \mathbb{D}^k \times A_r \to V_S^0$ is as above then, for every $z \in \mathbb{D}^k$, by property (iii), $(\pi_S^0 \circ f)(z, A_r)$ extends to a disc, tangent to $\mathcal{F}$ and therefore cutting $Sing(\mathcal{F})$ at a finite set of points. It follows that $f(z, \partial A_r)$ bounds, in the corresponding fibre of V_S^0, a subset isomorphic to $\overline{\mathbb{D}} \setminus \{\text{finite set}\}$. By restricting f, we see that each point in that finite set still gives a S-vanishing end of the fibre. Thus, after completion, the map $f : \mathbb{D}^k \times A_r \to V_S$ extends to a family of discs in the fibres of V_S, and this can be used to define the holomorphic structure of V_S (so that $f : \mathbb{D}^k \times A_r \to V_S$ will extend to a *holomorphic* family of discs in V_S, lifting the *meromorphic* family of discs in X).

By construction, we still have a submersion $Q_S : V_S \to S$, a section $q_S : S \to V_S$ giving basepoints, and a meromorphic map

$$\pi_S : V_S \dashrightarrow X$$

which extends π_S^0. Note that $V_S \setminus V_S^0$ is an analytic subset of V_S, which contains the indeterminacies of π_S but which may be strictly larger. The restriction of π_S to a fibre of V_S, after removal of indeterminacies, sends $(V_S \setminus V_S^0) \cap \{fibre\}$ to $Sing(\mathcal{F})$.

Finally, let $\widetilde{L_{s,S}}$ be the universal covering of $\widehat{L_{s,S}}$ (with basepoint s), and let

$$U_S = \bigcup_{s \in S} \widetilde{L_{s,S}}$$

be the fiberwise universal covering of V_S, called **S-covering tube**. Again, this is a complex manifold. To see this, we need to verify that if an embedded cycle in a fibre of V_S is not homotopic to zero in the fibre, then its displacement in nearby fibres enjoys the same property. This gives the Hausdorff property of U_S, which is all we need [Ily], [Br2], [Br3]. Now, suppose that the displacement $\gamma' \subset \widehat{L_{s',S}}$ of an embedded cycle $\gamma \subset \widehat{L_{s,S}}$ is homotopic to zero in the fibre. Then γ' bounds a disc in $\widehat{L_{s',S}}$, and the same obviously holds for every displacement $\gamma'' \subset \widehat{L_{s'',S}}$ close to γ'. But, by Proposition 2.1 and the definition of V_S, we see that γ also bounds a disc in $\widehat{L_{s,S}}$, so that it is homotopic to zero in the fibre.

Let us observe that here we may apply Proposition 2.1 (which requires the almost embedding hypothesis) for the following reasons. First of all, up to a small deformation of γ we may suppose that γ and its small displacements γ' are contained in V_S^0, where π_S is a holomorphic immersion. Over γ, π_S may be injective or not. In the former case, we can evidently apply Proposition 2.1 (to the restriction of π_S to a neighbourhood of γ). In the latter case, the closed curve $\gamma \subset \widehat{L_{s,S}^0}$ projects into L_s^0 to a closed curve with selfintersections and with finite, nontrivial holonomy. It is however easy to see that for a generic s' (more precisely: s' in the Zariski-open subset of S where the cardinality of the holonomy is maximal) the projection of $\gamma' \subset \widehat{L_{s',S}^0}$ into $L_{s'}^0$ is still injective, and therefore we can apply Proposition 2.1. See also the proof of Lemma 3.1 below for a related argument.

We also have a submersion $P_S : U_S \to S$, a section $p_S : S \to U_S$ giving basepoints, and a meromorphic map

$$\Pi_S : U_S \dashrightarrow X$$

which factorizes as $\Pi_S = \pi_S \circ F_S$, where $F_S : U_S \to V_S$ is a local diffeomorphism induced by the coverings $\widetilde{L_{s,S}} \to \widehat{L_{s,S}}$. Set $U_S^0 = F_S^{-1}(V_S^0)$, and note that its complement is an analytic subset of U_S.

Remarks. These constructions are similar to those of [Br3], following the scheme

holonomy covering $\to$ completion $\to$ universal covering.

Let us however observe the following two differences with [Br3]:

(a) Here we are working with foliated meromorphic maps which are not necessarily immersions (outside their indeterminacy sets). The reason is instrumental: in [Br3] we were interested in constructing a hopefully nontrivial leafwise Poincaré metric, so it was better to add to L_s^0 the minimal amount as possible of points (hence to work with the smallest class of maps) in order to avoid parabolic leaves. Here, on the contrary, we want to construct entire curves tangent to $\mathcal{F}$, so that parabolicity is welcome. Note that this difference with [Br3] exists even in the "absolute" case, i.e. $k = n - 1$.

(b) Here we have not defined S-vanishing ends of L_s^0, neither (completed) leaves of $\mathcal{F}$. The reason is that the notion of "S-vanishing end of L_s^0" is somewhat ambiguous, due to the fact that the holonomy group of L_s^0 possibly does not preserve $(S, s) \subset (T, s)$. For instance, it makes no sense to say that a parabolic end of L_s^0 has trivial or finite holonomy on S, and more generally it makes no sense to speak of holonomy on S of a parabolic end of L_s^0: to do so we need to join the basepoint s to the end with a path in L_s^0, but then the resulting holonomy will depend on the homotopy class of the chosen path. This ambiguity disappears when we pass to $\widehat{L_{s,S}^0}$: a parabolic end of $\widehat{L_{s,S}^0}$ is associated not only to a parabolic end of L_s^0, but also to a homotopy class of paths in L_s^0 from the basepoint to the end. In other words, the preimage of a parabolic end of L_s^0 under the covering $\widehat{L_{s,S}^0} \to L_s^0$ has several connected components, someones are still parabolic ends (the covering is finite), someothers not (the covering is infinite); and, among those which are parabolic ends, someones are S-vanishing, someothers not.

We also insist on the fact that the coverings $\widehat{L_{s,S}}$ (and consequently $\widetilde{L_{s,S}}$) *do* depend on S, and not only on s: if $S_1, S_2 \subset T$ are two different subdiscs and $s \in S_1 \cap S_2$, then $\widehat{L_{s,S_1}}$ and $\widehat{L_{s,S_2}}$ may be very different (and consequently, for example, it may happen that $\widetilde{L_{s,S_1}}$ is parabolic whereas $\widetilde{L_{s,S_2}}$ is hyperbolic). However, we may compare U_S and U_T (or, more generally, U_{S_1} and U_{S_2} when $S_1 \subset S_2$). Indeed, for every $s \in S$, $G_{s,T}$ is a subgroup of $G_{s,S}$, so that we have a covering $\widehat{L_{s,T}^0} \to \widehat{L_{s,S}^0}$. These coverings glue together to a fiberwise covering $V_T^0|_S \to V_S^0$. A T-vanishing end of $\widehat{L_{s,T}^0}$ is clearly mapped to a S-vanishing end of $\widehat{L_{s,S}^0}$, hence the covering above extends to a map $\widehat{L_{s,T}} \to \widehat{L_{s,S}}$. But this map may fail to be a covering, for the same reasons evoked in remark (b) above: the preimage in $\widehat{L_{s,T}^0}$ of a S-vanishing end of $\widehat{L_{s,S}^0}$ may have several connected components, and only some of them (possibly none) are T-vanishing ends of $\widehat{L_{s,T}^0}$. Moreover, even if a connected component is a T-vanishing end, the corresponding map from $\widehat{L_{s,T}}$ to $\widehat{L_{s,S}}$ may acquire a ramification, because the order of the holonomy on T may be larger than the order of the holonomy on S. Thus, we still have a holomorphic map $V_T|_S \to V_S$, as well as $U_T|_S \to U_S$, but these maps generally speaking are not fiberwise coverings.

Finally, and closing this circle of considerations, it is worth noting that all these constructions can be done also in the "limit case" $\dim S = 0$, i.e. S is a single point s. Then $\widehat{L_{s,\{s\}}^0} = L_s^0$, and $\widehat{L_{s,\{s\}}}$ is obtained by compactifying each analytic end of L_s^0. Here, an end of a leaf is **analytic** if it accumulates to a single point of X (necessarily singular for the foliation); otherwise the end is called **transcendental**.

The following Lemma is one of the motivations for the previous relative constructions. Note that it is well possible that for *every* $s \in S$ the full holonomy (on T) of L_s^0 is nontrivial. However, this cannot happen for the S-holonomy:

Lemma 3.1. *Let S_0 be the subset of S corresponding to leaves without holonomy on S:*

$$S_0 = \{s \in S | G_{s,S} = \pi_1(L_s^0, s)\}.$$

Then $S \setminus S_0$ is thin.

Proof. It is a simple adaptation to our relative context of a classical argument in foliation theory [God, page 96]. Just observe that, for every $\gamma \in \pi_1(L_s^0, s)$, the fixed point set $\{s' \in S| \; Hol(\gamma)(s') = s' \}$ is either the full (S, s) or a proper analytic subset of it. □

Let us now look at the convexity properties of the S-covering tube U_S, following [Br5]. From now on we shall restrict to the case in which the manifold X is *projective*. The results that we need are resumed in the next Proposition.

Proposition 3.1. *The following holds*:
(i) *If there exists a fibre of U_S isomorphic to $\mathbb{C}P^1$, then all the fibres are isomorphic to $\mathbb{C}P^1$ and*

$$U_S \simeq S \times \mathbb{C}P^1.$$

(ii) *The set of fibres of U_S isomorphic to $\mathbb{C}$ is either complete pluripolar or full, and in the latter case*

$$U_S \simeq S \times \mathbb{C}.$$

Proof. The first statement follows from Proposition 2.1, exactly as in [Br3, Lemma 2.3]. Consider now the case in which all the fibres are isomorphic to $\mathbb{C}$ or $\mathbb{D}$. If the set of hyperbolic fibres is not empty then, by [Br5], the fibrewise Poincaré metric on U_S has a plurisubharmonic variation; this implies that the set of parabolic fibres is complete pluripolar. If, otherwise, all the fibres are parabolic then we proceed as in [Br4, Theorem 1] in order to obtain (using the plurisubharmonicity result of [Br5]) the product structure $U_S \simeq S \times \mathbb{C}$. □

Probably these results can be proved also in the Kähler case, by adapting arguments from [Br2] and [Br3]. There are however some additional difficulties here, because we are in a relative context ($S = T$ in [Br3]) and moreover the map $\Pi_S : U_S \dashrightarrow X$ is not necessarily a foliated meromorphic immersion (as in [Br3]) and can contract divisors.

4. Lifting entire curves and proof of the Theorem

Notation and assumptions as in the previous section, with X projective. We now investigate the structure of the set $\mathcal{E}(S) \subset S$ through which there are entire curves in X tangent to $\mathcal{F}$. Let us define $\mathcal{P}(S) \subset S$ as the set of points over which the fibre of the covering tube U_S is parabolic ($\mathbb{C}$ or $\mathbb{C}P^1$). Of course,

$\mathcal{P}(S)$ is a subset of $\mathcal{E}(S)$, and we define $\mathcal{Z}(S)$ as their difference, so that we have the splitting

$$\mathcal{E}(S) = \mathcal{P}(S) \cup \mathcal{Z}(S).$$

Our aim is to prove the following fact.

Proposition 4.1. *$\mathcal{Z}(S)$ is a thin subset of S.*

The Theorem stated in the Introduction follows from Proposition 4.1 and Proposition 3.1 of the previous section. Indeed, $\mathcal{P}(S)$ is either complete pluripolar or full, and in the latter case U_S is a product of S with $\mathbb{C}$ or $\mathbb{C}P^1$, so that the map F of the Theorem will be $\Pi_S : U_S \dashrightarrow X$ or a restriction of it.

In order to prove Proposition 4.1, let us firstly observe that standard facts in complex analytic geometry (Barlet's cycles space, etc.) show that the subset of S through which there are rational or elliptic curves tangent to $\mathcal{F}$ is either a countable union of proper analytic subsets or the full S. In this last case, the holonomy tube V_S is a rational or elliptic fibration over a Zariski open dense subset, and the covering tube U_S is a product of S with $\mathbb{C}$ or $\mathbb{C}P^1$, so that $\mathcal{P}(S) = S$ and consequently $\mathcal{Z}(S) = \emptyset$. See also [Br4, §2] for a more detailed description of the elliptic case.

Therefore, from now on we shall suppose that through a very generic point of S there is no rational nor elliptic curve tangent to $\mathcal{F}$. Let us call a leaf L_s^0 **transcendental** if its closure in X is not an algebraic curve, and observe that if s is in $\mathcal{E}(S)$ and if L_s^0 is not transcendental then its closure is a rational or elliptic curve (possibly singular and with selfintersections). Hence, if we define

$$\mathcal{E}_0(S) = \{s \in \mathcal{E}(S) \mid L_s^0 \text{ is transcendental and without holonomy on } S\},$$

then by the previous considerations and by Lemma 3.1 $\mathcal{E}(S) \setminus \mathcal{E}_0(S)$ is thin.

For each $s \in \mathcal{E}_0(S)$ we have two possibilities for L_s^0:

(1) L_s^0 is isomorphic to $\mathbb{C}$ minus a discrete (possibly finite or empty) subset Γ; each point Γ corresponds to an analytic end of L_s^0, whereas the point at infinity of $\mathbb{C}$ corresponds to a transcendental end (nonisolated if Γ is infinite).

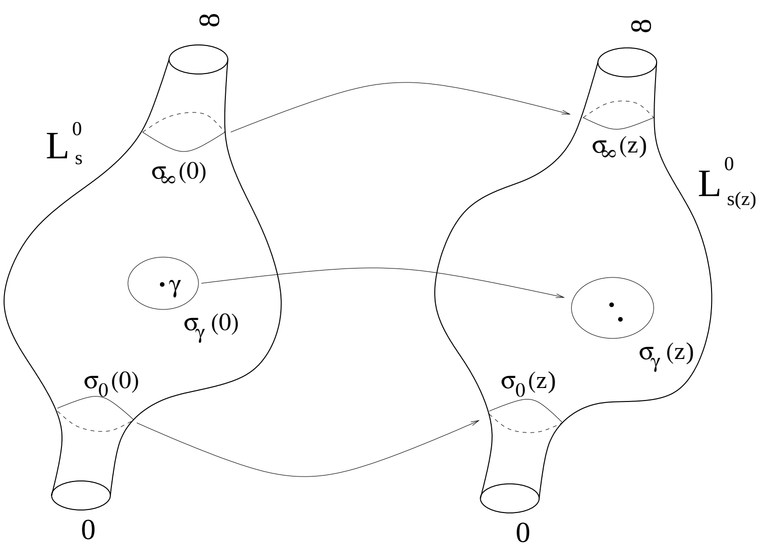
(2) L_s^0 is isomorphic to $\mathbb{C}^*$ minus a discrete subset Γ; each point of Γ corresponds to an analytic end of L_s^0, whereas the two points at infinity of $\mathbb{C}^*$ correspond to two transcendental ends.

Correspondingly, we decompose $\mathcal{E}_0(S) = \mathcal{E}_0^{(1)}(S) \cup \mathcal{E}_0^{(2)}(S)$. Next, for each $j = 1, 2$ let $\mathcal{E}_0^{(j)}(S)'$ be the set of points of $\mathcal{E}_0^{(j)}(S)$ around which $\mathcal{E}_0^{(j)}(S)$ is not thin. It is easy to see that the difference $\mathcal{E}_0^{(j)}(S) \setminus \mathcal{E}_0^{(j)}(S)'$ is thin.

Proposition 4.1 is therefore a consequence of the next Lemma.

Lemma 4.1. $\mathcal{E}_0^{(1)}(S)' \cup \mathcal{E}_0^{(2)}(S)' \subset \mathcal{P}(S).$

Proof. We give the proof for $\mathcal{E}_0^{(2)}(S)'$, the one for $\mathcal{E}_0^{(1)}(S)'$ being totally similar. We will prove, more precisely, that for every $s \in \mathcal{E}_0^{(2)}(S)'$ the fibre of V_S over s is isomorphic to $\mathbb{C}^*$.



Let us firstly consider the fibre of V_S^0 over s, which is, by the absence of holonomy, the leaf $L_s^0 \simeq \mathbb{C}^* \setminus \Gamma$. Call ∞ and 0 the two transcendental ends of L_s^0. Let $f_\infty : \mathbb{D}^k \times A_r \to V_S^0$ be a holomorphic embedding, sending fibres to fibres, such that $\sigma_\infty(0) = f_\infty(0, \partial A_r)$ is a "circle around ∞" in L_s^0, i.e. an oriented circle bounding a region containing the end ∞ (because this end is possibly nonisolated, such a region may contain other ends, it doesn't matter). The map $f_\infty|_{\{0\}\times A_r}$, followed by $\pi_S^0 : V_S^0 \to X$, cannot be extended to a disc, for the transcendency of the end ∞. Therefore, by Proposition 2.1, the same non-extension holds for $f_\infty|_{\{z\}\times A_r}$ for z outside a thin subset of $\mathbb{D}^k$. When $f_\infty(z, A_r)$ is in a fibre over $\mathcal{E}_0^{(2)}(S)$, this means that $\sigma_\infty(z) = f_\infty(z, \partial A_r)$ is still an oriented circle around a transcendental end (or more) of the fibre. Similar considerations can be done starting with a holomorphic embedding f_0 such that $\sigma_0(0) = f_0(0, \partial A_r)$ is a circle around 0 in L_s^0. Hence, for some non-thin subset $B \subset \mathbb{D}^k$ we have that, for every $z \in B$, $\sigma_\infty(z)$ and $\sigma_0(z)$ are in a fibre $L^0_{s(z)}$ over $\mathcal{E}_0^{(2)}(S)$, where they are circles around transcendental ends.

Take now a point $\gamma \in \Gamma$, and let $f_\gamma : \mathbb{D}^k \times A_r \to V_S^0$ be an embedding such that $\sigma_\gamma(0) = f_\gamma(0, \partial A_r)$ is a circle around γ in L_s^0. Up to a shift of f_∞, resp. f_0, closer to ∞, resp. 0, we may suppose that the three oriented circles $\sigma_\infty(0)$, $\sigma_0(0)$ and $\sigma_\gamma(0)$ bound three disjoint regions on L_s^0. The same must hold for the three circles $\sigma_\infty(z)$, $\sigma_0(z)$ and $\sigma_\gamma(z)$, when z is sufficiently close to 0 and the corresponding fibre $L^0_{s(z)}$ is over $\mathcal{E}_0^{(2)}(S)$: this is a consequence of the fact that $L^0_{s(z)}$ has genus zero. Thus, for every $z \in B$, close to 0, $\sigma_\infty(z)$ is a circle around one transcendental end (say ∞), $\sigma_0(z)$ is a circle around the other transcendental end (0), and $\sigma_\gamma(z)$ bounds a region which can contain only analytic ends (possibly more than one).

Therefore, $f_\gamma|_{\{z\}\times A_r}$ followed by π_S^0 can be extended to a disc, for every $z \in B$ close to 0, and by Proposition 2.1 we see that γ is a S-vanishing end. This proves that the fibre of V_S over s is equal to $L_s^0 \cup \Gamma \simeq \mathbb{C}^*$. $\square$

Remark. In fact, at least one of the two sets $\mathcal{E}_0^{(j)}(S)'$, $j = 1, 2$, must be empty, otherwise V_S would have at the same time a non-thin subset of $\mathbb{C}$-fibres and a non-thin subset of $\mathbb{C}^*$-fibres, which is impossible [Br4].

5. Appendix: towards a pluripolar Lang conjecture

In the eighties of the last century Serge Lang did the following conjecture [Lan, §IV.5]: given a complex projective manifold of general type Y, there exists a proper subvariety $Z \subset Y$ such that every (nonconstant) entire curve in Y is actually contained in Z. This conjecture is still largely open, but in recent years several positive results appeared for relatively large classes of surfaces: we refer to [Br1] for a survey and some bibliography. These results are based on a link with foliations which will be recalled below. We shall discuss here a weak version of Lang's conjecture for surfaces, in which the algebraic subset Z is replaced by a pluripolar subset of Y. Of course, a pluripolar subset may be much bigger than an algebraic one, but it is still something small in Y (for instance, it has measure zero).

The starting point is the following fact, which may be attributed to Green-Griffiths, Demailly, Siu, McQuillan: see [Dem], [McQ], and references therein.

Proposition 5.1. *Let Y be a smooth complex projective surface of general type. Then there exists a finite collection of complex projective manifolds $X_1, ..., X_n$, each one equipped with a foliation by curves $\mathcal{F}_i$ and a surjective map $\pi_i : X_i \to Y$, such that the following holds. If $f : \mathbb{C} \to Y$ is an entire curve, then for some $i \in \{1, ..., n\}$, f can be lifted to X_i as an entire curve $g : \mathbb{C} \to X_i$ (i.e. $f = \pi_i \circ g$) which is tangent to $\mathcal{F}_i$ and not completely contained in $Sing(\mathcal{F}_i)$.*

Let us briefly sketch the proof, following mostly [McQ].

Over Y we have a tower of $\mathbb{C}P^1$-bundles $P_1, P_2, ...$, the so-called jet-bundles: P_1 is the projective tangent bundle PTY, and each P_j is a $\mathbb{C}P^1$-bundle over P_{j-1}, obtained by taking higher order derivatives. On each P_j there is a tautological line bundle $L_j \in Pic(P_j)$. Because Y is of general type, by Riemann-Roch formula and Bogomolov vanishing theorem one finds that L_j is big for some j sufficiently large, which now will be fixed (strictly speaking, this is perhaps not completely correct, and we need to twist L_j with some positive combination of the pull-backs of L_i, $i < j$ [McQ] [Dem], but this does not affect substantially the subsequent discussion). This means that if $L_0 \in Pic(Y)$ is ample then for some m sufficiently big the line bundle $L_j^{\otimes m} \otimes pr_j^*(L_0^{-1})$ (where pr_j is the projection from P_j to Y) is effective, thus represented by a divisor Z in P_j.

Any entire curve $f : \mathbb{C} \to Y$ can be lifted, by taking derivatives, to P_j, $f^{(j)} : \mathbb{C} \to P_j$. One can associate to $f^{(j)}$ a closed positive current $\Phi^{(j)}$, which satisfies McQuillan's tautological inequality: $c_1(L_j) \cdot [\Phi^{(j)}] \leq 0$. We also have $c_1(pr_j^*(L_0)) \cdot [\Phi^{(j)}] = c_1(L_0) \cdot (pr_j)_*[\Phi^{(j)}] > 0$ because L_0 is ample, and therefore $[Z] \cdot [\Phi^{(j)}] < 0$. This inequality implies that $f^{(j)}(\mathbb{C})$ is actually *contained* in the support $|Z|$ of Z.

Let $W \subset P_j$ be (one of) the irreducible component(s) of $|Z|$ which contains $f^{(j)}(\mathbb{C})$, and let us distinguish several cases:

(1) W is *horizontal*, i.e. its projection to P_{j-1} is surjective. Then W is naturally equipped with a (tautological) foliation $\mathcal{F}$, and $f^{(j)}$ is (tautologically) tangent to $\mathcal{F}$. Up to desingularisation, W will be one of the X_i, and $\mathcal{F}$ the corresponding $\mathcal{F}_i$. It could happen, however, that $f^{(j)}(\mathbb{C})$ is completely contained in $Sing(\mathcal{F})$, a situation that we want to avoid. This case will be treated later.

(2) W is *vertical*, i.e. its projection to P_{j-1} is still an hypersurface $W' \subset P_{j-1}$. Then $f^{(j-1)}(\mathbb{C})$ is contained in W'. If W' is horizontal, we are in the same situation as in (1). If W' is vertical, we project again. Continuing in this way, either we get the same situation as in (1), in some lower order jet-space P_k, or W projects to Y to a curve C, containing $f(\mathbb{C})$. In this last case we take on Y any foliation tangent to C, and we are done (with $X_i = Y$).

(3) Returning to (1), let us consider the case in which $f^{(j)}(\mathbb{C}) \subset S$, where S is an irreducible component of $Sing(\mathcal{F})$. The codimension of S in P_j is at least 3, but by projecting to lower order jet-spaces P_k (or to Y) we finally obtain an hypersurface which contains $f^{(k)}(\mathbb{C})$ (or $f(\mathbb{C})$). As in (1) or (2), we then obtain a foliation to which $f^{(h)}$ or f is tangent. The problem may reappear, but after a finite number of steps we avoid in this way to fall completely into the singular set of the foliation.

Remark that all the foliations thus constructed are intrinsically defined starting from Z, they do not depend on f. This completes the proof of the above Proposition.

We are therefore lead to the following problem: we have a surface of general type Y, a projective manifold X with a foliation by curves $\mathcal{F}$ and a surjective map $\pi : X \to Y$, and we would like to bound the size of $\pi(\mathcal{E}_{nc})$, where $\mathcal{E}_{nc}$ is the set of points of $X^0 = X \setminus Sing(\mathcal{F})$ through which there is an entire curve tangent to $\mathcal{F}$ and whose projection to Y is not constant. Lang's conjecture is equivalent to the assertion that $\pi(\mathcal{E}_{nc})$ is an algebraic curve (a finite union of rational and elliptic curves).

Let $T \subset X$ be a disc transverse to $\mathcal{F}$, and express $\mathcal{E}(T)$ as in the Introduction:

$$\mathcal{E}(T) = \left[\cup_{j=0}^{\infty} \mathcal{P}(S_j) \right] \cup \mathcal{R}.$$

In order to prove the pluripolar Lang conjecture ("there exists a pluripolar subset of Y which contains all the entire curves") it would be sufficient to prove that, for every j, *the parabolic fibres of the covering tube U_{S_j} are sent by $\pi \circ \Pi_{S_j}$ to a pluripolar subset of Y* (unless $\pi \circ \Pi_{S_j}$ is constant on the fibres of U_{S_j}, in which case those fibres do not produce entire curves in Y).

This is somewhat related to *measure hyperbolicity* [Lan]. Indeed, let us firstly consider the case $\mathcal{P}(S_j) = S_j$. Then $U_{S_j} = S_j \times \mathbb{C}$ or $S_j \times \mathbb{C}P^1$, and, by measure hyperbolicity of Y, the map $\pi \circ \Pi_{S_j}$ is either constant on the fibres or it has rank 1, in which case its image is a *single* entire curve. Next, suppose that $\mathcal{P}(S_j)$ is thin, and take subdiscs $R_i \subset S_j$, $i \in \mathbb{N}$, whose union covers S_j. Because $\mathcal{P}(S_j)$ is complete pluripolar, $\mathcal{P}(S_j) \cap R_i$ is either complete pluripolar or full, but the latter option is again set out by measure hyperbolicity. By continuing in this way, we are therefore reduced to the case in which $\mathcal{P}(S_j)$ is complete pluripolar but not thin, on every neighbourhood of every point of it. This means that each parabolic fibre of U_{S_j} is accumulated by a "large" set of parabolic fibres. One can expect that this should imply that the generic rank of $\pi \circ \Pi_{S_j}$ along parabolic fibres (and hence everywhere) is 1, giving the pluripolarity (and even thinness) of the image.

All of this is still conjectural, but, at least, let us observe the following true corollary of our results. Given a surface of general type Y, by a *meromorphic family of curves* on Y we mean a connected complex manifold U equipped with a holomorphic map $P : U \to B$, all of whose fibres are irreducible curves, and a meromorphic map $\pi : U \dashrightarrow Y$ which does not contract to a point a generic P-fibre, and which has maximal rank at a generic point. Let $\mathcal{P} \subset B$ be the set of points of B over which the P-fibre is parabolic, and suppose that $\mathcal{P}$ is Zariski-dense in B. Such a family of curves can be lifted to the jet-bundles P_j over Y, giving maps $\pi^{(j)} : U \dashrightarrow P_j$. Obviously, if $P^{-1}(\mathcal{P})$ is sent by $\pi^{(j)}$ into some subvariety $Z \subset P_j$, then the full U is sent into the same Z. Looking at the proof of the Proposition above, we therefore see that the family U can be lifted to some X_i, as a family of curves tangent to $\mathcal{F}_i$. Then, by our Theorem and measure hyperbolicity of Y:

Corollary 5.1. *$\mathcal{P}$ is a pluripolar subset of B.*

For instance, we deduce from this result that on a surface of general type we cannot have a nontrivial *real analytic family* of entire curves. By this, we mean a real analytic map $\pi' : M \to Y$ such that: 1) $M = (0,1) \times \mathbb{R}^2$, and each fibre $M_t = \{t\} \times \mathbb{R}^2$ is equipped with a complex structure, varying analytically with t, so that $M_t \simeq \mathbb{C}$; 2) $\pi'|_{M_t}$ is holomorphic, for every t. Indeed, such a real map can be complexified, giving rise to a holomorphic family of curves on Y, with B a disc and $\mathcal{P}$ an interval in B.

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