J. Math. Kyoto Univ.
1-1 (1961) 129-146.

Remarks on the canonical representation of strictly stationary processes

By

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(Communicated by Prof. K. Itô, July 10, 1961)

§ 1. Introduction.

In this note we will consider some problems on the non-linear representation of strictly stationary processes. These problems were treated by N. Wiener [1] and M. Rosenblatt [2], [3].

To begin with, we shall introduce some terminologies necessary for our representation as a natural extension of those introduced for the linear representation of Gaussian processes [4].

Let $X=\{X(t, \tilde{\omega}), -<t<\infty]$ be a strictly stationary process on a probability space $\tilde{\Omega}(\tilde{\boldsymbol{B}}, \tilde{\boldsymbol{P}})$. Let $dB=\{dB(t, \omega), -\infty<t<\infty\}$ be a Wiener random measure (abbreviated hereafter as W. r. m). On a probability space $\Omega(\boldsymbol{B}, \boldsymbol{P})$, which may or may not be identical with $\tilde{\Omega}(\tilde{\boldsymbol{B}}, \tilde{\boldsymbol{P}})$.

Consider a system of functions $\boldsymbol{f}=\{f_n, h=0.1, 2, \cdots\}$, where each f_n is a symmetric L_2-function defined on the negative domain $(-\infty, 0]^n$. We assume that $\sum_{n=0}^{\infty} n! \|f_n\|^2$ is finite, $\|f_n\|$ being the L_2-norm of f_n. In this paper such a system $\boldsymbol{f}$ is said to belong to $\mathcal{K}$. For $\boldsymbol{f}\in\mathcal{K}$, we define the process $Y=\{Y(t, \omega), -\infty<t<\infty\}$ as follows,

$$Y(t, \omega) = \sum_{n=0}^{\infty} \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} f_n(t_1-t, \cdots, t_n-t)dB(t_1, \omega) \cdots dB(t_n, \omega) \quad (1)$$

where the n-th term is the orthogonal n-th Multiple Wiener Integral [1], [5]. It is clear that Y is $\boldsymbol{M}_2$-continuous and strictly stationary.

Definition 1. If Y is a version of X, then we call the stochastic process of the right side of (1) a (non-linear) *representation* of X, and the system $\boldsymbol{f}$ the *kernel* of the representation.

We define the degree of representation by $sup\,\{n\,;\,\|f_n\|>0\}$. If $\|f_n\|=0$ for all n, we define its degree by -1 for convenience.

Definition 2. The representation is called *properly canonical* if and only if

$$\boldsymbol{L}_2(dB,\,t) = \boldsymbol{L}_2(Y,\,t) \qquad \text{for all } t\,.$$

$\boldsymbol{L}_2(dB,\,t)$ is the set of all random variables measurable with respect to $\boldsymbol{B}_t(dB)$, with zero mean and finite variance, $\overline{\boldsymbol{B}}_t(dB)$ being the completion of Borel field generated by $[\{B(u,\,\omega)-B(v,\,\omega),\ u,\,v\leq t\} \cup \{\text{all null set in } \boldsymbol{B}\}]$, and $\boldsymbol{L}_2(Y,\,t)$ is the set of all random variables measurable with respect to $\overline{\boldsymbol{B}}_t(Y)$, with zero mean and finite variance, $\overline{\boldsymbol{B}}_t(Y)$ being the completion of the Borel field generated by $[\{Y(s,\,\omega),\ s\leq t\} \cup \{\text{all null set in } \boldsymbol{B}\}]$. In $\boldsymbol{L}_2(dB,\,t)$ as well as in $\boldsymbol{L}_2(Y,\,t)$ we identify two elements which coincicide up to measure zero.

Definition 3. The representation is called *canonical* if and only if

$$\boldsymbol{L}_2(dB,\,t) \perp (\boldsymbol{L}_2(Y) \ominus \boldsymbol{L}_2(Y,\,t)) \qquad \text{for all } t\,. \tag{2}$$

where $\boldsymbol{L}_2(Y)=\boldsymbol{L}_2(Y,\,\infty)$.
Intuitively speaking, the condition (2) means that we do not need the future information of the given stationary process Y to construct the W. r. m. dB. It is obvious that any properly canonical representation is canonical.

Two representations of a process with the same kernel are of the same type, namely, if one is canonical (or properly canonical), then it is so with the other. Thus we can also use the terminology such as "the kernel is canonical (or properly canonical)"

Definition 4. X is called *purely non-deterministic* if and only if

$$\bigcap_t \boldsymbol{L}_2(X,\,t) = \{0\} \quad \text{and} \quad \boldsymbol{L}_2(X) \neq \{0\}\,.$$

Any process with a representation is purely non-deterministic except a trivial case. Furthermore it can be proved that an ergodic and strictly stationary Markovian process, provided that if $\boldsymbol{P}(t, a, E)$ is the transition probability, then $\int \boldsymbol{P}(t, a, db) f(b)$ is continuous in a for an bounded continuous function f.[1] To prove this it is enough to show that if $\xi(\omega) \in \bigcap_t \boldsymbol{L}_2(X, t)$,

$$\lim_{t\to-\infty} E(\xi(\omega)/\boldsymbol{B}_t(X)) = E(\xi(\omega))$$

and we can assume that $\xi(\omega)$ is of the form

$$\xi(\omega) = f_1(X(t_1, \omega)) f_2(X(t_2, \omega)) \cdots f_n(X(t_n, \omega)), \qquad (t_1 < t_2 < \cdots < t_n)$$

f_i being contnuous, since the linear combination of these functions are dense in $\boldsymbol{L}_2(X, \infty)$; in fact

$$E(\xi(\omega)/\boldsymbol{B}_t(X)) = \int \cdots \int f_1(x_1) f_2(x_2) \cdots f_n(x_n) \boldsymbol{P}(t_n - t_{n-1}, x_{n-1}, dx_n) \cdots \boldsymbol{P}(t_l - t, X(t, \omega), dx_1)$$

$$\rightarrow \int \cdots \int f_1(x_1) f_2(x_2) \cdots f_n(x_n) \boldsymbol{P}(t_n - t_{n-1}, x_{n-1}, dx_n) \cdots \boldsymbol{P}(t_2 - t_1, x_1, dx_2) \varphi(dx_1)$$

$$= E(\xi(\omega))$$

where φ is the invariant distribution which is equal to the limit distribution of $\boldsymbol{P}(t, a, E)$ as $t \to \infty$ by the ergodicity of our process.

The main purpose of this note is the determination of all canonical representations of a stationary process which has a properly canonical one. We shall discuss this in § 2.

In § 3 we shall clarify the relation between our (non-linear) representation and the ordinary linear representations in case the processes in question are Gaussian. An interesting result from this relation is that the degree of the canonical representation of Gaussian processes is either 1 (linear representation) or infinite.

In § 4 we shall mention two examples of stationary processes in connection with the representation. The first one is as follows. Let $\{P(t, \tilde{\omega}), -\infty < t < \infty\}$ be Poisson process with parameter λ. Then $Y(t, \tilde{\omega}) \equiv P(t, \tilde{\omega}) - P(t-1, \tilde{\omega}) - \lambda$ is a strictry stationary and

1) This was given by K. Ito.

purely non-deterministic, but has no canonical representation. The second example is a Markov process which has a properly canonical representation.

In conclusion the author wishes to express her sincere thanks to Professor K. Ito for his kind guidance and valuable suggestions.

§ 2. Canonical Representation.

Firstly we shall give a method to transform a W. r. m. into a new one by a random change of the sign of increments.

Let $\Psi(\tau, \omega)$ be a (τ, ω)-measurable function whose values are either 1 or -1, and assume that it is measurable with respect to $\boldsymbol{B}_\tau(dB)$ ($\equiv \mathfrak{B}[B(u, \omega)-B(v, \omega),\ u, v \leq \tau]$) for each τ. Putting $\bar{B}(t, \omega)-\bar{B}(s, \omega)=\int_s^t \Psi(\tau, \omega)\, dB(\tau, \omega)$ (stochastic integral [6]), we have

Lemma 1. $d\bar{B}=\{d\bar{B}(t, \omega),\ -\infty < t < \infty\}$ is a W. r. m.

Proof. Consider first the simple case in which $\Psi(\tau, \omega)$ is of the following form:

$$\Psi(\tau, \omega) = a_i(\omega) \quad \text{for} \quad \tau_i \leq \tau < \tau_{i+1},\quad i=1, 2, \cdots, n\,.$$

For $s, t \in [\tau_i, \tau_{i+1})$,

$$\begin{aligned} &\boldsymbol{P}(\bar{B}(t, \omega)-\bar{B}(s, \omega) < c/\boldsymbol{B}_s(dB)) \\ &= \boldsymbol{P}(a_i(\omega)=1 \quad \text{and} \quad B(t, \omega)-B(s, \omega) < c/\boldsymbol{B}_s(dB)) \\ &+ \boldsymbol{P}(a_i(\omega)=-1 \quad \text{and} \quad B(t, \omega)-B(s, \omega) > c/\boldsymbol{B}_s(dB)) \\ &= \boldsymbol{P}(a_i(\omega)=1/\boldsymbol{B}_s(dB))\boldsymbol{P}(B(t, \omega)-B(s, \omega) < c) \\ &+ \boldsymbol{P}(a_i(\omega)=-1/\boldsymbol{B}_s(dB))\boldsymbol{P}(B(t, \omega)-B(s, \omega) > -c) \\ &= \int_{-\infty}^{c} \frac{1}{\sqrt{2\pi(t-s)}} e^{-x^2/2(t-s)} dx\,. \end{aligned}$$

Hence, it is clear that $d\bar{B}$ is a W. r. m., since $\bar{B}(t, \omega)-\bar{B}(s, \omega)$ is measurable with respect to $\boldsymbol{B}_t(dB)$.

Given a general $\Psi(\tau, \omega)$, there exists a sequence of functions $\Psi_n(\tau, \omega)$, $n=1, 2, \cdots$, of the form mentioned above such that $\int_{-n}^{n} E(\Psi(\tau, \omega)-\Psi_n(\tau, \omega))^2 d\tau < \frac{1}{n}$. Using this fact we can easily see that $d\bar{B}$ is the limit of a sequence of W. r. m.'s. Therefore $d\bar{B}$ is a W. r. m, as we wished.

Thoughout this section we assume that $X \equiv \{X(t, \tilde{\omega}), -\infty < t < \infty\}$ is a strictly stationary process with zero mean and has a properly canonical representation with the kernel $\boldsymbol{g} = (g_1, g_2, \cdots)$, namely X has a version Z such that

$$Z(t, \omega) = \sum_{n=1}^{\infty} \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} g_n(t_1 - t, \cdots, t_n - t) dB(t_1, \omega) \cdots dB(t_n, \omega) \quad (1)$$

$$\boldsymbol{L}_2(dB, t) = \boldsymbol{L}_2(Z, t). \quad (2)$$

We shall note that if T_τ is a shift operator on $\boldsymbol{L}_2(dB)$ defined by

$$T_\tau(B(t, \omega) - B(s, \omega)) = B(t+\tau, \omega) - B(s+\tau, \omega)$$

and if $\tilde{T}_\tau$ is on $\boldsymbol{L}_2(Z)$ defined by

$$\tilde{T}_\tau(Z(s, \omega)) = (s+\tau, \omega)$$

then (1) and (2) will imply $T_\tau = \tilde{T}_\tau$, while the relation (1) implies $T_\tau > \tilde{T}_\tau$ only.

Lemma 2. If a process Y is a version of X, then we can construct a W. r. m. dB^* from Y such that

$$Y(t, \omega^*) = \sum_{n=1}^{\infty} \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} g_n(t_1 - t, \cdots, t_n - t) dB^*(t_1, \omega^*) \cdots dB^*(t_n, \omega^*)$$

and $\boldsymbol{L}_2(Y, t) = \boldsymbol{L}_2(dB^*, t)$ for all t.

Proof. By the relation (2), we can construct dB from Z as $B(t, \omega) - B(s, \omega) = f_{s,t}(Z(\tau, \omega), \tau \leq t)$. Hence it follows that $\{f_{t,s}(Z(\tau, \omega^*), \tau \leq t), -\infty < s < t < \infty\}$ is also a W. r. m. since Y is a version of Z. Denoting this W. r. m. with dB^*, we obtain Lemma 2 at once.

For any set $E \in \boldsymbol{B}_0(dB)$, we define $\bar{\chi}_E(\omega)$ by

$$\bar{\chi}_E(\omega) = 1 \quad \text{on} \quad E, \qquad = -1 \quad \text{on} \quad E^c.$$

This random variable $\bar{\chi}_E(\omega)$ can be expanded by Multiple Wiener Integrals [5. Th. 4. 2] as

$$\bar{\chi}_E(\omega) = \sum_{n=0}^{\infty} \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} \varphi_{E,n}(t_1, \cdots, t_n) dB(t_1, \omega) \cdots dB(t_n, \omega)$$

where $\varphi_{E,n}$ is a symmetric function.

Let $\mathfrak{F}$ denote the set $\{\varphi_E=(\varphi_{E,0}, \varphi_{E,1}, \cdots); E\in \boldsymbol{B}_0(dB)\}$. $\mathfrak{F}$ does not depend on the W. r. m. dB by the definition and if φ belongs to $\mathfrak{F}$, then $-\varphi=(-\varphi_0, -\varphi_1, \cdots)$ does also.

For any $\varphi\in\mathfrak{F}$ and for any $\boldsymbol{a}, \boldsymbol{b}\in\mathcal{K}$, we shall define the relation $\boldsymbol{a}=\boldsymbol{b}\circ\varphi$ as follows. Take a W. r. m. dB and define $d\bar{B}$ by

$$\bar{B}(t, \omega)-\bar{B}(s, \omega) = \sum_{n=0}^{\infty}\int_s^t dB(\tau, \omega)\Big[\int_{-\infty}^{\tau}\cdots\int_{-\infty}^{\tau}\varphi_n(t_1-\tau, \cdots t_n-\tau)\, dB(t_1, \omega)\cdots dB(t_n, \omega)\Big],$$

$d\bar{B}$ proves to be a W. r. m. by Lemma 1. If it holds that

$$\sum_{n=0}^{\infty}\int_{-\infty}^{0}\cdots\int_{-\infty}^{0} a_n(t_1, \cdots, t_n)dB(t_1, \omega)\cdots dB(t_n, \omega)$$
$$=\sum_{n=0}^{\infty}\int_{-\infty}^{0}\cdots\int_{-\infty}^{0} b_n(t_1, \cdots, t_n)d\bar{B}(t_1, \omega)\cdots d\bar{B}(t_n, \omega),$$

then we say that $\boldsymbol{a}=\boldsymbol{b}\circ\varphi$. This definition is independent of the special choice of the W. r. m. dB.

Thus our main Theorem reads as follows.

Theorem 1. A necessary ane sufficient condition for $\boldsymbol{f}\in\mathcal{K}$. to be the kernel of canonical representation of the process X mentioned above is that there exists φ in $\mathfrak{F}$ such that $\boldsymbol{f}=\boldsymbol{g}\circ\varphi$.

Proof. Let $\boldsymbol{f}$ be the kernel of a canonical represention of X.

$$Y(t, \omega)\equiv\sum_{n=0}^{\infty}\int_{-\infty}^{t}\cdots\int_{-\infty}^{t} f_n(t_1-t, \cdots, t_n-t)dB(t_1, \omega)\cdots dB(t_n, \omega) \quad (3)$$

is a canonical representation of X, so that

$$\boldsymbol{L}_2(dB, t)\perp(\boldsymbol{L}_2(Y)\ominus\boldsymbol{L}_2(Y, t)).$$

Using Lemma 2, we can construct dB^* such that

$$Y(t, \omega) - \sum_{n=1}^{\infty}\int_{-\infty}^{t}\cdots\int_{-\infty}^{t} g_n(t_1-t, \cdots, t_n-t)dB^*(t_1, \omega)\cdots dB^*(t_n, \omega)$$

and that

$$\boldsymbol{L}_2(Y, t) = \boldsymbol{L}_2(dB^*, t).$$

Thus we have

$$\boldsymbol{L}_2(dB, t)\perp(\boldsymbol{L}_2(dB^*)\ominus\boldsymbol{L}_2(dB^*, t)) \quad (4)$$

Expand dB^* by Multiple Wiene Integral as

$$B^*(t, \omega) - B^*(s, \omega) = \sum_{n=1}^{\infty} \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} a_{t,s,n}(t_1, \cdots, t_n) dB(t_1, \omega) \cdots dB(t_n, \omega)$$

$$= \int_{-\infty}^{t} A_{t,s}(\tau, \omega) dB(\tau, \omega) \tag{5}$$

where $A_{t,s}(\tau, \omega) = \sum_{n=1}^{\infty} n! \int_{-\infty}^{\tau} \int_{-\infty}^{t_{n-1}} \cdots \int_{-\infty}^{t_2} a_{t,s,n}(t_1, \cdots, t_{n-1}, \tau) dB(t_1, \omega) \cdots dB(t_{n-1}, \omega)$.

On the other hand we have, by (4)

$$\int_{-\infty}^{s} A_{t,s}(\tau, \omega) dB(\tau, \omega) = E(B^*(t, \omega) - B^*(s, \omega) / \boldsymbol{B}_s(dB)) = 0 . \tag{6}$$

and for $s < u < v < t$

$$E(B^*(t, \omega) - B^*(s, \omega) / \boldsymbol{B}_v(dB)) - E(B^*(t, \omega) - B^*(s, \omega) / \boldsymbol{B}_u(dB))$$
$$= B^*(v, \omega) - B^*(u, \omega) \tag{7}$$

Thus we obtain

$$B^*(t, \omega) - B^*(s, \omega) = \int_s^t A_{t,s}(\tau, \omega) dB(\tau, \omega) \tag{8}$$

by (5) and (6), and

$$\int_u^v A_{t,s}(\tau, \omega) dB(\tau, \omega) = \int_u^v A_{v,u}(\tau, \omega) dB(\tau, \omega) \tag{9}$$

by (5) and (7). This implies

$$A_{t,s}(\tau, \omega) = A_{v,u}(\tau, \omega) \tag{10}$$

for almost all (τ, ω) in $[u, v) \times \boldsymbol{\Omega}$. Therefore we can extend $A_{t,s}$ to a (τ, ω)-measurable function $\Phi(\tau, \omega)$, defined on $R^1 \times \boldsymbol{\Omega}$ such that for any $s < t$

$$B^*(t, \omega) - B^*(s, \omega) = \int_s^t \Phi(\tau, \omega) dB(\tau, \omega) . \tag{11}$$

It is obvious that $\Phi(\tau, \omega)$ is measurable with respect to $\boldsymbol{B}_\tau(dB)$ for any fixed τ.

Next we will show that $\Phi^2(\tau, \omega) = 1$.

First we have

$$\int_s^t E(\Phi^2(\tau, \omega) / \boldsymbol{B}_s(dB)) d\tau = t - s \tag{12}$$

comparing both side with $E\{(B^*(\tau, \omega)-B^*(s, \omega))^2/\boldsymbol{B}_s(dB)\}$ by (11) and (4). Furthermore, $E(\Phi^2(\tau, \omega)/\boldsymbol{B}_s(dB))$ is a measurable function of (s, τ, ω), because it is measurable in (τ, ω) for any fixed s and it is continuous in s for any fixed (τ, ω). Fixing s arbitrary, we obtain, from (12),

$$E(\Phi^2(\tau, \omega)/\boldsymbol{B}_s(dB)) = 1 \tag{13}$$

for almost all ω and for almost all $\tau \in [s, \infty)$. Appealing to Fubini's theorem, we have

$$\text{“}E(\Phi^2(s+\tau, \omega)/\boldsymbol{B}_s(dB)) = 1 \text{ for almost all } s \text{ and for almost all } \omega\text{”}. \tag{14}$$

for almost all $\tau \in [0, \infty)$. It is clear that, for almost all ω, $\Phi^2(\tau, \omega)$ is locally integrable in τ and hence

$$\int_a^b |\Phi^2(\tau, \omega)-\Phi^2(\tau+v, \omega)|\, d\tau \to 0 \qquad \text{as} \quad v \to 0\,.$$

On the other hand, we have

$$\int_a^b |\Phi^2(\tau, \omega)-\Phi^2(\tau+v, \omega)|\, d\tau \leq 2\int_{a-1}^{b+1} \Phi^2(\tau, \omega)\, d\tau$$

for v small, and noting that the random variable of the right side belongs to $\boldsymbol{L}_1(\boldsymbol{\Omega})$ and using the Lebesgue bounded convergence theorem, we have

$$E\int_a^b |\Phi^2(\tau, \omega)-\Phi^2(\tau+v, \omega)|\, d\tau \to 0\,, \qquad \text{as} \quad v \to 0 \tag{15}$$

Let $\{\tau_n, n=1, 2, \cdots\}$ be a seqvence of those values of τ which satisfy (14) such that $\tau_n \to 0$. Now it follows that

$$\begin{aligned}
&\int_a^b E|1-\Phi^2(s, \omega)|\, ds \\
&= \int_a^b E|E(\Phi^2(s+\tau_n, \omega)/\boldsymbol{B}_s(dB))-\Phi^2(s, \omega)|\, ds \\
&= \int_a^b E|E(\Phi^2(s+\tau_n, \omega)-\Phi^2(s, \omega)/\boldsymbol{B}_s(dB))|\, ds \\
&\leq \int_a^b E\{E(|\Phi^2(s+\tau_n, \omega)-\Phi^2(s, \omega)|/\boldsymbol{B}_s(dB))\}\, ds \\
&= E\Big(\int_a^b |\Phi^2(s+\tau_n, \omega)-\Phi^2(s, \omega)|\, ds \xrightarrow[n\to\infty]{} 0
\end{aligned}$$

by (15), which implies

$$\Phi^2(u, \omega) = 1 \qquad \text{for almost all } \omega \tag{16}$$

for almost all u.

Expanding $\Phi(s, \omega)$ by Multiple Wiener Integrals and noting that $\Phi(s, \omega)$ is measurable with respect to $\boldsymbol{B}_s(dB)$, (11) can be written as

$$B^*(t, \omega) - B^*(s, \omega) = \sum_{n=0}^{\infty} \int_s^t dB(u, \omega) \Big[\int_{-\infty}^{u} \cdots \int_{-\infty}^{u} \varphi_n(t_1, \cdots, t_{n-1}, u) dB(t_1, \omega) \cdots dB(t_n, \omega) \Big] \tag{17}$$

where φ_n is symmetric on $(t_1, \cdots, t_{n-1})$. Operating T_τ on the both side of (17), we have

$$B^*(t+\tau, \omega) - B^*(s+\tau, \omega)$$
$$= \sum_{n=0}^{\infty} \int_{s+\tau}^{t+\tau} dB(u, \omega) \int_{-\infty}^{u} \cdots \int_{-\infty}^{u} \varphi_n(t_1-\tau, \cdots, t_n-\tau, u-\tau) dB(t_1, \omega) \cdots dB(t_{n-1}, \omega)$$

Replacing $t+\tau$ and $s+\tau$ with t and s respectively in this expression and comparing it with (17), we obtain

$$\varphi_n(t_1, \cdots, t_{n-1}, u) = \varphi_n(t_1-\tau, \cdots, t_{n-1}-\tau, u-\tau)$$

for almost all $(t_1, \cdots, t_{n-1}, u)$. Therefore we have

$$\text{"}\varphi_n(t_1, \cdots, t_{n-1}, u) = \varphi_n(t_1-\tau, \cdots, t_{n-1}-\tau, u-\tau)$$
$$\text{for almost all } \tau \text{ and for almost all } (t_1, \cdots, t_{n-1})\text{"} \tag{18}$$

for almost all u. Taking a value u_0 of u which satisfies (18) and (16), and defining ψ_n by

$$\psi_n(t_1-u_0, \cdots, t_n-u_0) = \varphi_{n+1}(t_1, \cdots, t_n, u_0)$$

we have

$$\psi_n(t_1-u_0-\tau, \cdots, t_n-u_0-\tau) = \varphi_{n+1}(t_1, \cdots, t_n, u_0+\tau) \tag{19}$$

for almost all τ and for almost all $(t_1, \cdots, t_n)$, and therefore

$$\Phi(u_0+\tau, \omega) = \sum_{n=0}^{\infty} \int_{-\infty}^{\tau} \cdots \int_{-\infty}^{\tau} \psi_n(t_1-u_0-\tau, \cdots, t_n-u_0-\tau) dB(t_1, \omega) \cdots dB(t_n, \omega). \tag{20}$$

for almost all τ and for almost all ω.

Defining $\Psi(\tau, \omega)$ as a measurable version of $\sum_{n=0}^{\infty}\int_{-\infty}^{\tau}\cdots\int_{-\infty}^{\tau}\psi_n(t_1-\tau, \cdots, t_n-\tau)dB(t_1, \omega)\cdots dB(t_n, \omega)$, we get

$$\Psi(\tau, \omega) = \Phi(\tau, \omega) \qquad \text{for almost all } \omega\,. \tag{21}$$

for almost all τ, and since $\Psi(\tau, \omega)$ is a strictly stionary process by the definition

$$\boldsymbol{P}(\Psi(\tau, \omega) = 1 \ \text{ or } \ -1) = \boldsymbol{P}(\Psi(u_0, \omega) = 1 \ \text{ or } \ -1) = 1\,. \tag{22}$$

Hence the system $\boldsymbol{\psi}=(\psi_0, \psi_1, \cdots)$ belongs to the class $\mathfrak{J}$ introduced before, and we have $\boldsymbol{f}=\boldsymbol{g}\circ\boldsymbol{\psi}$.

Assume tonversely that $\boldsymbol{f}=\boldsymbol{g}\circ\boldsymbol{\psi}$ and put

$$Y(t, \omega) \equiv \sum_{n=1}^{\infty}\int_{-\infty}^{t}\cdots\int_{-\infty}^{t} f(t_1-t, \cdots, t_n-t)dB(t_1, \omega)\cdots dB(t_n, \omega) \tag{23}$$

and

$$\bar{B}(t, \omega)-\bar{B}(s, \omega) = \sum_{n=0}^{\infty}\int_{s}^{t}dB(\tau, \omega)\int_{-\infty}^{\tau}\cdots\int_{-\infty}^{\tau}\psi_n(t_1-\tau, \cdots, t_n-\tau)dB(\tau_1, \omega)\cdots dB(t_n, \omega)\,,$$

we shall get

$$Y(t, \omega) = \sum_{n=1}^{\infty}\int_{-\infty}^{t}\cdots\int_{-\infty}^{t} g(t_1-t, \cdots, t_n-t)d\bar{B}(t_1, \omega)\cdots d\bar{B}(t_n, \omega)$$

by the definition of $\boldsymbol{g}\circ\boldsymbol{\psi}$, and this proved to be a properly canonical representation of X, and it implies that (23) is a representation of X. It is therefore enough to prove

$$\boldsymbol{L}_2(dB, t) \perp (\boldsymbol{L}_2(d\bar{B}) \ominus \boldsymbol{L}_2(d\bar{B}, t))$$

in order to complete the proof of our theorem.

Take any element Z in $\boldsymbol{L}_2(d\bar{B}) \ominus \boldsymbol{L}_2(d\bar{B}, t)$. Then Z can be written as

$$Z = \int_{-\infty}^{\infty}\eta(\tau, \omega)d\bar{B}(\tau, \omega)$$

where $\eta(\tau, \omega)$ is measurable with respect to $\boldsymbol{B}_\tau(d\bar{B})$, since $Z \in \boldsymbol{L}_2(d\bar{B})$. Since $Z \perp \boldsymbol{L}_2(d\bar{B}, t)$ and

$$Z' \equiv \int_{-\infty}^{t}\eta(\tau, \omega)d\bar{B}(\tau, \omega) \qquad \in \boldsymbol{L}_2(d\bar{B}, t)\,,$$

we have, by the property of stochastic integral,

$$0 = E(Z' \cdot Z) = \int_{-\infty}^{t} E(\eta^2(\tau, \omega))\, d\tau$$

namely $\eta(\tau, \omega)=0$ for almost all $(\tau, \omega) \in [-\infty, t] \times \Omega$, so that

$$Z = \int_{t}^{\infty} \eta(\tau, \omega)\, d\bar{B}(\tau, \omega) = \int_{t}^{\infty} \eta(\tau, \omega) \Psi(\tau, \omega)\, dB(\tau, \omega)$$

where $\Psi(\tau, \omega)$ is measurable with respect to $\boldsymbol{B}_{\tau}(dB)$. Since $\eta(\tau, \omega)$ is measurable with respect to $\boldsymbol{B}_{\tau}(d\bar{B})$, it is also measurable with respect to $\boldsymbol{B}_{\tau}(dB)$. Therefore Z is orthogonal to any elemnt Z'' of the form

$$Z'' = \int_{-\infty}^{t} \zeta(\tau, \omega)\, dB(\tau, \omega)$$

by the property of stochastic integral, where $\zeta(\tau, \omega)$ is measurable with respect to $\boldsymbol{B}_{\tau}(dB)$. Since Z'' is a general etement of $\boldsymbol{L}_2(dB, t)$, the proof is now completed.

In connection with this theorem there arises an open question whether or not $\boldsymbol{L}_2(dB, t)$ is strictly larger than $\boldsymbol{L}_2(d\bar{B}, t)$.

§ 3. The Representation of Gaussian Processes.

In this section we will discuss the non-linear representation of Gaussian processes. Let X be a stationary Gaussian process, continuous in the mean square sense with zero mean and finite variance. Let $\mathcal{M}(X, t)$ denote the closed linear manifold spanned by $\{X(\tau, \omega),\ \tau \leq t\}$ and put $\mathcal{M}(X) = \mathcal{M}(X, \infty)$. Replacing $\boldsymbol{L}_2(X, t)$ and $\boldsymbol{L}_2(X)$ with $\mathcal{M}(X, t)$ and $\mathcal{M}(X)$ respectively in Definition 4 in § 1, we shall introduce the concept "*purely non-deterministic in linear sense*". It is well known that X has a moving average representation (representation of degree 1)

$$Y(t, \omega) = \int_{-\infty}^{t} g(t-s)\, dB(s, \omega) \tag{1}$$

if X is purely non-deterministic. Replacing $\boldsymbol{L}_2(X, t)$, $\boldsymbol{L}_2(X)$, $\boldsymbol{L}_2(dB, t)$ and $\boldsymbol{L}_2(dB)$ with $\mathcal{M}(X, t)$ $\mathcal{M}(X)$, $\mathcal{M}(dB, t)$ and $\mathcal{M}(dB)$ respectively in Definitions 2 and 3 in § 1, we shall introduce the concepts "*properly canonical in linear sense*" and "*canonical in linear sense*".[2]

2) This definition of canonical representation is equivalent to Levy's one that is $E(Y(t)/\mathfrak{M}(Y, s)) = \int^{s} g(t-u)\, dB(u, \omega)$ for $s < t$.

It should be noted that the proviso "*in linear sense*" is omitted in the ordinary theory of linear representation of Gaussian processes.

Now we shall state some relations between two kinds of concepts mensioned above.

(A) X is purely non-deterministic if and only if X is purely non-deterministic in linear sense.

(B) In the representation of degree 1, four conditions, (a) *properly canonical in linear sense,* (b) *properly canonical,* (c) *canonical* and (d) *canonical in linear sense* are equivalent to each other.

The proof of (A) is easy. We can prove (B) by showing the implications

$$(a) \to (b) \to (c) \to (d) \to (a).$$

We shall now prove the implication (d) $\to$ (a) only, because other implications can be proved at once. Let $Y(t, \omega)=\int^t g(u-t)dB(u, \omega)$ be a properly canonical representation in linear sense and let $Z(t, \omega)=\int^t f(u-t)dB(u, \omega)$ be a canonical representation in linear sense. Since both are versions of X,

$$\begin{aligned} & E\{\hat{E}[Y(t, \omega)/\mathcal{M}_\tau(Y)]\hat{E}[Y(s, \omega)/\mathcal{M}_\tau(Y)]\} \\ &= E\{\hat{E}[Z(t, \omega)/\mathcal{M}_\tau(Z)]\hat{E}[Z(s, \omega)/\mathcal{M}_\tau(Z)]\} \qquad (\tau \leq t, s) \end{aligned}$$

Because of canonicality we deduce from this

$$\int^\tau g(u-t)g(u-s)du = \int^\tau f(u-t)f(u-s)du\,,$$

namely, $g(u-t)g(u-s)=f(u-t)f(u-s)$ for almost all $u \leq \min.(t, s)$. Taking $EX^2(0, \omega)>0$ into account, we obtain $f(t)=\delta g(t)$ for almost all t with a censtant δ equal to ± 1, which completes the proof of (d) $\to$ (a).

It follows at once from (B) that if (1) is not properly canonical in linear sense, as in $Y(t, \omega)\equiv B(t, \omega)-B(t-1, \omega)$, then $\boldsymbol{L}_2(Y, t)\subsetneqq \boldsymbol{L}_2(dB, t)$.

As a result of Theorem 1 in §2 we can determine all the canonical representations of a Gaussian process X which is purely non-deterministic. Let (1) be the properly canonical representation of X in linear sense. Then we have

Theorem 2. $\boldsymbol{f}$ is the kernel of a canonical representation of X if and only if there exists φ in $\mathfrak{F}$ such that

$$\begin{aligned} & f_n(t_1, \cdots, t_n) \\ & = \frac{1}{n} \sum_{i=1}^{n} g(t_i) \chi_{(-\infty, t_i]^{n-1}}(t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n) \varphi_{n-1}(t_1 - t_i, \\ & \qquad \cdots, t_{i-1} - t_i, t_{i+1} - t_i, \cdots, t_n - t_i) . \qquad (2) \end{aligned}$$

for almost everywhere in $(-\infty, 0]^n$, where $\chi_A(\cdot)$ is the indicator function of the set A.

Using this, we will determine the degree of canonical representation of Gaussian processes.

Theorem 3. The degree of canonical representation of Gaussian process is either 1 or ∞.

Proof. Because of

$$\|f_n\|^2 = \frac{1}{n} \|g\|^2 \|\varphi_{n-1}\|^2 \qquad (3)$$

it is enough to show that if $\varphi_E = (\varphi_{E,0}, \varphi_{E,1}, \cdots, \varphi_{E,N}, 0, 0 \cdots)$ for finite N, then $N=0$.

Let $\{\theta_n, n=1, 2, \cdots\}$ be the complete orthonormal system of $\boldsymbol{L}_2(-\infty, 0]$. Then $\{\xi_n(\omega) \equiv \int_{-\infty}^{0} \theta_n(t) dB(t, \omega), n=1, 2, \cdots\}$ are $N(0, 1)$-distributed and mutually independent random variables. By the definition of $\varphi_E = (\varphi_{E,0}, \cdots, \varphi_{E,N}, 0, 0 \cdots)$, we have.

$$\bar{X}_E(\omega) = \sum_{n=0}^{N} \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} \varphi_{E,n}(t_1, \cdots, t_n) dB(t_1, \omega) \cdots dB(t_n, \omega) .$$

More precisely ahd expressing this by $\xi_n(\omega)$, $n=1, 2, \cdots$ we get,

$$\bar{X}_E(\omega) = \sum_{p=0}^{N} \sum_{n} \sum_{p_1 + \cdots + p_n = p} \sum_{\alpha_1 \cdots \alpha_n} a_{p_1 \cdots p_n}^{\alpha_1 \cdots \alpha_n} \prod_{\nu=1}^{n} H_{p_\nu}\left(\frac{1}{\sqrt{2}} \xi_n(\omega)\right) \qquad (4)$$

We shall expand $\bar{X}_E(\omega)$ as a power series of $\xi_1(\omega)$:

$$\bar{X}_E(\omega) = \sum_{\nu=0}^{N} C_\nu(\xi_1(\omega), \xi_3(\omega), \cdots) \xi_1^\nu(\omega) \qquad (5)$$

Hence for fixed $(\xi_2(\omega), \xi_3(\omega), \cdots)$ it becomes a polynominal of $\xi_1(\omega)$

of degree N at most. Since the value of the left side of (5) is either 1 or -1, the degree of this polynomial must be 0. Thus

$$\bar{\chi}_E(\omega) = C_0(\xi_2(\omega), \xi_3(\omega), \cdots) \tag{6}$$

Using (4), C_0 can be expanded as a power series of $\xi_2(\omega)$ again, say,

$$\bar{\chi}_E(\omega) = \sum_{\nu=1}^{N} C_\nu^{(1)}(\xi_3(\omega), \xi_4(\omega), \cdots)\xi_2^\nu(\omega) \tag{7}$$

By the same argument we conclude

$$\bar{\chi}_E(\omega) = C_0^{(1)}(\xi_3(\omega), \xi_4(\omega), \cdots) . \tag{8}$$

Repeating this method, we obtain that $\bar{\chi}_E(\omega)$ is the function of $\xi_n(\omega), \xi_{n+1}(\omega), \cdots$ for any large n, and therefore $\bar{\chi}_E(\omega)$ is a constant by the $0-1$ low. This implies that $N=0$ as we wanted.

§ 4. Examples

Example 1. Let $P \equiv \{P(t, \tilde{\omega}),\ -\infty < t < \infty\}$ be a Poisson process with parameter λ and put

$$Y(t, \tilde{\omega}) = P(t, \tilde{\omega}) - P(t-1, \tilde{\omega}) - \lambda .$$

We shall show that Y has no canonical representation. Firstly we shall note that $\boldsymbol{B}_\tau(Y) = \boldsymbol{B}_\tau(dP)$. Almost all sample paths of Y are step functions whose jumps are either 1 or -1. Denote with D_m the set of

$$\{\tilde{\omega} : m > {}^{\exists}t > -m,\ P(t, \tilde{\omega}) - P(t-0, \tilde{\omega}) = P(t-1, \tilde{\omega}) - P(t-1-0, \tilde{\omega}) = 1\}$$

Then we have

$$D_m \subset \bigcap_{n=1}^{\infty} \left\{\tilde{\omega} ; m \geq {}^{\exists}\frac{k}{n} \geq -m,\ P\left(\frac{k}{n}, \tilde{\omega}\right) - P\left(\frac{k-1}{n}, \tilde{\omega}\right) \neq 0 ,\right.$$
$$\left. P\left(\frac{k-n}{n}, \tilde{\omega}\right) - P\frac{k-n-1}{n}, \tilde{\omega}\Big) \neq 0\right\} . \tag{1}$$

By the independence of Poisson processes we have for $n\,(>2)$.

$$\tilde{\boldsymbol{P}}\left\{P\left(\frac{k}{n}, \tilde{\omega}\right) - P\left(\frac{k-1}{n}, \tilde{\omega}\right) \neq 0,\ P\left(\frac{k-n}{n}, \tilde{\omega}\right) - P\frac{k-u-1}{n}, \tilde{\omega}\Big) \neq 0\right\}$$
$$= (1-e^{-\lambda/n})^2 . \tag{2}$$

Therefore the probability of the n-th set of the right side of (1) is less than $2(m+1)n(1-e^{-\lambda/n})^2$. Hence $\tilde{\boldsymbol{P}}(D_m)=0$, so that $\tilde{\boldsymbol{P}}(\bigcup_m D_m)=0$. This implies that for almost all sample paths there is no value of t such that

$$P(t, \tilde{\omega})-P(t-0, \tilde{\omega})=P(t-1, \tilde{\omega})-P(t-1-0, \tilde{\omega}) = 1 .$$

For any fixed s and t, consider

$$Z_n(\tilde{\omega}) \equiv \sum_{i=1}^{n} [\{Y(\tau_i, \tilde{\omega})-Y(\tau_{i-1}, \tilde{\omega})\} \vee 0]$$

$$\tau_i \equiv s+\frac{i(t-s)}{n} . \qquad (3)$$

It is clear that $Z_n(\tilde{\omega})$ increases to $P(t, \tilde{\omega})-P(s, \tilde{\omega})$ for almost all $\tilde{\omega}$ as $n\to\infty$. Hence $P(t, \tilde{\omega})-P(s, \tilde{\omega})$ is a measurable function with respect to $\bar{\boldsymbol{B}}_t(Y)$, namely $\boldsymbol{L}_2(dP, t)\subset \boldsymbol{L}_2(Y, t)$. The inverse inclusion relation is obvious.

It follows from this fact that

$$E(Y(t, \tilde{\omega})/\boldsymbol{B}_s(Y)) = P(s, \tilde{\omega})-P(t-1, \tilde{\omega})-\lambda(s-t+1) \qquad (4)$$

$$E[\{E(Y(t, \tilde{\omega})/\boldsymbol{B}_s(Y))-E(Y(t, \tilde{\omega})/\boldsymbol{B}_{s'}(Y))\}^2/\boldsymbol{B}_{s'}(Y)] = \lambda(s-s') \qquad (5)$$

for $t\geq s\geq s'\geq t-1$. Now suppose that Y had a canonical representation, say

$$Z(t, \omega) = \sum_{n=1}^{\infty} \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} f_n(t_1-t, \cdots , t_n-t)dB(t_1, \omega) \cdots dB(t_n, \omega)$$

$$= \int_{-\infty}^{t} F(t, u, \omega)dB(u, \omega)$$

Since Z is a version of Y, we should have

$$F^2(t, u, \omega) = \begin{cases} \lambda & \text{for } t\geq u\geq t-1 \\ 0 & \text{for } u<t-1 \end{cases}$$

using (5). Hence the value of F is either $\sqrt{\lambda}$ or $-\sqrt{\lambda}$ and

$$Z(t, \omega) = \int_{t-1}^{t} F(t, u, \omega)dB(u, \omega) .$$

Using the same argument as we used in Lemma 1, the probability law of $Z(t, \omega)$ would be $N(0, \lambda)$, which is absurd.

Example 2. Consider the Markov process with generator

$$\mathcal{G} = A(x-B)^2\frac{d^2}{dx^2}+(Dx+E)\frac{d}{dx} \qquad (x>B).$$

It is known that under the condition

$$A>0, \quad DB+E>0, \quad D<A \tag{6}$$

there exists one and only one invariant measure, which has density function p, [7].

$$p(y) = \text{const.}\,(y-B)^{D/A-2}\exp\left(-\frac{DB+E}{A}\frac{1}{y-B}\right) \qquad (y>B). \tag{7}$$

Then there exists a stationary Markov process $X=\{X(t, \omega), -\infty<t<\infty\}$ with generator $\mathcal{G}$. X has finite second momet, provided

$$D<0, \qquad -D>A>0. \tag{8}$$

Hereafter we will assume the condition (8). From X we can construct a W. r. m. dB [8] such that

$$X(t, \omega) = X(s, \omega)+\int_s^t (DX(\tau, \omega)+E)\,d\tau+\int_s^t \alpha(X(\tau, \omega)-B)\,dB(\tau, \omega) \tag{9}$$

where $\alpha=\sqrt{2A}$. This implies that $X(t, \omega)$ is measurable with respect to the Boreld field generated by $\{B(u, \omega)-B(v, \omega), s\leq v\leq u\leq t\}$ and $\bar{\boldsymbol{B}}_s(X)$. According to [2. Th. 3] we shall show that $X(t, \omega)$ is measurable with respect to $\bar{\boldsymbol{B}}_t(dB)$. Consider two processes X_a and X_b starting a and b at time 0, $(a>b>B)$, which satisfy stochastic integral equation.

$$X_i(t, \omega) = i+\int_0^t (DX_i(\tau, \omega)+E)\,d\tau+\int_0^t \alpha(X_i(\tau, \omega)-B)\,dB(\tau, \omega) \qquad i = a, b. \tag{10}$$

respectively, putting $Y(t, \omega)=X_a(t, \omega)-X_b(t, \omega)$, Y satisfies the following equation, for $t\geq 0$

$$Y(t+\tau, \omega) = Y(t, \omega)+\int_t^{t+\tau} DY(s, \omega)\,ds+\int_t^{t+\tau} Y(s, \omega)\,dB(s, \omega). \tag{11}$$

Solving (11), we have

$$Y(t, \omega) = (a-b)\exp\left[(D-A)t+\alpha(B(t, \omega)-B(0, \omega))\right]. \tag{12}$$

Hence

$$E|X_a(t, \omega)-X_b(t, \omega)| = |b-a|e^{Dt} . \tag{13}$$

On the other hand, solving the stochastic integral equation

$$\eta(t, \tilde{\omega}) = \eta(0, \tilde{\omega})+\int_0^t (D\eta(\tau, \tilde{\omega})+E)d\tau+\int_0^t \alpha(\eta(\tau, \tilde{\omega})-B)d\tilde{B}(\tau, \tilde{\omega}) . \tag{14}$$

where $\eta(0, \tilde{\omega})$ is independent to $\{\tilde{B}(u, \tilde{\omega})-\tilde{B}(u, \omega),\ 0\leq v< u\}$, we get a Markov process $\{\eta(t, \tilde{\omega}),\ t\geq 0\}$ and it is seen that η is a version of $\{X(t, \omega),\ t\geq 0\}$. This implies that $X(0, \omega)$ is also independent of $\{B(u, \omega)-B(v, \omega),\ 0\leq v< u\}$. Using (13),

$$\lim_{t\to\infty} E|X(t, \omega)-X_a(t, \omega)| = \lim_{t\to\infty} E[(|X(t, \omega)-X_a(t, \omega)|/X(0, \omega))] = 0 . \tag{15}$$

Denoting with $\boldsymbol{B}_{(0,t)}(dB)$ the Borel field generated $\{B(u, \omega)-B(v, \omega),\ 0\leq v< u\leq t\}$, we get

$$E|E(X(t, \omega)/\boldsymbol{B}_{(0,t)}(dB))-X_a(t, \omega)|\leq E[E(|X(t, \omega)-X_a(t, \omega)|/\boldsymbol{B}_{(0,t)}(dB))] = E|X(t, \omega)-X_a(t, \omega)|\to 0 . \tag{16}$$

Combining (15) with (16) concludes that $E|E(X(t, \omega)/\boldsymbol{B}_{(0,t)}(dB))-X(t, \omega)|$ tends to 0, which was to be proved.

Using equation (9), we obtain the expansion of X (properly canonical representation),

$$X(t, \omega) = -\frac{E}{D}-\sum_{n=1}^{\infty}\int_{-\infty}^{t}\cdots\int_{-\infty}^{t}\frac{\alpha^n}{n!}\left(\frac{E}{D}+B\right)e^{-D[(t_1-t)\wedge(t_2-t)\wedge\cdots\wedge(t_n-t)]} dB(t_1, \omega)\cdots dB(t_n, \omega) .$$

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REFERENCES

[1] N. Wiener, Non-linear problems in random theory (1958).

[2] M. Rosenblatt, Stationary Processes as Shifts of Functions of Independent Random Variables, Jour. Math. and Mech., 8 (159), 665–681.

[3] M. Rosenblatt, Stationary Markov Chains and Independent Random Variables, Jour. Math and Mech., 9 (1960), 945–950.

[4] T. Hida, Canonical representation of Gaussian processes and their application, Mem. Coll. Sci. Univ. Kyoto, 33 (1960), 109-155.
[5] K. Ito, Multiple Wiener Integral, Jour. Math. Soc. Japan, 3 (1951), 157-169.
[6] K. Ito, Stochastic integral. Proc. Imp. Acad. Tokyo, 20 (1944), 519-524.
[7] H. Tanaka, On Limiting Distributions for One-demensional Diffusion Processes, Bull. Math. Stat. 7 (1957), 84-91.
[8] J. L. Doob, Stochastic Processes, New York. J. Wiley (1953).