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On the imbedding of the Schwarzschild space-time I.

By

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Introduction.

The Schwarzschild space-time is a four-dimensional Riemannian space with the line element

$$ds^2 = \left(1-\frac{2m}{r}\right)dt^2-\left(1-\frac{2m}{r}\right)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta d\varphi^2),$$

where m is a positive constant, and the domains of variables are

$$-\infty < t < +\infty, \quad 2m < r < +\infty, \quad 0 \leqq \theta \leqq \pi, \quad 0 \leqq \varphi < 2\pi .$$

This was obtained as a solution of the Einstein gravitational equation and may satisfactorily describe the behaviour of the solar system. The properties of this space-time have frequently been studied so far both from the physical and the mathematical points of view.

It is well known that the Schwarzschild space-time is *of class two* in the sense of imbedding, that is, it can be imbedded in a six-dimensional pseudo-Euclidean spase [1]*. The following are the already established expressions for the imbedding, which were derived by Kanser [3] and Fronsdal [2] in an intuitive way.

(1) *Kasner imbedding*:

$$ds^2 = dz_1^2+dz_2^2-dz_3^2-dz_4^2-dz_5^2-dz_6^2,$$
$$z_1 = (1-2m/r)^{1/2}\cos t, \quad z_2 = (1-2m/r)^{1/2}\sin t,$$

* This is a special case of the fact that a spherically symmetric space time is of class two at most [4].

$$z_3 = f(r)\,, \quad \text{where } (df/dr)^2 = (2mr^3+m^2)/r^3(r-2m)\,,$$
$$z_4 = r\sin\theta\sin\varphi\,, \quad z_5 = r\sin\theta\cos\varphi\,, \quad z_6 = r\cos\theta\,.$$

(2) *Fronsdal imbedding*:

$$ds^2 = dz_1^2 - dz_2^2 - dz_3^2 - dz_4^2 - dz_5^2 - dz_6^2\,,$$
$$z_1 = 2(1-1/r)^{1/2}\sinh(t/2)\,, \quad z_2 = 2(1-1/r)^{1/2}\cosh(t/2)\,,$$
$$z_3 = g(r)\,, \quad \text{where } (dg/dr)^2 = (r^2+r+1)/r^3\,,$$
$$z_4 = r\sin\theta\sin\varphi\,, \quad z_5 = r\sin\theta\cos\varphi\,, \quad z_6 = r\cos\theta\,.$$

(Note that in the paper [2] the unit is taken so that $2m=1$.)

So far as we know, no further progress has been made in the study of the subject under consideration. The purpose of our work is to make a *thorough* investigation of the imbedding problem of the Schwarzschild space-time. Particularly, in the end of this paper, we shall determine all possible types of the imbedding function for a special case, which includes the above (1) and (2).

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§ 1. Preliminaries.

1. First of all, for the sake of convenience to later references we shall give a table of some quantities of the Schwarzschild space-time If the coordinates t, r, θ and φ are denoted by x^0, x^1, x^2 and x^3 respectively, and if we put $\gamma = 1-2m/r$, then the quantities are as follows. The fundamental tensor g_{ij} is given by

$$\begin{aligned} &g_{00} = \gamma\,, \quad g_{11} = -\gamma^{-1}\,, \quad g_{22} = -r^2\,, \quad g_{33} = -r^2\sin^2\theta\,, \\ &g_{ij} = 0 \quad \text{for } i \neq j\,; \end{aligned} \tag{1.1}$$

The Christoffel's symboles Γ^i_{jk} are given by

$$\begin{aligned} &\Gamma^0_{01} = \Gamma^0_{10} = mr^{-2}\gamma^{-1}\,, \quad \Gamma^1_{00} = mr^{-2}\gamma\,, \\ &\Gamma^1_{11} = -mr^{-2}\gamma^{-1}\,, \quad \Gamma^1_{22} = -r\gamma\,, \quad \Gamma^1_{33} = -r\gamma\sin^2\theta\,, \\ &\Gamma^2_{12} = \Gamma^2_{21} = r^{-1}\,, \quad \Gamma^3_{13} = \Gamma^3_{31} = r^{-1}\,, \\ &\Gamma^2_{33} = -\sin\theta\cos\theta\,, \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta\,, \\ &\text{other } \Gamma^i_{jk} = 0\,; \end{aligned} \tag{1.2}$$

Let R_{ijkl} be the components of the curvature tenser. Then $R_{ij}{}^{kl}=g^{ka}g^{lb}R_{ijab}$ are given by

$$(1.3)\quad \begin{cases} R_{01}{}^{01} = R_{23}{}^{23} = -2mr^{-3}, \\ R_{02}{}^{02} = R_{03}{}^{03} = R_{12}{}^{12} = R_{13}{}^{13} = mr^{-3}, \\ R_{ij}{}^{kl} = 0, \text{ if at least three of indices are different.} \end{cases}$$

2. The Schwarzschild space-time has, as immediately seen from the above table, the following properties:

[I] *There exists a coordinate system* (x^i), $i=0, 1, 2, 3$, *with respect to which the components of the fundamental tensor and the curvature tensor are such that*

$$g_{ij} = 0, \quad \text{if } i \neq j;$$
$$R_{ij}{}^{kl} = 0, \quad \text{if at least three of indices are different.}$$

[II] *The Ricci tensor vanishes*: $R_{ij}=g^{ab}R_{iajb}=0$.

For a while, we shall not confine our consideration to the Schwarzschild space-time, but more generally deal with the four-dimensional Riemannian spaces V^4 having those two properties only.

If we put

$$R_{ij}{}^{ij*} = S_{ij}, \quad i \neq j,$$

then it follows from the property [II] that

$$\sum_{j=0}^{3} S_{ij} = 0, \qquad i = 0, 1, 2, 3,$$

and this is, as easily seen, equivalent to

$$S_{01}+S_{02}+S_{03} = 0, \quad S_{01} = S_{23}, \quad S_{02} = S_{13}, \quad S_{03} = S_{12}.$$

Because of these identities, we can now classify those V^4 into the following four classes.

[A] None of S_{01}, S_{02} and S_{03} vanishes, and they are different one another.

[B] None of S_{01}, S_{02} and S_{03} vanishes, and two of them, say S_{02} and S_{03}, are equal each other, while the rest S_{01} differs from the other two.

[C] One of S_{01}, S_{02} and S_{03}, say S_{01}, vanishes and the other two S_{02} and S_{03} differ from zero.

[D] All of S_{ij} vanish.

* In this paper we do not use the summation convention.

The Schwarzschild space-time belongs, by virtue of (1. 3), to the class [**B**]. The condition of [**D**] implies that the V^4 is flat, and hereafter we exclude this class from the following consideration.

3. Suppose that a V^4 is imbedded in a six-dimensional pseudo-Euclidean space E^6 of some signature. We denote the (indefinite) inner product of the E^6 by the symbol $\langle , \rangle$. Let (x^i) be a point in V^4 and the vector $z(x)$ its image in E^6 by the imbedding map. The map being isometric, we have

$$\langle dz(x),\ dz(x)\rangle = \sum_{i,j} g_{ij}(x)dx^i dx^j ,$$

or this is equivalent to

$$\langle z_i , z_j\rangle = g_{ij} , \tag{1.4}$$

where $z_i = z_i(x) = \partial z/\partial x^i$, and geometrically $z_i(x)$ are tangent vectors in E^6 to $z(V^4)$ at $z(x)$. Let $n_\sigma(x)$, $\sigma=4, 5$ be two mutually orthogonal unit normals in E^6 to $z(V^4)$ at $z(x)$:

$$\langle n_\sigma , n_\sigma\rangle = e_\sigma(=\pm 1) , \qquad \sigma = 4, 5 ,$$
$$\langle n_4 , n_5\rangle = 0 ,$$
$$\langle n_\sigma , z_i\rangle = 0 , \qquad i = 0, 1, 2, 3 , \quad \sigma = 4, 5 .$$

As is well known, the expressions of the covariant derivatives $\nabla_j z_i$ and $\nabla_j n_\sigma$ in terms of z_i and n_σ are given by the so-called *Gauss* and *Weingarten formulas*

$$\nabla_j z_i = \sum_\sigma e_\sigma b_{\sigma ij} n_\sigma , \tag{1.5}$$

$$\nabla_j n_\sigma = -\sum_{i,k} g^{ik} b_{\sigma ij} z_k + \sum_\tau e_\tau \nu_{\tau\sigma j} n_\tau , \tag{1.6}$$

and the coefficients $b_{\sigma ij}(=b_{\sigma ji})$ and $\nu_{\tau\sigma j}(=-\nu_{\sigma\tau j})$ in the expressions (1. 5) and (1. 6) necessarily satisfy the following *Gauss, Codazzi* and *Ricci equations* [1].

$$\sum_\sigma e_\sigma(b_{\sigma ik}b_{\sigma jl} - b_{\sigma il}b_{\sigma jk}) = R_{ijkl} , \tag{1.7}$$

$$\nabla_k b_{\sigma ij} - \nabla_j b_{\sigma ik} = \sum_\tau e_\tau(b_{\tau ij}\nu_{\tau\sigma k} - b_{\tau ik}\nu_{\tau\sigma j}) , \tag{1.8}$$

$$\nabla_k \nu_{\tau\sigma j} - \nabla_j \nu_{\tau\sigma k} = \sum_{i,l} g^{il}(b_{\tau ik}b_{\sigma jl} - b_{\tau ij}b_{\sigma kl}) . \tag{1.9}$$

These equations are essentially important because of the fundamental theorem of imbedding, stated as follows: a V^4 can be imbedded in an E^6, if and only if there exist quantities $b_{\sigma ij}$ $(=b_{\sigma ji})$, $\nu_{\tau\sigma j}$ $(=-\nu_{\sigma\tau j})$ and $e_\sigma=\pm 1$, satisfying the above three equations.

4. Let $\bar{n}_\sigma$, $\sigma=4, 5$ be another set of normals and let $\bar{e}_\sigma$, $\bar{b}_{\sigma ij}$ and $\bar{\nu}_{\tau\sigma j}$ be the corresponding quantities. If we put

$$\bar{n}_\tau = \sum_\sigma c_{\tau\sigma} n_\sigma\,, \qquad \tau = 4, 5\,,$$

then the coefficients $c_{\tau\sigma}$ must satisfy the orthogonality conditions

$$\sum_\rho e_\rho c_{\tau\rho} c_{\sigma\rho} = \bar{e}_\tau \delta_{\tau\sigma}\,. \tag{1.10}$$

The following is an example of such alterations of normals:

$$\begin{pmatrix}\bar{n}_4\\ \bar{n}_5\end{pmatrix} = \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}n_4\\ n_5\end{pmatrix} \text{ or } \begin{pmatrix}\cosh\alpha & \sinh\alpha\\ \sinh\alpha & \cosh\alpha\end{pmatrix}\begin{pmatrix}n_4\\ n_5\end{pmatrix}, \tag{1.11}$$

where the alternative is to be chosen according to $e_4 e_5=\pm 1$, and α may be a function of coordinates. It is easily verified that the coefficients in (1.11) satisfy the conditions (1.10). For the later reference we give here the transformation formula of $b_{\sigma ij}$ and $\nu_{\tau\sigma j}$ corresponding to (1.11):

$$\begin{pmatrix}\bar{b}_{4ij}\\ \bar{b}_{5ij}\end{pmatrix} = \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}b_{4ij}\\ b_{5ij}\end{pmatrix} \tag{1.12}$$

$$\text{or } \begin{pmatrix}\cosh\alpha & \sinh\alpha\\ \sinh\alpha & \cosh\alpha\end{pmatrix}\begin{pmatrix}b_{4ij}\\ b_{5ij}\end{pmatrix} \text{ respectively,}$$

$$\bar{\nu}_j = \nu_j - e_5 \partial_j \alpha\,, \quad \text{in both cases,} \tag{1.13}$$

where $\nu_j=\nu_{45j}$ and $\bar{\nu}_j=\bar{\nu}_{45j}$.

§ 2. The Gauss equation.

1. In this section we deal with the Gauss equation (1.7), for such a V^4 satisfying the assumptions [I] and [II]. We shall show that only few components of the tensors $b_{\sigma ij}$ remain non-trivial under an additional assumption, which is, of course, satisfied by the Schwarzschild space-time.

If we put

$$b_{\sigma ij}{}^{kl} = b_{\sigma i}{}^k b_{\sigma j}{}^l - b_{\sigma i}{}^l b_{\sigma j}{}^k\,, \qquad \Big(b_{\sigma i}{}^k = \sum_j g^{jk} b_{\sigma ij}\Big)\,, \tag{2.1}$$

then the Gauss equation (1. 7) is written as

(2. 2) $$R_{ij}{}^{pq} = e_4 b_{4ij}{}^{pq} + e_5 b_{5ij}{}^{pq} .$$

Let $\overline{pq}$ be the ordered complement of pq in 0123, (by which we mean that if $\overline{pq}=rs$, then $pqrs$ is a permutation of 0123 and $(p-q)(r-s)>0$)*. Multiplying (2. 2) by $(-1)^{p+q}b_{4ik}{}^{\overline{pq}}$ and summing up for $p, q=0, 1, 2, 3$ and $p<q$, we have

$$\sum_{p<q}(-1)^{p+q}R_{ij}{}^{pq}b_{4ik}{}^{\overline{pq}} = e_4\sum_{p<q}(-1)^{p+q}b_{4ij}{}^{pq}b_{4ik}{}^{\overline{pq}} + e_5\sum_{p<q}(-1)^{p+q}b_{5ij}{}^{pq}b_{4ik}{}^{\overline{pq}}$$

On the left hand-side we apply the property [I] in the last section, while on the right hand-side Laplace's expansion theorem in the determinant theory. Then we obtain

$$(-1)^{i+j}S_{ij}b_{4ik}{}^{\overline{ij}} = e_5\sum_{p<q}(-1)^{p+q}b_{5ij}{}^{pq}b_{4ik}{}^{\overline{pq}} .$$

Similarly we have

$$(-1)^{i+k}S_{ik}b_{5ij}{}^{\overline{ik}} = e_4\sum_{p<q}(-1)^{p+q}b_{4ik}{}^{pq}b_{5ij}{}^{\overline{pq}} .$$

It follows from these two relations that

(2. 3) $$(-1)^{j}e_4S_{ij}b_{4ik}{}^{\overline{ij}} = (-1)^{k}e_5S_{ik}b_{5ij}{}^{\overline{ik}} .$$

First, if we take $k=j$, then (2. 3) becomes

$$S_{ij}(e_4b_{4ij}{}^{\overline{ij}} - e_5b_{5ij}{}^{\overline{ij}}) = 0 ,$$

while the Gauss equation (2. 2) gives

$$e_4b_{4ij}{}^{\overline{ij}} + e_5b_{5ij}{}^{ij} = R_{ij}{}^{\overline{ij}} = 0 .$$

Therefore we get $S_{ij}=0$ or $b_{\sigma ij}{}^{\overline{ij}}=0$, $\sigma=4, 5$. Thus we have

(2. 4) $$b_{\sigma ij}{}^{\overline{ij}} = 0 \begin{cases} i, j: \text{ any} & \text{for [A], [B]} \\ \{i, j\}^{**} \neq \{0, 1\}, \{2, 3\} & \text{for [C]} . \end{cases}$$

Next, let i, j and k in (2. 3) be different one another. Then (2. 3) is written as

(2. 5) $$(-1)^{j}e_4S_{ij}b_{4kl}{}^{\overline{jl}} = (-1)^{k}e_5S_{ik}b_{5jl}{}^{\overline{kl}} ,$$

* For example, $\overline{01}=23$, $\overline{02}=13$, $\overline{21}=30$.

** The expression $\{i, j\}=\{0, 1\}$ means that $i=0$, $j=1$ and $i=1$, $j=0$.

by making use of the following identities

$$b_{\sigma ik}{}^{\overline{ij}} = g_{ii}g^{ll}b_{\sigma\overline{ij}}{}^{ik}\,, \qquad l \neq i, j, k\,.$$
$$b_{\sigma\overline{ij}}{}^{ik} = \operatorname{sgn}[(i-j)(k-l)(i-k)(j-l)]b_{\sigma kl}{}^{\overline{jl}}\,.$$

These are verified as follows: $b_{\sigma ik}{}^{\overline{ij}} = \pm b_{\sigma ik}{}^{kl} = \pm(b_{\sigma i}{}^{k}b_{\sigma k}{}^{l} - b_{\sigma i}{}^{l}b_{\sigma k}{}^{k})$ $= \pm(g_{ii}g^{kk}b_{\sigma}{}^{i}{}_{k}g_{kk}g^{ll}b_{\sigma}{}^{k}{}_{l} - g_{ii}g^{ll}b_{\sigma}{}^{i}{}_{l}g_{kk}g^{kk}b_{\sigma}{}^{k}{}_{k}) = \pm g_{ii}g^{ll}(b_{\sigma k}{}^{i}b_{\sigma l}{}^{k} - b_{\sigma k}{}^{k}b_{\sigma l}{}^{i})$ $= \pm g_{ii}g^{ll}b_{\sigma kl}{}^{ik} = g_{ii}g^{ll}b_{\sigma\overline{ij}}{}^{ik}$; and $b_{\sigma\overline{ij}}{}^{ik} = \operatorname{sgn}[(i-j)(k-l)]b_{\sigma kl}{}^{ik} =$ $\operatorname{sgn}[(i-j)(k-l)(i-k)(j-l)]b_{\sigma kl}{}^{\overline{jl}}$. Exchanging i and l in (2.5), we have

$$(2.6) \qquad (-1)^{j}e_4S_{lj}b_{4ki}{}^{\overline{ji}} = (-1)^{k}e_5S_{lk}b_{5ji}{}^{\overline{ki}}\,.$$

It follows from (2.3) and (2.6) that

$$(2.7) \qquad (S_{ij}S_{lk} - S_{lj}S_{ik})b_{\sigma ik}{}^{\overline{ij}} = 0\,, \qquad \sigma = 4, 5\,.$$

If we recall a property of S_{ij}: $S_{ij} = S_{\overline{ij}}$, (2.7) is reduced to

$$(S_{lk}{}^{2} - S_{ik}{}^{2})b_{\sigma ik}{}^{kl} = 0\,, \qquad i, k, l:\ \text{different}\,,\ \sigma = 4, 5\,,$$

from which we get another case-wise result:

$$(2.8) \qquad b_{\sigma ik}{}^{kl} = 0 \begin{cases} \textit{for [A], if } i, k \textit{ and } l \textit{ are different,} \\ \textit{for [B] and [C], if } i, k \textit{ and } l \textit{ are different} \\ \quad \textit{and } \{i, l\} \neq \{0, 1\},\ \{2, 3\}. \end{cases}$$

2. Let us consider the case [**A**]. The indices i, k and l being different and fixed, four equations $b_{\sigma ik}{}^{kl} = 0$ and $b_{\sigma ik}{}^{il} = 0$ ($\sigma = 4, 5$) obtained from (2.8) may be regarded as two systems of linear equations in $b_{\sigma i}{}^{l}$ and $b_{\sigma k}{}^{l}$:

$$\begin{cases} b_{\sigma k}{}^{k}b_{\sigma i}{}^{l} - b_{\sigma i}{}^{k}b_{\sigma k}{}^{l} = 0\,, \\ b_{\sigma k}{}^{i}b_{\sigma i}{}^{l} - b_{\sigma i}{}^{i}b_{\sigma k}{}^{l} = 0\,, \end{cases} \qquad \sigma = 4, 5\,.$$

Two determinants $b_{\sigma k}{}^{k}b_{\sigma i}{}^{i} - b_{\sigma k}{}^{i}b_{\sigma i}{}^{k} = b_{\sigma ki}{}^{ki}$ ($\sigma = 4, 5$) formed with the coefficients of the above systems never vanish at the same time because of $\sum e_\sigma b_{\sigma ki}{}^{ki} = S_{ki} \neq 0$. If both of them differ from zero, it follows immediately that

$$(2.9) \qquad b_{\sigma i}{}^{l} = b_{\sigma k}{}^{l} = 0\,, \qquad \sigma = 4, 5\,.$$

If $b_{4ki}{}^{ki} = 0$ and $b_{5ki}{}^{ki} \neq 0$, we replace the normal $\boldsymbol{n}_\sigma$ by the other $\bar{\boldsymbol{n}}_\sigma$ given in (1.11) where α is small. Then it is easily seen from

(1.12) that both $\bar{b}_{4ki}{}^{ki}$ and $\bar{b}_{5ki}{}^{ki}$ differ from zero so far as $\alpha \neq 0$. Therefore we have

$$\bar{b}_{\sigma i}{}^{l} = \bar{b}_{\sigma k}{}^{l} = 0\,, \qquad \sigma = 4, 5\,. \tag{2.10}$$

But, since $\bar{b}_{\sigma i}{}^{k}$ continuously depend on α, (2.10) must be true even when $\alpha = 0$. Thus we have the following

Proposition 2.1. *In the case* [**A**], *the second fundamental tensors* $b_{\sigma ij}$ *are necessarily diagonal*:

$$b_{\sigma ij} = 0\,, \qquad i \neq j\,, \quad \sigma = 4, 5\,.$$

3. Next, consider the case [**B**]. We shall show

Proposition 2.2. *In the case* [**B**] *the second fundamental tensors* $b_{\sigma i}{}^{k}$ *satisfy*

$$b_{\sigma i}{}^{k} = 0\,, \qquad i \neq k\,, \quad \{i, k\} \neq \{0, 1\},\ \{2, 3\}\,, \tag{2.11}$$

$$b_{\sigma 0}{}^{1} b_{\sigma 2}{}^{3} = 0\,, \tag{2.12}$$

$$b_{\sigma 0}{}^{1}(b_{\sigma 2}{}^{2} - b_{\sigma 3}{}^{3}) = 0\,, \quad b_{\sigma 2}{}^{3}(b_{\sigma 0}{}^{0} - b_{\sigma 1}{}^{1}) = 0\,, \tag{2.13}$$

$$(b_{\sigma 0}{}^{0} - b_{\sigma 1}{}^{1})(b_{\sigma 2}{}^{2} - b_{\sigma 3}{}^{3}) = 0\,, \tag{2.14}$$

where σ *takes* 4, 5.

Proof. The above argument for the case [A] remains valid so long as $\{i, l\} \neq \{0, 1\}$, $\{2, 3\}$ and $\{k, l\} \neq \{0, 1\}$, $\{2, 3\}$, and hence (2.11) holds as well. Next, take $i=0$ and $j=2$ in (2.4). Then we have $b_{\sigma 02}{}^{13} = 0$ and, by virtue of (2.11), this is (2.12) itself. If we take $i=0$, $j=2$ and $k=3$ in (2.3) and divide it by $S_{02} (= S_{03} \neq 0)$, then we have $e_4 b_{403}{}^{13} + e_5 b_{502}{}^{12} = 0$. On the other hand, the Gauss equation (2.2) gives $e_4 b_{402}{}^{12} + e_5 b_{502}{}^{12} = 0$. From these two, we get $e_4 b_{402}{}^{12} = e_4 b_{403}{}^{13}$, which is now reduced to $b_{40}{}^{1}(b_{42}{}^{2} - b_{43}{}^{3}) = 0$. Similarly we have $b_{50}{}^{1}(b_{52}{}^{2} - b_{53}{}^{3}) = 0$; thus the first of (2.13) is proved. By the similar way the second follows. Finally, from the Gauss equation (2.2) where $i=k=2, 3$ and $j=l=0, 1$, we have, in the present case,

$$\sum_{\sigma} e_\sigma b_{\sigma i}{}^{i} b_{\sigma j}{}^{j} = S_{ij} \qquad i = 2, 3\,;\ j = 0, 1\,. \tag{2.15}$$

Since $S_{20} = S_{30} = S_{21} = S_{31}$, we get

$$\sum_{\sigma} e_\sigma b_{\sigma i}{}^{i} (b_{\sigma 0}{}^{0} - b_{\sigma 1}{}^{1}) = 0\,, \qquad i = 2, 3\,.$$

From these two equations for $i=2, 3$ it follows that

$$(2.16) \qquad (b_{42}{}^2 b_{53}{}^3 - b_{43}{}^3 b_{52}{}^2)(b_{\sigma 0}{}^0 - b_{\sigma 1}{}^1) = 0 , \qquad \sigma = 4, 5 .$$

On the other hand, from (2.15) we can derive

$$e_4(b_{42}{}^2 b_{53}{}^3 - b_{43}{}^3 b_{52}{}^2) b_{40}{}^0 = S_{20} b_{53}{}^3 - S_{30} b_{52}{}^2 ,$$
$$e_5(b_{53}{}^3 b_{42}{}^2 - b_{52}{}^2 b_{43}{}^3) b_{50}{}^0 = S_{30} b_{42}{}^2 - S_{20} b_{43}{}^3 .$$

Since $S_{20} = S_{30} \neq 0$, these equations show that $b_{42}{}^2 b_{53}{}^3 - b_{43}{}^3 b_{52}{}^2 = 0$ implies $b_{\sigma 2}{}^2 - b_{\sigma 3}{}^3 = 0$, $\sigma = 4, 5$. From this fact and (2.16), the desired (2.14) follows immediately.

It should be noted that (2.12) i.e. $b_{40}{}^1 b_{42}{}^3 = b_{50}{}^1 b_{52}{}^3 = 0$ implies $b_{40}{}^1 = b_{50}{}^1 = 0$ or $b_{42}{}^3 = b_{52}{}^3 = 0$. In fact, if either $b_{40}{}^1$ or $b_{50}{}^1 \neq 0$, then, replacing the $\boldsymbol{n}_\sigma$ by a suitable $\bar{\boldsymbol{n}}_\sigma$, we may assume that both $\bar{b}_{40}{}^1$ and $\bar{b}_{50}{}^1 \neq 0$. On the other hand, since (2.12) is valid for any normals, we have $\bar{b}_{\sigma 0}{}^1 \bar{b}_{\sigma 2}{}^3 = 0$ and hence $\bar{b}_{42}{}^3 = \bar{b}_{52}{}^3 = 0$, which is equivalent to $b_{42}{}^3 = b_{52}{}^3 = 0$. The same is true for (2.13) and (2.14).

From the standpoint of the above considerations, we shall further classify the case [**B**]. First, by virtue of (2.12), the following three cases are possible:

(**a**) $\qquad b_{\sigma 0}{}^1 = b_{\sigma 2}{}^3 = 0 ,$

(**b**) $\qquad b_{\sigma 0}{}^1 \neq 0 , \quad b_{\sigma 2}{}^3 = 0 ,$

(**c**) $\qquad b_{\sigma 0}{}^1 = 0 , \quad b_{\sigma 2}{}^3 \neq 0 .$

Then (2.13) gives an additional property $b_{\sigma 2}{}^2 = b_{\sigma 3}{}^3$ to the case (**b**) and $b_{\sigma 0}{}^0 = b_{\sigma 1}{}^1$ to the case (**c**). For the later requirement we may subdivide the case (**a**) as follows:

(**a**$_1$) $\qquad b_{\sigma 0}{}^1 = b_{\sigma 2}{}^3 = 0 , \quad b_{\sigma 2}{}^2 = b_{\sigma 3}{}^3 ,$

(**a**$_2$) $\qquad b_{\sigma 0}{}^1 = b_{\sigma 2}{}^3 = 0 , \quad b_{\sigma 2}{}^2 \neq b_{\sigma 3}{}^3 .$

Then (2.14) gives an additional property $b_{\sigma 0}{}^0 = b_{\sigma 1}{}^1$ to the case (**a**$_2$). In the sequel we refer to the cases (**a**$_1$), (**a**$_2$), (**b**) and (**c**) as [**B**$_1$], [**B**$_3$], [**B**$_2$] and [**B**$_4$] respectively.

4. Finally, consider the case [**C**]. The argument for the case [**A**] is valid so long as $\{i, l\}$, $\{k, l\}$, $\{i, k\} \neq \{0, 1\}$, $\{2, 3\}$, but under this restriction any choice of indices is impossible.

Now we put on the V^4 a further assumption:

[III] *None of S_{ij} vanishes.*
Then only the cases [A] and [B] can take place. As is seen from (1.3), the Schwarzschild space-time satisfies this assumption.

At closing this section, we summarize the results.

Proposition 2.3. *If a space V^4 satisfying the assumptions* [I], [II] *and* [III] *is imbedded in an E^6, its second fundamental tensors $b_{\sigma i}{}^k$ belong to one of the following five classes:*

[A] $b_{\sigma i}{}^k = 0 \ (i \neq k)$, $b_{\sigma 0}{}^0 \neq b_{\sigma 1}{}^1$, $b_{\sigma 2}{}^2 \neq b_{\sigma 3}{}^3$ * ,

[B_1] $b_{\sigma i}{}^k = 0 \ (i \neq k)$, $b_{\sigma 2}{}^2 = b_{\sigma 3}{}^3$,

[B_2] $b_{\sigma i}{}^k = 0 \ (i \neq k, \{i, k\} \neq \{0, 1\})$, $b_{\sigma 0}{}^1 \neq 0$, $b_{\sigma 2}{}^2 = b_{\sigma 3}{}^3$,

[B_3] $b_{\sigma i}{}^k = 0 \ (i \neq k)$, $b_{\sigma 0}{}^0 = b_{\sigma 1}{}^1$, $b_{\sigma 2}{}^2 \neq b_{\sigma 3}{}^3$,

[B_4] $b_{\sigma i}{}^k = 0 \ (i \neq k, \{i, k\} \neq \{2, 3\})$, $b_{\sigma 2}{}^3 \neq 0$, $b_{\sigma 0}{}^0 = b_{\sigma 1}{}^1$.

§ 3. The Codazzi and Ricci equations.

1. In this section we deal with the Codazzi equation (1.8) and the Ricci equation (1.9) for such a V^4 satisfying the assumptions [I], [II] and [III]. Concerning the Ricci equation we shall find that the coefficients $\nu_{\tau\sigma i}$ may be taken as zero by a suitable choice of the normals. This fact will simplify the forthcoming discussion. While, by examing the Codazzi equation, we shall see that the cases [B_3] and [B_4] can not take place under a further assumption which is also satisfied by the Schwarzschild space.

First, we consider the Ricci equation (1.9). If we set $\nu_i = \nu_{45i}$ $(= -\nu_{54i})$, then (1.9) is written as

$$\partial_k \nu_j - \partial_j \nu_k = \sum_i b_{4k}{}^i b_{5ij} - \sum_i b_{4j}{}^i b_{5ik} .$$

In the cases [A], [B_1] and [B_3], the right hand-side always vanishes since the $b_{\sigma i}{}^k$ (and hence $b_{\sigma ik}$ too) are diagonal. In the case [B_2] the right hand-side also vanishes so long as $\{j, k\} \neq \{0, 1\}$. When $(j, k) = (0, 1)$, we have

$$\partial_1 \nu_0 - \partial_0 \nu_1 = g_{11}[b_{40}{}^1(b_{50}{}^0 - b_{51}{}^1) - b_{50}{}^1(b_{40}{}^0 - b_{41}{}^1)] .$$

On the other hand, the Gauss equation (2.2) gives

* The last two follow easily from the fact that S_{01}, S_{02} and S_{03} are different one another.

$$e_4 b_{40}{}^1 b_{42}{}^2 + e_5 b_{50}{}^1 b_{52}{}^2 = R_{02}{}^{12} = 0 ,$$
$$e_4 (b_{40}{}^0 - b_{41}{}^1) b_{42}{}^2 + e_5 (b_{50}{}^0 - b_{51}{}^1) b_{52}{}^2 = S_{02} - S_{12} = 0 ,$$
$$e_4 (b_{42}{}^2)^2 + e_5 (b_{52}{}^2)^2 = e_4 b_{42}{}^2 b_{43}{}^3 + e_5 b_{52}{}^2 b_{53}{}^3 = S_{23} \neq 0 .$$

From these it follows that $b_{40}{}^1(b_{50}{}^0 - b_{51}{}^1) - b_{50}{}^1(b_{40}{}^0 - b_{41}{}^1) = 0$, and hence we have $\partial_k \nu_j - \partial_j \nu_k = 0$ for any j and k. Similarly we have the same result for [$\mathbf{B}_4$]. Therefore, in each case we find a function ν such that $\nu_i = \partial_i \nu$. With this ν, let the normals $\boldsymbol{n}_\sigma$ replace by the other $\bar{\boldsymbol{n}}_\sigma$ in (1. 11) where we take $\alpha = e_5 \nu$. Then (1. 13) gives $\bar{\nu}_i = 0$. Thus we have

Proposition 3. 1. *For a space V^4 stated in the proposition* 2. 3., *there exists a set of normals $\boldsymbol{n}_\sigma$ such that the corresponding coefficients $\nu_{\tau\sigma i}$ vanish.*

Hereafter it is always understood that the normals are chosen as in the proposition.

2. Consider the Codazzi equation (1. 8). The normals being chosen as above, it is rewritten in the following form:

(3. 1) $$\partial_k b_{\sigma j}{}^i - \partial_j b_{\sigma k}{}^i = -\sum_l \Gamma_l{}^i{}_k b_{\sigma j}{}^l + \sum_l \Gamma_l{}^i{}_j b_{\sigma k}{}^l .$$

In the cases [$\mathbf{B}_3$] and [$\mathbf{B}_4$], if we take $i=0$, $j=0$ and $k=1$, (3. 1) reduces to $\partial_1 b_{\sigma 0}{}^0 = 0$, since $b_{\sigma 0}{}^1 = b_{\sigma 1}{}^0 = 0$ and $b_{\sigma 0}{}^0 = b_{\sigma 1}{}^1$. Hence $\partial_1 S_{01} = \partial_1(\sum e_\sigma b_{\sigma 0}{}^0 b_{\sigma 1}{}^1) = \partial_1(\sum e_\sigma (b_{\sigma 0}{}^0)^2) = 2 \sum e_\sigma b_{\sigma 0}{}^0 \partial_1 b_{\sigma 0}{}^0 = 0$. This is not the case of the Schwarzschild space-time, because $S_{01} = -2mr^{-3}$ from (1. 3). By this fact we are suggested to set an assumption

[**IV**] *The quantity S_{01} really depends on x^1.*

Under this assumption the cases [$\mathbf{B}_3$] and [$\mathbf{B}_4$] can not take place as seen above.

Proposition 3. 2. *If a space V^4 satisfying the assumptions* [**I**], [**II**], [**III**] *and* [**IV**] *is imbedded in an E^6, its second fundamental tensors $b_{\sigma i}{}^k$ belong to one of the three classes* [**A**], [$\mathbf{B}_1$] *and* [$\mathbf{B}_2$] *in the proposition* 2. 3.

§ 4. The integration of the Gauss formula.

1. We now return to the Schwarzschild space-time itself, so that the case [**A**] is excluded in the proposition 3. 2. The aim of this section is to integrate the Gauss formula (1.5) and to obtain an explicit form of z.

In the present case the Gauss formula (1.5) for $i \neq j$, $\{i, j\} \neq \{0, 1\}$ is written as

$$\partial_i \partial_j z - \sum_k \Gamma_{ij}^k \partial_k z = 0 .$$

Substituting from (1.2), we have the following five equations:

$$\partial_0 \partial_2 z = 0 , \quad \partial_0 \partial_3 z = 0 , \quad \partial_1 \partial_2 z = r^{-1} \partial_2 z ,$$
$$\partial_1 \partial_3 z = r^{-1} \partial_3 z , \quad \partial_2 \partial_3 z = \cot \theta \partial_3 z .$$

Integrating this set of equations (note that ∂_0, ∂_1, ∂_2 and ∂_3 mean $\partial/\partial t$, $\partial/\partial r$, $\partial/\partial \theta$ and $\partial/\partial \varphi$ respectively), we easily find

$$z = A(t, r) + r[B(\theta) + \sin \theta C(\varphi)] , \tag{4.1}$$

where A, B and C are some functions, which we are going to determine. On the other hand, we have another equation in z

$$\nabla_3 z_3 = \sin^2 \theta \nabla_2 z_2 . \tag{4.2}$$

This is obtained as follows: from $b_{\sigma 2}{}^2 = b_{\sigma 3}{}^3$, we have $b_{\sigma 33} = g_{33} g^{22} b_{\sigma 22} = \sin^2 \theta b_{\sigma 22}$ and hence from (1.5) the above (4.2) follows at once. (4.2) is expressed as

$$\partial_3 \partial_3 z + r\gamma \sin^2 \theta \partial_1 z + \sin \theta \cos \theta \, \partial_2 z = \sin^2 \theta (\partial_2 \partial_2 z + r\gamma \partial_1 z) ,$$

where $\gamma = 1 - 2m/r$. Substituting from (4.1), we get

$$C''(\varphi) + C(\varphi) = \sin \theta B''(\theta) - \cos \theta B'(\theta) .$$

Evidently both of the sides must be equal to a constant vector d. Then each side being separately integrated, we find

$$C(\varphi) = \sin \varphi a + \cos \varphi b + d ,$$
$$B(\theta) = -\sin \theta d + \cos \theta c + e ,$$

where a, b, c and e are some constant vectors. Putting these expressions into (4.1), we get a more precise expression of z:

$$z = y(t, r) + r(\sin \theta \sin \varphi a + \sin \theta \cos \varphi b + \cos \theta c) ,$$

where we put $y(t, r) = A(t, r) + re$. Often we abbreviate it as

(4. 3) $$z = y(t, r) + rS(\theta, \varphi),$$

where

(4. 4) $$S(\theta, \varphi) = \sin\theta \sin\varphi\, a + \sin\theta \cos\varphi b + \cos\theta c.$$

2. Recall the isometry condition (1. 4). Substituting the above expression (4. 3) into (1. 4), we have several relations

(4. 5) $$\begin{cases} \langle \partial_0 y, \partial_0 y \rangle = 1 - 2m/r, \\ \langle \partial_0 y, \partial_1 y + S \rangle = 0, \\ \langle \partial_1 y + S, \partial_1 y + S \rangle = -(1 - 2m/r)^{-1}, \end{cases}$$

(4. 6) $$\begin{cases} \langle \partial_2 S, \partial_2 S \rangle = -1, \\ \langle \partial_2 S, \partial_3 S \rangle = 0, \\ \langle \partial_3 S, \partial_3 S \rangle = -\sin^2\theta, \end{cases}$$

(4. 5) $$\begin{cases} \langle \partial_0 y, \partial_2 S \rangle = 0, \quad \langle \partial_0 y, \partial_3 S \rangle = 0, \\ \langle \partial_1 y + S, \partial_2 S \rangle = 0, \quad \langle \partial_1 y + S, \partial_3 S \rangle = 0. \end{cases}$$

From (4. 6) and (4 .4) we find

$$\langle a, a \rangle = \langle b, b \rangle = \langle c, c \rangle = -1,$$
$$\langle b, c \rangle = \langle c, a \rangle = \langle a, b \rangle = 0.$$

Take an orthogonal base $\{e_1, \cdots, e_6\}$ in E^6 such that

$$e_4 = a, \quad e_5 = b, \quad e_6 = c.$$

Then $S(\theta, \varphi)$ as well as $\partial_2 S$ and $\partial_3 S$ are always contained in the $e_4 e_5 e_6$-plane. Next (4. 7) shows that $\partial_0 y$ and $\partial_1 y$ are orthogonal to the plane and hence they are contained in the $e_1 e_2 e_3$-plane. Therefore by a translation in E^6 we may assume that y itself lies in the $e_1 e_2 e_3$-plane. Finally (4. 5) reduces to

(4. 8) $$\begin{cases} \langle \partial_0 y, \partial_0 y \rangle = \dfrac{r-2m}{r}, \quad \langle \partial_0 y, \partial_1 y \rangle = 0, \\ \langle \partial_1 y, \partial_1 y \rangle = -\dfrac{2m}{r-2m}. \end{cases}$$

§ 5. Solutions for the stationary imbedding.

1. Continuing the last section, we consider the Schwarzschild space-time. Let us again take up the Gauss equation (1. 7) and the

Codazzi equation (1. 8). Since $b_{\sigma i}{}^{k}=0$ $(i\neq k,\ \{i, k\}\neq\{0, 1\})$ and $b_{\sigma 2}{}^{2}=b_{\sigma 3}{}^{3}$ in the present case, non-trivial equations of (1. 7) and (1. 8) are as follows

$$(5.1)\quad \begin{cases} \sum e_\sigma(b_{\sigma 0}{}^{0}b_{\sigma 1}{}^{1}-b_{\sigma 0}{}^{1}b_{\sigma 1}{}^{0}) = -2mr^{-3}, \\ \sum e_\sigma b_{\sigma 0}{}^{0}b_{\sigma 2}{}^{2} = mr^{-3}, \\ \sum e_\sigma b_{\sigma 1}{}^{1}b_{\sigma 2}{}^{2} = mr^{-3}, \\ \sum e_\sigma b_{\sigma 2}{}^{2}b_{\sigma 2}{}^{2} = -2mr^{-3}, \\ \sum e_\sigma b_{\sigma 0}{}^{1}b_{\sigma 2}{}^{2} = 0, \end{cases}$$

$$(5.2)\quad \begin{cases} \partial_1 b_{\sigma 0}{}^{0}-\partial_0 b_{\sigma 1}{}^{0} = mr^{-1}\gamma^{-1}(b_{\sigma 1}{}^{1}-b_{\sigma 0}{}^{0}), \\ \partial_1 b_{\sigma 0}{}^{1}-\partial_0 b_{\sigma 1}{}^{1} = 0, \\ \partial_0 b_{\sigma 2}{}^{2} = r^{-1}b_{\sigma 0}{}^{1}, \\ \partial_1 b_{\sigma 2}{}^{2} = r^{-1}(b_{\sigma 1}{}^{1}-b_{\sigma 2}{}^{2}), \\ \partial_2 b_{\sigma i}{}^{k} = \partial_3 b_{\sigma i}{}^{k} = 0 \quad \text{for any } i, k. \end{cases}$$

Making use of these equations, we shall show

Proposition 5.1. *The* $b_{\sigma i}{}^{k}$ *always do not depend on* θ *and* φ, *while the* $b_{\sigma i}{}^{k}$ *do not depend on* t, *if and only if* $b_{\sigma 0}{}^{1}=0$.

Proof. The first part is evident from the last equations of (5.2). The proof of the second part is as follows. Suppose $\partial_0 b_{\sigma i}{}^{k}=0$, and we get $b_{\sigma 0}{}^{1}=0$ at once from the third equation of (5. 2). Conversely, suppose $b_{\sigma 0}{}^{1}=0$. Then the equations $\partial_0 b_{\sigma 1}{}^{1}=0$ and $\partial_0 b_{\sigma 2}{}^{2}=0$ are easily seen from (5. 2), so that we must only prove $\partial_0 b_{\sigma 0}{}^{0}=0$. From the first two equations of (5. 1), $b_{40}{}^{0}$ and $b_{50}{}^{0}$ are expressible by means of $b_{\sigma 1}{}^{1}$, $b_{\sigma 2}{}^{2}$ and r so long as $D\equiv b_{41}{}^{1}b_{52}{}^{2}-b_{42}{}^{2}b_{51}{}^{1}\neq 0$; then $\partial_0 b_{\sigma 0}{}^{0}=0$ is trivial. Suppose now $D=0$. Then we have necessarily $b_{\sigma 1}{}^{1}=-2b_{\sigma 2}{}^{2}$, while from the third and fourth equations of (5. 1) we have $b_{\sigma 2}{}^{2}=-2b_{\sigma 1}{}^{1}$; this is a contradiction.

2. In the following part of this section we treat the case [$\mathbf{B}_1$] only. By the reason of the above proposition the imbedding of the case [$\mathbf{B}_1$] may be called to *be stationary.* Our purpose is to detirmine all possible types of the stationary imbeddings.

Take $i=0$ and $j=1$ in the Gauss formula (1. 5), and we have

$$\nabla_0 \boldsymbol{z}_1 = \sum e_\sigma b_{\sigma 01}\boldsymbol{n}_\sigma = 0,$$

which is written as

$$\partial_0\partial_1 \boldsymbol{z} - mr^{-2}\gamma^{-1}\partial_0 \boldsymbol{z} = 0 .$$

Substituting from (4. 3) and integrating the obtained equation, we have

$$\boldsymbol{y} = \gamma^{1/2}\boldsymbol{T}(t) + \boldsymbol{R}(r) , \tag{5. 3}$$

where $\boldsymbol{T}$ and $\boldsymbol{R}$ are some functions of t and r respectively, Putting this expression into (4. 8), we get

$$\begin{cases} \langle \boldsymbol{T}', \boldsymbol{T}' \rangle = 1 , \quad mr^{-2}\gamma^{-1/2}\langle \boldsymbol{T}, \boldsymbol{T}' \rangle + \langle \boldsymbol{T}', \boldsymbol{R}' \rangle = 0 , \\ m^2 r^{-4}\gamma^{-1}\langle \boldsymbol{T}, \boldsymbol{T} \rangle + 2mr^{-2}\gamma^{-1/2}\langle \boldsymbol{T}, \boldsymbol{R}' \rangle + \langle \boldsymbol{R}', \boldsymbol{R}' \rangle = -\dfrac{2m}{r-2m} . \end{cases} \tag{5. 4}$$

Next we consider the Weingarten formula (1. 6) ; this is now written as

$$\partial_i \boldsymbol{n}_\sigma = -b_{\sigma i}{}^i \partial_i \boldsymbol{z} . \tag{5. 5}$$

If we integrate two equations for $i=2$ and 3, making use of $b_{\sigma 2}{}^2 = b_{\sigma 3}{}^3$ and $\partial_2 b_{\sigma 2}{}^2 = 0$, then we find that the $\boldsymbol{n}_\sigma$ is

$$\boldsymbol{n}_\sigma = -rb_{\sigma 2}{}^2 \boldsymbol{S} + \boldsymbol{A}_\sigma(t, r) ,$$

where $\boldsymbol{A}_\sigma$ ($\sigma=4, 5$) are some functions of t and r. Differentiating this expression and using (5. 5), we obtain

$$\boldsymbol{A}_\sigma = -\gamma^{1/2} b_{\sigma 0}{}^0 \boldsymbol{T}(t) + \boldsymbol{B}_\sigma(r) ,$$

where $\boldsymbol{B}_\sigma$ ($\sigma=4, 5$) are some functions of r. Thus we have

$$\boldsymbol{n}_\sigma = -\gamma^{1/2} b_{\sigma 0}{}^0 \boldsymbol{T}(t) + \boldsymbol{B}_\sigma(r) - rb_{\sigma 2}{}^2 \boldsymbol{S}(\theta, \varphi) .$$

Let those expressions of $\boldsymbol{z}$ and $\boldsymbol{n}_\sigma$ carry in the Gauss formula (1. 5) for $i=j=0$. Then a direct calculation gives

$$\gamma^{1/2}\boldsymbol{T}'' - mr^{-2}\gamma(mr^{-2}\gamma^{-1/2}\boldsymbol{T} + \boldsymbol{R}') = -\gamma^{1/2}\beta \boldsymbol{T} + \boldsymbol{B}_0 ,$$

where we put $\beta = \sum e_\sigma (b_{\sigma 0}{}^0)^2$ and $\boldsymbol{B}_0 = \sum e_\sigma b_{\sigma 00} \boldsymbol{B}_\sigma$. It should be noted that both β and $\boldsymbol{B}_0$ are functions of r only. Differentiating the above equation and making the inner product with $\boldsymbol{T}'$, we get

$$\langle \boldsymbol{T}'', \boldsymbol{T}' \rangle = m^2 r^{-4} - \gamma\beta .$$

The left hand-side is a function of t only, while the right depends upon r only, so that their common value must be a constant. Consequently we obtain an equation

(5. 6) $$\boldsymbol{T}''' - \kappa \boldsymbol{T}' = 0\,, \qquad \kappa: \text{constant.}$$

3. As is well known, the equation (5. 6) has different aspects in each case where κ is <0, $=0$, >0, so we shall discuss each case separately.

i) The case where $\kappa = -1/k^2 < 0$. In this case the solution is given by

$$\boldsymbol{T}(t) = k \sin \frac{t}{k} \boldsymbol{a} + k \cos \frac{t}{k} \boldsymbol{b} + \boldsymbol{c}\,,$$

where $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are some constant vectors. We may assume that the additive constant $\boldsymbol{c}$ is zero because it can be included in the undetermined function $\boldsymbol{R}(r)$, so we drop the $\boldsymbol{c}$ hereafter. From the first equation of (5. 4) we find

$$\langle \boldsymbol{a}, \boldsymbol{a} \rangle = \langle \boldsymbol{b}, \boldsymbol{b} \rangle = 1\,, \quad \langle \boldsymbol{a}, \boldsymbol{b} \rangle = 0\,.$$

Choose the base vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ so that $\boldsymbol{e}_1 = \boldsymbol{a}$, $\boldsymbol{e}_2 = \boldsymbol{b}$. Then $\boldsymbol{T}(t)$ is expressed as

(5. 7) $$\boldsymbol{T}(t) = k \sin \frac{t}{k} \boldsymbol{e}_1 + k \cos \frac{t}{k} \boldsymbol{e}_2\,,$$

where

$$\langle \boldsymbol{e}_1, \boldsymbol{e}_1 \rangle = \langle \boldsymbol{e}_2, \boldsymbol{e}_2 \rangle = +1\,.$$

Next we seek the expression of $\boldsymbol{R}(r)$. If we carry (5. 7) into the second equation of (5. 4), then we find $\langle \boldsymbol{e}_1, \boldsymbol{R}' \rangle = \langle \boldsymbol{e}_2, \boldsymbol{R}' \rangle = 0$, and hence $\boldsymbol{R}'$ keeps the constant direction $\boldsymbol{e}_3$. Then we may assume that $\boldsymbol{R}(r)$ itself keeps the constant direction e_3. If we set $\boldsymbol{R}(r) = f(r)\boldsymbol{e}_3$ and put it in the last equation of (5. 4), after a short calculation we have

$$(f')^2 \langle \boldsymbol{e}_3, \boldsymbol{e}_3 \rangle = -\frac{m^2k^2 + 2mr^3}{r^3(r-2m)}\,.$$

The right hand-side being always negative, we know

$$(f')^2 = \frac{m^2k^2 + 2mr^3}{r^3(r-2m)}\,, \quad \langle \boldsymbol{e}_3, \boldsymbol{e}_3 \rangle = -1\,.$$

Thus $\boldsymbol{z}$ is completely determined.

The imbedding of Kasner (see Introduction) *is contained in this type as a special case where* $k=1$.

ii) The case where $\kappa=0$. In this case the solution is given by

$$T(t) = t^2\boldsymbol{a}+t\boldsymbol{b}\,, \tag{5.8}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are some constant vectors and an additive constant is previously omitted by the same reason as in **i**). Substituting from (5.8) in the first equation of (5.4), we have

$$\langle \boldsymbol{a}, \boldsymbol{a}\rangle = \langle \boldsymbol{a}, \boldsymbol{b}\rangle = 0\,, \quad \langle \boldsymbol{b}, \boldsymbol{b}\rangle = +1\,.$$

Take a vector $\boldsymbol{c}$ in the $\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$-plane such that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly independent and satisfy the following conditions

$$\langle \boldsymbol{a}, \boldsymbol{c}\rangle = +1\,, \quad \langle \boldsymbol{b}, \boldsymbol{c}\rangle = 0\,, \quad \langle \boldsymbol{c}, \boldsymbol{c}\rangle = 0\,.$$

The existence of such a vector $\boldsymbol{c}$ is easily proved. If we set

$$\boldsymbol{R}(r) = f_1(r)\boldsymbol{a}+f_2(r)\boldsymbol{b}+f_3(r)\boldsymbol{c}$$

then from the last two equations of (5.4) we find

$$f_2'(r) = 0\,, \quad 2f_3'(r)+(\gamma^{1/2})' = 0\,, \quad f_1'(r) = 2r\gamma^{-1/2}\,.$$

Integrating these, we get

$$f_1(r) = 2\int r\gamma^{-1/2}dr\,, \quad f_2(r) = 0\,, \quad f_3(r) = -\frac{1}{2}\gamma^{1/2}\,,$$

(where we have taken additive constants equal to zero, since they can be canceled by a suitable translation in E^6). Thus $\boldsymbol{y}$ is now completely determined:

$$\boldsymbol{y} = (\gamma^{1/2}t^2+2f(r))\boldsymbol{a}+\gamma^{1/2}t\boldsymbol{b}-\frac{1}{2}\gamma^{1/2}\boldsymbol{c}\,,$$

where

$$f(r) = \int r\gamma^{-1/2}dr\,.$$

In this expression, however, the vector $\boldsymbol{a}$ is not orthogonal to $\boldsymbol{c}$. In order to improve this defect, we put

$$\boldsymbol{e}_1 = \boldsymbol{a}+\frac{1}{2}\boldsymbol{c}\,, \quad \boldsymbol{e}_2 = \boldsymbol{b}\,, \quad \boldsymbol{e}_3 = \boldsymbol{a}-\frac{1}{2}\boldsymbol{c}\,.$$

Then these e_1, e_2 and e_3 are orthogonal to each other and

$$\langle e_1, e_1\rangle = +1, \quad \langle e_2, e_2\rangle = +1, \quad \langle e_3, e_3\rangle = -1.$$

With this base, the expression of y is as follows:

$$y = \left[\frac{1}{2}\gamma^{1/2}(t^2-1)+f(r)\right]e_1+\gamma^{1/2}te_2+\left[\frac{1}{2}\gamma^{1/2}(t^2+1)+f(r)\right]e_3$$

iii) The case where $\kappa=1/k^2>0$. In this case, by a discussion similar to **i)** we have

$$T(t) = k\sinh\frac{t}{k}e_1+k\cosh\frac{t}{k}e_2, \quad R(r) = g(r)e_3,$$

where

$$\langle e_1, e_1\rangle = +1, \quad \langle e_2, e_2\rangle = -1,$$
$$(f')^2\langle e_3, e_3\rangle = \frac{m^2k^2-2mr^3}{r^3(r-2m)}.$$

Here arises a particular situation. If $k^2>16m^2$, then the term $(m^2k^2-2mr^3)/r^3(r-2m)$ takes both positive and negative values; consequently the sign of $\langle e_3, e_3\rangle$ is indeterminate. This fact shows that the function z corresponding to such a value of k does not give an imbedding map of the *whole* V^4 into an E^6 with a fixed signature. On the other hand, if $k^2\leqq 16m^2$, the term is always negative since $2m<r<+\infty$. Therefore we may set

$$(f')^2 = \frac{2mr^3-m^2k^2}{r^3(r-2m)}, \quad \langle e_3, e_3\rangle = -1.$$

Thus we get another set of the imbedding functions. *Fronsdal's imbedding* (see Introduction) *corresponds to a special case* $k=4m$ *of this type.*

Consequently, we now establish the following result:

Theorem. *Any stationary imbedding* (*i.e. the case* [$\mathbf{B}_1$]) *of the Schwarzschild space into a six-dimensional pseudo-Euclidean space is given by one of the following expressions*:

(i) $$ds^2 = dz_1^2+dz_2^2-dz_3^2-dz_4^2-dz_5^2-dz_6^2,$$
$$z_1 = k(1-2m/r)^{1/2}\sin(t/k), \quad k\neq 0,$$
$$z_2 = k(1-2m/r)^{1/2}\cos(t/k),$$

$z_3 = f(r),$ *where* $(df/dr)^2 = (2mr^3 + m^2k^2)/r^3(r-2m)$,

$z_4 = r \sin\theta \sin\varphi$, $z_5 = r\sin\theta\cos\varphi$, $z_6 = r\cos\theta$;

(ii) $ds^2 = dz_1^2 - dz_2^2 - dz_3^2 - dz_4^2 - dz_5^2 - dz_6^2$,

$z_1 = k(1-2m/r)^{1/2}\sinh(t/k)$, $-4m \leqq k \leqq 4m,\quad k \neq 0$,

$z_2 = k(1-2m/r)^{1/2}\cosh(t/k)$,

$z_3 = g(r),$ *where* $(dg/dr)^2 = (2mr^3 - m^2k^2)/r^3(r-2m)$,

z_4, z_5, z_6 : *the same as above* ;

(iii) $ds^2 = dz_1^2 + dz_2^2 - dz_3^2 - dz_4^2 - dz_5^2 - dz_6^2$,

$z_1 = \frac{1}{2}(t^2-1)(1-2m/r)^{1/2} + h(r)$,

$z_2 = t(1-2m/r)^{1/2}$, *where* $dh/dr = r(1-2m/r)^{-1/2}$,

$z_3 = \frac{1}{2}(t^2+1)(1-2m/r)^{1/2} + h(r)$,

z_4, z_5, z_6 : *the same as above.*

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