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On a compactification of an open Riemann surface and its application

By

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Introduction

In this paper, we shall give a compactification (denoted by $R^*_{\mathfrak{F}}$) of an open Riemann surface R ($\notin 0_G$) such that HB-functions on R are extended continuously onto $R^*_{\mathfrak{F}}$. Likely as the Royden's compactification [15], the ideal boundary $R^*_{\mathfrak{F}}-R$ has the compact part (denoted by $\Delta_{\mathfrak{F}}$) with an important role with respect to HB-functions. After H. L. Royden, we shall call it the harmonic boundary of R, and it will be remarked in § 4 as the hyper Stone space (cf. [13]). In § 1, the compactification will be carried out by means of some family consisting of bounded continuous function on R. In § 2, some properties of $\Delta_{\mathfrak{F}}$ will be studied. In § 3, we shall study the generalized harmonic measure on R in relation to subsets of the harmonic boundary $\Delta_{\mathfrak{F}}$, where the generalized harmonic measure ω is characterized as follows: 1) $\omega \in HBP$, 2) $0<\omega<1$ and 3) $\omega \wedge (1-\omega)=0$ (cf. [5]). We shall define the harmonic measure Ω_α with respect to a compact subset α of $\Delta_{\mathfrak{F}}$ by the same manner as did in [7] and we shall show that Ω_α is the generalized harmonic measure and conversely a generalized harmonic measure is the harmonic measure with respect to a compact set of $\Delta_{\mathfrak{F}}$. And further we shall define the outer harmonic measure with respect to any subset of $\Delta_{\mathfrak{F}}$. We shall see that the outer harmonic measure is the Caratheodory outer measure with respect to the subsets of $\Delta_{\mathfrak{F}}$. In § 4, we shall introduce the integral representation of an HB-function. With respect to HD-functions, the integral representation has been studied by M. Nakai [14]. We shall treat the integral representation of an HB-function in relation to the generalized harmonic measure. With respect to an unbounded HP-function, some results will be stated. In §5, we shall be concerned with the harmonic boundary of the class 0_{HB_n} (cf. [8], [3], [12]). The HB-minimal function will be characterized by the harmonic measure with respect to an isolated point of $\Delta_{\mathfrak{F}}$. The Bader-Parreau, Matsumoto's Theorem [2], [11] will be studied concerning to the harmonic boundary $\Delta_{\mathfrak{F}}$.

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1. Compactification of $R\,(\notin 0_G)$.

Let $K: |z|<1$ be a conformal image of the universal covering surface R^∞ of R and let $T(z)$ be the mapping from K onto R. We denote by $\mathfrak{F}$ the family of real-valued bounded, continuous functions on R each of which has the radial limits in K for almost all $e^{i\theta}$. $\mathfrak{F}$ is a normed space with a norm $\|f\|=\sup_R|f|$ $(f\in\mathfrak{F})$. The completeness of this space is verified by the following

Proposition 1.1. *Let $\{f_n\}$ be the Cauchy sequence with respect to the above norm. Then there exists the function $f(\in\mathfrak{F})$ such as $\|f-f_n\|\to 0$ $(n\to\infty)$.*

Proof. It is evident that $\{f_n\}$ is an uniformly convergent sequence in the narrow sense. Let f be the limit function of the sequence. We can see easily that f belongs to $\mathfrak{F}$. (q.e.d.)

From this, we know that $\mathfrak{F}$ is a normed ring with uniform norm $\|f\|=\sup_R|f|$ $(f\in\mathfrak{F})$. Let $\mathfrak{F}_0$ be the subfamily of $\mathfrak{F}$ defined as follows: $f(\in\mathfrak{F})$ belongs to $\mathfrak{F}_0$ if and only if $\lim_{r\to1} f(T(re^{i\theta}))=0$ for almost all $e^{i\theta}$. It is evident that $\mathfrak{F}_0$ is an ideal of $\mathfrak{F}$. We denote by $\mathfrak{F}_c$ the family of functions $(\in\mathfrak{F})$ whose carriers are compact respectively. $\mathfrak{F}_c$ is an ideal of $\mathfrak{F}$ and $\mathfrak{F}_c\subset\mathfrak{F}_0$. Let $\mathfrak{M}$ be the family consisting of all maximal ideal of $\mathfrak{F}$ and we put in $\mathfrak{M}$

the closure topology by the method of Gelfand [4]. Thus we have the compact Hausdorff space $R^*_{\mathfrak{F}}$. It is clear that $M_a=\{f\in\mathfrak{F};\ f(a)=0,\ (a\in R)\}$ is a maximal ideal and $M_a\neq M_{a'}$ for different points a, a' in R. Now we have a topologcal mapping T from R into $R^*_{\mathfrak{F}}$ such as $M_a=T(a)$. We can see easily that the image $T(R)$ of R is open and dense in $R^*_{\mathfrak{F}}$. From now on, $T(R)$ is denoted by R again. $R^*_{\mathfrak{F}}-R$ is called the ideal boundary of an open Riemann surface R and is denoted by $\Gamma_{\mathfrak{F}}$. $\Gamma_{\mathfrak{F}}$ consists of the maximal ideals containing the ideal $\mathfrak{F}_c$. The subset of $\Gamma_{\mathfrak{F}}$, consisting of the maximal ideals each of which contains $\mathfrak{F}_0$, is denoted by $\Delta_{\mathfrak{F}}$. We call $\Delta_{\mathfrak{F}}$ the harmonic boundary of R after H. L. Royden.

Proposition 1.2. *Let f belong to $\mathfrak{F}$. Then $f(T(e^{i\theta}))$ is the measurable function with respect to $\theta(0\leqq\theta<2\pi)$, where $f(T(e^{i\theta}))=\lim_{r\to 1}f(T(re^{i\theta}))$ for almost all $e^{i\theta}$.*

From this, we have the following

Proposition 1.3. *$f(\in\mathfrak{F})$ has the following decomposition: $f=u+\varphi$ $(u\in HB,\ \varphi\in\mathfrak{F}_0)$, and the decomposition is unique. With respect to the norm, $\|f\|\geqq\|u\|$ holds.*

Proof. It is clear that

$$u=\frac{1}{2\pi}\int_0^{2\pi}f(T(e^{i\theta}))\frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)}d\theta,$$

consequently $\|f\|\geqq\|u\|$ holds. (q.e.d.)

On the Royden's compactification, we can see that the class HBD becomes a normed ring with the norm $\|u\|=\sqrt{D_R(u)}+\sup_R|u|$ $(u\in HBD)$, provided that the multiplication is defined as follows: the multiplicative structure is defined by the projection π such as $f(\in BD)\xrightarrow{\pi}u$ $(\in HBD)$, where u is the harmonic component of the orthogonal decomposition of f, (Royden. [15]). From this normed ring HBD with the above multiplicative structure, we can construct the compact Hausdorff space $\mathfrak{H}$ by the method of Gelfand. The space $\mathfrak{H}$ is homeomorph to Δ (Royden's harmonic boundary) [15]. We shall show later on that the same relations hold between HB-space and $\Delta_{\mathfrak{F}}$. For this purpose, we note the following

Theorem (Littlewood [9]). *Let*

$$w(z) = \int_{|a|<1} \log \left| \frac{1-\bar{a}z}{z-a} \right| d\mu(a)$$

where μ *is a positive mass distribution in* $|z|<1$, *such that*

$$\int_{|a|<1} (1-|a|)\, d\mu(a) < +\infty$$

Then $\lim_{r\to 1} w(re^{i\theta})=0$ *for almost all* $e^{i\theta}$, *and* $\lim_{r\to 1} \int_0^{2\pi} w(re^{i\theta})=0$.

Theorem (Littlewood [10]). *Let* $u(z)$ *be subharmonic in* $|z|<1$, *such that*

$$\int_0^{2\pi} |u(re^{i\theta})|\, d\theta = 0(1), \quad 0 \leqq r < 1,$$

then $u(z)=v(z)-w(z)$, *where* $v(z)$ *is harmonic in* $|z|<1$, *such that*

$$\int_0^{2\pi} |v(re^{i\theta})|\, d\theta = 0(1), \quad 0 \leqq r < 1,$$

and

$$w(z) = \int_{|a|<1} \log \left| \frac{1-\bar{a}z}{z-a} \right| d\mu(a),$$

where μ *is a positive mass distribution in* $|z|<1$, *such that*

$$\int_{|a|<1} (1-|a|)\, d\mu(a) < +\infty .$$

Hence for almost all $e^{i\theta}$, $\lim_{r\to 1} u(re^{i\theta})=u(e^{i\theta})$ $(\neq \infty)$ *exists.* $v(z)$ *is the least harmonic majorant of* u, *such that if* $v_\rho^*(z)$ *be harmonic in* $|z|<\rho<1$, *such that* $v_\rho^*=u$ *on* $|z|=\rho$, *then* $\lim_{\rho\to 1} v_\rho^*(z)=v(z)$.

From this, we know that the bounded subharmonic functions belong to $\mathfrak{F}$. Hence the bounded superharmonic functions belong to $\mathfrak{F}$.

Proposition 1.4. *Let* M_{p^*} *is the family of all HBD-functions such as* $M_{p^*}=\{u\in HBD;\ u(p^*)=0,\ (p^*\in\Delta_{\mathfrak{F}})\}$. *Then* M_{p^*} *is a maximal ideal of the normed ring HBD, that is,* M_{p^*} *corresponds to a point of* $\mathfrak{H}$. *Conversely, a point* $\tilde{M}_q$ *of* $\mathfrak{H}$ *corresponds to a point of* $\Delta_{\mathfrak{F}}$, *that is,* $\tilde{M}_q=M_{q^*}$ *for some point* $q^*(\in\Delta_{\mathfrak{F}})$.

Proof. Let u be any element of M_{p^*}. Then $4uv=(u+v)^2-(u-v)^2$ for any $v\in HBD$ and by the Royden's decomposition we have the following

$$(u+v)^2 = w_1+\varphi_1$$
$$(u-v)^2 = w_2+\varphi_2, \qquad (w_1,\ w_2\in HBD,\ \varphi_1,\ \varphi_2\in \bar{K}).$$

Thus we know that $4uv=(w_1-w_2)+(\varphi_1-\varphi_2)$. We note that φ_1 and φ_2 are subharmonic respectively, hence φ_1 and φ_2 belong to $\mathfrak{F}$. To be exact, φ_1 and φ_2 belong to $\mathfrak{F}_0$. From this we know that $\varphi_1-\varphi_2$ vanishes at $p^*(\in\Delta_{\mathfrak{F}})$. Therefore w_1-w_2 vanishes at p^*. This means that M_{p^*} is an ideal of the normed ring HBD, because $w_1-w_2=\pi(uv)$ (π : Proj.). That M_{p^*} is maximal is evident. Thus we know that M_{p^*} corresponds to a point of $\mathfrak{H}$. Next, let $\tilde{M}_q$ be any point of $\mathfrak{H}$. Suppose that there is no any point of $\Delta_{\mathfrak{F}}$ such as a common zero point of the functions belonged to $\tilde{M}_q$. Then there exists an HBD-function u in $\tilde{M}_q$ such that u is positive on $\Delta_{\mathfrak{F}}$. This is easily verified by means of the compactness of $\Delta_{\mathfrak{F}}$. This function u has a positive infimum on R (cf. §2, Lemma 2. 1). Consequently u is positive at each point of Δ. This is absurd. Thus we know that $\tilde{M}_q$ corresponds to some point $q^*\in\Delta_{\mathfrak{F}}$. (q.e.d.)

Now we note that HB space is a normed ring with a norm $\|u\|=\sup_R|u|$ $(u\in HB)$, provided that the multiplication is defined as follows: the multiplicative structure is defined by the projection π such as $f(\in\mathfrak{F})\xrightarrow{\pi}u(HB)$, where u is the harmonic component of the decomposition of f. Thus we have the following

Proposition 1. 5. *$\tilde{\mathfrak{H}}$ is homeomorph to $\Delta_{\mathfrak{F}}$, where $\tilde{\mathfrak{H}}$ is a compact Hausdorff space constructed from the normed ring HB with the above multiplicative structure.*

Let T be a correspondence from $\Delta_{\mathfrak{F}}$ onto Δ as follows: $q^*\in\Delta_{\mathfrak{F}}\xrightarrow{T}M_{q^*}\in\Delta$. Then we have the following

Proposition 1. 6. *The correspondence T is one-valued continuous mapping.*

Proof. Let σ be an open subset of Δ (Δ is subspace of R^*). In the following, we shall show that $\sigma_{\mathfrak{F}}=T^{-1}(\sigma)$ is open with respect to $\Delta_{\mathfrak{F}}$. Since $\Delta-\sigma$ is closed

$$S=\bigcap_{M\in\Delta-\sigma} M$$

does not be contained in any maximal ideal belonging to σ, where M is a maximal ideal as a point of $\Delta-\sigma$. Consequently $S\cap HBD$ also does not be contained in any maximal ideal belonging to σ. Next, we consider an ideal $S_{\mathfrak{F}}$ such as

$$S_{\mathfrak{F}}=\bigcap_{M_{\mathfrak{F}}\in\Delta_{\mathfrak{F}}-\sigma_{\mathfrak{F}}} M_{\mathfrak{F}},$$

where $M_{\mathfrak{F}}$ is a maximal ideal as a point of $\Delta_{\mathfrak{F}}-\sigma_{\mathfrak{F}}$. We show that $S_{\mathfrak{F}}\cap HBD$ coincides with $S\cap HBD$. Indeed, $S\cap HBD$ is contained in any maximal ideal M of $\Delta-\sigma$, while $M\cap HBD$ coincides with $M_{\mathfrak{F}}\cap HBD$ by the Proposition 1.4, where $M_{\mathfrak{F}}$ is any one in $T^{-1}(M)$, consequently $(S\cap HBD)\subset S_{\mathfrak{F}}\cap HBD$. Conversely $S_{\mathfrak{F}}\cap HBD$ is contained in any maximal ideal $M_{\mathfrak{F}}$ of $\Delta_{\mathfrak{F}}-\sigma_{\mathfrak{F}}$, while $M_{\mathfrak{F}}\cap HBD$ coincides with $M(=T(M_{\mathfrak{F}}))$. This shows that $S\cap HBD\supset(S_{\mathfrak{F}}\cap HBD)$. Thus we know that $S\cap HBD=S_{\mathfrak{F}}\cap HBD$. Now let $\tilde{M}_{\mathfrak{F}}$ be any maximal ideal as a point in $\sigma_{\mathfrak{F}}$. Then $T(\tilde{M}_{\mathfrak{F}})$ $(=\tilde{M}\in\sigma)$ does not contain $S\cap HBD$, consequently $\tilde{M}_{\mathfrak{F}}$ does not contain $S_{\mathfrak{F}}\cap HBD$, because $\tilde{M}_{\mathfrak{F}}\cap HBD=T(\tilde{M}_{\mathfrak{F}})=\tilde{M}(\in\sigma)$. Thus we see that $\sigma_{\mathfrak{F}}$ is an open subset of $\Delta_{\mathfrak{F}}$. (q.e.d.)

2. Properties of the harmonic boundary $\Delta_{\mathfrak{F}}$.

Lemma 2.1. *Every HB-function u attains its maximum and minimum on $\Delta_{\mathfrak{F}}$.*

Proof. Let $\inf_R u=\lambda$, then the infimum of $\tilde{u}(=u-\lambda)$ is zero. For any HB-function v, $\tilde{u}v$ belongs to $\mathfrak{F}$. We decompose $\tilde{u}v$ such as $\tilde{u}v=w+\varphi$, where $w\in HB$ and $\varphi\in\mathfrak{F}_0$. We note that w is either constantly zero or non-constant function on R, because $\inf_R \tilde{u}=0$. Indeed,

$$w(T(re^{i\theta}))=\frac{1}{2\pi}\int_0^{2\pi}(\tilde{u}v)(T(e^{i\theta}))\frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)}d\varphi,$$

consequently $\underline{M}\tilde{u}\leq w\leq\bar{M}\tilde{u}$ on R, where $\underline{M}=\inf_R v$ and $\bar{M}=\sup_R v$.

From this, we know that $\mathfrak{P}=\{\tilde{u}f;\, f$ varies on $\mathfrak{F}\}$ is a principal ideal of $\mathfrak{F}$ and furthermore $\mathfrak{P}\cap\mathfrak{F}_0$ is also an ideal of $\mathfrak{F}$. Therefore $\tilde{u}$ is contained in some maximal ideal of $\Delta_{\mathfrak{F}}$, that is, $\tilde{u}$ vanishes at the point of $\Delta_{\mathfrak{F}}$. Thus we know that u takes its minimum on $\Delta_{\mathfrak{F}}$. From this we know that $c-u$ vanishes at some point of $\Delta_{\mathfrak{F}}$, where $c=\sup_R u$. (q.e.d.)

Proposition 2.1. *Let D be a non-compact subregion of R whose relative boundary ∂D consists of an at most countable number of analytic Jordan curves not accumulating in R. Then $\bar{D}-\overline{\partial D}$ meets $\Delta_{\mathfrak{F}}$, provided that $D\notin SO_{HB}$, where $\bar{D}$ and $\overline{\partial D}$ are respectively the closure of D and ∂D with respect to $R^*_{\mathfrak{F}}$.*

Proof. Let $\omega=I_D[1]$ (cf. [3]). Now we define the subharmonic function $\tilde{u}$ such as $\tilde{u}=\omega$ on D and $=0$ on $R-D$. Then $\tilde{u}$ belongs to $\mathfrak{F}$ by the Littlewood's Theorem. Let $\tilde{u}=v+\varphi$ be the decomposition of $\tilde{u}$, where $v\in HB$, $\varphi\in\mathfrak{F}_0$. It is easily verified that v is the least harmonic majorant of $\tilde{u}$. Consequently $\sup_R v=1$ and from the Lemma 2.1 we know that there is a point $p^*\in\Delta_{\mathfrak{F}}$ such as $v(p^*)=1$. From this, we know that $\tilde{u}(p^*)=1$ because $\varphi(p^*)=0$. This shows that p^* belongs to $\bar{D}-\overline{\partial D}$.

Lemma 2.2. *Every bounded subharmonic (superharmonic) function attains its maximum (minimum) on $\Delta_{\mathfrak{F}}$.*

Proof. Let u be a subharmonic function. We decompose u such as $u=v+\varphi$, where $v\in HB$ and $\varphi\in\mathfrak{F}_0$. If $v\equiv 0$, then $u=\varphi$ $(\leqq 0)$, because $u\leqq v$. From this we know that $\sup_R u=\sup_R \varphi=0$ and this is attained on $\Delta_{\mathfrak{F}}$. If $v\not\equiv 0$, v attains its maximum at some point p^* of $\Delta_{\mathfrak{F}}$. Then $u(p^*)$ is the maximum of u, because $\sup_R u\leqq \sup_R v+\sup_R \varphi=v(p^*)$. (q.e.d.).

Lemma 2.3. *Let u_1 and u_2 be HB-functions on R, then for $p^*\in\Delta_{\mathfrak{F}}$*

$$(u_1\vee u_2)(p^*)=\max[u_1(p^*),\ u_2(p^*)]$$
$$(u_1\wedge u_2)(p^*)=\min[u_1(p^*),\ u_2(p^*)],$$

where $u_1\vee u_2$ is the least harmonic majorant and $u_1\wedge u_2$ is the greatest harmonic minorant of u_1 and u_2.

Proof. Let $f(p)=\min[u_1(p), u_2(p)]$ $(p\in R)$, then $f(p)$ is bounded and super-harmonic on R, consequently f is continuously extended onto $R^*_{\mathfrak{F}}$ because $f\in\mathfrak{F}$, On the other hand, $\tilde{f}(p)=\min[u_1(p), u_2(p)]$ $(p\in R^*_{\mathfrak{F}})$ is continous on $R^*_{\mathfrak{F}}$. Thus we know that $f(p)=\tilde{f}(p)$ on $R^*_{\mathfrak{F}}$. Let $u(p)+\varphi(p)$ be the decomposition of $\tilde{f}$, where $u\in HB$ and $\varphi\in\mathfrak{F}_0$. Then $u(p^*)=\tilde{f}(p^*)$ on $\Delta_{\mathfrak{F}}$, consequently $(u_1\wedge u_2)(p^*)=\min[u_1(p^*), u_2(p^*)]$ on $\Delta_{\mathfrak{F}}$, because $u(p)=(u_1\wedge u_2)(p)$. In the same manner, we can prove that $(u_1\vee u_2)(p^*)=\max[u_1(p^*), u_2(p^*)]$ on $\Delta_{\mathfrak{F}}$. (q.e.d.)

Lemma 2.4. *Let e_1 and e_2 be the compact subsets of $\Delta_{\mathfrak{F}}$ disjoint respectively. Then there exists an HBP-function u such as $u=1$ on e_1 and $=0$ on e_2.*

Proof. We can construct an *HBP*-function u such as $u=0$ on e_2 and >0 on e_1 (cf. [6]). Since e_1 is compact, the infimum of u with respect to e_1 is positive. According to the Lemma 2.3, $(u/c)\wedge 1$ is the function that answers to the Lemma, where $c=\inf_{e_1} u$.

3. Generalized harmonic measures.

Theorem. 3.1. *Let α be a compact set $(\neq\phi)$ of $\Delta_{\mathfrak{F}}$ and β its complementaly set $(\neq\phi)$ in $\Delta_{\mathfrak{F}}$. Then there exists a function Ω^*_α defined in $R^*_{\mathfrak{F}}$ such that*

i) *Ω^*_α is upper semi-continuous on $R^*_{\mathfrak{F}}$ and $\Omega^*_\alpha\in HBP$ in R*

ii) *$\Omega^*_\alpha=1$ on α, $=0$ on β*

iii) *Ω^*_α is the harmonic measure in R, that is, $\Omega_\alpha\wedge(1-\Omega_\alpha)=0$, where Ω_α is the restriction of Ω^*_α to R. (we call Ω_α the harmanic measure with respect to α).*

Proof. Let H_α be a family of *HBP* functions such as

$$H_\alpha=\{u\in HBP;\ u\leqq 1 \quad \text{and} \quad =1 \text{ on } \alpha\}\ .$$

Then we know that $u_1\wedge u_2$ belongs to H_α for any u_1 and u_2 of H_α, by means of the Lemma 2.3. Consequently,

$$\Omega^*_\alpha=\inf_{u\in H_\alpha} u(p) \qquad (p\in R^*_{\mathfrak{F}})$$

is an *HBP* function (may be constantly zero) (cf. [7]), and Ω_α^* is obtained as the limit function of the non-increasing sequence consisting of the elements of H_α. In the following, we shall show that Ω_α^* is the function that answers to the above Theorem. From Lemma 2.4, we know that there exists a function $u\in H_\alpha$ such as, for arbitrarily given $p^*(\in\beta)$, $u(p^*)=0$. This shows that $\Omega_\alpha^*=$ on β, that is, Ω_α^* satisfies the condition ii). Next we show that $\sup_R \Omega_\alpha=1$ provided that $\Omega_\alpha\not\equiv 0$. Suppose that $\sup_R \Omega_\alpha=c(<1)$. Then $\Omega_\alpha\leqq cu$ for any $u\in H_\alpha$ by Lemma 2.1, consequently $\Omega_\alpha\leqq c\Omega_\alpha$. This is absurd, that is, $\sup_R \Omega_\alpha=1$ provided that $\Omega\not\equiv 0$. Let e be a set of $\Delta_{\mathfrak{F}}$ such as $e=\{p^*\in\alpha\,;\ \Omega_\alpha(p^*)=1\}$ then e is a compact subset of α. Suppose that Ω_α takes a positive value λ at some point $q^*(\in\alpha-e)$. Then q^* does not be contained in $\overline{\beta\cup e}$, consequently there exists an *HBP*-function U such as $U(q^*)=1$ and $=0$ on $\overline{\beta\cup e}$ by the Lemma 2.4. Let $\tilde{U}=U\wedge 1$, then $(\tilde{U}\vee\Omega_\alpha)>\Omega_\alpha$ on R. Indeed, $\tilde{U}\vee\Omega_\alpha=1$ at q^* by the Lemma 2.3, while $\Omega_\alpha(q^*)=\lambda(<1)$. Noting that $(\tilde{U}\vee\Omega_\alpha)<u_n$ for every n, we conclude that

$$\Omega_\alpha<(\tilde{U}\vee\Omega_\alpha)\leqq\lim_{n\to\infty}u_n=\Omega_\alpha,$$

where $\{u_n\}$ is a non-increasing sequence such as $u_n\in H_\alpha$ for every n and $u_n\downarrow\Omega_\alpha$. This is absurd. Thus we know that Ω_α vanishes on $\alpha-e$, provided that $\alpha-e\neq\phi$. From this, we can see that $\Omega_\alpha\wedge(1-\Omega_\alpha)=0$ on R. (q.e.d.)

Corollary 3.2. *Let α be a compact subset of $\Delta_{\mathfrak{F}}$ and Ω_α be its harmonic measure. Then $\alpha-e$ belongs to the closure of β provided that $\alpha-e\neq\phi$, where β is the complementary set of α with respect to $\Delta_{\mathfrak{F}}$.*

Corollary 3.2′. *Let α be a compact subset of $\Delta_{\mathfrak{F}}$. Then there exists a simultaneously open and closed set $\tilde{\alpha}$ in α such as $\Omega_\alpha^*=\Omega_{\tilde{\alpha}}^*$ on R, provided that $\Omega_\alpha>0$.*

Now we define the harmonic measure with respect to an open set of $\Delta_{\mathfrak{F}}$. Let α be an open set of $\Delta_{\mathfrak{F}}$. Then we call $1-\Omega_\beta$ the harmonic measure with respect to α, where $\beta=\Delta_{\mathfrak{F}}-\alpha$.

Theorem 3.2. *Let α be an open subset of $\Delta_{\mathfrak{F}}$ and let $\bar{\alpha}$ be* *its closure. Then* $\bar{\alpha}$ *is either a simultaneously open and closed set of* $\Delta_{\mathfrak{F}}$ *or* $\Delta_{\mathfrak{F}}$ *itself. Cousequently* $\Omega_{\alpha}>0$, *provided that* $\alpha\neq\phi$.

Proof. Let β be a complementary set of $\alpha(\neq\phi)$ with respect to $\Delta_{\mathfrak{F}}$. In the case $\Omega_{\beta}>0$, the $\bar{\alpha}$ is simultaneously open and closed set of $\Delta_{\mathfrak{F}}$ by Corollary 3.2. Next, we suppose that $\Delta_{\mathfrak{F}}-\bar{\alpha}\neq\phi$. Then there exists a point q^* in β such as $q^*\notin\bar{\alpha}$, consequently there exists an *HBP*-function u such as $u=0$ on $\bar{\alpha}$ and $=1$ at q^*. From this we know that $\Omega_{\beta}\geqq u\wedge 1>0$. This shows that $\bar{\alpha}=\Delta_{\mathfrak{F}}$ provided that $\Omega_{\beta}=0$.

Proposition 3.1. *Let* $\{\alpha_n\}$ *be the family of open subsets of* $\Delta_{\mathfrak{F}}$ *and let* $\gamma=\bigcup_{n=1}^{\infty}\alpha_n$. *Then*

$$\Omega_{\gamma}\leqq\sum_{n=1}^{\infty}\Omega_{\alpha_n}\qquad on\quad R.$$

Proof. We assume that $\Omega_{\alpha_n}<1$ for every n and furthermore $\sum_{n=1}^{\infty}\Omega_{\alpha_n}$ converges. In the other cases, this Proposition is trivial. From the Corollary 3.2 and Theorem 3.2, we know that $\Omega_{\gamma}=\Omega_{\bar{\gamma}}$ and $\Omega_{\alpha_n}=\Omega_{\bar{\alpha}_n}$ on R for every n. Suppose that $\Omega_{\bar{\gamma}}(p_0)-\sum_{n=1}^{\infty}\Omega_{\bar{\alpha}_n}(p_0)=\varepsilon>0$ for some point p_0 in R. Then

$$\tilde{D}=\left\{p\in R\,;\ \Omega_{\bar{\gamma}}(p)-\sum^{\infty}\Omega_{\bar{\alpha}_n}(p)>\frac{\varepsilon}{2}\right\}$$

is non-compact set in R. Let D be a component of $\tilde{D}$, then $D\notin SO_{HB}$. Consequently $(\bar{D}-\overline{\partial D})\cap\Delta_{\mathfrak{F}}$ is non-empty by the Proposition 2.1. On the other hand, we can see that $(\bar{D}-\overline{\partial D})\cap\Delta_{\mathfrak{F}}$ is empty by the following reason. Suppose that $q^*(\in\Delta_{\mathfrak{F}})$ is contained in $\bar{D}-\overline{\partial D}$. If $q^*\in\bigcup_{n=1}^{\infty}\bar{\alpha}_n$, then $q^*\in\bar{\alpha}_n$ for some $\bar{\alpha}_n$, consequently $\sup_D\Omega_{\bar{\alpha}_n}=1$ by the Theorem 3.2. This is incompatible with $\Omega_{\bar{\gamma}}(p)-\sum^{\infty}\Omega_{\bar{\alpha}_n}(p)>\frac{\varepsilon}{2}$ in D. If $q^*\notin\bigcup^{\infty}\bar{\alpha}_n$, then $q^*\notin\gamma$, that is, $q^*\notin\bar{\gamma}$ or $\in\bar{\gamma}-\gamma$. In the first case, $\Omega_{\bar{\gamma}}(q^*)=0$, consequently $\inf_D\Omega_{\bar{\gamma}}=0$. In the second case we can see easily that $\bar{D}-\overline{\partial D}$ contains some point p^* belonged to γ. This point p^* belongs to $\bigcup^{\infty}\bar{\alpha}_n$, and this is incomatible with $\Omega_{\bar{\gamma}}(p)-\sum^{\infty}\Omega_{\bar{\alpha}_n}(p)>\frac{\varepsilon}{2}$ in D. Thus we know that

$(\bar{D}-\overline{\partial D})\cap\Delta_{\mathfrak{F}}$ is empty. This is absurd, that is, $\Omega_\gamma \leqq \sum_{n=1}^{\infty}\Omega_{\alpha_n}$.

(q.e.d.)

Now we define the outer harmonic measure μ_γ with respect to any subset γ of $\Delta_{\mathfrak{F}}$. Let $\mathfrak{G}_\gamma$ be the family of open subsets of $\Delta_{\mathfrak{F}}$ each of which contains γ respectively, and let H_γ be the family $\{\Omega_\alpha\}$, where α varies on $\mathfrak{G}_\gamma$. We call the lower evelope of H_γ, that is,

$$\mu_\gamma(p) = \inf_{\alpha\in\mathfrak{G}_\gamma}\Omega_\alpha(p) \qquad (p\in R)\,,$$

the outer harmonic measure with respect to γ.

Proposition 3.2. *μ_γ is the harmonic measure, that is,* $\mu_\gamma\wedge(1-\mu_\gamma)=0$.

Proof. Let α_1 and α_2 be open sets containing γ respectively. Then

$$\begin{aligned}\Omega_{\alpha_1\cap\alpha_2} = \Omega_{\overline{\alpha_1\cap\alpha_2}} &= 1 \qquad \text{on} \qquad \overline{\alpha_1\cap\alpha_2}\\ &= 0 \qquad \text{on} \qquad \Delta_{\mathfrak{F}}-\overline{\alpha_1\cap\alpha_2}\end{aligned}$$

by Theorem 3.2. Therefore $\Omega_{\alpha_1\cap\alpha_2}=\Omega_{\alpha_1}\wedge\Omega_{\alpha_2}$, because $\overline{\alpha_1\cap\alpha_2}=\bar{\alpha}_1\cap\bar{\alpha}_2$, From this we can see that μ_γ is the limit function of a non-increasing sequence consisting of the elements of H_γ (cf. [1], [7]). Consequently μ_γ is the *HBP*-function on R. Next we prove that $\sup_R \mu_\gamma=1$, provided that $\mu_\gamma\not\equiv 0$. Let $\mu_\gamma=\lim_{n\to\infty}\Omega_{\alpha_n}$, where $\{\Omega_{\alpha_n}\}$ is a non-increasing sequence such as $\Omega_\alpha\in H_\gamma$ for every n. Suppose that $0<\sup_R\mu_\gamma=c<1$. Since $\mu_\gamma\leqq\Omega_{\alpha_n}$ for every n, $\mu_\gamma=0$ on $\Delta_{\mathfrak{F}}-\bar{\alpha}_n$ for every n. From this we know that

$$\mu_\gamma \leqq c\Omega_{\alpha_n}$$

for every n by Lemma 2.1 and Theorem 3.2. Thus we have $\mu_\gamma\leqq c\mu_\gamma$. This is absurd, that is, $\sup_R\mu_\gamma=1$. Let e be the compact set such as $e=\{p^*\in\Delta_{\mathfrak{F}}\,;\,\mu_\gamma(p^*)=1\}$. We note that $\mu_\gamma=0$ on $\Delta_{\mathfrak{F}}-\bar{\alpha}_n$ for every n, consequently $\mu_\gamma=0$ on $\bigcup_{n=1}^{\infty}(\Delta_{\mathfrak{F}}-\bar{\alpha}_n)$, that is, μ_γ vanishes on $\overline{\bigcup^{\infty}(\Delta_{\mathfrak{F}}-\bar{\alpha}_n)}$. Thus we know that $e\subset\bigcap^{\infty}\bar{\alpha}_n$. Next we show that e is a simultaneously open and closed set in $\Delta_{\mathfrak{F}}$. Suppose that μ_γ does not vanish at some point $q^* \in \bigcap^{\infty} \bar{\alpha}_n - e$, then there exists an *HBP*-function u such as $u=1$ at q^*, $=0$ on $e \cup (\Delta_{\mathfrak{F}} - \bigcap^{\infty} \bar{\alpha}_n)$ by Lemma 2.4. The function $\mu_\gamma \vee U$ (where $U = u \wedge 1$) is larger than μ_γ, but is smaller than Ω_{α_n} for every n, because $\Omega_{\alpha_n} = 1$ on $\bar{\alpha}_n$ and $=0$ on $\Delta_{\mathfrak{F}} - \bar{\alpha}_n$, while $\mu_\gamma \vee U = 0$ on $\Delta_{\mathfrak{F}} - \bar{\alpha}_n$, and $\leqq 1$ on $\bar{\alpha}_n$. From this, we have that

$$\mu_\gamma = \lim_{n\to\infty} \Omega_{\alpha_n} \geqq \mu_\gamma \wedge U > \mu_\gamma .$$

This is absurd. Thus we conclude that $\mu_\gamma \wedge (1 - \mu_\gamma) = 0$. (q.e.d.)

Proposition 3.3. *The* μ_γ *is the harmonic measure with respect to* $\bigcap^{\infty} \bar{\alpha}_n$

Proof. It is evident that $\Omega_{\bar{\alpha}_n} \geqq \Omega_{\bigcap^{\infty} \bar{\alpha}_n} \geqq \mu_\gamma$. Therefore $\mu_\gamma \geqq \Omega_{\bigcap^{\infty} \bar{\alpha}_n} \geqq \mu_\gamma$ as $n \to \infty$, that is, $\mu_\gamma = \Omega_{\bigcap^{\infty} \bar{\alpha}_n}$. (q.e.d.)

Lemma 3.2. *Let* $\gamma = \bigcup_{n=1}^{\infty} \gamma_n$, *where* γ_n *is any subset of* $\Delta_{\mathfrak{F}}$ *for every* n. *Then*

$$\mu_\gamma \leqq \sum_{n=1}^{\infty} \mu_{\gamma_n} \qquad on \quad R .$$

Proof. Let G be any compact subregion in R. Then, for any fixed $\varepsilon(>0)$ and any γ_n, there exists a certain open subset α_n of $\Delta_{\mathfrak{F}}$ such as $\gamma_n \subset \alpha_n$ and

$$\mu_{\gamma_n} \leqq \Omega_{\alpha_n} < \mu_{\gamma_n} + \frac{\varepsilon}{2^n}$$

in G. Thus we know that

$$\mu_\gamma \leqq \Omega_{\bigcup^{\infty} \alpha_n} \leqq \sum_{n=1}^{\infty} \Omega_{\alpha_n} < \sum_{n=1}^{\infty} \mu_{\gamma_n} + \varepsilon$$

in G by the Proposition 3.1. Therefore $\mu_\gamma \leqq \sum^{\infty} \mu_{\gamma_n}$ in G and G is an arbitrarily fixed subregion in R, we know that this inequality holds on R. (q.e.d.)

Thus we know that the outer harmonic measure μ_γ is the Caratheodory outer measure with respect to the subsets of $\Delta_{\mathfrak{F}}$.

4. Integral representation of an *HB*-function and quasi-bounded component of an *HP*-function.

We already proved that μ_γ is the harmonic measure with respect to a simultaneously open and closed set e in $\bigcap^\infty \bar{\alpha}_n$, provided that $\mu_\gamma \not\equiv 0$. Now we note that $e \cap \gamma$ is not empty. Suppose that $e \cap \gamma$ is empty. Then $e \cap \bar{\gamma}$ is empty, because e is simultaneously open and closed in $\Delta_{\mathfrak{F}}$. According to the Lemma 2.4, there exists an *HBP*-function u such as $u=1$ on e and $=0$ on $\bar{\gamma}$. Let σ be an open set in $\Delta_{\mathfrak{F}}$ such as $\sigma=\left\{p^* \in \Delta_{\mathfrak{F}}\,;\ u(p^*)<\frac{1}{2}\right\}$. Then $\sigma \supset \gamma$. Let $\{\alpha_n\}_{n=1}^\infty$ be the family of open sets in $\Delta_{\mathfrak{F}}$ such as $\Omega_{\alpha_n} \downarrow \mu_\gamma$ as $n \to \infty$. Then we can see easily that $\Omega_{\sigma \cap \alpha_n} \downarrow \mu_\gamma$ as $n \to \infty$, because every $\sigma \cap \alpha_n$ contains γ. Therefore μ_γ coincides with the harmonic measure with respect to $\bigcap^\infty \overline{(\sigma \cap \alpha_n)}$. This is absurd, because $\bigcap^\infty \overline{(\sigma \cap \alpha_n)}$ does not contain the set e. Thus we know that $e \cap \gamma \neq \phi$ provided that $\mu_\gamma \not\equiv 0$.

Lemma 4.1. *Let γ be any subset of $\Delta_{\mathfrak{F}}$ such as $\mu_\gamma > 0$ and let e be a simultaneously open and closed set in $\Delta_{\mathfrak{F}}$ such as $\mu_\gamma = \Omega_e$. Then $e \subset \bar{\gamma}$ and $\mu_{\gamma-e} \equiv 0$.*

Proof. Suppose that $e-\bar{\gamma}$ be non-empty. Let $q^* \in e-\bar{\gamma}$. Then there exists an *HBP*-function u such as $u=1$ at q^* and $=0$ on $\bar{\gamma}$ by the Lemma 2.4. Let σ be an open set of $\Delta_{\mathfrak{F}}$ such as $\sigma=\left\{p^* \in \Delta_{\mathfrak{F}}\,;\ u(p^*)<\frac{1}{2}\right\}$. From this, we see that $\Omega_{\sigma \cap \alpha_n} \downarrow \mu_\gamma$ as $n \to \infty$, that is, μ_γ is the harmonic measure with respect to $\bigcap^\infty \overline{(\sigma \cap \alpha_n)}$ by the Proposition 3.3. But $\bigcap^\infty \overline{(\sigma \cap \alpha_n)}$ does not contain the $q^* (\in e)$. This is absurd. Thus we know that $e \subset \bar{\gamma}$. Next we show that $\mu_{\gamma-e} \equiv 0$. Evidently $\gamma - e \subset \bigcap^\infty \bar{\alpha}_n - e$, consequently $\mu_{\gamma-e} \leqq \mu_{(\bigcap^\infty \bar{\alpha}_n - e)}$ $(=\mu_{\bigcap^\infty \bar{\alpha}_n} - \mu_e)$, that is, $\mu_{\gamma-e}=0$ (cf. Prop. 4.2).

Proposition 4.1. *Let α be an open set of $\Delta_{\mathfrak{F}}$. Then α is μ-measurable and $\mu_\alpha = \Omega_\alpha$.*

Proof. It is evident that $\mu_\alpha = \Omega_\alpha$. We shall prove the measurability. Let γ be any subset of $\Delta_{\mathfrak{F}}$. It is evident that

$\mu_\gamma \leqq \mu_{\gamma\frown\alpha} + \mu_{(\gamma-\gamma\frown\alpha)}$ by the Lemma 3.2. If any one of the right vanishes, then the left hand coincides with the right side. Therefore we assume that $\mu_{\gamma\frown\alpha}$ and $\mu_{(\gamma-\gamma\frown\alpha)}$ do not vanish respectively. According to the Corollary 3.2, there exists the simultaneously open and closed sets e, e_1 and e_2 such as $\mu_\gamma = \Omega_e$, $\mu_{\gamma\frown\alpha} = \Omega_{e_1}$ and $\mu_{(\gamma-\gamma\frown\alpha)} = \Omega_{e_2}$. It is clear that e_1 and e_2 are contained in e respectively. According to the Lemma 4.1, the e_2 is contained in $\Delta_{\mathfrak{F}} - \alpha$, because $e_2 \subset \gamma \frown \overline{(\Delta_{\mathfrak{F}} - \alpha)} \subset \Delta_{\mathfrak{F}} - \alpha = \overline{\Delta_{\mathfrak{F}} - \alpha}$. With respect to e_1, it is contained in $\bar{\alpha}$ by the Lemma 4.1. Noting that e_1 and e_2 are simultaneously open and closed respectively, we know that $e_1 \frown e_2 = \phi$. Thus we have the following

$$\mu_\gamma = \Omega_e \geqq \Omega_{e_1 \smile e_2} = \Omega_{e_1} + \Omega_{e_2} = \mu_{\gamma\frown\alpha} + \mu_{(\gamma-\gamma\frown\alpha)}. \qquad \text{(q.e.d.)}$$

Proposition 4.2. *Let α be a closed set of $\Delta_{\mathfrak{F}}$. Then α is measurable and $\mu_\alpha = \Omega_\alpha$.*

Proof. Since $\Delta_{\mathfrak{F}} = \alpha + (\Delta_{\mathfrak{F}} - \alpha)$ and $\Delta_{\mathfrak{F}} - \alpha$ is measurable, $\mu_{\Delta_{\mathfrak{F}}} = \mu_\alpha + \mu_{(\Delta_{\mathfrak{F}} - \alpha)}$. From this we have $\mu_\alpha = 1 - \mu_{(\Delta_{\mathfrak{F}} - \alpha)} = 1 - \Omega_{(\Delta_{\mathfrak{F}} - \alpha)}$. This shows that $\mu_\alpha = \Omega_\alpha$. (q.e.d.)

Proposition 4.3. *Let $\mu_\alpha = 0$, then $\alpha \subset \overline{(\Delta_{\mathfrak{F}} - \alpha)}$.*

Proof. Supose that $\beta = \alpha - \overline{(\Delta_{\mathfrak{F}} - \alpha)} \neq \phi$. Then β is an open subset of $\Delta_{\mathfrak{F}}$, because $\Delta_{\mathfrak{F}} = (\Delta_{\mathfrak{F}} - \alpha) \smile \alpha = \overline{(\Delta_{\mathfrak{F}} - \alpha)} \smile \alpha = \overline{(\Delta_{\mathfrak{F}} - \alpha)} \smile \beta$ and that $\overline{(\Delta_{\mathfrak{F}} - \alpha)} \frown \beta = \phi$. Therefore $\mu_\beta > 0$ by Theorem 3.2. On the contrary, $\mu_\beta = 0$ since $\beta \subset \alpha$. This is absurd, that is $\alpha \subset \overline{(\Delta_{\mathfrak{F}} - \alpha)}$. (q.e.d.)

Proposition 4.4. *Let u be an HB-function such as $u=0$ on $\Delta_{\mathfrak{F}}$ except for a null-set. Then $u \equiv 0$.*

Proof. According to Prepotion 4.3, u vanishes on $\Delta_{\mathfrak{F}}$ because u is continuous on $\Delta_{\mathfrak{F}}$. Thus we conclude that $u \equiv 0$ in R. (q.e.d.)

Proposition 4.5. *Let u and v be HB-functions such as $u=v$ ($u>v$) on $\Delta_{\mathfrak{F}}$ except for a null-set. Then $u \equiv v$ ($u>v$) on R.*

Proof. This is clear by Proposition 4.4 and Lemma 2.1. (q.e.d.)

Lemma 4.2. *Let γ be a measurable set of positive measure. Then $\mu_{\gamma-e}=\mu_{e-\gamma}=0$, that is, $\mu_\gamma=1$ on γ except for a null-set and $=0$ on $\Delta_{\mathfrak{F}}-\gamma$ except for a null-set provided that $\mu_\gamma>0$.*

Proof. It has been proved in Lemma 4.1 that $\mu_{\gamma-e}=0$. Now we show that $\mu_{e-\gamma}=0$. Since $e-\gamma\cap\bigwedge^{\infty}\bar{\alpha}_n-\gamma$, we know that $\mu_{e-\gamma}\leqq\mu_{(\bigcap^{\infty}\bar{\alpha}_n-\gamma)}$. And $\mu_{(\cap\bar{\alpha}_n-\gamma)}=\mu_{\cap\bar{\alpha}_n}-\mu_\gamma$ because γ is measurable and $\gamma\subset\bigwedge^{\infty}\bar{\alpha}_n$. Thus we know that $\mu_{e-\gamma}=0$. (q.e.d.)

Theorem 4.1. *Let u be an HBP-function. Then*

$$u(p)=\int_{\Delta_{\mathfrak{F}}}u(p^*)d\mu(p^*,\ p)$$

where μ is the outer harmonic measure.

Proof. Let $\{e_k\}_{k=1}^{n}$ be a partition of $\Delta_{\mathfrak{F}}$ each of which is μ-measurable and let $m_k=\inf_{e_k}u(p^*)$ for each k. Then

$$s(p)=\sum_{k=1}^{n}m_k\mu_{e_k}(p)\qquad(p\in R)$$

is an HBP-function on R by Proposition 3.2. Let Γ be the family of $s(p)$ corresponding to each partition of $\Delta_{\mathfrak{F}}$. Then Γ has the following property: there exists an element $s(p)$ of Γ such as $s(p)\geqq\max[s_1(p),\ s_2(p)]$ $(p\in R)$ for any given s_1 and s_2 of Γ. This is verified by means of Lemma 2.1 and Lemma 4.2. Therefore we know that $\sup_{s\in\Gamma}s(p)=U(p)$ is harmonic on R (cf. [1] p. 134). Thus we know that

$$U(p)=\int_{\Delta_{\mathfrak{F}}}u(p^*)d\mu(p^*,\ p)$$

is harmonic on R. It is clear that $s(p)\,(\in\Gamma)\leqq u(p)$ for any $s\in\Gamma$ by Lemma 4.2. Consequently $U\leqq u$. And we can see easily that U is identical with u. (q.e.d.)

Theorem 4.2. *Let f be a measurable function positive and bounded on $\Delta_{\mathfrak{F}}$, then*

$$U(p)=\int_{\Delta_{\mathfrak{F}}}f(p^*)d\mu(p^*,\ p)$$

is an HB-function on R and $U=f$ on $\Delta_{\mathfrak{F}}$ except for a null-set.

Proof. In the same manner as did in Theorem 4.1, we can verify that $U(p)$ is an HB-function on R. we shall prove the latter. Let $\{e_i^{(n)}\}_{i=1}^{k_n}$ be a partition of $\Delta_{\mathfrak{F}}$ such as

$$e_i^{(n)} = \left\{p^* \in \Delta_{\mathfrak{F}} ; \frac{i}{2^n} \leqq f(p^*) < \frac{i+1}{2^n}\right\} \quad (i=0, 1, \cdots, k_n : k_n=2M-1),$$

where $M=\sup_{\Delta_{\mathfrak{F}}} f(p^*)$. Then $\{s^{(n)}\}_{n=1}^{\infty}$ is a non-decreasing sequence and converges to $U(p)$, consequently

$$U(p) = \lim_{n\to\infty} s^{(n)}(p) = \lim_{n\to\infty} \int_{\Delta_{\mathfrak{F}}} s^{(n)}(p^*) d\mu(p^*, p) = \int_{\Delta_{\mathfrak{F}}} (\lim_{n\to\infty} s^{(n)}(p^*)) d\mu(p^*, p).$$

On the other hand, $U(p^*) \geqq s^{(n)}(p^*)$ on $\Delta_{\mathfrak{F}}$ consequently $U(p^*) \geqq \lim_{n\to\infty} s^{(n)}(p^*) = f(p^*)$ except for a null-set, From this we know that $U(p^*)=f(p^*)$ except for a null-set, because

$$U(p) - \int_{\Delta_{\mathfrak{F}}} f(p^*) d\mu(p^*, p) = \int_{\Delta_{\mathfrak{F}}} (U(p^*)-f(p^*)) d\mu(p^*, p) = 0.$$

(q.e.d.)

In the following, we shall treat the unbounded HP-functions. Let u be an unbounded HP-function. Then $u(p)=\lim_{n\to\infty} u_n(p)$, where $u_n(p)=\min[u(p), n]$ for every n. Let e_n be a set of $R^*_{\mathfrak{F}}$ such as $e_n=\{p \in R^*_{\mathfrak{F}} ; u_n(p)=n\}$. Then $e_n \supset e_{n+1}$ for each n, therefore $e_\infty=\bigcap_{n=1}^{\infty} e_n$ is non-empty, because every e_n is compact. We define the function $u^*(p)$ as follows:

$$u^*(p) = u(p) \quad \text{on} \quad R^*_{\mathfrak{F}} - e_\infty$$
$$= +\infty \quad \text{on} \quad e_\infty.$$

Then $u^*(p)$ is continuous on $R^*_{\mathfrak{F}}$ in the sense of $\overline{\lim}\, u(p) = \underline{\lim}\, u(p)$ on $R^*_{\mathfrak{F}}$. From now on, $u^*(p)$ is denoted by $u(p)$ again.

Theorem 4.3. *Let u be an unbounded HP-function on R. Then $u(p^*)$ $(p^* \in \Delta_{\mathfrak{F}})$ is integrable on $\Delta_{\mathfrak{F}}$. Consequently $e_\infty \cap \Delta_{\mathfrak{F}}$ is μ-measure zero.*

Proof. Let $u_n(p)=\min[u(p), n]$. Then $u_n(p)\uparrow u(p)$ on R and

$$\tilde{u}_n(p) = \int_{\Delta_{\mathfrak{F}}} u_n(p^*)d\mu(p^*, p) \qquad (n=1, 2, \cdots)$$

is a non-decreasing sequence. According to Fatou's lemma, we know that

$$+\infty > u(p) \geqq \underline{\lim}\, \tilde{u}_n(p) \geqq \int_{\Delta_{\mathfrak{F}}} (\underline{\lim}\, u_n(p^*))d\mu(p^*, p) = \int_{\Delta_{\mathfrak{F}}} u(p^*)d\mu(p^*, p)\,.$$

This shows that $u(p^*)$ $(p^*\in\Delta_{\mathfrak{F}})$ is integrable on $\Delta_{\mathfrak{F}}$ and $e_\infty\cap\Delta_{\mathfrak{F}}$ is μ-measure zero. (q.e.d.)

From this, we have a decomposition such as $u(p)=w(p)+\int_{\Delta_{\mathfrak{F}}} u(p^*)d\mu(p^*, p)$. In the following, we show that this decomposition coincides with the Parreau's decomposition.

Theorem 4.4. *$w(p)$ is the singular component of u and the integral term is the quasi-bounded component of u.*

Proof. Let $\tilde{u}(p)=\int_{\Delta_{\mathfrak{F}}} u(p^*)d\mu(p^*, p)$ and let $u_n(p)=\min[u(p), n]$ on R. Then we have the inequality

$$u(p) \geqq \tilde{u}(p) > u_n(p) \qquad \text{on} \quad R\,.$$

Let q^* be a point of $\Delta_{\mathfrak{F}}$ such as $q^*\notin e_\infty$, where $e_\infty=\{p^*\in\Delta_{\mathfrak{F}};\ u(p^*)=+\infty\}$. Then, for a suitable number N, $u(q^*)\geqq\tilde{u}(q^*)\geqq u_N(q^*)=u(q^*)$ by Lemma 2.3. Therefore $u(p^*)=\tilde{u}(p^*)$ on $\Delta_{\mathfrak{F}}$ except for e_∞. This shows that $w(p)=u(p)-\tilde{u}(p)$ is singular, because e_∞ is a null-set. (q.e.d.)

Theorem 4.5. *Let an HP-function w be singular. Then w vanishes at each point of $\Delta_{\mathfrak{F}}$.*

Proof. It is evident that w vanishes on $\Delta_{\mathfrak{F}}-e_\infty$, where. $e_\infty=\{p^*\in\Delta_{\mathfrak{F}};\ w(p^*)=+\infty)$. we shall show that $e_\infty=\phi$. Suppose that e_∞ is non-empty. Then a set of G,

$$G = \{p\in R;\ w(p) > c > 0\}$$

is non-empty in R and e_∞ is contained in $\bar{G}-\overline{\partial G}$. We note that e_∞ is a simultaneously open and closed set in $\Delta_{\mathfrak{F}}$ because $\overline{\partial G}\cap\Delta_{\mathfrak{F}}=\phi$. From this we know that μ_{e_∞} is positive. This is absurd, because $\mu_{e_\infty}=0$ by Theorem 4.3. (q.e.d.)

Finally we shall give the following

Theorem 4.6. *The harmonic boundary $\Delta_{\mathfrak{F}}$ is totally disconnected.*

Proof. Let σ be connected subset of $\Delta_{\mathfrak{F}}$. In the following, we shall see that σ is a single point. Suppose that σ has at least two points, say q_1^*, q_2^*. According to Lemma 2.4, there exists an *HBP*-function u such as $u(q_1^*)=1$ and $u(q_2^*)=0$. Now let G be an open set such as $G=\left\{p^*\in\Delta_{\mathfrak{F}}\,;\ u(p^*)>\frac{1}{2}\right\}$. Evidently $q_1^*\notin\bar{G}$, consequently $\sigma\cap\bar{G}$ and $\sigma-\sigma\cap\bar{G}$ are disjoint non-empty sets respectively and furthermore $\bar{G}$ is a simultaneously open and closed set in $\Delta_{\mathfrak{F}}$ from Theorem 3.2. This is absurd, because σ is connected. Thus we know that σ must be a single point, provided that σ is connected. This shows that $\Delta_{\mathfrak{F}}$ is totally disconnected. (q.e.d.)

Remark. The results in Theorem 3.2 and 4.2 seem to us curious, but on the other hand, from these results we see the similarity between $\Delta_{\mathfrak{F}}$ and the hyper Stone space (cf. [13] pp. 108~111). In the following, we shall show that $\Delta_{\mathfrak{F}}$ as the subspace of $R_{\mathfrak{F}}^*$ is the hyper Stone space. Let f_1 and f_2 be essentially bounded functions on $\Delta_{\mathfrak{F}}$, then we define that f_1 is equivalent to f_2, provided that $f_1=f_2$ except for a null-set. Under this stipulation, we denote by $M(\Delta_{\mathfrak{F}})$ a family of essentially bounded, measurable function on $\Delta_{\mathfrak{F}}$. Then we have the following theorem; the compact Hausdorff space H constructed from maximal ideals of $M(\Delta_{\mathfrak{F}})$ is the Stone space and furthermore is the hyper Stone space (cf. [13]). On the other hand, Theorem 4.2 shows that $M(\Delta_{\mathfrak{F}})$ is identical with $C(\Delta_{\mathfrak{F}})$, where $C(\Delta_{\mathfrak{F}})$ denotes a family of continuous function on $\Delta_{\mathfrak{F}}$, and that the compact Hausdorff space constructed from maximal ideals of $C(\Delta_{\mathfrak{F}})$ is identical with $\Delta_{\mathfrak{F}}$ because $\Delta_{\mathfrak{F}}$ is the compact space. Thus we know that $\Delta_{\mathfrak{F}}$ is the hyper Stone space.

5. 0_{HB_n} and $\Delta_{\mathfrak{F}}$

Let γ be a measurable set of $\Delta_{\mathfrak{F}}$ with positive measure. Now we define a partition $\sigma_1|\sigma_2$ of γ as follows: σ_1 and σ_2 are disjoint, measurable sets with positive measures respectively and have union γ. We call γ an indivisible set if it admits no partition.

Lemma 5.1. *Let γ be an indivisible set of $\Delta_{\mathfrak{F}}$. Then γ consists of an isolated point and a null-set.*

Proof. According to Proposition 3.2 and Lemma 4.2, we know that μ_γ coincides with μ_e, where e is a simultaneously open and closed set of $\Delta_{\mathfrak{F}}$ and $\mu_{e-\gamma}=\mu_{\gamma-e}=0$. From this, we know that e is an indivisible set. In the following, we shall prove that every HB-function is constant on e. Suppose that for some HB-function u, $(c_1=)\sup u > \inf u(=c_2)$ with respect to e. Then whether $e_1=\{p^*\in e\,;\ u(p^*)>c\}$ or $e_2=\{p^*\in e\,;\ u(p^*)<c\}$ is a null-set for any given $c(c_2<c<c_1)$. If e_1 is a null-set, e_1 would be contained in $\bar{e}_2$, because e is a simultaneously open and closed set in $\Delta_{\mathfrak{F}}$ and a null-set is contained the closure of its complementary set with respect to $\Delta_{\mathfrak{F}}$. This shows that $\sup_e u\leqq c\ (<c_1)$. This is absurd. Analogously we can see that e_2 must be positive measure. Thus we know that every HB-function is constant on e, that is, e consists of a single point. At the beginning of § 4, we have shown that $\gamma\cap e$ does not be empty, provided that $\mu_\gamma>0$. From these, we conclude that γ is union of a simultaneously open and closed set and a null-set. (q.e.d.)

Theorem 5.1. *Let $q^*(\in\Delta_{\mathfrak{F}})$ be a point with positiue measure. Then the $\mu_{\{q^*\}}$ is HB minimal. Conversely, every HB minimal function whose supremum is 1 is the μ-measure of an isolated point of $\Delta_{\mathfrak{F}}$.*

Proof. It is clear that $\mu_{\{q^*\}}$ is HB-minimal, because q^* is identical with the set e. We shall prove the inverse. Let $\omega(p)$ be an HB-minial function such as $\sup_R \omega=1$. Let e be a set of $\Delta_{\mathfrak{F}}$ such as $e=\{p^*\in\Delta_{\mathfrak{F}}\,;\ \omega(p^*)=1\}$. Now, for any HBP-function u, $(u/\|u\|)\wedge\omega$ is smaller than ω. Therefore $(u/\|u\|)\vee\omega=c\omega$, where

$0 \leqq c \leqq 1$. This shows that u is constant on e. From this, we know that every HB-function is constant on e, because $u = u \vee 0 + u \wedge 0$ Thus we conclude that e consists of a single point with positive mearure ω. Indeed ω vanishes on $\Delta_{\mathfrak{F}} - e$ and this is verified easily by Lemma 2. 4.

Theorem 5. 2. *$R \in 0_{HB_n} - 0_{HB_{n-1}}$ if and only if $\sigma(R) = n$, where $\sigma(R)$ denotes the number of the harmonic boundary points.*

Proof. In the same manner as did in [7], this is proved.

Theorem 5. 3. *There exist at least n generalized harmonic measure $\{\omega_i\}_{i=1}^{n}$ on R such as $\omega_i \wedge \omega_j = 0$ $(i \neq j)$, provided that $\Delta_{\mathfrak{F}}$ contains at least n points. The inverse is true also.*

Proof. The harmonic boundary $\Delta_{\mathfrak{F}}$ does not be an indivisible set, if not so, $\Delta_{\mathfrak{F}}$ would consist of a single point. Consequenly there exists a partition $\Delta_1 | \Delta_2$ of $\Delta_{\mathfrak{F}}$. Next, one or the other of Δ_1 and Δ_2 does not be an indivisible set, if not so, the R would be of the class $0_{HB_2} - 0_{HB_1}$. Thus this decomposition will be continued up to at least the $(n-1)$ th step, that is,

$$\Delta_{\mathfrak{F}} = \Delta^{(1)} \cup \Delta^{(2)} \cup \cdots\cdots\cdots \cup \Delta^{(n)},$$

where every $\Delta^{(i)}$ has positive measuse and disjoint respectively. From this, we have the generalized harmonic measure $\{\mu_{\Delta^{(i)}}\}_{i=1}^{n}$ such as $\mu_{\Delta^{(i)}} \wedge \mu_{\Delta^{(j)}} = 0$ $(i \neq j)$, where $\mu_{\Delta^{(i)}} \wedge \mu_{\Delta^{(j)}} = 0$ $(i \neq j)$ is clear from Lemma 4. 2.

Corollary. (Bader-Parreau [2], Matsumoto [12]) *$R \notin 0_{HB_n}$ if and only if there exist $n+1$ non-compact subregions $\{G_i\}$ such as $G_i \cap G_j = \phi$ $(i \neq j)$ and $\notin SO_{HB}$ respectively.*

Proof. It is easily verified by means of Theorem 5. 2, 5. 3 and Proposition 2. 1.

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Supplement

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We shall state briefly the relation the measure μ on $\Delta_{\mathfrak{F}}$ and the canonical measure $\tilde{\mu}$ on Δ which was introduced by M. Nakai [14]. We defined in [7] the harmonic measure Ω_σ with respect to a compact set σ of Δ. Now we define the harmonic measure Ω_α with respect to an open subset of Δ as follows: $\Omega_\alpha = 1 - \Omega_{\Delta-\alpha}$. Let γ be any subset in Δ and $\mathfrak{G}$ be a family of open subsets in Δ each of which contains the γ. We define the function Ω_γ on R such as $\Omega_\gamma(z) = \inf \Omega_\alpha(z)$, where α runs over $\mathfrak{G}$. Then we have the following properties: 1) *Ω_γ is the generalized harmonic measure,* 2) *Ω_γ is the Caratheodory outer measure,* 3) *the Borel sets are measurable with respect to Ω_γ.* Next, let γ be any subset in Δ and let $\gamma_{\mathfrak{F}}$ be the set $T^{-1}(\gamma)$ in $\Delta_{\mathfrak{F}}$, where T is the continuous mapping from $\Delta_{\mathfrak{F}}$ onto Δ (cf. Prop. 1.6). Then we have the following

Theorem. *Let* γ *be a Borel set in* Δ. *Then*

$$\mu_{\gamma_{\mathfrak{F}}}(z) = \Omega_{\gamma}(z) = \int_{\gamma} K(z, \zeta) d\tilde{\mu}(\zeta)$$

and

$$\int_{\Delta} f(\zeta) K(z, \zeta) d\tilde{\mu}(\zeta) = \int_{\Delta_{\mathfrak{F}}} f \circ T(\zeta_{\mathfrak{F}}) d\mu(z, \zeta_{\mathfrak{F}}),$$

where $f(\zeta)$ *is any bounded* $\tilde{\mu}$*-measurable function on* Δ.

Concerning to the above subjects, I shall state in detail in another place.