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Asymptotic behavior of solutions of non-autonomous system near sets

By

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I. Introduction

The relationships between solutions of an unperturbed system and solutions of a perturbed system have been discussed by many authors (cf. [2], [3], [5]). Hale also has discussed asymptotic behavior of solutions of differential-difference equations by using a Liapunov functional for an unperturbed system [4]. Markus has discussed the case where the perturbation term tends to zero as $t \to \infty$ [8], and Antosiewicz, Opial, Levin and Nohel have discussed the case where the perturbation terms are integrable [1], [6], [7], [10]. Recently the author has also discussed the asymptotic behavior of solutions of a perturbed system [11].

In this paper we shall discuss the asymptotic stability of a set, and by constructing a Liapunov function, we shall discuss the relationships between solutions of an unperturbed system and solutions of a perturbed system. As a special case of this paper, some results concerning an autonomous system have been reported at International Symposium on Nonlinear Vibrations in Kiev [12].

We shall discuss the stability of an arbitrary set and hence, as special cases, the stability in the sense of Liapunov and orbital stability are included in our case. Moreover the perturbation term

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in this paper is a combination of the case where it tends to zero as $t\to\infty$ and the case where it is integrable.

Now we consider a system of differential equations

$$(1)\qquad x'=F(t, x) \qquad \left(' = \frac{d}{dt}\right),$$

where x is an n-dimensional vector. Let M be a set in $I\times R^n$, where I is the interval $0\leqq t<\infty$ and R^n is the Euclidean n-space. In this paper we use the following notations: $\bar{A}$ is the closure of a set A and π_σ is the hyperplane such that $t=\sigma$. $M(\sigma)$ represents the set such that $M\cap\pi_\sigma$. $M(\sigma, \varepsilon)$ is the ε-neighborhood of $M(\sigma)$ in R^n and $d(x, A)$ is the distance between a point x and a set A, that is, $d(x, A)=\inf\{\|x-a\|, a\in A\}$ and $C_0(x)$ is the class of functions which satisfy locally a Lipschitz condition with respect to x. Let $x(t; x_0, t_0)$ be a solution of (1) through the point (t_0, x_0).

We assume that $F(t, x)$ of (1) is defined on $I\times R^n$ or in a suitable neighborhood of the set M and that $F(t, x)$ is continuous in its domain of definition.

II. Definitions

In this section we shall give the definitions of stabilities of the set M.

(i) M is said to be *a stable set of* (1), if for any $\varepsilon>0$, any $\alpha>0$ and $t_0\in I$, there exists a $\delta(t_0, \varepsilon, \alpha)$ such that if $d(x_0, M(t_0))<\delta$ and $\|x_0\|\leqq\alpha$, we have $d(x(t; x_0, t_0), M(t))<\varepsilon$ for all $t\geqq t_0$.

(ii) M is said to be *a uniform-stable set of* (1) *with respect to* t_0 (*or* α), if δ in (i) is independent of t_0 (or α).

(iii) M is said to be *a quasi-equiasymptotically stable set of* (1), if there is a $\delta_0(t_0, \alpha)$ and for any $\varepsilon>0$, $\alpha>0$ there exists a $T(t_0, \varepsilon, \alpha)>0$ such that if $d(x_0, M(t_0))<\delta_0$ and $\|x_0\|\leqq\alpha$, we have $d(x(t; x_0, t_0), M(t))<\varepsilon$ for $t\geqq t_0+T(t_0, \varepsilon, \alpha)$.

(iv) M is said to be *a uniform-quasi-asymptotically stable set of* (1) *with respect to* t_0 (*or* α), if δ_0 and T in (iii) are independent of t_0 (or α).

(v) M is said to be *an equiasymptotically stable set of* (1), if M is a stable set of (1) and a quasi-equiasymptotically stable set of (1).

(vi) M is said to be *a uniform-asymptotically stable set of* (1) *with respect to* t_0 (*or* α), if M is a uniform-stable set of (1) and a uniform-quasi-asymptotically stable set of (1) with respect to t_0 (or α).

(vii) M is said to be *a quasi-equiasymptotically stable set of* (1) *in the large*, if for any $\varepsilon>0$, and $\eta>0$ and any $\alpha>0$, there exist a $\beta(t_0, \eta, \alpha)>0$ and a $T(t_0, \varepsilon, \eta, \alpha)$ such that if $d(x_0, M(t_0)) \leqq \eta$ and $\|x_0\| \leqq \alpha$, we have $d(x(t; x_0, t_0), M(t)) \leqq \beta(t_0, \eta, \alpha)$ for all $t \geqq t_0$ and $d(x(t; x_0, t_0), M(t)) < \varepsilon$ for all $t \geqq t_0 + T(t_0, \varepsilon, \eta, \alpha)$, where β is a continuous function of t_0.

(viii) M is said to be *a uniform-quasi-asymptotically stable set of* (1) *in the large with respect to* t_0 (*or* α), if β and T in (vii) are independent of t_0 (or α).

(ix) M is said to be *an equiasymptotically stable set of* (1) *in the large*, if M is a stable set of (1) and a quasi-equiasymptotically stable set of (1) in the large.

(x) M is said to be *a uniform-asymptotically stable set of* (1) *in the large with respect to* t_0 (*or* α), if M is a uniform-stable set of (1) and a uniform-quasi-asymptotically stable set of (1) in the large with respect to t_0 (or α).

In the case where the system is autonomous and $M(t) = M(t')$ for arbitrary t and t', it is clear that the stability of a set M is uniform with respect to t_0.

III. Hypotheses on M

$M(t)$ is a set in R^n and we assume that $M(t)$ is not empty for any $t \in I$. Clearly we have

$$d(x, M(t)) = d(x, \overline{M(t)}) . \tag{2}$$

Lemma 1. *If a set M is a stable set of* (1), *for any* $x_0 \in \overline{M(t_0)}$ *we have*

$$x(t; x_0, t_0) \in \overline{M(t)} \qquad \text{for all} \quad t \geqq t_0 . \tag{3}$$

We assume that if (t, x) and (t', x) belong to any compact set in $I \times R^n$, there is a positive constant K depending on each compact set such that

$$|d(x, M(t)) - d(x, M(t'))| \leqq K |t - t'| . \tag{4}$$

In this case, from (2) and $\overline{M \cap \pi_t} = \bar{M} \cap \pi_t$, we can easily prove the following theorem.

Theorem 1. *If a set M satisfies the condition* (4), *each kind of stabilities of M is equivalent to one of $\bar{M}$.*

Lemma 2. *If $d(x, M(t))$ and $d(x, M(t'))$ satisfy the condition* (4), *for any $\varepsilon > 0$ we have*

$$(5) \qquad |d(x, M(t, \varepsilon)) - d(x, M(t', \varepsilon))| \leqq K|t-t'| .$$

Proof. It is clear that we have

$$(6) \qquad d(x, M(t)) \leqq \varepsilon \qquad \text{if} \quad x \in \overline{M(t, \varepsilon)}$$

and

$$(7) \qquad d(x, M(t)) = d(x, M(t, \varepsilon)) + \varepsilon \qquad \text{if} \quad x \bar{\in} \overline{M(t, \varepsilon)} .$$

In the case where $x \in \overline{M(t, \varepsilon)}$ and $x \in \overline{M(t', \varepsilon)}$, we have clearly

$$|d(x, M(t, \varepsilon)) - d(x, M(t', \varepsilon))| = 0 < K|t-t'| .$$

In the case where $x \bar{\in} \overline{M(t, \varepsilon)}$ and $x \bar{\in} \overline{M(t', \varepsilon)}$, we have

$$\begin{aligned} & |d(x, M(t, \varepsilon)) - d(x, M(t', \varepsilon))| \\ &= |d(x, M(t)) - \varepsilon - (d(x, M(t')) - \varepsilon)| \qquad \text{(by (7))} \\ &= |d(x, M(t)) - d(x, M(t'))| \\ &\leqq K|t-t'| . \end{aligned}$$

In other case, for example, we assume that $x \bar{\in} \overline{M(t, \varepsilon)}$ and $x \in \overline{M(t', \varepsilon)}$ and then we have

$$d(x, M(t, \varepsilon)) - d(x, M(t', \varepsilon)) = d(x, M(t)) - \varepsilon .$$

By the assumption (4), $d(x, M(t))$ is a continuous function of t and we have $d(x, M(t)) > \varepsilon$ and $d(x, M(t')) \leqq \varepsilon$. Therefore there is a t_1 such that $t < t_1 \leqq t'$ and $d(x, M(t_1)) = \varepsilon$. Hence we have

$$\begin{aligned} & |d(x, M(t, \varepsilon)) - d(x, M(t', \varepsilon))| \\ &= |d(x, M(t)) - d(x, M(t_1))| \\ &\leqq K|t-t_1| \\ &\leqq K|t-t'| . \end{aligned}$$

Lemma 3. *If for any* $\varepsilon>0$ *we define a function* $G(t, \zeta, \varepsilon)$ *such that*

$$G(t, \zeta, \varepsilon) = d(\zeta, M(t, \varepsilon)),$$

we have

$$(8) \qquad |G(t, \zeta, \varepsilon)-G(t', \zeta', \varepsilon')| \leqq d(\zeta, \zeta')+K|t-t'|+|\varepsilon-\varepsilon'|,$$

where K *is a positive constant when* (t, ζ), (t', ζ') *belong to a compact set in* $I\times R^n$.

Proof. We have

$$\begin{aligned} &|G(t, \zeta, \varepsilon)-G(t', \zeta', \varepsilon')| \\ &\leqq |G(t, \zeta, \varepsilon)-G(t, \zeta', \varepsilon)|+|G(t, \zeta', \varepsilon)-G(t', \zeta', \varepsilon)| \\ &\qquad +|G(t', \zeta', \varepsilon)-G(t', \zeta', \varepsilon')| \\ &\leqq |d(\zeta, M(t, \varepsilon))-d(\zeta', M(t, \varepsilon))|+|d(\zeta', M(t, \varepsilon))-d(\zeta', M(t', \varepsilon))| \\ &\qquad +|d(\zeta', M(t', \varepsilon))-d(\zeta', M(t', \varepsilon'))| \\ &\leqq d(\zeta, \zeta')+K|t-t'|+|\varepsilon-\varepsilon'|. \end{aligned}$$

IV. Equiasymptotically stable set in the large

In this section we shall discuss the equiasymptotically stable set in the large. We assume that $F(t, x)$ of (1) is defined and continuous on $I\times R^n$ and that a set M satisfies the condition (4). Now we shall construct a Liapunov function for the equiasymptotically stable set of (1) in the large. We need the following lemma.

Lemma 4. *Let* $A(t, \eta, \alpha, \varepsilon)$ *be defined, continuous and positive on* $0\leqq t$, $0\leqq\eta$, $0\leqq\alpha$, $0<\varepsilon$. *Then there exist four continuous functions* $g(\varepsilon)$, $h(\eta)$, $k(t)$, $l(\alpha)$ *such that* $g(\varepsilon)>0$ *for* $\varepsilon\neq 0$, $g(0)=0$, $h(\eta)>0$, $k(t)>0$, $l(\alpha)>0$ *and* $A(t, \eta, \alpha, \varepsilon)\geqq g(\varepsilon)h(\eta)k(t)l(\alpha)$.

By Massera's lemma [9], we can easily prove this lemma.

Theorem 2. *We assume that* $F(t, x)\in C_0(x)$. *If a set* M *is an equiasymptotically stable set of* (1) *in the large and solutions of* (1) *are equi-bounded, there exists a continuous Liapunov function* $V(t, x)$ *defined on* $I\times R^n$ *satisfying the following conditions*:

1° $V(t, x)=0$ *if* $(t, x)\in M$,

2° $a(d(x, M(t)))\leqq V(t, x)$, *where* $a(r)$ *is continuous increasing and positive definite*, $a(r)\to\infty$ *as* $r\to\infty$, 3° $V(t, x) \in C_0(t, x)$ *and*

$$V'(t, x) = \overline{\lim_{h \to 0^+}} \frac{1}{h} \{V(t+h, x+hF(t, x)) - V(t, x)\} \leqq -cV(t, x),$$

where c is an arbitrary positive constant.

Proof. Let $\Omega_{\sigma,\eta,\alpha}$ be a domain such that $0 \leqq t \leqq \sigma$, $d(x, M(t)) \leqq \eta$ and $||x|| \leqq \alpha$. If $(t_0, x_0) \in \Omega_{\sigma,\eta,\alpha}$, there are three positive numbers $\gamma(\sigma, \alpha)$, $\beta(\sigma, \eta, \alpha)$ and $T(\sigma, \eta, \alpha, \varepsilon)$ such that $||x(t; x_0, t_0)|| \leqq \gamma(\sigma, \alpha)$ for all $t \geqq t_0$, $d(x(t; x_0, t_0), M(t)) \leqq \beta(\sigma, \eta, \alpha)$ for all $t \geqq t_0$ and $d(x(t; x_0, t_0), M(t)) < \varepsilon$ for all $t \geqq t_0 + T(\sigma, \eta, \alpha, \varepsilon)$. For a suitable $\varepsilon_0 > 0$ we put

$$T(\sigma, \eta, \alpha, \varepsilon) = T(\sigma, \eta, \alpha, \varepsilon_0) \qquad \text{if} \quad \varepsilon \geqq \varepsilon_0 .$$

Let $D_{\sigma,\eta,\alpha,\varepsilon}$ be a domain such that $0 \leqq t \leqq \sigma + T(\sigma, \eta, \alpha, \varepsilon)$ and $d(x, M(t)) \leqq \beta(\sigma, \eta, \alpha)$. Since $F(t, x) \in C_0(x)$, there is a $L(\sigma, \eta, \alpha, \varepsilon) > 0$ such that if $(t, x) \in D_{\sigma,\eta,\alpha,\varepsilon}$, $(t, x') \in D_{\sigma,\eta,\alpha,\varepsilon}$, $||x|| \leqq \gamma(\sigma, \alpha)$ and $||x'|| \leqq \gamma(\sigma, \alpha)$, we have

$$||F(t, x) - F(t, x')|| \leqq L(\sigma, \eta, \alpha, \varepsilon) ||x - x'|| .$$

We put

$$\max_{\substack{0 \leqq t \leqq \sigma \\ ||x|| \leqq \gamma(\sigma, \alpha)}} ||F(t, x)|| = F^*(\sigma, \alpha)$$

and we may assume that $F^*(\sigma, \alpha) \geqq 1$. Let $K(\sigma, \alpha)$ represent K in (4), where $0 \leqq t \leqq \sigma$ and $||x|| \leqq \gamma(\sigma, \alpha)$. These numbers can be assumed to be continuous. If we put

$$(9) \quad A(\sigma, \eta, \alpha, \varepsilon) = F^*(\sigma, \alpha) \exp \{(c + L(\sigma, \eta, \alpha, \varepsilon)) T(\sigma, \eta, \alpha, \varepsilon)\} + K(\sigma, \alpha) + c\beta(\sigma, \eta, \alpha) \exp (c(T(\sigma, \eta, \alpha, \varepsilon) + \sigma)),$$

$A(\sigma, \eta, \alpha, \varepsilon)$ is defined and continuous on $0 \leqq \sigma$, $0 \leqq \eta$, $0 \leqq \alpha_0 \leqq \alpha$ (α_0: a suitable constant), $0 < \varepsilon$ and moreover $A(\sigma, \eta, \alpha, \varepsilon)$ is positive. Therefore by applying Lemma 4 to $1/A(\sigma, \eta, \alpha, \varepsilon)$, there are four continuous functions $g(\varepsilon)$, $h(\eta)$, $k(\sigma)$ and $l(\alpha)$ such that $g(\varepsilon) > 0$ for $\varepsilon \neq 0$, $g(0) = 0$, $h(\eta) > 0$, $k(\sigma) > 0$, $l(\alpha) > 0$ and that

$$(10) \qquad g(\varepsilon) A(\sigma, \eta, \alpha, \varepsilon) \leqq h(\eta) k(\sigma) l(\alpha) .$$

For $\varepsilon = 1/k$ ($k = k_0, k_0 + 1, \cdots$; k_0: a suitable integer), we represent $g(\varepsilon)$, $T(\sigma, \eta, \alpha, \varepsilon)$ and $G(t, \zeta, \varepsilon)$ in Lemma 3 by g_k, $T_k(\sigma, \eta, \alpha)$ and

$G_k(t, \zeta)$ respectively.

Now we define a function $V_k(t, x)$ as follows:

(11) $$V_k(t, x) = g_k \sup_{\tau \geqq 0} G_k(t+\tau, x(t+\tau ; x, t))e^{c\tau} .$$

Since M is a stable set of (1), we have from Lemma 1

(12) $$V_k(t, x) \equiv 0 \quad \text{if} \quad (t, x) \in M .$$

Moreover it is clear that we have

(13) $$g_k G_k(t, x) \leqq V_k(t, x) .$$

If $(t, x) \in \Omega_{\sigma,\eta,\alpha}$, we have

(14) $$V_k(t, x) \leqq g_k \beta(\sigma, \eta, \alpha) \exp(cT_k(\sigma, \eta, \alpha)) \leqq \frac{1}{c} h(\eta)k(\sigma)l(\alpha) ,$$

because $g_k\beta(\sigma, \eta, \alpha) \exp(cT_k(\sigma, \eta, \alpha)) \leqq g_k \frac{1}{c} A(\sigma, \eta, \alpha, 1/k)$ and we have (10).

Next we shall show that $V_k(t, x) \in C_0(t, x)$. We suppose that $(t, x) \in \Omega_{\sigma,\eta,\alpha}$, $(t', x') \in \Omega_{\sigma,\eta,\alpha}$ and $t \leqq t'$. Moreover we assume that $V_k(t, x) = g_k G_k(t+\tau, x(t+\tau ; x, t))e^{c\tau}$ and that $t+\tau \geqq t'$. If we put $t+\tau = t'+\tau'$, $X = x(t' ; x, t)$ and if we represent $T_k(\sigma, \eta, \alpha)$ etc. briefly by T_k etc., we have

$$\begin{aligned}
&V_k(t, x) - V_k(t', x') \\
&\leqq g_k G_k(t+\tau, x(t+\tau ; x, t))e^{c\tau} - g_k G_k(t'+\tau', x(t'+\tau' ; x', t'))e^{c\tau'} \\
&\leqq g_k e^{c\tau} \{G_k(t+\tau, x(t+\tau ; x, t)) - G_k(t'+\tau', x(t'+\tau' ; x', t'))\} \\
&\qquad\qquad + g_k G_k(t'+\tau', x(t'+\tau' ; x', t'))(e^{c\tau} - e^{c\tau'}) \\
&\leqq g_k e^{c\tau} \| x(t+\tau ; x, t) - x(t'+\tau' ; x', t') \| + g_k \beta e^{c\tau'}(e^{c(\tau-\tau')} - 1) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 3)} \\
&\leqq g_k e^{c\tau} \| x(t'+\tau' ; X, t') - x(t'+\tau' ; x', t') \| + g_k \beta e^{c\tau'}(e^{c(\tau-\tau')} - 1) \\
&\leqq g_k e^{c\tau} e^{L_k \tau'} \| X - x' \| + g_k \beta e^{c\tau'}(e^{c(t'-t)} - 1) \qquad (\tau - \tau' = t' - t) \\
&\leqq g_k e^{(c+L_k)T_k} \{ \| X - x \| + \| x - x' \| \} + g_k \beta e^{cT_k} c e^{c\sigma}(t' - t) \\
&\leqq g_k e^{(c+L_k)T_k} \{ \| x - x' \| + F^*(t' - t) \} + c g_k \beta e^{c(T_k+\sigma)}(t' - t) \\
&\leqq g_k e^{(c+L_k)T_k} \| x - x' \| + g_k \{ e^{(c+L_k)T_k} F^* + c e^{c(T_k+\sigma)} \beta \} (t' - t) \\
&\leqq h(\eta)k(\sigma)l(\alpha) [\| x - x' \| + | t - t' |] \qquad \text{(by (9) and (10))}.
\end{aligned}$$

On the other hand, in the case where $t+\tau < t'$, we have

$$
\begin{aligned}
&V_k(t,x)-V_k(t',x')\\
&\leqq g_kG_k(t+\tau,x(t+\tau;x,t))e^{c\tau}-g_kG_k(t',x(t';x',t'))\\
&\leqq g_kG_k(t+\tau,x(t+\tau;x,t))(e^{c\tau}-1)+g_k\{G_k(t+\tau,x(t+\tau;x,t))\\
&\qquad\qquad -G_k(t',x(t';x',t'))\}\\
&\leqq g_k\{\beta(e^{c(t'-t)}-1)+K(t'-(t+\tau))+\|x(t+\tau;x,t)-x(t';x',t')\|\}\\
&\qquad\qquad \text{(by Lemma 3 and } \tau<t'-t)\\
&\leqq g_k\{\beta ce^{c\sigma}(t'-t)+K(t'-t)+\|x(t+\tau;x,t)-x(t;x,t)\|\\
&\qquad\qquad +\|x(t;x,t)-x(t';x',t')\|\}\\
&\leqq g_k\{\beta ce^{c\sigma}(t'-t)+K(t'-t)+F^*(t'-t)+\|x-x'\|\}\\
&\leqq g_k\|x-x'\|+g_k\{\beta ce^{cT_k}+K+F^*\}(t'-t)\\
&\leqq h(\eta)k(\sigma)l(\alpha)[\|x-x'\|+|t-t'|].
\end{aligned}
$$

Now we assume that $V_k(t',x')=g_kG_k(t'+\tau',x(t'+\tau';x',t'))e^{c\tau'}$ and that $t'+\tau'=t+\tau$. Then we have $\tau\geqq\tau'$ and

$$
\begin{aligned}
&V_k(t,x)-V_k(t',x')\\
&\geqq g_kG_k(t+\tau,x(t+\tau;x,t))e^{c\tau}-g_kG_k(t'+\tau',x(t'+\tau';x',t'))e^{c\tau'}\\
&\geqq -g_ke^{c\tau'}\|x(t+\tau;x,t)-x(t'+\tau';x',t')\|\\
&\geqq -g_ke^{c\tau'}\|x(t'+\tau';X,t')-x(t'+\tau';x',t')\|\\
&\geqq -g_ke^{c\tau'}e^{L_k\tau'}\|X-x'\|\\
&\geqq -g_ke^{(c+L_k)T_k}\{\|x-x'\|+\|X-x\|\}\\
&\geqq -g_ke^{(c+L_k)T_k}\{\|x-x'\|+F^*(t'-t)\}\\
&\geqq -h(\eta)k(\sigma)l(\alpha)[\|x-x'\|+|t-t'|].
\end{aligned}
$$

Therefore if $(t,x)\in\Omega_{\sigma,\eta,\alpha}$ and $(t',x')\in\Omega_{\sigma,\eta,\alpha}$, we have

$$
(15)\qquad |V_k(t,x)-V_k(t',x')|\leqq h(\eta)k(\sigma)l(\alpha)[\|x-x'\|+|t-t'|].
$$

If $x'=x(t';x,t)$ and $t'=t+h$ $(h>0)$, we have

$$
\begin{aligned}
V_k(t',x')&=g_k\sup_{\tau'\geqq 0}G_k(t'+\tau',x(t'+\tau';x',t'))e^{c\tau'}\\
&=g_k\sup_{\tau\geqq h}G_k(t+\tau,x(t+\tau;x,t))e^{c\tau}e^{-ch}\qquad (t+\tau=t'+\tau')\\
&\leqq g_k\sup_{\tau\geqq 0}G_k(t+\tau,x(t+\tau;x,t))e^{c\tau}e^{-ch}\\
&\leqq V_k(t,x)e^{-ch},
\end{aligned}
$$

whence we have

$$
\frac{V_k(t',x')-V_k(t,x)}{h}\leqq V_k(t,x)\frac{e^{-ch}-1}{h}.
$$

Therefore if $h \to 0^+$ we have

$$V_k'(t, x) \leqq -c V_k(t, x). \tag{16}$$

Now we define a function $V(t, x)$ as follows; namely

$$V(t, x) = \sum_{k=k_0}^{\infty} \frac{1}{2^{k-k_0+1}} V_k(t, x). \tag{17}$$

If $(t, x) \in \Omega_{\sigma, \eta, \alpha}$, by (14) we have

$$V(t, x) \leqq \frac{1}{c} h(\eta) k(\sigma) l(\alpha) \sum_{k=k_0}^{\infty} \frac{1}{2^{k-k_0+1}} \leqq \frac{1}{c} h(\eta) k(\sigma) l(\alpha) \tag{18}$$

and hence (17) is convergent. Moreover σ, η and α are arbitrary and therefore $V(t, x)$ is defined for all (t, x). From (12), it is clear that we have

$$V(t, x) = 0 \quad \text{if} \quad (t, x) \in M. \tag{19}$$

We shall show the existence of a function $a(r)$ which satisfies the condition 2°. First, we have

$$\begin{aligned} V(t, x) &\geqq \frac{1}{2} V_{k_0}(t, x) \geqq \frac{1}{2} g_{k_0} G_{k_0}(t, x) \qquad \text{(by (13))} \\ &\geqq \frac{1}{2} g_{k_0} d\left(x, M\left(t, \frac{1}{k_0}\right)\right) \\ &\geqq \frac{1}{2} g_{k_0} \left\{ d(x, M(t)) - \frac{1}{k_0} \right\} \end{aligned}$$

and so $V(t, x) \to \infty$ as $d(x, M(t)) \to \infty$. Second, if $(t, x) \in \overline{M(t, 1/k)}$ and $(t, x) \bar{\in} \overline{M(t, 1/k+1)}$, we have

$$\begin{aligned} V(t, x) &\geqq \frac{1}{2^{k-k_0+3}} V_{k+2}(t, x) \\ &\geqq \frac{1}{2^{k-k_0+3}} g_{k+2} d(x, M(t, 1/k+2)) \\ &\geqq \frac{1}{2^{k-k_0+3}} g_{k+2} (d(x, M(t, 1/k+1)) + \left(\frac{1}{k+1} - \frac{1}{k+2}\right) \\ &\geqq \frac{1}{2^{k-k_0+3}} g_{k+2} \left(\frac{1}{k+1} - \frac{1}{k+2}\right), \end{aligned}$$

whence we can see the existence of a function $a(r)$.

From (15) and (16), when $(t, x)\in\Omega_{\sigma,\eta,\alpha}$ and $(t', x')\in\Omega_{\sigma,\eta,\alpha}$, we have

$$(20)\quad |V(t, x)-V(t', x')|\leqq h(\eta)k(\sigma)l(\alpha)[\|x-x'\|+|t-t'|]\quad \text{(by (15))}$$

and

$$V'(t, x)\leqq -cV(t, x)\qquad \text{(by (16))}.$$

Therefore this function $V(t, x)$ is the desired one and the theorem is proved.

As a special case, *if $M(t)$ is contained in a compact set Q in R^n for all $t\in I$*, there exists a sphere $S_\xi: \|x\|\leqq\xi$ such that $M(t)\subset S_\xi$ for all $t\in I$ and moreover for any $\alpha>0$ there is an η such that $d(x, M(t))\leqq\eta$ contains the sphere $\|x\|\leqq\alpha$. Therefore δ, β and T in the definitions (i) and (vii) are determined independently of α. Thus in this case, the equiasymptotic stability of M becomes to the uniform-asymptotic stability of M with respect to α. Moreover in this case, *necessarily the solutions of* (1) *are equi-bounded.* It is sufficient that we take a domain $\Omega_{\sigma,\eta}$ such that $0\leqq t\leqq\sigma$, $d(x, M(t))\leqq\eta$ in place of $\Omega_{\sigma,\eta,\alpha}$ in the proof of Theorem 2. The function $A(\sigma, \eta, \alpha, \varepsilon)$ does not depend on α and so we have three functions $g(\varepsilon)$, $h(\eta)$, $k(\sigma)$ such that $g(\varepsilon)A(\sigma, \eta, \varepsilon)\leqq h(\eta)k(\sigma)$. Therefore if $(t, x)\in\Omega_{\sigma,\eta}$ and $(t', x')\in\Omega_{\sigma,\eta}$, we have in place of (20)

$$(21)\qquad |V(t, x)-V(t', x')|\leqq h(\eta)k(\sigma)[\|x-x'\|+|t-t'|].$$

If a set M is a quasi-equiasymptotically stable set of (1) in the large, then it is not necessary that $V(t, x)=0$ if $(t, x)\in M$. Therefore we have the following theorem.

Theorem 3. *We assume that $F(t, x)\in C_0(x)$. If a set M is a quasi-equiasymptotically stable set of* (1) *in the large and solutions of* (1) *are equi-bounded, there exists a continuous non-negative Liapunov function $V(t, x)$ defined on $I\times R^n$ satisfying the following conditions*:

1° $a(d(x, M(t)))\leqq V(t, x)$, *where $a(r)$ is continuous increasing and $a(r)>0$ for $r\neq 0$, $a(r)\to\infty$ as $r\to\infty$,*

2° $V(t, x)\in C_0(t, x)$ *and $V'(t, x)\leqq -cV(t, x)$, where c is an arbitrary positive constant.*

Now we shall discuss the case where a set M is a uniform-asymptotically stable set of (1) in the large with respect to t_0 and solutions of (1) are uniformly bounded. In this case β and T in the definition are independent of t_0 and hence we can replace $\beta(\sigma, \eta, \alpha)$, $T(\sigma, \eta, \alpha, \varepsilon)$ and $\gamma(\sigma, \alpha)$ by $\beta(\eta, \alpha)$, $T(\eta, \alpha, \varepsilon)$ and $\gamma(\alpha)$ respectively. Moreover by the uniform stability, we can assume that $\beta(\eta, \alpha)\to 0$ as $\eta\to 0$. From (14), we have

$$(22)\qquad \begin{cases} V_k(t, x)\leqq g_k\beta(\eta, \alpha)e^{cT_k(\eta,\alpha)} \\ \qquad\qquad \leqq \beta(\eta, \alpha)g_k e^{cT_k(\eta,\alpha)}\,. \end{cases}$$

On the other hand, by (9) we have

$$F^*(0, \alpha)e^{cT(\eta,\alpha,\varepsilon)}e^{L(0,\eta,\alpha,\varepsilon)T(\eta,\alpha,\varepsilon)}\leqq A(0, \eta, \alpha, \varepsilon)$$

and hence we have

$$(23)\qquad g_k e^{cT_k(\eta,\alpha)}\leqq g_k A(0, \eta, \alpha, 1/k)\leqq h(\eta)k(0)l(\alpha)\,,$$

because $F^*(0, \alpha)\geqq 1$ by the assumption and $\exp(L(0, \eta, \alpha, \varepsilon)T(\eta, \alpha, \varepsilon))\geqq 1$. From the definition of $V(t, x)$, if $(t, x)\in\Omega_{\sigma,\eta,\alpha}$, we have

$$(24)\qquad V(t, x)\leqq\beta(\eta, \alpha)h(\eta)k(0)l(\alpha)\,.$$

The right-hand side of (24) is independent of σ and hence if $d(x, M(t))\leqq\eta$ and $\|x\|\leqq\alpha$, (24) is valid and moreover $\beta(\eta, \alpha)\to 0$ as $\eta\to 0$. Therefore we can see that there is a continuous function $b(r, s)$ such that $b(r, s)\to 0$ as $r\to 0$ and that

$$V(t, x)\leqq b(d(x, M(t)), \|x\|)\,.$$

Theorem 4. *We assume that $F(t, x)\in C_0(x)$. If a set M is a uniform-asymptotically stable set of* (1) *in the large with respect to t_0 and if solutions of* (1) *are uniformly bounded, there exists a continuous Liapunov function $V(t, x)$ defined on $I\times R^n$ satisfying the following conditions:*

1° $V(t, x)=0$ *if* $(t, x)\in M$,

2° $a(d(x, M(t)))\leqq V(t, x)\leqq b(d(x, M(t)), \|x\|)$, *where $a(r)$ is the same as in Theorem* 2 *and $b(r, s)$ is continuous and $b(r, s)\to 0$ as $r\to 0$,*

3° $V(t, x)\in C_0(t, x)$ *and* $V'(t, x)\leqq -cV(t, x)$, *where c is an arbitrary positive constant.*

In this case, we can see easily from the proof of Theorem 2 that if $(t, x) \in \Omega_{t,\eta,\alpha}$ and $(t, x') \in \Omega_{t,\eta,\alpha}$, we have

$$|V(t, x) - V(t, x')| \leqq h(\eta) l(\alpha) k(t) \|x - x'\| .$$

As a special case, now we consider *the case where the Lipschitz constant of* $F(t, x)$ *is independent of* t, that is, if $\|x\| \leqq \alpha$, $\|x'\| \leqq \alpha$, $d(x, M(t)) \leqq \eta$ and $d(x', M(t)) \leqq \eta$, there is a constant $P(\eta, \alpha)$ such that

$$\|F(t, x) - F(t, x')\| \leqq P(\eta, \alpha) \|x - x'\| \tag{25}$$

for all $t \in I$. Adding this assumption to the assumptions in Theorem 4, we can replace L in the proof of Theorem 2 by $L(\eta, \alpha)$. Therefore $A(\sigma, \eta, \alpha, \varepsilon)$ is replaced by $F^*(\sigma, \alpha) \exp \{(c + L(\eta, \alpha)) T(\eta, \alpha, \varepsilon)\} + K(\sigma, \alpha) + c\beta(\eta, \alpha) \exp \{c(T(\eta, \alpha, \varepsilon) + \sigma)\}$. When $(t, x) \in \Omega_{\sigma,\eta,\alpha}$ and $(t, x') \in \Omega_{\sigma,\eta,\alpha}$, the Lipschitz constant of $V_k(t, x)$ with respect to x is considered as $g_k e^{(c + L(\eta,\alpha)) T_k(\eta,\alpha)}$. Since we have

$$g_k \exp \{(c + L(\eta, \alpha)) T_k(\eta, \alpha)\} \leqq g_k A(0, \eta, \alpha, 1/k) \leqq h(\eta) k(0) l(\alpha) ,$$

we can obtain

$$|V(t, x) - V(t, x')| \leqq h(\eta) k(0) l(\alpha) \|x - x'\| . \tag{26}$$

This inequality does not depend on σ and therefore this inequality is valid for all t when $\|x\| \leqq \alpha$, $\|x'\| \leqq \alpha$, $d(x, M(t)) \leqq \eta$ and $d(x', M(t)) \leqq \eta$.

Remark 1. When a set M is a uniform-asymptotically stable set of (1) in the large with respect to both t_0 and α, β and T are independent of t_0 and α, and hence corresponding to (22) and (23), we have

$$V_k(t, x) \leqq \beta(\eta) g_k e^{cT_k(\eta)}$$

and

$$g_k e^{cT_k(\eta)} \leqq h(\eta) k(0) l(\alpha_0) .$$

Therefore in the same way in Theorem 4, we can see that there is a continuous function $b(r)$ such that $b(r) \to 0$ as $r \to 0$ and that $V(t, x) \leqq b(d(x, M(t)))$.

Remark 2. In the same way, we can obtain a Liapunov function for the uniform-quasi-asymptotically stable set of (1) in the large with respect to t_0 and α. In this case, a Liapunov function satisfies the conditions

1° $a(d(x, M(t))) \leqq V(t, x) \leqq b(d(x, M(t)))$, where $a(r)$ is the same as in Theorem 2 and $b(r)$ is continuous,

2° $V(t, x) \in C_0(t, x)$ and $V'(t, x) \leqq -cV(t, x)$, where c is an arbitrary positive constant.

V. Uniform-asymptotically stable set

In this section we shall discuss the uniform-asymptotically stable set of (1) with respect to both t_0 and α. In this case we assume that $F(t, x)$ of (1) is defined and continuous in a domain

$$D: 0 \leqq t < \infty, \quad d(x, M(t)) \leqq H \qquad (H > 0 \text{ constant})$$

and that $F(t, x) \in C_0(x)$.

By the uniform stability, there is a $\delta(H) > 0$ such that if $d(x_0, M(t_0)) < \delta(H)$, we have $d(x(t; x_0, t_0), M(t)) < H$ for all $t \geqq t_0$. And let H' be a positive number such that $H' < \min(\delta(H), \delta_0)$, where δ_0 is the positive number which appears in Definition (iv). Then we can show the existence of a Liapunov function in the domain $0 \leqq t < \infty$, $d(x, M(t)) \leqq H'$. To do that, it is sufficient that we take k_0 sufficiently large and $\eta = H'$ in the proof of Theorem 2. Therefore we have the following theorem.

Theorem 5. *If all the solutions of* (1) *are uniformly bounded and a set M is a uniform-asymptotically stable set of* (1) *with respect to both* t_0 *and* α, *there exists a continuous Liapunov function* $V(t, x)$ *defined in a domain* $0 \leqq t < \infty$, $d(x, M(t)) \leqq H'$ *satisfying the following conditions*:

1° $V(t, x) = 0$ *if* $(t, x) \in M$,

2° $a(d(x, M(t))) \leqq V(t, x) \leqq b(d(x, M(t)))$, *where* $a(r)$ *is continuous increasing, positive definite and* $b(r)$ *is continuous,* $b(r) \to 0$ *as* $r \to 0$,

3° $V(t, x) \in C_0(t, x)$ *and* $V'(t, x) \leqq -cV(t, x)$, *where* c *is an arbitrary positive constant.*

VI. Eventual stability

Now we consider a perturbed system

$$x' = F(t, x) + G_1(t, x) + G_2(t, x), \tag{27}$$

where $G_i(t, x)$ are defined and continuous on $I \times R^n$ or on D.

Definitions. A set M is said to be *an eventually stable set of* (1), if for every $\varepsilon > 0$ and every $\alpha \geqq \alpha_0$ ($\alpha_0 \geqq 0$: a suitable constant), there exists an $S(\varepsilon, \alpha) \geqq 0$ such that Definition (i) in the section 2 is satisfied for all those $t_0 \geqq S(\varepsilon, \alpha)$. Similar definitions apply for eventual stabilities for (ii) through (x). For example, we shall say that a set M is *an eventually uniform-stable set of* (1) *with respect to* t_0, if for any $\varepsilon > 0$ and any $\alpha \geqq \alpha_0$, there exist an $S(\varepsilon, \alpha) \geqq 0$ and $\delta(\varepsilon, \alpha) > 0$ such that if $\|x_0\| \leqq \alpha$, $d(x_0, M(t_0)) < \delta(\varepsilon, \alpha)$ and $t_0 \geqq S(\varepsilon, \alpha)$, we have $d(x(t; x_0, t_0), M(t)) < \varepsilon$ for all $t \geqq t_0$. We shall say that a set M is *an eventually uniform-asymptotically stable set of* (1) *with respect to* t_0, if M is an eventually uniform-stable set of (1) with respect to t_0 and if for any $\varepsilon > 0$ and any $\alpha \geqq \alpha_0$, there exist a $\delta_0(\alpha)$, a $T(\varepsilon, \alpha)$ and $S_1(\varepsilon, \alpha)$ such that if $d(x_0, M(t_0)) < \delta_0(\alpha)$, $\|x_0\| \leqq \alpha$ and $t_0 \geqq S_1(\varepsilon, \alpha)$, we have $d(x(t; x_0, t_0), M(t)) < \varepsilon$ for all $t \geqq t_0 + T(\varepsilon, \alpha)$ (cf. [4]).

Lemma 5. *We assume that $F(t, x)$ of* (1) *is defined on D, $F(t, x) \in C_0(x)$ and all the solutions of* (1) *are uniformly bounded and that a set M is a uniform-asymptotically stable set of* (1) *with respect to both t_0 and α. Moreover we assume that $G_i(t, x)$ of* (27) *are defined on D and that all the solutions of* (27) *are also uniformly bounded, that is, for any $\alpha \geqq \alpha_0 \geqq 0$ there is a $\gamma^*(\alpha)$ such that $\|x^*(t; x_0, t_0)\| \leqq \gamma^*(\alpha)$ if $\|x_0\| \leqq \alpha$, where $x^*(t; x_0, t_0)$ is a solution of* (27).

If $u(t; x_0, t_0)$ is the solution of the equation of the first order

$$\begin{cases} u' = -cu + h(H')l(\gamma^*(\alpha))k(t)[g_1(t, \gamma^*(\alpha)) + g_2(t, \gamma^*(\alpha))] \\ u(t_0) = V(t_0, x_0), \end{cases} \tag{28}$$

where $h(H')l(\gamma^*(\alpha))k(t)$ is given in (20), *$V(t, x)$ is the function given in Theorem* 5, *$d(x, M(t)) \leqq H'$ and $g_i(t, \alpha) = \max_{\|x\| \leqq \alpha} \|G_i(t, x)\|$, and if*

$u(t; x_0, t_0) \leqq a(H')$ for $t \geqq t_0$, where $a(r)$ is the function given in Theorem 5, *then the solution $x^*(t; x_0, t_0)$ of* (27) *such that $\|x_0\| \leqq \alpha$ satisfies the relation*

$$a(d(x^*(t; x_0, t_0), M(t))) \leqq u(t; x_0, t_0) \qquad \text{for} \quad t \geqq t_0 .$$

Proof. If $(t, x) \in \Omega_{\sigma,\eta,\alpha}$ and $(t, x') \in \Omega_{\sigma,\eta,\alpha}$, we have by (20)

$$| V(t, x) - V(t, x') | \leqq h(\eta) k(\sigma) l(\alpha) \| x - x' \| .$$

For this lemma, we may put $\eta = H'$ and $x^*(t; x_0, t_0)$ of (27) such that $\|x_0\| \leqq \alpha$ satisfies $\|x^*(t; x_0, t_0)\| \leqq \gamma^*(\alpha)$. Thus we consider the function $V(t, x)$ given in Theorem 5 in the domain such that $0 \leqq t < \infty$, $\|x\| \leqq \gamma^*(\alpha)$, $d(x, M(t)) \leqq H'$ and hence if (t, x) and (t, x') belong to this domain we have

$$| V(t, x) - V(t, x') | \leqq h(H') l(\gamma^*(\alpha)) k(t) \| x - x' \| .$$

Therefore it follows that

$$\begin{aligned} & V'(t, x^*(t; x_0, t_0)) \\ & \leqq -c V(t, x^*(t; x_0, t_0)) + h(H') l(\gamma^*(\alpha)) k(t) [\| G_1(t, x^*(t; x_0, t_0)) \| \\ & \qquad + \| G_2(t, x^*(t; x_0, t_0)) \|] , \end{aligned}$$

if $d(x^*(t; x_0, t_0), M(t)) \leqq H'$. Thus, if $u(t; x_0, t_0)$ is the solution of (28), we have $V(t, x^*(t; x_0, t_0)) \leqq u(t; x_0, t_0)$. Since $a(d(x^*(t; x_0, t_0), M(t))) \leqq V(t, x^*(t; x_0, t_0))$, we can prove the lemma.

Now we assume that $G_i(t, x)$ satisfy the conditions

(29) $$k(t) \| G_1(t, x(t)) \| \to 0 \qquad \text{as} \quad t \to \infty$$

and

(30) $$\int_{t_0}^{\infty} k(t) \| G_2(t, x(t)) \| dt < \infty ,$$

where $x(t)$ is an arbitrary continuous bounded function defined for $t \geqq t_0$ and belongs to the domain of definition of $G_i(t, x)$.

Theorem 6. *We assume that $F(t, x)$ of* (1) *is defined on D, $F(t, x) \in C_0(x)$ and all the solutions of* (1) *are uniformly bounded and that a set M is a uniform-asymptotically stable set of* (1) *with respect to both t_0 and α. Moreover we assume that $G_i(t, x)$ of* (27) *are defined on D, continuous and satisfy the conditions* (29), (30) *and that all the solutions of* (27) *are uniformly bounded. Then the set M is an eventually uniform-asymptotically stable set of* (27) *with respect to* t_0.

Proof. By the assumptions, there exists a continuous Liapunov function $V(t, x)$ satisfying the conditions in Theorem 5, and if $(t, x)\in\Omega_{t,H',\alpha}$ and $(t, x')\in\Omega_{t,H',\alpha}$, we have

$$|V(t, x)-V(t, x')|\leqq h(H')l(\alpha)k(t)\|x-x'\| \qquad \text{(by (20))}.$$

Let $x^*(t; x_0, t_0)$ be a solution of (27) such that $\|x_0\|\leqq\alpha$. Then there is a $\gamma^*(\alpha)$ such that $\|x^*(t; x_0, t_0)\|\leqq\gamma^*(\alpha)$ for all $t\geqq t_0$. Now we consider the equation (28) in Lemma 5. By (29) and (30), we have

$$(31) \qquad k(t)g_1(t, \gamma^*(\alpha))\rightarrow 0 \quad \text{as} \quad t\rightarrow\infty$$

and

$$(32) \qquad \int_{t_0}^{\infty} k(t)g_2(t, \gamma^*(\alpha))dt<\infty .$$

The solution of (28) is of form as follows:

$$u(t) = e^{-c(t-t_0)}V(t_0, x_0)+e^{-ct}\int_{t_0}^{t} e^{cs}h(H')l(\gamma^*(\alpha))k(s)[g_1(s, \gamma^*(\alpha)) + g_2(s, \gamma^*(\alpha))]ds .$$

From this, we have

$$u(t)\leqq e^{-c(t-t_0)}b(d(x_0, M(t_0)))+Je^{-ct}\int_{t_0}^{t} e^{cs}k(s)g_1(s, \gamma^*(\alpha))ds + Je^{-ct}\int_{t_0}^{t} e^{cs}k(s)g_2(s, \gamma^*(\alpha))ds ,$$

where $J=J(H', \alpha)=h(H')l(\gamma^*(\alpha))$.

If corresponding to any $\varepsilon>0$, we choose $\delta(\varepsilon)$ and $S(\varepsilon, \alpha)$ so that

$$b(r)<\frac{a(\varepsilon)}{3}, \qquad \text{if} \quad r<\delta(\varepsilon),$$

$$\sup_{t\geqq S(\varepsilon,\alpha)} k(t)g_1(t, \gamma^*(\alpha))<\frac{a(\varepsilon)c}{3J},$$

$$\int_{S(\varepsilon,\alpha)}^{\infty} k(s)g_2(s, \gamma^*(\alpha))ds<\frac{a(\varepsilon)}{3J},$$

we have $u(t)<a(\varepsilon)$ when $d(x_0, M(t_0))<\delta(\varepsilon)$, $\|x_0\|\leqq\alpha$ and $t\geqq t_0>S(\varepsilon, \alpha)$ (cf. [4]). From Lemma 5, we have for $t\geqq t_0$

$$a(d(x^*(t;x_0,t_0),M(t)))\leqq u(t)<a(\varepsilon).$$

Therefore we have $d(x^*(t;x_0,t_0),M(t))<\varepsilon$. This means that a set M is an eventually uniform-stable set of (27) with respect to t_0.

By (30), there is a $p(\alpha)>0$ such that $k(t)g_1(t,\gamma^*(\alpha))\leqq p(\alpha)$. Moreover for any $\rho>0$, there are $T_i(\rho,\alpha)$ $(i=1,2)$ such that if $t\geqq T_1(\rho,\alpha)$,

$$e^{-\frac{ct}{2}}<\frac{a(\rho)c}{6Jp(\alpha)}$$

and that if $t\geqq T_2(\rho,\alpha)$, $\max_{\frac{t}{2}\leqq s\leqq t}k(s)g_1(s,\gamma^*(\alpha))<\frac{a(\rho)c}{6J}$. For $t\geqq\max_i T_i$ we have

$$Je^{-ct}\int_{t_0}^{t}e^{cs}k(s)g_1(s,\gamma^*(\alpha))ds<\frac{a(\rho)}{3},$$

because

$$Je^{-ct}\int_{t_0}^{t}e^{cs}k(s)g_1(s,\gamma^*(\alpha))ds$$
$$\leqq\frac{1}{c}J\{p(\alpha)e^{-\frac{ct}{2}}+\max_{\frac{t}{2}\leqq s\leqq t}k(s)g_1(s,\gamma^*(\alpha))\}.$$

For a fixed $\varepsilon=\varepsilon_0$, we put $S(\varepsilon_0,\alpha)=S_1(\alpha)$ and $\delta(\varepsilon_0)=\delta_0$. Corresponding to any $\rho>0$, we choose $S_2(\rho,\alpha)\geqq S_1(\alpha)$ so large that

$$\int_{S_2(\rho,\alpha)}^{\infty}k(s)g_2(s,\gamma^*(\alpha))ds<\frac{a(\rho)}{6J}$$

and then choose $T_3(\rho,\alpha)$ so large that

$$\exp\{-c(S_1(\alpha)+T_3(\rho,\alpha)-S_2(\rho,\alpha))\}\int_{S_1(\alpha)}^{\infty}k(s)g_2(s,\gamma^*(\alpha))ds<\frac{a(\rho)}{6J}.$$

Then by following the proof of Hale's theorem in [4], for any $t_0\geqq S_1(\alpha)$, $t\geqq t_0+T_3(\rho,\alpha)$ we have

$$Je^{-ct}\int_{t_0}^{t}e^{cs}k(s)g_2(s,\gamma^*(\alpha))ds<\frac{a(\rho)}{3}.$$

Moreover there is a $T_4(\rho)$ such that $e^{-c(t-t_0)}b(d(x_0,M(t_0)))<a(\rho)/3$ for $d(x_0,M(t_0))<\delta_0$ and $t\geqq t_0+T_4(\rho)$, and hence for any $t_0\geqq S_1(\alpha)$, $t\geqq t_0+T(\rho,\alpha)$ and $d(x_0,M(t_0))<\delta_0$ we have $u(t)<a(\rho)$ from which we have $d(x^*(t;x_0,t_0),M(t))<\rho$, where $T(\rho,\alpha)=\max(T_1,T_2,T_3,T_4)$. Thus the proof is completed.

As a sufficient condition, we have the following theorem.

Theorem 7. *We assume that $F(t, x)$ of* (1) *is defined on D and all the solutions of* (1) *are uniformly bounded. Suppose that there exists a continuous Liapunov function $V(t, x)$ satisfying the conditions in Theorem* 5 *and that if $\|x\|\leqq\alpha$ and $\|x'\|\leqq\alpha$, we have*

$$|V(t, x)-V(t, x')|\leqq\omega(\alpha)k(t)\|x-x'\|.$$

Moreover we assume that $G_i(t, x)$ of (27) *are defined on D, continuous and satisfy the conditions*

$$k(t)\|G_1(t, x(t))\|\to 0 \quad \text{as} \quad t\to\infty$$

and

$$\int_{t_0}^{\infty}k(t)\|G_2(t, x(t))\|dt<\infty,$$

where $x(t)$ is the same in (29), (30), *and that all the solutions of* (27) *are uniformly bounded.*

Then the set M is a uniform-asymptotically stable set of (1) *with respect to both t_0 and α and M is an eventually uniform-asymptotically stable set of* (27).

As a special case of Theorem 6, if we have the condition (25), from (26) we have

$$|V(t, x)-V(t, x')|\leqq h(H')k(0)l(\alpha)\|x-x'\|$$

when $\|x\|\leqq\alpha$, $\|x'\|\leqq\alpha$, and therefore the conditions which are satisfied by $G_i(t, x)$ are as follows:

$$G_1(t, x(t))\to 0 \quad \text{as} \quad t\to\infty \tag{29'}$$

and

$$\int_{t_0}^{\infty}\|G_2(t, x(t))\|dt<\infty. \tag{30'}$$

Remark. If a set M is a uniform-asymptotically stable set of (1) with respect to both t_0 and α, and if $M(t)$ is contained in a compact set Q in R^n for all $t\in I$, then necessarily the solutions of (1) in consideration are uniformly bounded. Moreover if we assume (25), $P(\eta, \alpha)$ of (25) is determined depending only on η. Therefore from (26) we have

$$|V(t, x)-V(t, x')| \leqq h(H')k(0)l(\alpha_0)||x-x'||.$$

And the functions g_1 and g_2 in Lemma 5 are independent of α. Therefore we do not need assume the boundedness of solutions of (27).

VII. Asymptotic behavior of solutions of perturbed system

In this section we assume that $F(t, x)$ of (1) is defined on $I\times R^n$, $F(t, x)\in C_0(x)$ and that $G_1(t, x)$ and $G_2(t, x)$ are defined on $I\times R^n$ and continuous. Now we consider a perturbed system (27), where $G_1(t, x)$ and $G_2(t, x)$ are perturbation terms. We assume that all the solutions of (1) are equi-bounded. Moreover we assume that a set M is an equiasymptotically stable set of (1) in the large and that $G_1(t, x)$ and $G_2(t, x)$ satisfy the conditions (29) and (30) respectively, where $k(t)$ is the function which appear for the equiasymptotic stability of M in the large.

Theorem 8. *Under the assumptions above, if every solution of* (27) *is bounded, every solution of* (27) *approaches* M *as* $t\rightarrow\infty$.

Proof. We consider a solution $x^*(t; x_0, t_0)$ of (27). Then there is a positive constant γ such that for all $t\geqq t_0$

$$||x^*(t; x_0, t_0)||<\gamma.$$

Since the solution of (1) is bounded and approaches the set M, we have a positive number N such that the set $||x||\leqq N$ includes at least one point of $M(t)$ for every $t\in I$. Therefore we can assume that γ is so large that the domain $||x||\leqq\gamma$ includes a point of $M(t)$ for every $t\in I$. From the assumptions, there exists a continuous Liapunov function $V(t, x)$ defined on $I\times R^n$ satisfying the conditions in Theorem 2. Now we consider this function $V(t, x)$ only in the domain $I\times S_\gamma$ (S_γ: $||x||\leqq\gamma$). If $x\in S_\gamma$, we have $d(x, M(t))\leqq 2\gamma$, and therefore by (20) we have

$$|V(t, x)-V(t, x')| \leqq h(2\gamma)l(\gamma)k(t)||x-x'||,$$

if $x\in S_\gamma$ and $x'\in S_\gamma$. Thus we have

$$V'(t, x^*(t; x_0, t_0)) \leqq -cV(t, x^*(t; x_0, t_0))+Kk(t)(g_1(t)+g_2(t)),$$

where $K=h(2\gamma)l(\gamma)$ and $g_i(t)=\max_{\|x\|\leqq\gamma}\|G_i(t, x)\|$. By the assumptions on $G_1(t, x)$ and $G_2(t, x)$, we have

$$k(t)g_1(t)\to 0 \qquad \text{as} \quad t\to\infty$$

and

$$\int_{t_0}^{\infty} k(t)g_2(t)dt < \infty .$$

If we let $u(t\,;x_0, t_0)$ be the solution of the equation

$$\begin{cases} u' = -cu+Kk(t)(g_1(t)+g_2(t)) \\ u(t_0) = V(t_0, x_0)\,, \end{cases}$$

we have

$$V(t, x^*(t\,;x_0, t_0))\leqq u(t\,;x_0, t_0) \qquad \text{for all} \quad t\geqq t_0 .$$

In the same way in Theorem 6, we can see that $u(t\,;x_0, t_0)\to 0$ as $t\to\infty$, and therefore $V(t, x^*(t\,;x_0, t_0))\to 0$ as $t\to\infty$. By the condition 2° which $V(t, x)$ satisfies, we have

$$d(x^*(t\,;x_0, t_0), M(t))\to 0 \qquad \text{as} \quad t\to\infty ,$$

which implies that $x^*(t\,;x_0, t_0)$ approaches M as $t\to\infty$.

As a special case, we have the following corollary.

Corollary. *We assume that* $F(t, x)$ *is defined and continuous on* $I\times R^n$ *and that* $F(t, x)$ *satisfies the condition* (25). *Moreover we assume that* $G_i(t, x)$ *are defined and continuous on* $I\times R^n$ *and that* $G_i(t, x)$ *satisfy the conditions* (29′) *and* (30′). *If all the solutions of* (1) *are uniformly bounded and every solution of* (27) *is bounded and if a set* M *is a uniform-asymptotically stable set of* (1) *in the large with respect to* t_0, *every solution of* (27) *approaches* M *as* $t\to\infty$.

In this case, we have by (26)

$$|V(t, x)-V(t, x')|\leqq h(2\gamma)l(\gamma)k(0)\|x-x'\|$$

and hence it is sufficient that $G_i(t, x)$ satisfy the conditions (29′) and (30′).

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