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On weak measurability in functional spaces

By

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§1. Introduction.

The fundamental work of R. A. Minlos [9] concerning the existence of measures in nuclear spaces seems to make it possible to get an approach to a new foundation of the theory of stochastic processes. To achieve this end it is necessary to prove the weak measurability (see Definition in §3) of some subspaces of the space of tempered distributions of L. Schwartz [12].

The aim of the present paper is to give a first step in this direction. In §2 we recall some facts in topological vector space theory which will be used in the sequel. In §3 we give the definition of the weak measurability of a set in the dual of a locally convex vector space. For the weak measurability of linear subspaces two simple criteria are obtained there. In §4 examples of weakly measurable subspaces of the space of distributions are treated. We shall see that practically almost all functional spaces of usual use in analysis are weakly measurable.

I should thank here Prof. K. Itô for his kind leading and encouragement. The origin of this work was the question asked by him whether the space of the derivatives of locally Hölder continuous functions is weakly measurable or not[1]. Applications of the present work to the theory of stochastic processes will be treated by him. I should be grateful to Prof. H. Yoshizawa also for his showing constant interest, for the helpful discussions and for his critical reading of the manuscript.

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1) See §4, Proposition 6 and Proposition 7.

§ 2. Preliminaries from topological vector space theory.

We recall here in this paragraph those fundamentals from topological vector space theory (see [2] and [3]) which are needed in the subsequent paragraphs. When we speak of a vector space, we always mean a vector space over the field of real numbers or over the field of complex numbers; when the field of scalars is not specified, it is understood that the definitions and results are valid in both cases. When several vector spaces intervene in the same statement, it is understood that they have the same field of scalars.

Let F and G be two vector spaces. Suppose that there is given a bilinear form $\langle x, y\rangle$ on the product $F\times G$ $(x\in F,\ y\in G)$.

DEFINITION 1. The vector spaces F and G are said to constitute a *dual pair* (with respect to the bilinear form $\langle x, y\rangle$), if the following conditions are satisfied:

(i) For any $x\neq 0$ in F, there exists $y\in G$ such that $\langle x, y\rangle\neq 0$,
(ii) For any $y\neq 0$ in G, there exists $x\in F$ such that $\langle x, y\rangle\neq 0$.

Let E be a locally convex (Hausdorff) vector space and let E' be the (topological) dual space of E, i.e. the space of all the continuous linear forms on E. Denote by $\langle x, x'\rangle$ the canonical bilinear form $x'(x)$, i.e.,

$$\langle x, x'\rangle = x'(x). \tag{1}$$

Then, E and E' constitute a dual pair with respect to the bilinear form (1).[2]

Suppose that there is given a dual pair F, G. By conditions (i) and (ii) in the above definition, F (resp. G) can be canonically embedded in the algebraic dual G^* of G (resp. in the algebraic dual F^* of F). We identify, by this embedding, the space F (resp. G) and the corresponding subspace of G^* (resp. F^*). Hence the elements of G (resp. of F) are considered as functions on F (resp. on G).

2) In this case, the condition (i) is a consequence of Hahn-Banach's extension theorem ([2] I, p. 101 and p. 110).

DEFINITION 2. Let F, G be a dual pair. We denote by $\sigma(F, G)$ (resp. $\sigma(G, F)$) the weakest topology in F (resp. in G) that makes all the elements of G (resp. of F) continuous. We call this topology $\sigma(F, G)$ (resp. $\sigma(G, F)$) the *weak topology* of F defined by G (resp. the weak topology of G defined by F).

By conditions (i) and (ii) in Definition 1, $\sigma(F, G)$ is always a locally convex Hausdorff topology in F. When the dual pair consists of a locally convex vector space E and its dual E', $\sigma(E, E')$ is nothing but the usual *weakened topology*[3] of E; and $\sigma(E', E)$ is the *weak topology of the dual E'*.

A subset D of a vector space is called a *balanced* set is $\lambda x \in D$ whenever $|\lambda| \leq 1$ and $x \in D$. It is called a *disc* if it is balanced and convex.

DEFINITION 3. Let F, G be a dual pair. Let $\mathfrak{S}$ be a family of discs in G. The topology in F defined as the topology of uniform convergence on each set in $\mathfrak{S}$ is called the *$\mathfrak{S}$-topology.*

If a family $\mathfrak{S}$ consists of *bounded*[4] sets and if $\mathfrak{S}$ is a covering of G, i.e. $\bigcup_{D \in \mathfrak{S}} D = G$, then the $\mathfrak{S}$-topology is a locally convex Hausdorff topology in F and is stronger than $\sigma(F, G)$.

Let $\mathfrak{T}$ be a locally convex Hausdorff topology in F. We denote specifically by $F_{\mathfrak{T}}$ the space F with the topology $\mathfrak{T}$. The dual of $F_{\mathfrak{T}}$ is denoted by $F'_{\mathfrak{T}}$.

MACKEY-ARENS' THEOREM. *In order that $F'_{\mathfrak{T}} = G$ (under the identification mentioned before), it is necessary and sufficient that $\mathfrak{T}$ coincides with an $\mathfrak{S}$-topology such that $\mathfrak{S}$ consists of weakly compact*[5] *discs in G and is a covering of G.* (See [2] II, p. 68.)

DEFINITION 4. Let F, G be a dual pair. For a subset A of F, we denote by A^0 the set of those elements y in G which satisfy the inequality

$$\mathfrak{R}\langle x, y\rangle \leq 1$$ [6]

3) In French terminology, "la topologie affaiblie".
4) See [2] II, p. 4.
5) Compact with respect to $\sigma(G, F)$.
6) $\mathfrak{R}$ stands for "the real part of". It is easy to see that if A is a disc this inequality can be replace by $|\langle x, y\rangle| \leq 1$.

for any x in A. The set A^0 is called the *polar* of A.

Interchanging the roles of F and G we define the polar B^0 of a subset B of G.

DEFINITION 5. Let A be a subset of F and A^0 be the polar of A. Then, the polar $(A^0)^0$ of A^0 is called the *bipolar* of A and is denoted by A^{00}.

BIPOLAR THEOREM. *Let F, G be a dual pair and A be a subset of F. Then the bipolar A^{00} is the samllest weakly closed convex set that contains A and the origin* 0. *In particular, if A is a weakly closed disc, we have that*

$$A^{00} = A .$$

(See [2] II, p. 52 Proposition 3.)

Let E be a locally convex vector space and E' be its dual. Take a disc neighbourhood V of 0 in E. It is evident that V^0 is then weakly compact because of Tychonoff's theorem. But the contrary is not true in general, i.e. the polar B^0 of a weakly compact disc B in E' is not necessarily a neighbourhood of 0 in E. If E is *tonnelé* (see [2] II, p. 1), however, this contrary holds, i.e.

PROPOSITION 1. *Let E be a locally convex vector space which is tonnelé. Then, every weakly compact subset of the dual E' is an equicontinuous set.* (See [2] II, p. 65 Theorem 1.)

The following fact will be of later use.

PROPOSITION 2. *In a topological vector space, let A be a convex set with non-empty interior $\mathring{A}$. Then, every point of A is adherent to $\mathring{A}$.* (See [2] I, p. 51 Corollary 1 to Proposition 15.)

§ 3. Measurability criteria.

Let E be a locally convex vector space and E' be its dual. The purpose of this paragraph is to obtain criteria for weak measurability (see Definition 6 below) of subspaces of the dual space E'.

For an element $x \in E$ and a real number α, we denote by $S_{x,\alpha}$ the closed half space of E' defined by the inequality

$$\mathfrak{R} \langle x, x' \rangle \leq \alpha .$$

DEFINITION 6. Let $\mathfrak{S}(E')$ be the totality of closed half spaces $S_{x,\alpha}$ defined above. We denote by $\mathfrak{B}(E')$ the *Borel field on* E' *generated by* $\mathfrak{S}(E')$, i.e. the smallest Borel field containing $\mathfrak{S}(E')$. A subset of E' is called *weakly measurable* if it belongs to the Borel field $\mathfrak{B}(E')$.

Since E and E' constitute a dual pair, the elements of E are considered as functions on the dual E'. Then it is clear that the Borel field $\mathfrak{B}(E')$ defined above is just the smallest Borel field that makes all the elements of E measurable functions on E'.

Let us denote by $\mathfrak{T}_0$ the (original) topology of the space E and consider another topology $\mathfrak{T}$ on E which is weaker than the original topology $\mathfrak{T}_0$ and compatible with the vector structure of E but is neighther necessarily locally convex nor Hausdorff. Since $\mathfrak{T}$ is assumed to be weaker than $\mathfrak{T}_0$, a linear form on E which is continuous with respect to the new topology $\mathfrak{T}$ is *à fortiori* continuous with respect to the original one $\mathfrak{T}_0$ and therefore an element of E'. Hence $E'_{\mathfrak{T}}$ is a subspace of E'. Using the notations we get the following

THEOREM 1. *Let E be a separable locally convex vector space and E' be its dual. A subspace G of E' is weakly measurable if it can be written in the form*

$$G = E'_{\mathfrak{T}}$$

for a topology $\mathfrak{T}$ on E such that

(i) *$\mathfrak{T}$ is weaker than the original topology $\mathfrak{T}_0$*

(ii) *$\mathfrak{T}$ is metrizable*[7] *and compatible with the vector structure of E.*

PROOF. We shall show that G belongs to $\mathfrak{S}_{\delta\sigma}$, i.e. G is a countable union of those sets each of which is a countable intersection of sets in $\mathfrak{S}(E')$. First take a point $a \in E$. Taking its polar we have that

$$a^0 = S_{a,1}. \tag{2}$$

7) More generally, we might only assume that $\mathfrak{T}$ satisfies the first axiom of countability. We need not assume that $\mathfrak{T}$ is a Hausdorff topology.

Take an $\mathfrak{T}$-open set D in E. Since $\mathfrak{T}$ is weaker than $\mathfrak{T}_0$, D is $\mathfrak{T}_0$-open also. $\mathfrak{T}_0$ being a separable topology and D being $\mathfrak{T}_0$-open, D contains a countable number of points a_ν ($\nu=1, 2, 3, \cdots$) which are dense in D. Let A be the set consisting of these vectors a_ν ($\nu=1, 2, 3, \cdots$). A being contained and dense in D, we have that

$$D^0 = A^0 = \bigcap_{\nu=1}^{\infty} a_\nu^0 ,$$

since the elements of E' are continuous. Hence, by (2), we see that D^0 is in $\mathfrak{E}_\delta$, i.e. D^0 is a countable insersection of those sets which are in $\mathfrak{E}(E')$. Now since $\mathfrak{T}$ is a metrizable topology, there exists a countable number of $\mathfrak{T}$-open sets D_n ($n=1, 2, 3, \cdots$) that constitute a fundamental system of neighbourhoods of 0 in $E_{\mathfrak{T}}$. By the above argument, we know that each $D_n^0 \in \mathfrak{E}_\delta$. To complete the proof, therefore, it is enough to show that

$$G = \bigcup_{n=1}^{\infty} D_n^0 . \tag{3}$$

Take any element x' in $G=E'_{\mathfrak{T}}$. x' being $\mathfrak{T}$-continuous and $\langle 0, x'\rangle=0$, x' should belong to some D_n^0 since $\{D_n\}$ constitute a fundamental system of neighbourhoods of 0. Hence we have that

$$G \subseteq \bigcup_{n=1}^{\infty} D_n^0 . \tag{4}$$

On the other hand if an element x' of E' is in some D_n^0, x' should be continous relative to $\mathfrak{T}$. In fact, for any $\varepsilon > 0$, $x' \in D_n^0$ implies that $|\langle x, x'\rangle| \leq \varepsilon$ for any $x \in \varepsilon D_n$. εD_n being a $\mathfrak{T}$-open set containing 0 since $\mathfrak{T}$ is compatible with the vector structure of E, the inequality implies that x' is continuous with respect to $\mathfrak{T}$. Thus we have that

$$G \supseteq \bigcup_{n=1}^{\infty} D_n^0 . \tag{5}$$

From (4) and (5) we get (3). This completes the proof.

To assure the weak measurability of a given subspace of G of E', we cannot use Theorem 1 as a criterion unless we can easily find a topology $\mathfrak{T}$ on E that satisfies the conditions stated in the theorem. Therefore we shall give another criterion which concerns directly the subspace G.

THEOREM 2. *Let E be a locally convex vector space which is separable and tonnelé. E' be its dual. Then, a subspace G of E' is weakly measurable if there exists a family $\mathfrak{S}$ of discs in E' that satisfies the following conditions*:

(i) *$\mathfrak{S}$ consists of a countable number of discs K_n $(n=1, 2, 3, \cdots)$ each of which is contained in G and weakly compact.,*

(ii) *$\mathfrak{S}$ constitute a covering of G, i.e. $G=\bigcup_{n=1}^{\infty} K_n$.*

PROOF. According to Proposition 1 in the preceeding paragraph, each disc K_n is an equicontinuous set since K_n is weakly compact and the space F is tonnelé. Therefore the polar K_n^0 is a neighbourhood of 0 in E. Denote by V_n the interior of K_n^0. V_n is not empty since K_n^0 is a neighbourhood. By Proposition 2 in the preceeding paragraph, we know that V_n is dense in K_n^0. Since E is separable and since V_n is open, we can find a countable number of points a_ν $(\nu=1, 2, 3, \cdots)$ in V_n which are dense in V_n. Thus we have that

$$K_n^{00} = V_n^0 = \bigcap_{\nu=1}^{\infty} a_\nu^0 .$$

But since K_n is weakly compact, according to Bipolar Theorem in the proceeding paragraph, we have that $K_n^{00}=K_n$. Therefore we have, as in the proof of Theorem 1, that $K_n \in \mathfrak{F}_\delta$. Hence $G=\bigcup_{n=1}^{\infty} K_n \in \mathfrak{F}_{\delta\sigma}$. This completes the proof.

REMARK. Theorem 1 and Theorem 2 are essentially equivalent. In fact, if we know beforehand that the space E and the subspace G or E' constitute a dual pair with respect to the restriction of the canonical bilinear form (1) on $E\times E'$ to $E\times G$, we can deduce Theorem 2 directly from Theorem 1 referring to Mackey-Arens' Theorem in the preceeding paragraph, since the $\mathfrak{S}$-topology is metrizable because $\mathfrak{S}$ consists of a countable number of discs. (To assure that this $\mathfrak{S}$-topology is weaker than the original topology of E, we required that E is tonnelé.) In turn, we can reduce Theorem 1 to Theorem 2 if we restrict ourselves to spaces which are tonnelé, because equicontinuous sets are always weakly compact.

THEOREM 3. *Let E and F be two locally convex vector spaces. Let E' and F' be their dual spaces. And let $u: E \rightarrow F$ be a continuous linear mapping. Then, the transposed mapping $u': F' \rightarrow E'$ of u is a weakly measurable mapping*[8].

PROOF. Let $S_{x,\alpha}$ be an element of $\mathfrak{S}(E')$. Then, by the defining formula

$$\langle u(x), y' \rangle = \langle x, u'(y') \rangle \qquad (x \in E, \ y' \in F')$$

of the transposed mapping u', we see that

$$u'^{-1}(S_{x,\alpha}) = S_{u(x),\alpha}.$$

From this we have that

$$u'^{-1}(\mathfrak{S}(E')) \subseteq \mathfrak{B}(F'). \tag{6}$$

Now, let $\mathfrak{B}$ be the totality of those subsets A of E' which satisfy

$$u'^{-1}(A) \in \mathfrak{B}(F').$$

It is easy to see that $\mathfrak{B}$ is a Borel field. But since, by (6), $\mathfrak{B}$ contains $\mathfrak{S}(E')$ and $\mathfrak{B}(E')$ is the smallest Borel field containing $\mathfrak{S}(E')$, we have that $\mathfrak{B} \supseteq \mathfrak{B}(E')$, i.e.

$$u'^{-1}(\mathfrak{B}(E')) \subseteq \mathfrak{B}(F').$$

This completes the proof.

COROLLARY. *Let F be a locally convex vector space. Let E be a dense subspace of F and introduce a stronger locally convex topology into E. Then, we can canonically regard F' as a subspace of E'.*

Under these assumptions, if a subset A of F' is in $\mathfrak{B}(E')$, then A is in $\mathfrak{B}(F')$ also[9].

PROOF. Let $u: E \rightarrow F$ be the injection mapping which is continuous by assumption. Then, u' is nothing but canonical embedding $F' \rightarrow E'$ stated in the proposition. Hence, by the above theorem, we have that

$$A = u'^{-1}(A) \in \mathfrak{B}(F').$$

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8) When E and F are reflexive, every continuous linear mapping $v: F' \rightarrow E'$ is written in the form $v = u'$. This is the case when $E = \mathfrak{D}$ and $F = \mathcal{S}$, or when $E = \mathfrak{D}$, $F = \mathfrak{E}$.

9) This is the case when $E = \mathfrak{D}$ and $F = \mathcal{S}$, or when $E = \mathfrak{D}$, $F = \mathfrak{E}$. (See [12])

§ 4. Examples.

From now on we shall be concerned exclusively with subspaces of the space of distributions of L. Schwartz [12] defined on the n-dimensional space $\boldsymbol{R}^n$. Let us recall some notations defined in [12].

$\mathfrak{D}$: the space of indefinitely continuously differentiable functions defined on $\boldsymbol{R}^n$ and with compact supports.

$\mathfrak{D}^m$: the space of m times continuously differentiable functions defined on $\boldsymbol{R}^n$ and with compact supports. When $m=0$, this is the space $\mathcal{C}$ of continuous functions with compact supports.

$\mathcal{S}$: the space of indefinitely continuously differentiable functions defined on $\boldsymbol{R}^n$ and rapidly decreasing.

$\mathcal{S}^m$: the space of m times continuously differentiable rapidly decreasing functions. When $m=0$, this is the space of rapidly decreasing continuous functions.

$\mathcal{E}$: the space of indefinitely continuously differentiable functions with arbitrary supports.

$\mathcal{E}^m$: the space of m times continuously differentiable functions. When $m=0$, this is the space C of continuous functions.

We suppose that these spaces are associated with their standard topologies (see [12]). These spaces are thus locally convex vector spaces. We notice here only the facts that they are *tonnelé* and that it easy to see that they are *separable*. To them we add the following.

L^p $(1\leq p<\infty)$: the space of measurable functions whose p-th powers are integrable with the usual topology defined by the norm

$$||f||_p=\left(\int|f(x)|^p dx\right)^{\frac{1}{p}}.$$

The corresponding dual spaces are:

$\mathfrak{D}'$: the dual of $\mathfrak{D}$, i.e. the space of distributions.

$\mathfrak{D}'^m$: the dual of $\mathfrak{D}^m$, i.e. the space of distributions of order $\leq m$. When $m=0$, this is the space $\mathcal{C}'$ of (locally bounded) measures.

$\mathcal{S}'$: the dual of $\mathcal{S}$, i.e. the space of tempered (or slowly increasing) distributions.

$\mathcal{S}'^m$: the dual of $\mathcal{S}^m$, i.e. the space of tempered distributions of order $\leq m$. When $m=0$, this is the space of slowly increasing measures.

$\mathcal{E}'$: the dual of $\mathcal{E}$, i.e. the space of distributions with compact supports.

$\mathcal{E}'^m$: the dual of $\mathcal{E}^m$, i.e. the space of distributions of order $\leq m$ with compact supports. When $m=0$, this is the space of measures with compact supports.

$L^q \left(1<q\leq\infty\,;\ \frac{1}{p}+\frac{1}{q}=1\right)$: the dual of L^p.

As an application of Theorem 1 in the preceeding paragraph we can easily see the following

Proposition 1. *$\mathcal{E}'^m$, $\mathcal{E}'$, $\mathcal{S}'^m$, L^p $(1<p\leq\infty)$ and $\mathcal{S}'$ are in $\mathfrak{B}(\mathfrak{D}')$; $\mathcal{E}'^m$, $\mathcal{E}'$, $\mathcal{S}'^m$, L^p $(1<p\leq\infty)$ are in $\mathfrak{B}(\mathcal{S}')$; $\mathcal{E}'^m$ is in $\mathfrak{B}(\mathcal{E}')$*[10].

For, these spaces are obtained as dual spaces by introducing suitable metrizable topologies into $\mathfrak{D}$, $\mathcal{S}$ or $\mathcal{E}$.

In particular (the case $m=0$), we get the following

Corollary. *The set of all measures with compact supports is in $\mathfrak{B}(\mathcal{E}')$, $\mathfrak{B}(\mathcal{S}')$ and $\mathfrak{B}(\mathfrak{D}')$; the set of all slowly increasing measure is in $\mathfrak{B}(\mathcal{S}')$ and $\mathfrak{B}(\mathfrak{D}')$.*

Meanwhile, the topology of $\mathcal{C}\,(=\mathfrak{D}^0)$ being not metrizable, we cannot apply Theorem 1 directly to assure that the space $\mathcal{C}'$ of all measures is in $\mathfrak{B}(\mathfrak{D}')$. We have, however, the following

Proposition 2. *$\mathcal{C}'$ is in $\mathfrak{B}(\mathfrak{D}')$.*

Proof. Let $\alpha_m(x)$ be an element of $\mathfrak{D}$ such that $\alpha_m(x)=1$ when $|x|\leq m$ and $\alpha_m(x)=0$ when $|x|\geq m+1$. And consider the following sequence of linear mappings $\Phi_m: \mathfrak{D}'\to\mathfrak{D}'$ defined by

$$\Phi_m(T) = \alpha_m T,\ T\in\mathfrak{D}' \quad (\text{multiplication by } \alpha_m).$$

These mappings being the transposes of the continous mappings $\Phi_m: \mathfrak{D}\to\mathfrak{D}$ and $\mathcal{E}'^0$ being in $\mathfrak{B}(\mathfrak{D}')$ we have that $\Phi_m^{-1}(\mathcal{E}'^0)\in\mathfrak{B}(\mathfrak{D}')$ by

10) The weak measurability of L^1 will be established later (see Proposition 9).

Theorem 3. But since it is easy to see that $\mathcal{C}'=\bigcap_{m=1}^{\infty}\Phi_m^{-1}(\mathcal{E}'^0)$, we get that $\mathcal{C}'\in\mathfrak{B}(\mathfrak{D}')$. Specifically $\mathcal{C}'$ is in $\mathfrak{F}_{\delta\sigma\delta}$.

As for the weak measurability of sets of positive measures we have

PROPOSITION 3. *The set of all positive measures is in* $\mathfrak{B}(\mathfrak{D}')$. *The set of all positive slowly increasing measures is in* $\mathfrak{B}(\mathcal{S}')$ *and* $\mathfrak{B}(\mathfrak{D}')$. *The set of all positive measures with compact supports is in* $\mathfrak{B}(\mathcal{E}')$, $\mathfrak{B}(\mathcal{S}')$ *and* $\mathfrak{B}(\mathfrak{D}')$.

PROOF. Since it is easy to see that the set $\mathfrak{D}_+$ of positive elements of $\mathfrak{D}$ is separable and since a positive distribution is a positive measure (see [12] I, p. 28), we have that the set $\mathcal{C}'_+$ of all positive measures can be expressed in the form

$$\mathcal{C}'_+ = \bigcap_{n=1}^{\infty} S_{-\varphi_n,0}\,,$$

where $\{\varphi_n\}$ is a countable dense set in $\mathfrak{D}_+$. This shows that $\mathcal{C}'_m$ is in $\mathfrak{B}(\mathfrak{D}')$. The remaining things also can be proven by analoguous arguments or by Theorem 3.

PROPOSITION 4. $\mathfrak{D}$ *is in* $\mathfrak{B}(\mathcal{E}')$, $\mathfrak{B}(\mathcal{S}')$ *and* $\mathfrak{B}(\mathfrak{D}')$. $\mathcal{S}$ *is in* $\mathfrak{B}(\mathcal{S}')$ *and* $\mathfrak{B}(\mathfrak{D}')$. $\mathcal{E}$ *is in* $\mathfrak{B}(\mathfrak{D}')$.

PROOF. By Corollary to Theorem 3, it is enough to prove that $\mathfrak{D}$, $\mathcal{S}$, $\mathcal{E}$ are in $\mathfrak{B}(\mathfrak{D}')$. Let $\{D_k\}$ $(k=1, 2, 3, \cdots)$ be the set of all partial differentiations. D_k being weakly measurable (Theorem 3) and L^2 being in $\mathfrak{B}(\mathfrak{D}')$, we have that $\mathfrak{D}_{L^2}=\bigcap_{k=1}^{\infty}D_k^{-1}(L^2)$ is in $\mathfrak{B}(\mathfrak{D}')$. Here, $\mathfrak{D}_{L^2}$ is the space of those distributions whose derivative of any order is in L^2. By Sobolev's theorem (see [12] II, §6) we know that $\mathfrak{D}_{L^2}$ consist of indefinitely continuously differentiable functions. Therefore we have that $\mathfrak{D}=\mathcal{E}'\cap\mathfrak{D}_{L^2}$ is in $\mathfrak{B}(\mathfrak{D}')$.

Next, let Φ_m be the mappings defined in the proof of Proposition 2. Then we get that $\mathcal{E}=\bigcap_{m=1}^{\infty}\Phi_m^{-1}(\mathfrak{D})\in\mathfrak{B}(\mathfrak{D}')$.

Finally consider the mappings $\Psi_{m,k}:\mathfrak{D}'\rightarrow\mathfrak{D}'$ defined by

$$\Psi_{m,k}(T) = (1+|x|^2)^m D_k T\,, \qquad T\in\mathfrak{D}'\,.$$

$\Psi_{m,k}$ being weakly measurable (Theorem 3), we have that $\bigcap_{m,k}\Psi_{m,k}^{-1}(L^\infty)$

$\in \mathfrak{B}(\mathfrak{D}')$. Thus we have $\mathcal{S}=\mathcal{E}\cap\bigcap_{m,k}\Psi_{m,k}^{-1}(L^{\infty})\in\mathfrak{B}(\mathfrak{D}')$.

As an application of Theorem 2, let us prove the following

PROPOSITION 5. *For a* (fixed) *positive number* α, *let* H_{α} *be the space of those continuous functions such that*

$$\sup_{\substack{|x|\leq k\\|y|\leq k}}\frac{|f(x)-f(y)|}{|x-y|^{\alpha}}<+\infty \tag{7}$$

for any positive integer k. (The value of the left hand side may depend on f and k.) *Then* H_{α} *is in* $\mathfrak{B}(\mathfrak{D}')$.

PROOF. First let us consider the space $\tilde{H}_{\alpha}$ of those continuous functions which satisfy the folltwing conditions:

1°) There exists a positive number A (which may depend on f) such that

$$|f(x)-f(y)|\leq A|x-y|^{\alpha}\qquad\text{for all } x,\ y\in\boldsymbol{R}^{n}$$

2°) There exists a positive number B (which may depend on f) such that

$$|f(x)|\leq B\qquad\text{for all } x\in\boldsymbol{R}^{n}$$

We shall show thet $\tilde{H}_{\alpha}$ is in $\mathfrak{B}(\mathfrak{D}')$.

For positive integers m and n, denote by $K_{m,n}$ the set of those elements of $\tilde{H}_{\alpha}$ which satisfy the inequalities:

$$|f(x)-f(y)|\leq m|x-y|^{\alpha}, \tag{8}$$

$$|f(x)|\leq n. \tag{9}$$

It is clear that

$$\tilde{H}_{\alpha}=\bigcup_{m,n}K_{m,n}.$$

Therefore, according to Theorem 2, it is enough to prove that each $K_{m,n}$ is a $\sigma(\mathfrak{D}',\mathfrak{D})$-compact. To do this, introduce in the space C of all continuous functions on $\boldsymbol{R}^{n}$ the topology $\mathcal{T}_{c}$ of uniform convergence on every compact set in $\boldsymbol{R}^{n}$. Since each $K_{m,n}$ is an equi-continuous and uniformly bounded family of functions by (8) and (9), $K_{m,n}$ is a relatively compact set in C with respect to $\mathcal{T}_{c}$ according to Ascoli-Arzela's Theorem ([1] p. 43). Moreover it is easy to see that $K_{m,n}$ is $\mathcal{T}_{c}$-closed. Hence $K_{m,n}$ is $\mathcal{T}_{c}$-compact.

But since the induced topology of $\sigma(\mathfrak{D}', \mathfrak{D})$ in C is weaker than $\mathfrak{T}_c$, $K_{m,n}$ is $\sigma(\mathfrak{D}', \mathfrak{D})$-compact. Thus we get the weak measurability of $\tilde{H}_\alpha$.

Now consider the mappings defined in the proof of Proposition 2. It is clear that

$$H_\alpha = \bigcap_{m=1}^{\infty} \Phi_m^{-1}(\tilde{H}_\alpha).$$

Thus we get that H_α is in $\mathfrak{B}(\mathfrak{D}')$. This completes the proof.

COROLLARY. *The space of slowly increasing continuous functions which satisfy* (7) *is in* $\mathfrak{B}(\mathcal{S}')$.

PROOF. This is clear from the proposition above and Corollary to Theorem 3 in the proceeding paragraph.

For the proof of the weak measurability of the space of derivatives of functions we require the following

LEMMA. *Let D be an elliptic*[11] *differential operator with constant coefficients. Then we have that:*

1°) *Every solution $U \in \mathfrak{D}'$ of the equation*

$$DU = 0$$

in an open set $\Omega \subseteq \boldsymbol{R}^n$ is real-analytic in Ω.

2°) *$D: \mathfrak{D}' \to \mathfrak{D}'$ is an onto mapping,* i.e.

$$D\mathfrak{D}' = \mathfrak{D}'.$$

3°) *$D: \mathcal{S}' \to \mathcal{S}'$ is an onto mapping,* i.e.

$$D\mathcal{S}' = \mathcal{S}'.$$

1°) is classical (see [11] and [12]). 2°) and 3°) are true even for any differential operators with constant coefficients. (For the proof of 2°), see [5] or [10]. For 3°), see [7].)

PROPOSITION 6. *Let H_α be the space of functions is the preceeding proposition and let D be an elliptic*[11] *differential operator with constant coefficients. Then the image DH_α of H_α by D is in $\mathfrak{B}(\mathfrak{D}')$.*

PROOF. Since the image of a measurable set by an measurable mapping is not necessarily measurable, we should start again with

11) When $n=1$, any differential operator (with constant coefficients) is elliptic.

$\tilde{H}_\alpha$ defined in the proof of the preceeding proposition. We know that

$$\tilde{H}_\alpha = \bigcup_{\mu,\nu} K_{\mu,\nu}$$

with $K_{\mu,\nu}$ weakly compact discs.

Let Φ_m be the mapping defined in the proof of Proposition 2. Since the operators D and Φ_m are weakly continuous, $\Phi_m(DK_{\mu,\nu})$ is weakly compact. Hence, by Theorem 2, we have that $\Phi_m(D\tilde{H}_\alpha) = \bigcup_{\mu,\nu} \Phi_m(DK_{\mu,\nu})$ is in $\mathfrak{B}(\mathfrak{D}')$. Therefore, for the proof, it is enough to show that

$$DH_\alpha = \bigcap_{m=1}^{\infty} \Phi_m^{-1}(\Phi_m(D\tilde{H}_\alpha)).$$

Let us notice that

$$\Phi_m(D\tilde{H}_\alpha) = \Phi_m(DH_\alpha).$$

In fact we have that

$$\Phi_m(D\tilde{H}_\alpha) \subseteq \Phi_m(DH_\alpha) = \Phi_m(D\Phi_{m+2}(H_\alpha)) \subseteq \Phi_m(D\tilde{H}_\alpha).$$

Thus the equality should hold everywhere. Therefore we are to show that

$$DH_\alpha = \bigcap_{m=1}^{\infty} \Phi_m^{-1}(\Phi_m(DH_\alpha)).$$

The relation that $DH_\alpha \subseteq \bigcap_{m=1}^{\infty} \Phi_m^{-1}(\Phi_m(DH_\alpha))$ is clear. Thus we are only to show that

$$DH_\alpha \supseteq \bigcap_{m=1}^{\infty} \Phi_m^{-1}(\Phi_m(DH_\alpha)).$$

Let T be any element in $\bigcap_{m=1}^{\infty} \Phi_m(\Phi_m(DH_\alpha))$. Then, for every m, there exists an element $f_m \in H_\alpha$ such that

(10) $$T = Df_m$$

holds in the open ball $B_m = \{x\,;\,|x| < m\}$. Meanwhile, by 2°) of the lemma above, we have an expression of the form

(11) $$T = DS$$

with S a distribution. Thus it is enough to show that $S \in H_\alpha$. From (10) and (11) we have that

$$D(S - f_m) = 0 \quad \text{in } B_m.$$

Therefore, by 1°) of the lemma, $S-f_m$ is analytic in B_m. Thus we get that

$$S=(S-f_m)+f_m$$

is Hörder continuous of order α in B_m. Since $\boldsymbol{R}^n=\bigcup_{m=1}^{\infty}B_m$, we have that

$$S\in H_\alpha .$$

This completes the proof.

PROPOSITION 7. *Using the notations in the above proposition we have that $D(H_\alpha\cap\mathcal{S}')$ is in $\mathfrak{B}(\mathcal{S}')$.*

PROOF. Since we have already that DH_α and $\mathcal{S}'$ are in $\mathfrak{B}(\mathfrak{D}')$, $DH_\alpha\cap\mathcal{S}'$ is in $\mathfrak{B}(\mathcal{S}')$ according to Corollary to Theorem 3. It is therefore enough to prove that

$$D(H_\alpha\cap\mathcal{S}')=DH_\alpha\cap\mathcal{S}' .$$

But since the relation that $D(H_\alpha\cap\mathcal{S}')\subseteq DH_\alpha\cap\mathcal{S}'$ is clear, we have only to show that

$$D(H_\alpha\cap\mathcal{S}')\supseteq DH_\alpha\cap\mathcal{S}' .$$

Let T be an element on $DH_\alpha\cap\mathcal{S}'$. Then T can be expressed in the forms

$$T=Df,\qquad f\in H_\alpha, \tag{12}$$

and

$$T=DS,\qquad S\in\mathcal{S}' \tag{13}$$

by 3°) of the lemma. From (12) and (13) we get that

$$D(S-f)=0 .$$

Since $S-f$ is an analytic function by 1°) of the lemma, we have that

$$S=(S-f)+f\in H_\alpha .$$

This shows that $T=DS\in D(H_\alpha\cap\mathcal{S}')$. This completes the proof.

PROPOSITION 8. *The space C of all continuous functions on $\boldsymbol{R}^n$ is in $\mathfrak{B}(\mathfrak{D}')$.*

Proof. Let $\tilde{C}$ be the totality of those functions which are bounded and uniformly continuous on the whole $\boldsymbol{R}^n$. We first prove that $\tilde{C}$ is in $\mathfrak{B}(\mathfrak{D}')$. For each positive integer m let β_m be an element of $\mathfrak{D}(\boldsymbol{R}^n)$ such that $\beta_m(x)=1$ if $|x|\leq\frac{1}{m+1}$, $\beta_m(x)=0$ if $|x|\geq\frac{1}{m}$ and $0\leq\beta_m(x)\leq 1$ everywhere. Putting $\gamma_m(x, y)=\beta_m(x-y)$, we define a sequence of mappings $\Psi_m: \mathfrak{D}'(\boldsymbol{R}^{2n})\to\mathfrak{D}'(\boldsymbol{R}^{2n})$ by

$$\Psi_m(S) = \gamma_m\cdot S \text{ (multiplication by } \gamma_m), \qquad S\in\mathfrak{D}'(\boldsymbol{R}^{2n}).$$

And we define another mapping $\Theta: \mathfrak{D}'(\boldsymbol{R}^n)\to\mathfrak{D}'(\boldsymbol{R}^{2n})$ by

$$\Theta(T) = T\otimes 1-1\otimes T, \qquad T\in\mathfrak{D}'(\boldsymbol{R}^n),$$

where 1 denotes the function on $\boldsymbol{R}^n$ with value identically equal to 1, and $\otimes$ stands for the tensor product (or direct product). Ψ_m and Θ are weakly measurable according to Theorem 3.

Now let B be the closed unit ball in $L^\infty(\boldsymbol{R}^{2n})$. Since B is a $\sigma(L^\infty, L^1)$-compact disc (Banach's theorem)[12] and since $\sigma(L^\infty, L^1)$ is stronger than the induced topology of $\sigma(\mathfrak{D}', \mathfrak{D})$, B is a $\sigma(\mathfrak{D}', \mathfrak{D})$-compact disc. Therefore we have that B is in $\mathfrak{B}(\mathfrak{D}'(\boldsymbol{R}^{2n}))$ (see the proof of Theorem 2). Let us prove the following relation:

$$\tilde{C} = L^\infty(\boldsymbol{R}^n)\cap\left\{\bigcap_{n=1}^{\infty}\bigcup_{m=1}^{\infty}\Theta^{-1}\left(\Psi_m^{-1}\left(\frac{1}{n}B\right)\right)\right\}. \tag{14}$$

Since we know already that L^∞ is in $\mathfrak{B}(\mathfrak{D}')$, this relation proves that $\tilde{C}$ is in $\mathfrak{B}(\mathfrak{D}')$. Now let f be an element of $\tilde{C}$. Since f is uniformly continuous, for any n there exists an m such that

$$|x-y|\leq\frac{1}{m} \text{ implies } |f(x)-f(y)|\leq\frac{1}{n}.$$

Since $\beta_m(x-y)$ vanishes when $|x-y|\geq\frac{1}{m}$, we have that

$$\beta_m(x-y)\cdot|f(x)-f(y)|\leq\frac{1}{n} \quad \text{for all } (x, y)\in\boldsymbol{R}^{2n}.$$

Or, equivalently

$$\gamma_m(f\otimes 1-1\otimes f)\in\frac{1}{n}B.$$

12) The closed unit ball of the dual space of a Banach dual space is weakly compact.

Since n is arbitrary and since $\tilde{C}$ consists of bounded functions, this shows that

$$\tilde{C} \subseteq L^{\infty}(\boldsymbol{R}^n) \cap \left\{ \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \Theta^{-1}\left(\Psi_m^{-1}\left(\frac{1}{n} B\right)\right)\right\}.$$

Conversely, let f be an element of the right hand side. Then, for any n there exist an m such that

$$\gamma_m(f \otimes 1 - 1 \otimes f) \in \frac{1}{n} B.$$

Since f is a function in $L^{\infty}(\boldsymbol{R}^n)$, writing m instead of $m+1$ we get that

$$(15) \qquad |x-y| \leq \frac{1}{m} \quad \text{implies} \quad |f(x)-f(y)| \leq \frac{1}{n}$$

for almost all $(x, y) \in \boldsymbol{R}^{2n}$. Therefore the only remaining thing is to prove that f becomes continuous after a correction of its value on a set of measure zero in $\boldsymbol{R}^n$. From (15) we get that for any m and n there exist a set $M_{m,n}$ whose complement is of measure zero and for each $x \in M_{m,n}$ there corresponds a set $N_{m,n}^x$ whose complement is also of measure zero such that for any $x \in M_{m,n}$ and $y \in N_{m,n}^x$

$$|x-y| \leq \frac{1}{m} \quad \text{implies} \quad |f(x)-f(y)| \leq \frac{1}{n}.$$

Putting $M = \bigcap_{m,n} M_{m,n}$ and putting, for $x \in M$, $N^x = \bigcap_{m,n} N_{m,n}^x$, we get that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(16) \quad |x-y| \leq \delta \quad \text{implies } |f(x)-f(y)| \leq \varepsilon \text{ for } x \in M,\ y \in M^x.$$

Here both the complement of M and that of each N^x are of measure zero. Now for any positive integer n, let e_n be an element of $\mathfrak{D}$ such that $e_n(x) \geq 0$ on $\boldsymbol{R}^n$, $e_n(x) = 0$ if $|x| \geq \frac{1}{n}$ and $\int e_n(x)dx = 1$. And put

$$f_n = f * e_n \quad \text{(convolution)}.$$

Then each f_n belongs to $\mathfrak{E}$ and f_n converges to f in $\mathfrak{D}'$ as $n \to \infty$ (see [12] II, p. 23). Meanwhile, we shall prove that $\{f_n\}$ is a Cauchy sequence in the uniform. In fact, when $x \in M$, we have that

$$(17) \qquad |f_n(x)-f_m(x)| \leq |f_n(x)-f(x)|+|f(x)-f_m(x)| \\ \leq \int_{|x|\leq 1/n} |f(x)-f(x-y)|e_n(x)dx + \int_{|x|\leq 1/m} |f(x)-f(x-y)|e_m(x)dx.$$

For any $\varepsilon > 0$, if we take an n_0 large enough, we have that

$$|f(x)-f(x-y)| \leq \varepsilon/2 \quad \text{when} \quad |x-y| \leq \frac{1}{n_0} \quad \text{and} \quad y \in N^x.$$

But since the complement of N^x is of measure zero, from (17) we get that

$$(18) \qquad |f_n(x)-f_m(x)| \leq \varepsilon \qquad \text{for} \quad m, n \geq n_0$$

when $x \in M$. Since M is dense in $\boldsymbol{R}^n$ and $\{f_m\}$ consists of continuous functions, (18) should in fact hold everywhere. Thus $\{f_n\}$ is a Cauchy sequence in the uniform norm and therefore converges uniformly to a continuous function $\tilde{f}$. But since the topology of uniform convergence is stronger than the topology of $\mathfrak{D}'$, f_n converges *à fortiori* to $\tilde{f}$ in $\mathfrak{D}'$. Hence the two limits f and $\tilde{f}$ should coincide in $\mathfrak{D}'$. This means that f and $\tilde{f}$ coincide except on a set of measure zero. Thus we get the relation (14).

Now let Φ_m be the mapping defined in the proof of Proposition 2. Then it is clear that

$$C = \bigcap_{m=1}^{\infty} \Phi_m^{-1}(\tilde{C}),$$

since a continuous function is bounded and uniformly continuous on every compact set. This completes the proof.

COROLLARY. *$\mathfrak{D}^m$, $\mathcal{S}^m$, $\mathcal{E}^m$ are in $\mathfrak{B}(\mathfrak{D}')$. $\mathfrak{D}^m$, $\mathcal{S}^m$, $\mathcal{E}^m \cap \mathcal{S}'$ are in $\mathfrak{B}(\mathcal{S}')$. $\mathfrak{D}^m$ is in $\mathfrak{B}(\mathcal{E}')$.*

PROOF. Let $D_1 \cdots, D_l$ be the partial differentiations of order $\leq m$. They are weakly measurable (Theorem 3). Then we get that $\mathcal{E}^m = C \cap \{\bigcap_{j=1}^{l} D_j^{-1}(C)\} \in \mathfrak{B}(\mathfrak{D}')$. Hence $\mathfrak{D}^m = \mathcal{E}' \cap \mathcal{E}^m \in \mathfrak{B}(\mathfrak{D}')$. Finally, let $\Psi_{k,j} = (1+|x|^2)^k D_j$. Then we get that $\mathcal{S}^m = \mathcal{E}^m \cap \{\bigcap_{k=1}^{\infty} \bigcap_{j=1}^{l} \Psi_{j,k}^{-1}(L^\infty)\} \in \mathfrak{B}(\mathfrak{D}')$. The remaining things are clear from Corollary to Theorem 3.

PROPOSITION 9. *The space L^1 of all summable function on $\boldsymbol{R}^n$ is in $\mathfrak{B}(\mathfrak{D}')$.*

PROOF. Let A be the closed unit ball in $L^1(\boldsymbol{R}^n)$. Since $L^1=\bigcup_{n=1}^{\infty} nA$, it is enough to prove that A is in $\mathfrak{B}(\mathfrak{D}')$. We shall give the proof by dividing it into several steps.

Step. 1. $\mathfrak{D}\cap A$ *is in* $\mathfrak{B}(\mathfrak{D}')$. Let A_0 be the open unit ball of L^1 and B_0 be the open unit ball of L^2.[13] Let us prove first the following

LEMMA. *There exists a sequence of strictly positive functions $\{g_m\}$ in $\mathfrak{E}\cap B_0$ such that the following two conditions on a function f are equivalent.*

(i) *f is in $\mathfrak{D}\cap A_0$.*

(ii) *There exists an m such that f can be written in the form $f=g_m h_m$ with h_m in $\mathfrak{D}\cap B_0$.*

Let P be the totality of those function in $\mathfrak{E}\cap B_0$ which are positive everywhere on $\boldsymbol{R}^n$. Since the space $\mathfrak{E}$ satisfies the second axiom of countability, we can find a countable number of elements $\{g_m\}$ in P which are dense[14] in P. Let us prove that this sequence $\{g_m\}$ has the property stated in the lemma. In fact, if a function f can be written in the form $f=g_m h_m$ with $h_m\in\mathfrak{D}\cap B_0$ for an m, then it is clear that f is in $\mathfrak{D}$ and

$$\|f\|_1\leq\|g_m\|_2\cdot\|h_m\|_2<1$$

by Schwarz' inequality. Thus f is in $\mathfrak{D}\cap A_0$. Conversely, let f be an element of $\mathfrak{D}\cap A_0$. We shall prove that f satisfies (ii) in the lemma. Let θ be an element of $\mathfrak{E}\cap L^1$ which is everywhere positive on $\boldsymbol{R}^n$ (e.g. $\theta(x)=(1+|x|^2)^{-(n+1)/2}$). For $\delta>0$, we put

$$f_\delta(x) = \{|f(x)|^2+\delta^2(\theta(x))^2\}^{1/4}.$$

It is clear that f_δ is an element of $\mathfrak{E}$ and everywhere positive. Furthermore, we see that

13) i.e. $A_0=\{f\,;\|f\|_1<1\}$, $B_0=\{f\,;\|f\|_2<1\}$.

14) Here, we require only that $\{g_m\}$ is dense in P with respect to the topology of compact convergence.

$$||f_\delta||_2^2 = \int \{|f(x)|^2 + \delta^2(\theta(x))^2\}^{1/2} dx$$
$$\leq \int \{|f(x)| + \delta\cdot\theta(x)\} dx = ||f||_1 + \delta||\theta||_1 .$$

Therefore, if we take a sufficiently small δ, we get that $||f_\delta||_2 < 1$ since $||f||_1 < 1$. Thus f_δ is in P. Now, let us denote by K the support of f. Since $f_\delta^2 > |f|$ and K is compact, we have that

$$\rho = \operatorname*{Min}_{x \in K} (f_\delta(x)^2 - |f(x)|) > 0 . \tag{19}$$

But since $\{g_m\}$ is dense in P, we can find an m such that

$$\operatorname*{Max}_{x \in K} |f_\delta(x)^2 - g_m(x)^2| \leq \frac{\rho}{2} . \tag{20}$$

From (19) and (20) we get that

$$g_m(x)^2 - |f(x)| \geq \frac{\rho}{2}$$

when $x \in K$. But since $f(x)=0$ outside of K and $g_m(x) > 0$ everywhere, we get that

$$|f(x)|/g_m(x)^2 < 1 \tag{21}$$

everywhere on $\boldsymbol{R}^n$. Now put $h_m = f/g_m$. Then, it is clear that h_m is in $\mathfrak{D}$. Hence, the only remaining thing is to show that $||h_m||_2 < 1$. Using (21), we get that

$$||h_m||_2^2 = \int |f(x)| \frac{|f(x)|}{g_m(x)^2} dx \leq \int |f(x)| dx = ||f||_1 < 1 .$$

This completes the proof of the lemma.

Now, let $\Gamma_m : \mathfrak{D}' \to \mathfrak{D}'$ be the mapping defined by

$$\Gamma_m(T) = \frac{1}{g_m} T , \qquad T \in \mathfrak{D}' .$$

Γ_m is weakly measurable (Theorem 3). By the lemma, it is clear that

$$\mathfrak{D} \cap A_0 = \bigcup_{m=1}^{\infty} \Gamma_m^{-1}(\mathfrak{D} \cap B_0) . \tag{22}$$

Meanwhile, since B is $\sigma(L^2, L^2)$-compact and the induced topology by $\sigma(\mathfrak{D}', \mathfrak{D})$ is weaker than $\sigma(L^2, L^2)$, B is $\sigma(\mathfrak{D}', \mathfrak{D})$-compact and hence $B \in \mathfrak{B}(\mathfrak{D}')$ (see the proof of Theorem 2). Therefore $B_0 = \bigcup_{n=1}^{\infty}\left(1-\frac{1}{n}\right)B$ is in $\mathfrak{B}(\mathfrak{D}')$. But since we know already that $\mathfrak{D}$ is in $\mathfrak{B}(\mathfrak{D}')$, we see that $\mathfrak{D} \cap B_0$ is in $\mathfrak{B}(\mathfrak{D}')$. Thus (22) shows that $\mathfrak{D} \cap A_0$ is in $\mathfrak{B}(\mathfrak{D}')$. Therefore

$$\mathfrak{D} \cap A = \bigcap_{n=1}^{\infty}\left(1+\frac{1}{n}\right)(\mathfrak{D} \cap A_0) \in \mathfrak{B}(\mathfrak{D}') .$$

Step 2. *$\mathcal{E} \cap A$ is in $\mathfrak{B}(\mathfrak{D}')$.* Let Φ_m be the measurable mapping defined in the proof of Proposition 2 (with $0 \leq \alpha_m(x) \leq 1$ everywhere). Then, we see that

$$\mathcal{E} \cap A = \bigcap_{m=1}^{\infty} \Phi_m^{-1}(\mathfrak{D} \cap A) . \tag{23}$$

In fact, if $f \in \mathcal{E} \cap A$, it is clear that $\Phi_m(f) \in \mathfrak{D} \cap A$ for any m since $\alpha_m \in \mathfrak{D}$ and $0 \leq \alpha_m(x) \leq 1$. Conversely, if a distribution f is in the right hand side of (23), it is clear that $f \in \mathcal{E}$ and for any m

$$\int_{|x| \leq m} |f(x)|\, dx \leq 1 .$$

Letting $m \to \infty$, we get that $||f||_1 = \int |f(x)|\, dx \leq 1$. Thus $f \in \mathcal{E} \cap A$.

Step 3. *A is in $\mathfrak{B}(\mathfrak{D}')$.* Let e_m be an element of $\mathfrak{D}$ such that $e_m(x) \geq 0$ on $\boldsymbol{R}^n$, $e_m(x) = 0$ if $|x| \geq \frac{1}{m}$ and $\int e_m(x)dx = 1$. And let f be in L^1. It is well known that $||f * e_m||_1 \leq ||f||_1$ and $f * e_m$ converges to f in the L^1-norm as $m \to \infty$. Now, let Ψ_m be the mapping $\mathfrak{D}' \to \mathfrak{D}'$ defined by

$$\Psi_m(T) = T * e_m , \qquad T \in \mathfrak{D}' .$$

We shall prove that

$$A = \left\{\bigcap_{m=1}^{\infty} \Psi_m^{-1}(\mathcal{E} \cap A)\right\} \cap \left\{\bigcap_{n=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{\mu,\nu \geq N} (\Psi_\mu - \Psi_\nu)^{-1}\left(\frac{1}{n}(\mathcal{E} \cap A)\right)\right\} . \tag{24}$$

Since Ψ_m and $\Psi_\mu - \Psi_\nu$ are weakly measurable (Theorem 2)[15], this formula proves that A is in $\mathfrak{B}(\mathfrak{D}')$. Now, let f be an element of A. Since, for each m, $f * e_m$ is in $\mathcal{E}$ and $||f * e_m||_1 \leq ||f||_1 \leq 1$, f is in $\bigcap_{m=1}^{\infty} \Psi_m^{-1}(\mathcal{E} \cap A)$. And since $f * e_m$ converges to f in L^1, $\{f * e_m\}$ is

15) See the foot note 8) also.

a Cauchy sequence in L^1. From these we see that f is in the right hand side of (24). Conversely, let T be a distribution in the right hand side of (24). Then, $\|T*e_m\|_1 \leq 1$ and $\{T*e_m\}$ constitutes a Cauchy sequence in L^1. Since L^1 is complete, $T*e_m$ converges to a summable function f in the L^1-norm. But since $\|T*e_m\|_1 \leq 1$ for all m, we get that $\|f\|_1 \leq 1$. Since the topology of $\mathfrak{D}'$ is weaker than that of L^1, $T*e_m$ converges to f in $\mathfrak{D}'$. On the other hand, it is known that $T*e_m$ converges to T in $\mathfrak{D}'$. Hence the two limits f and T should coincide. This means that $T=f\in A$. This completes the proof.

COROLLARY. *L^1 is in $\mathfrak{B}(\mathcal{S}')$. The totality L^1_{loc} of locally summable functions is in $\mathfrak{B}(\mathfrak{D}')$.*

PROOF. The first assertion is clear according to Corollary to Theorem 3. The second follows from the formula: $L^1_{\mathrm{loc}} = \bigcap_{m=1}^{\infty} \Phi_m^{-1}(L^1)$, where Φ_m is the mapping defined in the proof of Proposition 2.

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