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On compactiffications

By

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In the theory of Tychonoff spaces, the existence of (Hausdorff) compactification is of great significance because of the fact that it is a characteristic property of Tychonoff spaces as well as the property that they are uniformizable. Some developments of the theory have been made through utilization of compactifications, as we can see in the recent literatures.

In the present paper, we shall make a systematic treatment of the properties of Tychonoff spaces in connection with the properties of their compactifications, with a view to visualizing those properties in a unificative fashion and establishing a general background for the concepts in the theory of Tychonoff spaces.

§1 is devoted to the preliminary results which will be used in the sequel. In §2, some properties of a Tychonoff space X will be characterized by the properties of its compactification BX. More precisely, we shall characterize some topological properties of X in terms of the properties of X as a dense subspace of BX. In §3, we shall be concerned with the properties of the product $X\times BX$. Suggesting by the author's theorem [27] which states that the paracompactness of X is equivalent to the normality of $X\times\beta X$ (βX: the Stone-Čech compactification), it may be expected that some topological properties of X can be characterized by simple properties of $X\times BX$. Our main results in the present paper are concerned with this subject, and we shall characterize a number of topological properties of X by modifications of the normality proposed on the product $X\times BX$. For example, a collectionwise normal space X will be characterized by the property that $F\times\beta X$ is normally embedded in $X\times\beta X$ for any closed subspace F of X. It will be shown that a space X is second countable if and only if $X\times BX$ is perfectly normal for some compactification BX of X. On the other while, Dowker's results [5, Lemma 3 and Theorem 4] can be stated as follows: X is normal and countably paracompact if and only if $X\times M$ is normal for any compact metrizable space M (Theorem 3.13). This suggests the possibility of characterizing some properties of a Tychonoff space X in terms of the properties of the product of X with some compact metrizable space. Some related results on this subject will be given in the last part of § 3. In § 4, we shall show that Michael's problem on the paracompactness of a metrizable space and a paracompact space can be reduced to ask whether the product of a metrizable space and a paracompact space is normal, and discuss some related problems.

§ 1. Preliminary.

All spaces mentioned in this paper are Tychonoff spaces unless the topology is explicitly represented. A compactification BX of a space X is a compact (Hausdorff) space containing X as a dense subspace. The Stone-Čech compactification βX is characterized among compactifications of X by the property that every bounded continuous function on X has a continuous extension over βX. It is the largest compactification of X in the sense that each compactification BX of X is a continuous image of βX, as the following theorem shows.

Theorem 1.1. *Any compactification BX of X is the image of βX under a unique continuous mapping φ that keeps X pointwise fixed and that $\varphi(\beta X-X)=BX-X$.*

For the proof, see [3, P. 831].

Theorem 1.2. *If f is any continuous mapping of a space X into a compact space Y, then f has a continuous extension f^* over βX, which carries βX into Y.*

The proof is in [18, P. 153]. (C. f. [25, P. 476].)

Let $C(X)$ denote the set of all continuous functions (real-valued) on X and let $C^*(X)$ denote the set of all bounded continuous functions on X. A continuous function $f \in C(X)$ defines a continuous mapping of X into R^*, where R^* denotes the one point compactification of the real number space R, and it has a continuous extension f^* over βX by virtue of the preceding theorem. The set $\{x \in \beta X : f^*(x) \in R\}$ will be denoted by X_f.

As we shall be concerned with the properties of dense subspace, we now state some propositions on dense subspace. Let X be a dense subspace of Y and let A be a subset of X. We shall denote by $Cl_X(A)$ ($Cl_Y(A)$) the closure of A taken in X (resp. in Y). Similarly, the interior of A will be denoted by $Int_X(A)$ or $Int_Y(A)$ according as it is taken in X or in Y. A subset U of X is said to be *regularly open* if $Int_X(Cl_X(U)) = U$. Let U be an open subset of X and let U^* be an open subset of Y such that $U = U^* \cap X$, then we shall say that U^* is an extension of U over Y. Put $U^{\varepsilon(Y)} = Y - Cl_Y(X - U)$, then $U^{\varepsilon(Y)}$ is an extension of U over Y. In fact, $U^{\varepsilon(Y)} \cap X = (Y - Cl_Y(X - U)) \cap X = X - Cl_Y(X - U) = X - (X - U) = U$. We call the set $U^{\varepsilon(Y)}$ the *proper extension* of U over Y. It is evident that $U \subset V$ implies $U^{\varepsilon(Y)} \subset V^{\varepsilon(Y)}$ in view of the definition of the proper extension.

Proposition 1.1. *Let X be a dense subspace of Y and let U be an open subset of X. Then, the proper extension $U^{\varepsilon(Y)}$ of U over Y is the largest extension of U over Y.*

Proof. Let U^* be any extension of U over Y. If $y \notin U^{\varepsilon(Y)}$, then $y \in Cl_Y(X - U)$. Therefore $0^*(y) \cap (X - U) \neq \emptyset$ and hence $0^*(y) \cap X \not\subset U^* \cap X$ for each open subset $0^*(y)$ of Y containing y, and it follows that $y \notin U^*$. Therefore we have $U^* \subset U^{\varepsilon(Y)}$. If U^* is the largest extension of U, then the reversed inclusion holds and we have $U^* = U^{\varepsilon(Y)}$.

Proposition 1.2. *Let X be a dense subspace of Y and let $U^{\varepsilon(Y)}$ be the proper extension of U over Y. Then $U^{\varepsilon(Y)}$ is regulary open if and only if U is regularly open, and we have $U^{\varepsilon(Y)} = Int_Y(Cl_Y(U))$ if U is regularly open.*

Proof. By an easy calculation, we see that $Int_X(Cl_X(A))=Int_Y(Cl_Y(A))\cap X$ for any subset A of X if X is dense in Y. If $U^{\varepsilon(Y)}$ is regularly open, then we have $U=U^{\varepsilon(Y)}\cap X=Int_Y(Cl_Y(U^{\varepsilon(Y)}))\cap X=Int_Y(Cl_Y(U^{\varepsilon(Y)}\cap X))\cap X=Int_X(Cl_X(U^{\varepsilon(Y)}\cap X))=Int_X(Cl_X(U))$. This shows that U is regularly open. Suppose conversely that U is a regularly open subset of X, then $U=Int_X(Cl_X(U))=Int_Y(Cl_Y(U))\cap X$ and hence $U^*=Int_Y(Cl_Y(U))$ is an extension of U over Y, which is clearly regularly open. Hence $U^*\subset U^{\varepsilon(Y)}$ by Proposition 1.1. We shall show that $U^*\supset U^{\varepsilon(Y)}$ which will complete the proof. Suppose on the contrary that $U^*\not\supset U^{\varepsilon(Y)}$, then $U^{\varepsilon(Y)}\not\subset Cl_Y(U^*)$ and since X is dense in Y we have $[U^{\varepsilon(Y)}\cap(Y-Cl_Y(U^*))]\cap X\neq\emptyset$. It follows that $U=U^{\varepsilon(Y)}\cap X\not\subset Cl_Y(U^*)$, which is contradictory. Therefore we have $U^*=U^{\varepsilon(Y)}$.

Proposition 1.3. *Let Y be a dense subspace of Z and let X be a dense subspace of Y. Let U, V denote any open subsets of X and Y respectively. Then the followings are valid.*

(1) $U^{\varepsilon(Z)}\cap Y=U^{\varepsilon(Y)}$.
(2) $[U^{\varepsilon(Y)}]^{\varepsilon(Z)}=U^{\varepsilon(Z)}$.
(3) $V^{\varepsilon(Z)}\subset[V\cap X]^{\varepsilon(Z)}$. If V is regularly open, then $V^{\varepsilon(Z)}=[V\cap X]^{\varepsilon(Z)}$.

Proof. (1) and (2) are evident by the following calculations. (1) $(Z-Cl_Z(X-U))\cap Y=Y-(Cl_Z(X-U)\cap Y)=Y-Cl_Y(X-U)=U^{\varepsilon(Y)}$. (2) $Z-Cl_Z(Y-U^{\varepsilon(Y)})=Z-Cl_Z[Y-(Y-Cl_Y(X-U))]=Z-Cl_Z(X-U)=U^{\varepsilon(Z)}$. To prove (3), note that V is an extension of $V\cap X$. By Proposition 1.1, we have $V\subset(V\cap X)^{\varepsilon(Y)}$, and hence $V^{\varepsilon(Z)}\subset(V\cap X)^{\varepsilon(Z)}$ by (2). If V is regularly open, then $V\cap X$ is also regularly open by virtue of the formula; $Int_X(Cl_X(A))=Int_Y(Cl_Y(A))\cap X$. Therefore $V=(V\cap X)^{\varepsilon(Y)}$ by Proposition 1.2, and hence we have $V^{\varepsilon(Z)}=(V\cap X)^{\varepsilon(Z)}$ by (2).

We now consider some properties of surroundings for X. A neighborhood V of the diagonal of $X\times X$ is said to be a *surrounding for X* if there exists a sequence of neighborhoods of the diagonal $\{V_n\}$ such that $V=V_1$, $V_n\circ V_n\subset V_{n-1}$*) for each n.

*) $V\circ V=\{(x, y)\in X\times X;\ (x, z)\in V \text{ and } (z, y)\in V \text{ for some } z\in X\}$.

Throughout the sequel, neighborhoods are assumed to be open. We call a surrounding V *stable* if there is a sequence $\{V_n\}$ such that $V_n^{\varepsilon(\beta X\times\beta X)}\cap\Delta(\beta X)=V^{\varepsilon(\beta X\times\beta X)}\cap\Delta(\beta X)$ for each n, where $\Delta(\beta X)$ denotes the diagonal of $\beta X\times\beta X$. In the following, we shall denote by $Z(f)$ the zero-set of $f\in C(X)$: $Z(f)=\{x\in X;\ f(x)=0\}$ and by $0(f)$ the complementary set of $Z(f)$. A partition of unity on a space X is a family $\Phi=\{\varphi_\lambda;\ \sum\varphi_\lambda=1\}$ of continuous function on X such that $0\leq\varphi_\lambda(x)\leq 1$, $\sum\varphi_\lambda(x)=1$ for each $x\in X$ and all but a finite number of members of Φ vanish outside some neighborhood of x for each $x\in X$. It is clear that $\{0(\varphi_\lambda)\}$ is a locally finite covering of X. If $\{0(\varphi_\lambda)\}$ is star-finite, then we shall say that Φ is a star-finite partition of unity on X. A partition of unity $\Phi=\{\varphi_\lambda;\ \sum\varphi_\lambda=1\}$ is *subordinate* to a covering $\{U_\alpha\}$ if each $0(\varphi_\lambda)$ is contained in U_α for some U_α. By virtue of the theorem of Dieudonné [4], for every point finite covering of a normal space there is a partition of unity subordinate to the covering (with the same index set). It will be shown that a star-finite partition of unity determines a stable surrounding.

Proposition 1.4. *The following conditions are equivalent.*

(1) *V is a stable surrounding for X.*
(2) *$Pr_{\beta X}[V^{\varepsilon(\beta X\times\beta X)}\cap\Delta(\beta X)]=X^*$ is a paracompact subspace of βX, where $Pr_{\beta X}$ denotes the projection of $\beta X\times\beta X$ onto βX.*
(3) *There is a star-finite partition of unity $\Phi=\{\varphi_\lambda:\ \sum\varphi_\lambda=1\}$ on X such that $W=\{(x,y)\in X\times X:\ \sum|\varphi_\lambda(x)-\varphi_\lambda(y)|<1\}\subset V$ and $Pr_{\beta X}[W^{\varepsilon(\beta X\times\beta X)}\cap\Delta(\beta X)]=Pr_{\beta X}[V^{\varepsilon(\beta X\times\beta X)}\cap\Delta(\beta X)]$.*

Proof. For the sake of convenience, we shall denote by V^ε the proper extension of V over $\beta X\times\beta X$: $V^\varepsilon=V^{\varepsilon(\beta X\times\beta X)}$. Suppose that V is a stable surrounding for X and let $\{V_n\}$ be a sequence of surroundings for X such that $V_1=V$, $V_n\circ V_n\subset V_{n-1}$ and $V_n^\varepsilon\cap\Delta(\beta X)=V^\varepsilon\cap\Delta(\beta X)$ for each n. It is easy to see that $(V_n^\varepsilon\circ V_n^\varepsilon)\cap(X\times X)=V_n\circ V_n$, and we have $V_n^\varepsilon\circ V_n^\varepsilon\subset(V_n\circ V_n)^\varepsilon$ in view of Proposition 1.1. It follows that $V_n^\varepsilon\circ V_n^\varepsilon\subset V_{n-1}^\varepsilon$ for each n. If we put $V_n^*=V_n^\varepsilon\cap(X^*\times X^*)$, then $\{V_n^*\}$ is a sequence of neighborhoods of the diagonal $\Delta(X^*)$ of $X^*\times X^*$ such that $V_n^*\circ V_n^*\subset V_{n-1}^*$ for each

n. Let d be the pseudo-metric on X^* defined by the family $\{V_n^*\}$ such that $\{(x, y) \in X^* \times X^* : d(x, y) < 1\} \subset V_1^*$, and let τ denote the topology of X^* induced by the pseudo-metric d, then the space (X^*, τ) is a paracompact space by virtue of the theorem due to A. H. Stone [24]. Put $U(x) = \{y \in X^* : d(x, y) < 1/2^2\}$, then $Cl_{\beta X}(U(x)) \subset X^*$ as we now verify: Let r be a point of $Cl_{\beta X}(U(x))$, and let d_x denote the restriction of $d(x, y)$ on $\{x\} \times X^*$. By virtue of Theorem 1.1, d_x has a continuous extension d_x^* over βX. Evidently, $d_x^*(r) \leq 1/2^2 < 1/2$ and hence there is a neighborhood (in βX) $W(r)$ of r such that $d_x^*(y) = d(x, y) < 1/2$ for each $y \in W(r) \cap X^*$. It is clear that $(W(r) \cap X^*) \times (W(r) \cap X^*) \subset V_2' \circ V_2' \subset V_1^* \subset V^\varepsilon$; where $V_2' = \{(x, y) \in X^* \times X^* : d(x, y) < 1/2\}$, and we have $W(r) \times W(r) \subset V^\varepsilon$ by Proposition 1.1. It follows that $r \in X^*$, and consequently $Cl_{\beta X}(U(x)) \subset X^*$ for each $x \in X$. Now, let us consider an open covering $\{U(x)\}_{x \in X^*}$ of (X^*, τ) and let $\{U_\lambda\}$ be a locally finite open refinement of $\{U(x)\}_{x \in X^*}$. Since τ is weaker than the original topology (induced topology of βX on X^*) of X^*, $\{U_\lambda\}$ is a locally finite open covering of X^* with respect to the original topology of X^*, and it is evident that $Cl_{\beta X}(U_\lambda) \subset X^*$. To prove the paracompactness of X^* let $\{G_\alpha\}$ be any open covering of X^*. Then, each $Cl_{\beta X}(U_\lambda)$ is covered by a finite number of G_α's, say $G_1, \cdots, G_m$. Put $H_{\lambda, k} = U_\lambda \cap G_k$ $(1 \leq k \leq m)$, and construct a finite collection of open subsets $H_{\lambda, k}$ for each λ in this fashion. Then, the family $\{H_{\lambda, k}\}$ is an open locally finite refinement of $\{G_\alpha\}$ as may easily be seen. It follows that X^* is paracompact. This proves the implication (1) $\Rightarrow$ (2).

Nextly, we prove the implication (2) $\Rightarrow$ (3). Assume that X^* is paracompact, then it is a topological sum of σ-compact spaces, since X^* is open in βX, and therefore each open covering of X^* has an open star-finite refinement. (Note that X^* is a locally compact paracompact space, in this case.) Since X^* is paracompact, $V^* = V^{\varepsilon(\beta X \times \beta X)} \cap (X^* \times X^*)$ is a surrounding for X^* and there is a sequence $\{V_n^*\}$ of neighborhoods of the diagonal $\Delta(X^*)$ of $X^* \times X^*$ such that $V_1^* = V^*$, $V_n^* \circ V_n^* \subset V_{n-1}^*$ for each n. Consider an open covering $\{V_2^*(x)\}_{x \in X^*}$, where $V_2^*(x) = \{y \in X^* : (x, y) \in$

$V_2^*\}$, and let $\Phi^*=\{\varphi_\lambda^*:\sum\varphi_\lambda^*=1\}$ be a star-finite partition of unity on X^* which is subordinate*) to the covering $\{V_2^*(x)\}_{x\in X^*}$. Since $0(\varphi_\lambda)\subset V_2^*(x)$ for some $x\in X^*$, the union of all $0(\varphi_\mu)$ for which $0(\varphi_\mu)\cap 0(\varphi_\lambda)\neq\emptyset$ is contained in some $V_1^*(x)$. It follows that $W^*=\{(x, y)\in X^*\times X^*: \sum|\varphi_\lambda^*(x)-\varphi_\lambda^*(y)|<1\}$ is contained in V^*. Let φ_λ denote the restriction of φ_λ^* on X, then $\Phi=\{\varphi_\lambda:\sum\varphi_\lambda=1\}$ is obviously a desired one. In fact, $W^*\subset V^*$ implies that $W=\{(x, y)\in X\times X: \sum|\varphi_\lambda(x)-\varphi_\lambda(y)|<1\}\subset V$ and it follows that $W^{\varepsilon(\beta X\times\beta X)}\subset V^{\varepsilon(\beta X\times\beta X)}$. Therefore $\Delta(X^*)\subset W^*\cap\Delta(\beta X)\subset W^{\varepsilon(\beta X\times\beta X)}\cap\Delta(\beta X)\subset V^{\varepsilon(\beta X\times\beta X)}\cap\Delta(\beta X)=\Delta(X^*)$, and hence we have $Pr_{\beta X}[W^{\varepsilon(\beta X\times\beta X)}\cap\Delta(\beta X)]=Pr_{\beta X}[V^{\varepsilon(\beta X\times\beta X)}\cap\Delta(\beta X)]$.

Finally, we prove that (3) implies (1). Let $\Phi=\{\varphi_\lambda:\sum\varphi_\lambda=1\}$ be a star-finite partition of unity on X. Let φ_λ^* denote the extension of φ_λ over βX and put $0(\varphi_\lambda^*)=\{p\in\beta X: \varphi_\lambda^*(p)>0\}$. Since $\{0(\varphi_\lambda)\}$ is star-finite, $\{0(\varphi_\lambda^*)\}$ is also star-finite. Let X^* be the subspace of βX consisting of all points p of βX such that all but a finite number of φ_λ^* vanish outside some neighborhood of p. Obviously X^* is an open subspace of βX containing X, and $\{0(\varphi_\lambda^*)\cap X^*\}$ is a star-finite covering of X^*. Let φ_λ' denote the restriction of φ_λ^* on X^*, then $\Phi'=\{\varphi_\lambda'\}$ is a star-finite partition of unity on X^*. Put $V_n=\{(x, y)\in X\times X: \sum|\varphi_\lambda(x)-\varphi_\lambda(y)|<1/2^n\}$ and put $V_n^*=\{(x', y')\in X^*\times X^*: \sum|\varphi_\lambda'(x')-\varphi_\lambda'(y')|<1/2^n\}$, then it is clear that $V_n^*\cap(X\times X)=V_n$ and we have $V_n^*\cap(X\times X)=V_n\subset V_n^{\varepsilon(\beta X\times\beta X)}\cap(X\times X)=[V_n^{\varepsilon(\beta X\times\beta X)}\cap(X^*\times X^*)]\cap(X\times X)$. Therefore $V_n^{\varepsilon(\beta X\times\beta X)}\cap(X^*\times X^*)\supset V_n^*\supset\Delta(X^*)$ by virtue of Proposition 1.1 and 1.3. Thus, we have $V_n^{\varepsilon(\beta X\times\beta X)}\cap\Delta(\beta X)\supset\Delta(X^*)$ for each n. To prove the reversed inclusion, let (p, q) be a point of $V_n^{\varepsilon(\beta X\times\beta X)}$, then $U^*(p)\times W^*(q)\subset V_n^{\varepsilon(\beta X\times\beta X)}$ for some neighborhood (in βX) $U^*(p)$ and $W^*(q)$ of p and q. Let x be a point of $U^*(p)\cap X$, then we have $\sum|\varphi_\lambda(x)-\varphi_\lambda(y)|<1/2^n$ for each $y\in W^*(q)\cap X$. Let $\varphi_1, \cdots, \varphi_m$ be a finite set of all members of Φ such that $\varphi_k(x)\neq 0$ $(1\leq k\leq m)$, then $y\in\bigcup_{k=1}^m 0(\varphi_k)$ for each $y\in W^*(q)\cap X$ and hence we have $W^*(q)\cap X\subset\bigcup_{k=1}^m 0(\varphi_k)$. If q does not belong to X^*,

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*) A partition of unity Φ is subordinate to a covering $\{U_\alpha\}$ of X if and only if each member of Φ vanishes outside some member of $\{U_\alpha\}$.

then $W^*(q)\cap X$ intersects infinitely many $0(\varphi_\lambda)$'s and hence $\bigcup_{k=1}^m 0(\varphi_k)$ intersects infinitely many $0(\varphi_\lambda)$'s. It follows that $\{0(\varphi_\lambda)\}$ is not star-finite which is contradictory. Therefore we have $q\in X^*$. Similarly, we see that $p\in X^*$ and hence we have $(p, q)\in X^*\times X^*$. Therefore $V_n^{\varepsilon(\beta X\times\beta X)}\subset X^*\times X^*$ and it follows that $V_n^{\varepsilon(\beta X\times\beta X)}\cap\Delta(\beta X)\subset(X^*\times X^*)\cap\Delta(\beta X)=\Delta(X^*)$. Thus, we have $V_n^{\varepsilon(\beta X\times\beta X)}\cap\Delta(\beta X)=\Delta(X^*)=V^{\varepsilon(\beta X\times\beta X)}\cap\Delta(\beta X)$ for each n. The proof is completed.

In view of the above proposition, we see that a surrounding V for X is stable if and only if $X^*=Pr_{\beta X}[V^{\varepsilon(\beta X\times\beta X)}\cap\Delta(\beta X)]$ is a topological sum of σ-compact spaces. Now, let us agree to call V a *strongly stable surrounding for* X if X^* is a σ-compact space. Then, it may easily be seen from the proof of the above proposition that the following proposition hold true.

Proposition 1.5. *The following conditions are equivalent.*
(1) *V is a strongly stable surrounding for X.*
(2) *$X^*=Pr_{\beta X}[V^{\varepsilon(\beta X\times\beta X)}\cap\Delta(\beta X)]$ is a σ-compact subspace of βX,*
(3) *There is a countable star-finite partion of unity $\Phi=\{\varphi_n:\sum\varphi_n=1\}$ on X such that $W=\{(x, y)\in X\times X:\sum|\varphi_n(x)-\varphi_n(y)|<1\}\subset V$ and $Pr_{\beta X}[W^{\varepsilon(\beta X\times\beta X)}\cap\Delta(\beta X)]=Pr_{\beta X}[V^{\varepsilon(\beta X\times\beta X)}\cap\Delta(\beta X)]$.*

It is to be noticed that in a connected space every stable surrounding for X is strongly stable. As is well known, a metrizable space X is characterized by the fact that there exists a countable family of surroundings $\{V_n\}$ for X such that $\bigcap_{n=1}^\infty V_n=\Delta(X)$. Similar characterization of second countable spaces may be obtained in terms of the strongly stable surrounding.

Proposition 1.6. *A space X is second countable if and only if there is a countable family $\{V_n\}$ of strongly stable surroundings for X such that $\bigcap_{n=1}^\infty V_n=\Delta(X)$.*

Proof. Recall that a second countable space is metrizable and is a Lindelöf space (c. f. [18], P. 49, P. 125). Let $d(x, y)$ be a metric on X, and consider a covering $\{V_n(x)\}_{x\in X}$, where $V_n(x)=\{y\in X: d(x, y)<1/2^n\}$. By virtue of the theorem due to K. Morita [19], there is a countable star-finite partition of unity

$\Phi^{(n)}=\{\varphi_k^{(n)}: \sum\varphi_k^{(n)}=1\}$ on V subordinate to the covering. Put $V^{(n)}=\{(x,y)\in X\times X: \sum|\varphi_k^{(n)}(x)-\varphi_k^{(n)}(y)|<1\}$, then $V^{(n)}$ is a strongly stable surrounding for X, for each n. It is easy to see that $\bigcap_{n=1}^{\infty}V^{(n)}=\Delta(X)$, and the necessity of the condition is proved. The converse will be proved is §3. (See, Remark of Theorem 3. 4.)

We finally state a result on the completeness of a uniform space, which will be used in the next section. In [26], the author proved that a uniform space $(X, \{V_\alpha\})$ is complete if and only if $\Delta(X)=\bigcap_\alpha Int_{\beta X\times\beta X}(Cl_{\beta X\times\beta X}(V_\alpha))$. This is equivalent to the following

Theorem 1. 3. *A uniform space* $(X, \{V_\alpha\})$ *is complete if and only if* $\Delta(X)=\bigcap_\alpha V_\alpha^{\varepsilon(\beta X\cdot\beta X)}$.

§2. Characterization of topology (I)

In this section, we shall characterize some topological properties of a space X in terms of the properties of X as a dense subspece of its compactification. E. Čech [3] proved that a space X is normal if and only if $Cl_{\beta X}(F)\cap Cl_{\beta X}(C)=\emptyset$ for each paire of disjoint closed subsets F, G of X. In his paper [15], E. Hewitt introduced the notion of pseudo-compact spaces and proved tnat X is pseudocompact if and only if no closed C_δ-set of βX is contained in $\beta X-X$. Recently, S. Mrówka [22] has given a characterization of Lindelöf spaces, which is similar to that of real compact spaces (Q-spaces in the sense of E. Hewitt [15]). He introduced the notion of Q-closed subset: A set $X\subset S$ is said to be Q-closed in S if and only if for each $p\subset S-X$ there is a continuous function $f\in C(S)$ such that $f(p)=0$ and $f(q)\neq 0$ for each $q\in X$. Then, he proved that X is a Lindelöf space if and only if X is Q-closed in each of its compactification, and that X is real compact if and only if X is Q-closed in βX. G. I. Kac [17] has given a characterization of topological comploteness by the properties of the Stone-Čech compactification. Further, some development of the theory in this direction has been made in [13] and [14] by M. Henriksen and J. R. Isbell. In [26], [27] and [29], some characterizations of paracompactness has been given and used to solve some topological problems. We now state those characterization of topological properties with some new results.

Theorem 2.1. *The following conditions on a space X are equivalent.*

(1) *X is normal.*

(2) *$Cl_{\beta X}(F)\cap Cl_{\beta X}(G)=\emptyset$ for each paire of disjoint closed subsets F, G of X.*

(3) *If $\{U_n\}$ is a finite open overing of X, then $\{U_n^{\varepsilon(\beta X)}\}$ covers βX.*

Proof. It will be suffice to prove the equivalence of (1) and (3). If $\beta X-\bigcup U_n^{\varepsilon(\beta X)}=C\neq\emptyset$, then there is a point $p\in C$ such that $p\in Cl_{\beta X}(U_i)$ for some i, since $\{U_n\}$ is a finite covering and since X is dense in βX. We may assume without loss of generallity that $\{U_n\}$ is a minimal covering: That is, no proper subfamily of $\{U_n\}$ can cover X. Evidently, $p\notin\bigcup U_n^{\varepsilon(\beta X)}$ implies that $p\in Cl_{\beta X}(X-U_j)$ for each j, and hence $p\in Cl_{\beta X}(\bigcap_{j\neq i}(X-U_j))=\bigcap_{j\neq i}Cl_{\beta X}(X-U_j))$. Put $F=\bigcap_{j\neq i}(X-U_j)$ and put $G=X-U_i$, then F and G are disjoint closed subsets of X such that $Cl_{\beta X}(F)\cap Cl_{\beta X}(G)\neq\emptyset$. It follows that X is not normal. Suppose conversely that X is not normal, then there are two disjoint closed subsets F and G of X such that $Cl_{\beta X}(F)\cap Cl_{\beta X}(G)\neq\emptyset$. Put $U_1=X-F$ and put $U_2=X-G$, then $U_1^{\varepsilon(\beta X)}\cup U_2^{\varepsilon(\beta X)}$ doese not cover βX as may easily be seen from the definitiod of the proper extensions.

Theorem 2.2. *A space X is locally compact if and only if $BX-X$ is compact for any compactification BX of X.*

Proof. For each $x\in X$, there is a neighborhood $U(x)$ of x such that $Cl_X(U(x))$ is compact, if X is locally compact. Then, it is clear that no point of $BX-X$ is contained in $Cl_{BX}(U(x))$ and therefore X is open in BX. Hence, $BX-X$ is compact. Conversely, if $BX-X$ is compact, then there is for each $x\in X$ a neighborhood $U(x)$ of x such that $Cl_{BX}(U(x))\cap(BX-X)=\emptyset$. Evidently, $Cl_{BX}(U(x))=Cl_X(U(x))$ is a compact neighborhood of $x\in X$, and X is therefore locally compact.

Theorem 2.3. *X has unique uniform structure if and only if $\beta X-X$ is at most one point.*

Proof. By the theorem due to R. Doss [7], X has unique uniform structure if and only if for each paire of functionally separated closed subsets of X one at least is compact. If F and G are functionally separated subsets of X, then $Cl_{\beta X}(F)\cap Cl_{\beta X}(G)=\emptyset$, and the present theorem is an immediate consequence of the above characterization due to Doss.

A space X is said to be pseudo-compact if every continuous function on X is bounded.

Theorem 2.4. *X is pseudo-compact if and only if $\beta X-X$ contains no closed G_δ-set of βX.*

Proof. If C is a closed G_δ-set of βX, which is contained in $\beta X-X$, then there is a continuous function $f\in C(\beta X)$ such that $C=\boldsymbol{Z}(f)=\{p\in\beta X: f(p)=0\}\subset\beta X-X$. It follows that there is for each n a point $x_n\in X$ such that $|f(x_n)|<1/n$, and $1/|f|$ is clearly an unbounded continuous function on X. Conversely, if $g(x)$ is an unbounded continuous function on X, then $f(x)=1/\max[|g(x)|,1]$ is a continuous function on X having no zero point. Let f^* denote the extension of f over βX, then $\boldsymbol{Z}(f^*)\neq\emptyset$ and $\boldsymbol{Z}(f^*)$ is a closed G_δ-set of βX contained in $\beta X-X$.

A space X is said to be real compact [9] if it is complete relative to the weakest uniformity for X with respect to which every continuous function on X is uniformly continuous.

Theorem 2.5. *The following conditions on a space X are equivalent.*

(1) *X is real compact.*

(2) *For each point $p\in\beta X-X$, there is a closed G_δ-set C of βX such that $p\in C\subset\beta X-X$.*

(3) *For each point $p\in\beta X-X$, there is a countable star-finite partition of unity $\Phi=\{\varphi_n:\sum\varphi_n=1\}$ such that $Cl_{\beta X}(0(\varphi_n))\not\ni p$ for each n.*

Proof. Let R^* denote the one point compactification of the real number space R. Then, each $f\in C(X)$ has a continuous extension f^* over βX (into R^*) by Theorem 1.2. Put $X_f=\{p\in\beta X: f^*(p)\in R\}$ and put $C_f=\beta X-X_f$. By virtue of Theorem 1.3, X is real compact if and only if $\Delta(X)=\bigcap_{f\in C(X)}V_f^{\varepsilon(\beta X\times\beta X)}$, where

$V_f=\{(x, y)\in X\times X : |f(x)-f(y)|<1\}$. If $x\in X_f$, then there is an open neighborhood (in βX) $U^*(x)$ of x such that $|f^*(x)-f^*(y)|<1$ for each $y\in U^*(x)$. It follows that $(x, x)\in V_f^{\varepsilon(\beta X\times\beta X)}$ for each $x\in X_f$. Conversely, if $p\notin X_f$, then every neighborhood $U^*(p)$ of p containe points x, y of X such that $|f(x)-f(y)|>1$, hence $(p, p)\in Cl_{\beta X\times\beta X}(X-V_f)$ and consequently $(p, p)\notin V_f^{\varepsilon(\beta X\times\beta X)}$. Therefore we have $\Delta(X_f)=\Delta(\beta X)\cap V_f^{\varepsilon(\beta X\times\beta X)}$. On the other hand, it is easy to see that $\bigcap_{f\in C(X)}V_f^{\varepsilon}\subset\Delta(\beta X)$, and it follows that X is real compact if and only if $X=\bigcap_{f\in C(X)}X_f$ by Theorem 1.3. Evidently, C_f is a closed G_δ-set of βX contained in $\beta X-X$ and every closed G_δ-set contained in $\beta X-X$ is a C_f for some $f\in C(X)$. This proves the equivalence of (1) and (2). If f is a continuous function on X such that $Z(f^*)=C\subset\beta X-X$, then we can construct a countable star-finite partition of unity $\Phi=\{\varphi_n : \sum\varphi_n=1\}$ by letting $f_n=\max[1/n+1, \min(f, 1/n-1)]$, $g_n=|1/n-f_n|$ and $\varphi_n=g_n/\sum g_n$. It is clear that $Cl_{\beta X}(0(\varphi_n))\cap C=\emptyset$ for each n. This proves that (2) implies (3). Finally, if $\Phi=\{\varphi_n : \sum\varphi_n=1\}$ is a countable (star-finite) partition of unity on X such that $Cl_{\beta X}(0(\varphi_n))\not\ni p$ for each n, then $f=\sum(1/2^n)\cdot\varphi_n$ is a continuous function on X such that $p\in Z(f^*)\subset\beta X-X$. Thus we see that (3) implies (2).

A spece X is said to be topologically complete if there is a uniformity for X relative to which X is complete.

Theorem 2.6. *X is topologically complete if and only if for each point $p\in\beta X-X$ there is a partition of unity $\Phi=\{\varphi_\lambda : \sum\varphi_\lambda=1\}$ such that $Cl_{\beta X}(0(\varphi_\lambda))\not\ni p$, for each λ.*

Proof. Suppose that $(X, \{V_\alpha\})$ is complete, then $\bigcap_\alpha V_\alpha^{\varepsilon}=\Delta(X)$ by virtue of Theorem 1.3, where V_α^{ε} denotes the proper extension of V_α over $\beta X\times\beta X$. There is a V_α such that $V_\alpha^{\varepsilon}\not\ni(p, p)$, for each $p\in\beta X-X$. Let $d(x, y)$ be a pseudo-metric on X such that $d(x, y)=1$ whenever $(x, y)\notin V_\alpha$, and let τ denote the topology of X induced by the pseudo-metric $d(x, y)$. Then the space (X, τ) is paracompact. Now, let us consider an open convering $\{U(x)\}_{x\in X}$ of (X, τ), where $U(x)=\{y\in(X, \tau) : d(x, y)<1/2^2\}$, and let $\{U_\lambda\}$ be an open locally finite refinement of $\{U(x)\}_{x\in X}$. Let $\Phi=\{\varphi_\lambda : \sum\varphi_\lambda=1\}$ be a partition of unity on (X, τ) subordinate to the covering $\{U_\lambda\}$. Since τ is weaker than the original topology of X, $\Phi=\{\varphi_\lambda : \sum\varphi_\lambda=1\}$ is also a partition of unity on X with respect to the original topology of X. We shall show that $p\notin Cl_{\beta X}(U(x))$ for each $x\in X$, which will imply that $p\notin Cl_{\beta X}(0(\varphi_\lambda))$ for each λ and the necessity of the condition will be thus proved. Suppose not, then there is $y\in U^*(p)\cap X$ such that $d(x, y)<1/2^2$ for each neighborhood $U^*(p)$ of p. Put $d_x(y)=d(x, y)$ and let d_x^* be the extension of d_x over βX, then $d_x^*(p)\leq 1/2^2<1/2$. There is an open neighborhood $W^*(p)$ of p such that $[W^*(p)\times W^*(p)]\cap(X\times X)=[W^*(p)\cap X]\times[W^*(p)\cap X]\subset V$. It follows that $(p, p)\in V_\alpha^\varepsilon$ which is contradictory. We now prove the sufficiency of the condition. If $\Phi=\{\varphi_\lambda : \sum\varphi_\lambda=1\}$ is a partition of unity on X such that $p\notin Cl_{\beta X}(0(\varphi_\lambda))$ for each λ, then $V=\{(x, y)\in X\times X: \sum|\varphi_\lambda(x)-\varphi_\lambda(y)|<1\}$ is a surrounding for X such that $(p, p)\notin V^\varepsilon$. To prove this, let $W^*(p)$ be any neighborhood of p and let x be any point of $W^*(p)\cap X$. Then, $\varphi_\lambda(x)\neq 0$ for all but a finite number of φ_λ's, and since $Cl_{\beta X}(0(\varphi_\lambda))\not\ni p$, there is a $y\in W^*(p)\cap X$ such that $(x, y)\notin V$. Therefore $(p, p)\in Cl_{\beta X\times\beta X}[(X\times X)-V]$ and consequently $(p, p)\notin V^\varepsilon$. It follows that X is topologically complete by Theorem 1. 3.

A space X is said to be a Lindelöf space if every open covering of X has a countable subcovering. It is well known that X is a Lindelöf space if and only if every open covering of X has a countable star-finite refinement (c. f. [20]).

Theorem 2. 7. *Let BX denote any compactification of X. Then, the following conditions are equivalent.*

(1) *X is a Lindelöf space.*

(2) *For each compact subset $C\subset BX-X$, there is a countable star-finite partition of unity $\Phi=\{\varphi_n : \sum\varphi_n=1\}$ on X such that $Cl_{BX}(0(\varphi_n))\cap C=\emptyset$ for each n.*

(3) *For each compact subset $C\subset BX-X$, there is a closed G_δ-set G of BX such that $C\subset G\subset BX-X$.*

(4) *For each compact subset $C\subset BX-X$, there is a countable family $\{G_n\}$ of compact subsets of BX such that $G_n\cap C=\emptyset$ for each n and $\bigcup_{n=1}^\infty G_n\supset X$.*

Proof. Assume that X is a Lindelöf space and let C be a compact subset of BX contained in $BX-X$. For each $x\in X$, let $U(x)$ be an open neighborhood of x such that $Cl_{BX}(U(x))\cap C=\emptyset$, and consider a covering $\{U(x)\}_{x\in X}$ of X. Let $\{W_n\}$ be a countable star-finite refinement of $\{U(x)\}_{x\in X}$. Since X is normal, there is a partition of unity $\Phi=\{\varphi_n: \sum\varphi_n=1\}$ such that $Cl_X(0(\varphi_n))\subset W_n$ for each n. It is evident that $Cl_{BX}(0(\varphi_n))\cap C=\emptyset$ for each member φ_n of the partition of unity. This proves the implication (1)$\Rightarrow$(2). Suppose that (2) is valid and let f_n be a continuous function on BX such that $0\leq f_n\leq 1$, $f_n=0$ on C and $f_n=1$ on $Cl_{BX}(0(\varphi_n))$. Put $f=\sum(1/2^n)\cdot f_n$, then $\boldsymbol{Z}(f)$ is a closed G_δ-set of BX such that $C\subset \boldsymbol{Z}(f)\subset BX-X$. This proves that (2) implies (3). The implication (3)$\Rightarrow$(4) is obvious. Finally, let $\{U_\alpha\}$ be any open covering of X and let U_α^ε denote the proper extension of U_α over BX. Put $C=BX-\bigcup_\alpha U_\alpha^\varepsilon$, then C is a compact subset of BX contained in $BX-X$. Evidently, each G_n is covered by a finite number of U_n, and therefore $\bigcup_{n=1}^\infty G_n$ is covered by a countable subfamily of $\{U_\alpha^\varepsilon\}$. Since $\bigcup_{n=1} G_n\supset X$, we see that X is covered by a countable subfamily of $\{U_\alpha\}$. It follows that X is a Lindelöf space. The proof is completed.

A space X is said to be paracompact if every open covering of X has a locally finite refinement.

Theorem 2.8. *Let BX denote any compactification of X. Then, the following conditions are equivalent.*

(1) *X is paracompact.*

(2) *For each compact subset $C\subset BX-X$, there is a partition of unity $\Phi=\{\varphi_\lambda: \sum\varphi_\lambda=1\}$ on X such that $Cl_{BX}(0(\varphi_\lambda))\cap C=\emptyset$ for each λ.*

(3) *For each compact subset $C\subset BX-X$, there is a family $\{G_\alpha\}$ of compact subsets of BX such that $G_\alpha\cap C=\emptyset$, $\bigcup G_\alpha\supset X$ and that there is for each $x\in X$ an open neighborhood (in BX) $U^*(x)$ of x which intersects finitely many G_α's.*

Proof. Assume that X is paracompact, and let C be a compact subset of $BX-X$. For each $x\in X$ let $U^*(x)$ be an open subset of BX containing x such that $Cl_{BX}(U^*(x))\cap C=\emptyset$, and consider a covering $\{U(x)\}_{x\in X}$ of X, where $U(x)=U^*(x)\cap X$. There is a partition of unity $\Phi=\{\varphi_\lambda : \sum\varphi_\lambda=1\}$ subordinate to the covering $\{U(x)\}_{x\in X}$. Since $0(\varphi_\lambda)\subset U(x)$ for some x, (for each λ), we have $Cl_{BX}(0(\varphi_\lambda))\bigcap C=\emptyset$ for each λ. This proves the implication (1)$\Rightarrow$(2). The implication (2)$\Rightarrow$(3) is obvious in view of the fact that if $\{H_\lambda\}$ is a locally finite family of subsets of X, then $\{Cl_X(H_\lambda)\}$ is also a locally finite family. To prove the implication (3)$\Rightarrow$(1), let us recall that X is paracompact if and only if every open covering of X has a locally finite refinement (c. f. [17, P. 156]). Let $\{U_\alpha\}$ be any open covering of X, and let U_α^e denote the proper extension of U_α over BX. Put $C=BX-\bigcup U_\alpha^e$, then C is a compact subset of $BX-X$. It is clear that each G_α is covered by a finite number of U_α^e's say $U_1^e, \cdots, U_n^e$, since G_α is compact. Put $H_{\alpha,k}=G_\alpha\cap U_k^e\cap X$, then each $H_{\alpha,k}$ is contained in some $U_\alpha=U_\alpha^e\cap X$ and $G_\alpha\cap X=\bigcup_{k=1}^n H_{\alpha,k}$, Constructing $H_{\alpha,k}$ for each G_α in this fashion, we have a locally finite refinement $\{H_{\alpha,k}\}$ of $\{U_\alpha\}$. It follows that X is paracompact.

A space X is said to be hereditarily paracompact if every subspace of X is paracompact. It is easy to see that X is hereditarily paracompact if and only if every open subspace of X is paracompact. Let E be any open subspace of X, and let BX be any compactification of X. Then $Cl_{BX}(E)$ is a compactification of E. Applying the above arguments to E and $Cl_{BX}(E)$, we obtain the following

Theorem 2.9. *Let BX denote any compactification of X. Then, the following conditions are equivalent.*

(1) *X is hereditarily paracompact.*

(2) *For each closed subset C of BX, there is a partition of unity $\Phi=\{\varphi_\lambda : \sum\varphi_\lambda=1\}$ on $X-C$ such that $Cl_{BX}(0(\varphi_\lambda))\cap C=\emptyset$ for each λ.*

(3) *For each closed subset C of BX, there is a family $\{G_\alpha\}$ of closed subsets of BX such that $C\cap G_\alpha=\emptyset$, $\bigcup G_\alpha\supset X-C$ and that there is for each $x\in X-C$ an open neighborhood (in BX) of x intersecting finitely many G_α's.*

Quite in a similar way, we have the following characterization of hereditarily Lindelöf spaces.

Theorem 2.10. *Let BX denote any compactification of X. Then the following conditions are equivalent.*
(1) *X is a hereditarily Lindelöf space: That is, every subspace of X is a Lindelöf space.*
(2) *Every closed subset C of BX is contained in a closed G_δ-set C^* of BX such that $X-C=X-C^*$.*
(3) *For each closed subset C of BX, there is a countable star-finite partition of unity $\Phi=\{\varphi_n: \sum\varphi_n=1\}$ on $X-C$ such that $Cl_{BX}(0(\varphi_n))\cap C=\emptyset$ for each n.*
(4) *For each closed subset C of BX, there is a countable family $\{G_n\}$ of closed subsets of BX such that $G_n\cap C=\emptyset$ for each n and $\bigcup_{n=1}^{\infty} G_n \supset X-C$.*

Corollary. *Every hereditarily Lindelöf space is perfectly normal.*

Proof. Suppose that X is a hereditarily Lindelöf space and let F be any closed subset of X, then $F^*=Cl_{BX}(F)$ is contained in a closed G_δ-set $G^*=\bigcap_{n=1}^{\infty} O_n^*$ of BX such that $X-F^*=X-G^*$ by virtue of the preceding theorem. Therefore $F=F^*\cap X=G^*\cap X=(\bigcap_{n=1}^{\infty} O_n^*)\bigcap X=\bigcap_{n=1}^{\infty}(O_n^*\cap X)$ and hence F is a closed G_δ-set of X. It follows that X is perfectly normal.

§3. Characterization of topology (II).

This section is devoted to the characterization of topological properties of X in terms of the properties of the product $X\times Z$ of X with some compact space Z. In most cases, we may take Z to be a compactification of X. However, for the sake of convenience, some of the following results will be stated letting Z to be a compact metrizable space or an arbitrary compact space.

From the work of Dowker [5] and the author [27], it may be expected that the modifications of the normality condition proposed on the product $X\times Z$ will yield some interesting properties of X. This will be discussed in detail, and we shall show that a number of important topological properties can be characterized by the modifications of the normality proposed on $X\times Z$.

Following Gillman and Jerison [9], we shall say that a subspace E of X is C^*-embedded in X if every bounded continuous function on E can be extended to a bounded continuous function on X. As is well known, every closed subspace of a normal space X is C^*-embedded in X.

Let $C'(X)$ be a subset of $C(X)$. If there is a function $f \in C'(X)$ such that $f(x)=1$ for each $x \in F$ and $f(x)=0$ for each $x \in G$, then we shall say that F and G are functionally separated by a member of $C'(X)$.

Theorem 3.1. *Let BX denote any compactification of X. Then, the following conditions are equivalent.*
(1) *X is paracompact.*
(2) *For each compact subset C of $BX-X$, there is a surrounding V for X such that $Cl_{X\times BX}(V)\cap(X\times C)=\emptyset$.*
(3)*) *$X\times BX$ is normal.*
(4) *If $G=X\times C$ is a closed subset of $X\times BX$ such that $G\cap\Delta(X)=\emptyset$, then G and $\Delta(X)$ are functionally separated (by a member of $C^*(X\times BX)$).*

Proof. We prove firstly the equivalence of (1) and (2). Assume that X is paracompact and let C be a compact subset of $BX-X$. Then, there is a partition of unity $\Phi=\{\varphi_\lambda : \sum \varphi_\lambda=1\}$ such that $Cl_{BX}(0(\varphi_\lambda))\cap C=\emptyset$ for each λ, by virtue of Theorem 2.8. Put $V=\{(x, y)\in X\times X : \sum|\varphi_\lambda(x)-\varphi_\lambda(y)|<1\}$, then V is a surrounding for X. We now prove that $Cl_{X\times BX}(V)\cap(X\times C)=\emptyset$. Suppose on the contrary that $Cl_{X\times BX}(V)\cap(X\times C)$ contains a point (x, p) of $X\times BX$, then there is a point (x', y') which belongs to V, in each neighborhood $U(x)\times U^*(p)$ of (x, p). Let us take $U(x)$ to be a neighborhood of x such that $U(x)$ intersects finitely many members of $\{0(\varphi_\lambda)\}$, say $0(\varphi_1), \cdots, 0(\varphi_n)$, then $\sum_{k=1}^{n}\varphi_k(x')=1$ for each $x'\in U(x)$. It is clear that $(x', y')\in V$ implies $y'\in\bigcup_{k=1}^{n}0(\varphi_k)$, and it follows that $p\in Cl_{BX}(\bigcup_{k=1}^{n}0(\varphi_k))=\bigcup_{k=1}^{n}Cl_{BX}(0(\varphi_k))$. But this is impossible, since $p\in C$ and since $C\cap Cl_{BX}(0(\varphi_k))=\emptyset$ for each k. Conversely, let V be a surrounding for X such that $Cl_{BX}(V)\cap(X\times C)=\emptyset$. Let d be a pseudo-metric on X such that $W=\{(x, y)\in$

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*) K. Morita has recently proved the equivalence of 1) and 3) independently. (*K. Morita, Paracompactness and product spaces, Fund. Math. Vol. 50 (1961) pp. 223–236*)

$X\times X: d(x, y)<1\}\subset V$, and let τ denote the topology of X induced by the pseudo-metric d. Then the space (X, τ) is paracompact. Put $W(x)=\{z\in(X, \tau): d(x, z)<1\}$, then it is clear that $Cl_{BX}(W(x))\cap C=\emptyset$ for each $x\in X$. Consider an open covering $\{W(x)\}_{x\in X}$ of (X, τ), and let $\Phi=\{\varphi_\lambda: \sum\varphi_\lambda=1\}$ be a partition of unity on (X, τ) subordinate to the covering $\{W(x)\}_{x\in X}$. Since τ is weaker than the original topology of X, Φ is a partition of unity on X with respect to the original topology of X. Thus, we see that there is a partition of unity $\Phi=\{\varphi_\lambda: \sum\varphi_\lambda=1\}$ on X such that $Cl_{BX}(0(\varphi_\lambda))\cap C=\emptyset$ for each λ. It follows that X is paracompact, in view of Theorem 2. 8.

The implication $(1)\Rightarrow(3)$ is well known [4], and the implication $(3)\Rightarrow(4)$ is evident. We shall prove that (4) implies (1). Let C be any compact subset of $BX-X$. We shall show that there is a partition of nnity $\Phi=\{\varphi_\lambda: \sum\varphi_\lambda=1\}$ on X such that $Cl_{BX}(0(\varphi_\lambda))\cap C=\emptyset$ for each λ, which will complete the proof by virtue of Theorem 2. 8. Let $F(x, p)$ be a continuous function on $X\times BX$ such that $F=1$ on $G=X\times C$ and $F=0$ on $\Delta(X)$. Let $F_x(p)$ denote the restriction of $F(x, p)$ on $\{x\}\times BX$, and put $d(x, y)=\|F_x(p)-F_y(p)\|=\operatorname{Sup}_{p\in BX}|F_x(p)-F_y(p)|$. Then $d(x, y)$ is a pseudo-metric on X. Let τ denote the topology of X induced by the pseudo-metric $d(x, y)$, and consider the space (X, τ) which is paracompact. Let $\Phi=\{\varphi_\lambda: \sum\varphi_\lambda=1\}$ be a partition of unity on X subordinate to the covering $\{U(x)\}_{x\in X}$ of (X, τ), where $U(x)=\{y\in(X, \tau): d(x, y)<1/2\}$. Since τ is weaker then the original topology of X, Φ is a partition of unity of X with respect to the original topology of X. We shall show that $Cl_{BX}(0(\varphi_\lambda))\cap C=\emptyset$ for each λ. It is clear that $d(x. y)<1/2$ implies $|F_x(y)|=|F_x(y)-F_y(y)|<1/2$. Therefore $F_x(p)\leq 1/2$ for each $p\in Cl_{BX}(U(x))$, since F_x is a continuous function on BX. On the other hand, $F(x, p)=F_x(p)=1$ for each $p\in C$. It follows that $Cl_{BX}(U(x))\cap C=\emptyset$ for each $x\in X$ and consequently $Cl_{BX}(0(\varphi_\lambda))\cap C=\emptyset$ for each λ. The proof is completed.

We now notice that $F(x, y)\in C(X\times Y)$ defines a continuous mapping of X into $C(Y)$ $(F: X\rightarrow C(Y))$ by letting $F\langle x\rangle=F_x\in C(Y)$, where F_x denotes the restriction of $F(x, y)$ on $\{x\}\times Y$. We shall denote by $\mathcal{C}_X(X\times Y)$ the subset of $C(X\times Y)$ consisting of all functions of $C(X\times Y)$ such that $F\langle X\rangle=\{F_x\}_{x\in X}$ is a separable subspace of $C(Y)$.

Theorem 3. 2. *Let BX denote any compactification of X. Then, the following conditions are equivalent.*

(1) *X is a Lindelöf space.*

(2) *For each compact subset C of $BX-X$, there is a strongly stable surrounding V for X such that $Cl_{X\times BX}(V)\cap(X\times C)=\emptyset$.*

(3) *If $G=X\times C$ is a closed subset of $X\times BX$ such that $G\cap\Delta(X)=\emptyset$, then G and $\Delta(X)$ are functionally separated by a member of $\mathcal{C}_X(X\times BX)$.*

Proof. In the first place, we prove the equivalence of (1) and (2). Assume that X is a Lindelöf space and let C be a compact subset of $BX-X$. Then, there is a countable star-finite partition of unity $\Phi=\{\varphi_n:\sum\varphi_n=1\}$ on X such that $Cl_{BX}(0(\varphi_n))\cap C=\emptyset$ for each n, by virtue of Theorem 2. 7. Put $V=\{(x,y)\in X\times X:\sum|\varphi_n(x)-\varphi_n(y)|<1\}$, then V is a strongly stable surrounding for X by Proposition 1. 5. By the similar argument done in the proof of Theorem 3. 1, we can see without difficulty that $Cl_{X\times BX}(V)\cap(X\times C)=\emptyset$. To prove the implication (2)⇒(1), let C be any compact subset of $BX-X$ and let V be a strongly stable surrounding for X such that $Cl_{X\times BX}(V)\cap(X\times C)=\emptyset$. By virtue of Proposition 1. 5, there is a countable star-finite partition of unity $\Phi=\{\varphi_n:\sum\varphi_n=1\}$ such that $W=\{(x,y)\in X\times X:\sum|\varphi_n(x)-\varphi_n(y)|<1\}\subset V$. It is clear that $Cl_{BX}(W(x))\cap C=\emptyset$ for each $W(x)$, where $W(x)=\{y\in X:(x,y)\in W\}$. Put $d(x,y)=\sum|\varphi_n(x)-\varphi_n(y)|$ and let τ denote the topology of X induced by the pseudo-metric d. Then, the space (X,τ) is secoud countable and hence it is a Lindelöf space Consider an open covering $\{W(x)\}_{x\in X}$ of (X,τ), where $W(x)=\{y\in X: d(x,y)<1\}$ $(=\{y\in X:(x,y)\in W\})$, and let $\{W_n\}$ be a countable subcovering of $\{W(x)\}_{x\in X}$. Since $Cl_{BX}(W(x))\cap C=\emptyset$ for each $W(x)$, $Cl_{BX}(W_n)\cap C=\emptyset$ for each n. Thus, we see that there is a countable covering $\{W_n\}$ of X such that $Cl_{BX}(W_n)\cap C=\emptyset$. It follows that X is a Lindelöf space by Theorem 2. 7. We nextly prove the implication (1)⇒(3). Let $G=X\times C$ be a closed subset of $X\times BX$ such that $G\cap\Delta(X)=\emptyset$, then C is a compact subset of $BX-X$. By virtue of Theorem 2.7, there is a closed G_δ-set C^* of BX such that $C\subset C^*\subset BX-X$. There is a continuous function $f^*\in C(BX)$ such that $0\leq f^*\leq 1$ and $C^*=\boldsymbol{Z}(f^*)=\{p\in BX: f^*(p)=0\}$. Now let us construct a partition of unity $\Phi=\{\varphi_n: \sum\varphi_n=1\}$ on X by letting

$$f_n^*(p) = \operatorname{Max}[1/n+1,\ \operatorname{Min}(f^*(p),\ 1/n-1)],$$
$$g_n^*(p) = |1/n-f_n^*(p)| \quad \text{and}$$
$$\varphi_n(x) = g_n(x)/\textstyle\sum_{n=1}^\infty g_n(x),$$

where g_n denotes the restriction of g_n^* on X. In view of the above definition of φ_n, it may easily be seen that φ_n has a continuous extension φ_n^* over BX and the extension φ_n^* is such that

$$\begin{aligned}\varphi_n^*(p) &= g_n^*(p)/g_{n-1}^*(p)+g_n^*(p)+g_{n+1}^*(p) &&\text{if } p\in 0(g_n^*) \text{ and}\\ &= 0 &&\text{if } p\notin 0(g_n^*).\end{aligned}$$

Define $F\in C(X\times BX)$ by letting $F(x, p)=\sum_{n=1}^\infty|\varphi_n(x)-\varphi_n^*(p)|$. then it is clear that $F=1$ on $X\times C$ and $F=0$ on $\Delta(X)$. To show that $F\in\mathcal{C}_X(X\times BX)$, let τ be the topology of X induced by the pseudo-metric $d(x, y)=\sum|\varphi_n(x)-\varphi_n(y)|$ on X. Then, the space (X, τ) is second countable, and hence there is a countable subset $\{x_i\}$ of X which is dense in (X, τ). The set $\{F_{x_i}\}$ is a dense subset of $F\langle X\rangle$. In fact, for any member F_y of $F\langle X\rangle$ and for each $\varepsilon>0$, we can choose a point $x_i\in X$ such that $d(x_i, y)<\varepsilon$. Then $\|F_y-F_{x_i}\|=\sup_{p\in BX}|(\sum|\varphi_n(y)-\varphi_n^*(p)|)-(\sum|\varphi_n(x_i)-\varphi_n^*(p)|)|\leq\sum|\varphi_n(x_i)-\varphi_n(y)|<\varepsilon$. Therefore, $F\in\mathcal{C}_X(X\times BX)$. Suppose conversely that there is a $F\in\mathcal{C}_X(X\times BX)$ such that $F=1$ on $X\times C$ and $F=0$ on $\Delta(X)$. Let τ denote the topology of X induced by the pseudo-metric $d(x, y)=\|F_x-F_y\|$. Then the space (X, τ) is second countable. Consider an open covering $\{U(x)\}_{x\in X}$ of (X, τ), where $U(x)=\{y\in(X, \tau): d(x, y)<1/2\}$, and let $\{U_n\}$ be a countable subcovering of $\{U(x)\}_{x\in X}$. It is clear that $d(x, y)<1/2$ implies $|F_x(y)-F_y(y)|<1/2$ and hence $|F(x, y)|<1/2$. Therefore $Cl_{BX}(U(x))\cap C=\emptyset$ for each $U(x)$ and hence $Cl_{BX}(U_n)\cap C=\emptyset$ for each n. Thus, we see that for any compact subset C of $BX-X$ there is a countable covering $\{U_n\}$ of X such that $Cl_{BX}(U_n)\cap C=\emptyset$ for each n. It follows that X is a Lindelöf space by Theorem 2.7.

Theorem 3.3. *A space X is metrizable if and only if $X \times BX$ is normal and $\Delta(X)$ is a closed G_δ-set of $X \times BX$, where BX is any compactification of X.*

Proof. Every metrizable space X is paracompact [24], hence $X \times BX$ is normal for any compactification BX of X. Let d be a metric on X and put $V_n = \{(x, y) \in X \times X: d(x, y) < 1/2^n\}$, then it is clear that $\bigcap_{n=1}^{\infty} V_n = \Delta(X)$. We shall show that $\bigcap_{n=1}^{\infty} V_n^{e(X \times BX)} = \Delta(X)$. Let (x, p) be a point of $X \times BX$ which is not contained in $\Delta(X)$, then there are two open subset $U^*(x)$ and $W^*(p)$ of BX containing x and p respectively such that $U^*(x) \cap W^*(p) = \emptyset$. There is a n such that $\{y \in X: (x, y) \in V_n\} \subset U^*(x)$, and we have $(x, y) \notin V_n$ for each $y \in W^*(p) \cap X$. It follows that $(x, p) \in Cl_{X \times BX}((X \times BX) - V_n)$ and hence $(x, p) \notin V_n^{e(X \times BX)}$. It follows that that $\Delta(X) = \bigcap_{n=1}^{\infty} V_n^{e(X \times BX)}$. Supppse conversely that $X \times BX$ is normal and $\Delta(X) = \bigcap_{n=1}^{\infty} U_n^*$, where U_n^* is an open subset of $X \times BX$. Then $G_n = (X \times BX) - U_n^*$ and $\Delta(X)$ are disjoint closed subsets of $X \times BX$. There is a continuous function $F(x, p)$ on $X \times BX$ such that $F = 1$ on G_n and $F = 0$ on $\Delta(X)$ and $0 \leq F \leq 1$. Put $d_n(x, y) = ||F_x - F_y||$. where F_x denotes the restriction of F on $\{x\} \times BX$. Put $V_n = \{(x, y) \in X \times X: d(x, y) < 1\}$, then $V_n \subset U_n^* \cap (X \times X)$ because $d_n(x, y) < 1$ implies $|F(x, y)| < 1$. Therefore $U_n^* \cap (X \times X)$ is a surrounding for X, and hence $\Delta(X)$ is an intersection of countable surroundings for X. It follows that X is metrizable [30].

Theorem 3.4. *The following conditions on a space X are equivalent.*

(1) *X is second countable.*

(2) *X is metrizable and separable.*

(3) *X is homeomorphic to a subspace of compact metric space.*

(4) *$X \times BX$ is perfectly normal for some compactification BX of X.*

Proof. The equivalence of (1), (2) and (3) is well known [18] and the implication. (3) $\Rightarrow$ (4) is evident. Therefore we have only to prove that (4) implies (1). Suppose that $X \times BX$ is perfectly normal and let $\{U_n\}$ be a countable family of open subset of $X \times BX$ such that $\bigcap_{n=1}^{\infty} \bar{U}_n = \Delta(X)$, where $\bar{U}_n = Cl_{X\times BX}(U_n)$. We put $U_n(x) = \{p \in BX : (x, p) \in U_n\}$ and $\bar{U}_n(x) = \{p \in BX : (x, p) \in \bar{U}_n\}$. Consider the product $BX \times BX$ and put $C_n = Pr_{BX}[\Delta(BX) - U_n^{\varepsilon(BX\times BX)}]$, $X_n = BX - C_n$. Since BX is perfectly normal, X_n is a σ-compact subspace of BX and hence it is a Lindelöf space. Now, let us consider the product $X_n \times BX$, which is clearly normal. Put $G_n = (X_n \times C_n) \cup ((X_n \times BX) - U_n^{\varepsilon(BX\times BX)})$, then G_n is a closed subset of $X_n \times BX$ such that $G_n \cap \Delta(X_n) = \emptyset$. There is a continuous function $F_n \in C(X_n \times BX)$ such that $0 \leq F_n \leq 1$, $F_n = 1$ on G_n and $F_n = 0$ on $\Delta(X_n)$. Let F_x^n denote the restriction of F_n on $\{x\} \times BX$, and consider an open covering $\{0(x)\}_{x\in X_n}$ of X_n, where $0(x) = \{y \in X_n : ||F_x^n - F_y^n|| < 1/2\}$. There is a countable star-finite partition of unity $\Phi = \{\varphi_k : \sum \varphi_k = 1\}$ subordinate to the covering $\{0(x)\}_{x\in X_n}$. Define a pseudo-metric d_n on X_n by letting $d_n(x, y) = \sum |\varphi_k(x) - \varphi_k(y)|$, and let τ_n denote the topology of X_n induced by the pseudo-metric d_n. Then, the space (X_n, τ_n) is second countable, and therefore $\Pi = \Pi_{n=1}^{\infty} (X_n, \tau_n)$ is second countable [2, Chap. 1, P. 72]. Let Pr_n denote the projection of Π onto (X_n, τ_n), and let X^* be a subspace of Π consisting of all points Q of Π such that $Pr_n(Q) = Pr_1(Q) \in X$ for each n. Then the space X^* is second countable. We shall show that X^* is homeomorphic with X, which will complete the proof. Let Q_x denote the point of X^* such that $Pr_n(Q_x) = x \in X$ for each n. Then, we have a one to one mapping θ of X onto X^* by letting $\theta(x) = Q_x$. Since τ_n is weaker than the original topology of X_n, for each n, the mapping θ is continuous. To prove that θ^{-1} is continuous, let $V(x)$ be any open neighborhood of $x \in X$. Then, there is a U_n such that $U_n(x) \subset V(x)^{\varepsilon(BX)}$ because $BX - V(x)^{\varepsilon(BX)}$ is compact and $\bigcap \bar{U}_n(x) = x$. We now prove that $W_n(x) = \{y \in (X_n, \tau_n) : d_n(x, y) < 1\} \subset U_n(x)$. Let $\varphi_1, \cdots, \varphi_m$ be the set of all members of Φ which do not vanish at x, and let $0(x_i)$ be a member of $\{0(x)\}_{x\in X}$ such that $0(x_i) \supset 0(\varphi_i)$ $(1 \leq i \leq m)$. Then $x \in 0(x_i)$ for each i, and $y \in W_n(x)$ implies $y \in 0(x_i)$ for some i, therefore $F_x^n(y) = F_x^n(y) - F_y^n(y) \leq |F_x^n(y) - F_{x_i}^n(y)| + |F_{x_i}^n(y) - F_y^n(y)| \leq ||F_x^n - F_{x_i}^n|| + ||F_{x_i}^n - F_y^n|| < 1/2 + 1/2 = 1$ and hence $y \in U_n(x)$. It follows that $W_n(x) \subset U_n(x)$. Put $W_n^* = X^* \cap ((\Pi_{m\neq n} (X_m, \tau_m)) \times W_n(x))$, then we have

$\theta^{-1}(W_n^*)\subset U_n(x)\cap X\subset V(x)^{\varepsilon(\beta X)}\cap X=V(x)$. Therefore θ^{-1} is continuous, and consequently θ is a homeomorphism. The proof is completed.

Remark. From the proof of the preceding theorem we see that if V_n is a strongly stable surrounding for X, for each n, and if $\Delta(X)=\bigcap_{i=1}^{\infty} V_n$, then X is second countable. (Put $X_n=Pr_{\beta X}[\Delta(\beta X)\bigcap V_n^{\varepsilon(\beta X\times\beta X)}]$.) This prove the sufficiency of the condition of Proposition 1.6.

We call a space X entirely normal if every neighborhood of the diagonal of $X\times X$ is a surrounding for X.

Theorem 3.5. *A space X is entirely normrl if and only if for any closed subset F of $X\times X$, $Cl_{X\times\beta X}(F)\cap\Delta(X)=\emptyset$ implies that F and $\Delta(X)$ are functionally separated by a member of $C(X\times\beta X)$.*

Proof. Assume that X is entirely normal and let F be a closed subset of $X\times X$ such that $Cl_{X\times\beta X}(F)\cap\Delta(X)=\emptyset$. Then $(X\times X)-F=V$ is a neighborhood of the diagonal $\Delta(X)$ of $X\times X$. There is a pseudo-metric d on X such that $d(x, y)=1$ for each $(x, y)\notin V$. Let d_x denote the restriction of $d(x, y)$ on $\{x\}\times X$. As we have noted above, d defines a mapping φ of X into $C^*(X)$ by letting $\varphi(x)=d_x\in C^*(X)$. The mapping φ is continuous by virtue of the triangular inequality of d. Applying Glicksberg's lemma [10, Lemma 2] to $d(x, y)\in C(X\times X)$, we can see that $d(x, y)$ has a continuous extension $d^*(x, p)$ over $X\times\beta X$. It is clear that $d^*=1$ on $Cl_{X\times\beta X}(F)$ and that $d^*=0$ on $\Delta(X)$. Conversely, let V be any neighborhood of the diagonal of $X\times X$ and let V^ε denote the proper extension of V over $X\times\beta X$. Put $E=(X\times X)-V$. Then $Cl_{X\times\beta X}(E)=(X\times\beta X)-V^\varepsilon$ and we have $Cl_{X\times\beta X}(E)\cap\Delta(X)=\emptyset$. Let $F(x, p)$ be a continuous function on $X\times\beta X$ such that $0\leq F\leq 1$, $F=0$ on $\Delta(X)$ and $F=1$ outside of V^ε. Put $d(x, y)=\|F_x-F_y\|=\operatorname*{Sup}_{p\in\beta X}|F_x(p)-F_y(p)|$, then $d(x, y)$ is a pseudo-metric on X, and it is easy to see that $\{(x, y)\in X\times X: d(x, y)<1\}\subset V$. Therefore V is a surrounding for X. It follows that X is entirely normal.

A space X is said to be collectionwise normal [1] if for every locally finite collection $\{F_\alpha\}$ of mutually non-intersecting closed subsets of X there is a collection $\{U_\alpha\}$ of mutually non-intersecting open subsets of X such that $F_\alpha \subset U_\alpha$ for each α.

Theorem 3.6. *A space X is collectionwise normal if and only if $G\times\beta X$ is C^*-embedded (normally embedded) in $X\times\beta X$ for every closed subspace G of X.*

Proof. Let F be a continuous function (bounded) on $G\times\beta X$, and let F_x denote the restriction of F on $\{x\}\times\beta X$. Define a mapping $\varphi: X\to C(\beta X)$ by letting $\varphi\langle x\rangle = F_x\in C(\beta X)$, then φ is a continuous mapping of G into $C(\beta X)$. By virtue of the theorem of Dowker [6, Th. 2], which states that a metric space is absolute retract for collectionwise normal space if and only if it is absolute retract for metric space and absolute G_δ, we can see without difficulty that the mapping φ can be extended to a continuous mapping φ^* of X into $C(\beta X)$*) By letting $F^*(x, p)=\varphi^*\langle x\rangle(p)$, we have a (real-valued) function F^* on $X\times\beta X$. The continuity of φ^* implies that F^* is a continuous function on $X\times\beta X$, and thus we see that F has a continuous extension F^* over $X\times\beta X$. Therefore $G\times\beta X$ is C^*-embedded in $X\times\beta X$. To prove the sufficiency of the contition, note first that X is a normal space under the assumption of the theorem. In fact, if G_1, G_2 are disjoint closed subsets of X, then the continuous function f on $(G_1\times\beta X)\cup(G_2\times\beta X)$ defined by letting $f(x, p)=1$ for each $(x, p)\in G_1\times\beta X$ and $f(x, p)=0$ for each $(x, p)\in G_2\times\beta X$ can be extended to a continuous function f^* on $X\times\beta X$. The restriction f_p^* of f^* on $X\times\{p\}$, where p is a point of βX, is a continuous function on X such that $f_p^*=1$ on G_1 and $f_p^*=0$ on G_2. It follows that X is a normal space. Now, let $\{G_\alpha\}$ be any locally finite collection of mutually non-intersecting closed subsets of X. Put $H_\alpha=\bigcup_{\beta\neq\alpha} G_\beta$, then G_α and H_α are disjoint closed subsets of X. Therefore $Cl_{\beta X}(G_\alpha)\cap Cl_{\beta X}(H_\alpha)=\emptyset$ by the norma-

*) Verification that $C(\beta X)$ is absolute retract for metric space is as follows: Let M be a metrizable space and let φ be a continuous mapping of a closed subspace G of M into $C(\beta X)$. By letting $F(x, p)=\varphi\langle x\rangle(p)$, we have a continuous function F on $G\times\beta X$. Since $M\times\beta X$ is normal, F has a continuous extension F^* over $X\times\beta X$. Let $F_x{}^*$ be the restriction of F^* on $\{x\}\times\beta X$, and put $\varphi^*\langle x\rangle=F_x{}^*$, then φ^* is a continuous extension of φ over X.

lity of X. Let g_α be a continuous function on βX such that $g_\alpha=1$ on $Cl_{\beta X}(H_\alpha)$, $g_\alpha=0$ on $Cl_{\beta X}(G_\alpha)$ and $0\leq g_\alpha\leq 1$. Define a bounded continuous function F on $(\bigcup G_\alpha)\times\beta X$ by letting $F(x, p)=\chi_\alpha(x)\otimes g_\alpha(p)$, where $\chi_\alpha(x)$ denotes the characteristic function of G_α. (That is, $F(x, p)=g_\alpha(p)$ if $x\in G_\alpha$.) From the assumption of the theorem, F has a continuous extension F^* over $X\times\beta X$. Let F_x^* denote the restriction of F^* on $\{x\}\times\beta X$, and put $U_\alpha=\{y\in X: ||F_x^*-F_y^*||<1/2^2$ for each $x\in G_\alpha\}$. Then $\{U_\alpha\}$ is the desired family of open subsets of X. Moreover, we can prove that $\{U_\alpha\}$ is a discrete collection of open subsets of X. (A family $\mathfrak{A}$ of subsets of X is said to be discrete if each point $x\in X$ has a neighborhood which intersects at most one member of $\mathfrak{A}$.) Let z be any point of X. In case that $||F_z^*-F_x^*||\leq 1/2$ for some $x\in G_\alpha$, we have $||F_z^*-F_y^*||\geq ||F_y^*-F_x^*||\sim||F_x^*-F_y^*||\geq 1/2$ for each $y\in G_\beta(\beta\neq\alpha)$, since $y\in G_\beta$ implies $||F_x^*-F_y^*||\geq|F_x^*(y)-F_y^*(y)|=F_x^*(y)=F(x, y)=1$. Therefore $W(z)=\{y\in X: ||F_z^*-F_y^*||<1/2^2\}$ does not intersects U_β for each $\beta\neq\alpha$. In another case, where $||F_z^*-F_x^*||>1/2$ for each $x\in\bigcup G_\alpha$, $W(z)=\{y\in X: ||F_z^*-F_y^*||<1/2^2\}$ does not intersects U_α for each α. For otherwise there would be a point $y\in U_\alpha$ such that $||F_z^*-F_y^*||<1/2^2$ and $||F_y^*-F_x^*||<1/2^2$ for some $x\in G_\alpha$. It follows that $||F_z^*-F_x^*||\leq||F_z^*-F_y^*||+||F_y^*-F_x^*||<1/2$, which is contradictory. Therefore X is a collectionwise normal space.

By the similar arguments done in the proof of Theorem 3.1 and Theorem 3.2, we can obtain the following characterizations of topologically complete spaces and real compact spaces respectively.

Theorem 3.7. *The following conditions on a space X are equivalent.*

(1) *X is topologically complete.*

(2) *For each point p of $\beta X-X$, there is a surroundidg V for X such that $Cl_{X\times\beta X}(V)\cap(X\times\{p\})=\emptyset$.*

(3) *Let p be any point of βX. If $X\times\{p\}\cap\Delta(X)=\emptyset$, then $X\times\{p\}$ and $\Delta(X)$ are functionally separated by a member of $C(X\times\beta X)$.*

Proof. The proof of the equivalence of (1) and (2) is entirely similar to that of Theorem 3.1. We shall show that (1) is equivalent to (3). Assume that X is topologically complete and let p

be a point of $\beta X-X$. Then, there is a partition of unity $\Phi=\{\varphi_\lambda : \sum\varphi_\lambda=1\}$ on X such that $p\notin Cl_{\beta X}(0(\varphi_\lambda))$ for each λ, by virtue of Theorem 2.6. Put $d(x, y)=\sum|\varphi_\lambda(x)-\varphi_\lambda(y)|$, then d has a continuous extension d^* over $X\times\beta X$ by virtue of the lemma due to Glicksberg [10], as we have shown in the proof of Theorem 3.5. It is easy to see that $d^*=1$ on $X\times\{p\}$ and that $d^*=0$ on $\Delta(X)$. This proves the implication (1)$\Rightarrow$(3). Conversely, if $(X\times\{p\})\cap\Delta(X)=\emptyset$, then p is a point of $\beta X-X$. We can construct a partition of unity $\Phi=\{\varphi_\lambda : \sum\varphi_\lambda=1\}$ such that $Cl_{\beta X}(0(\varphi_\lambda))\not\ni p$ for each λ, by the similar arguments done in the proof of the implication (4)$\Rightarrow$(1) of Theorem 3.1, and it follows that X is topologically complete by Theorem 2.6.

Theorem 3.8. *The following conditions on a space X are equivalent.*

(1) *X is real compact.*

(2) *For each point p of $\beta X-X$, there is a strongly stable surrounding V for X such that $Cl_{X\times\beta X}(V)\cap(X\times\{p\})=\emptyset$.*

(3) *Let p be any point of βX. If $(X\times\{p\})\cap\Delta(X)=\emptyset$, then $X\times\{p\}$ and $\Delta(X)$ are functionally separated by a member of $\mathfrak{C}_X(X\times\beta X)$*

Proof. The proofs of the implications (1)$\Leftrightarrow$(2) and (1)$\Rightarrow$(3) are entirely similar to those of (1)$\Leftrightarrow$(2) and (1)$\Rightarrow$(3) in Theorem 3.2. If (3) is valid, then we can construct a countable star-finite partition of unity $\Phi=\{\varphi_n : \sum\varphi_n=1\}$ such that $Cl_{\beta X}(0(\varphi_n))\not\ni p$ for each n, by the similar arguments done in the proof of the implication (3)$\Rightarrow$(1) of Theorem 3.2, and it follows that X is real compact by virtue of Theorem 2.5.

We now characterize some topological properties of a space X by the properties of the product of X with some compact space.

A space X is said to be countably paracompact if every countable open covering of X has a locally finite open refinement. *F.* Ishikawa [16] proved that X is countably paracompact if and only if for every countable descending chain of closed subsets

$\{F_n\}$ of X with empty intersection, there is a countable descending chain $\{U_n\}$ of open subsets of X whose closure have empty intersection such that $F_n \subset U_n$ for each n.

Theorem 3.9. *Let M be any compact metrizable space containing infinitely many points. Then X is countably paracompact if and only if G and $X \times C$ are separated by open subsets of $X \times M$ whenever G and $X \times C$ are disjoint closed subsets of $X \times M$.*

Proof. Assume that X is countably paracompact. Let $\{W_n\}$ be a countable family of open neighborhoods of $C \subset M$ such that $\bigcap_{n=1}^{\infty} W_n = C$ and $Cl_M(W_n) \subset W_{n-1}$ for each n, and let G be a closed subset of $X \times M$ such that $G \cap (X \times C) = \emptyset$. Put $F_n = Pr_X[(X \times Cl_M(W_n)) \cap G]$, then F_n is a closed subset of X, for each n, because Pr_X is a closed mapping. (Pr_X denotes the projection of $X \times M$ onto X.) Put $G_x = G \cap (\{x\} \times M)$, then there is a W_n such that $(X \times Cl_M(W_n)) \cap G_x = \emptyset$, for each $x \in X$. It follows that $\{F_n\}$ is a countable descending chain of closed subset of X with empty intersection. (We may assume that $F_n \neq \emptyset$ for each n.) By virtue of Ishikawa's characterization, there is a descending chain $\{U_n\}$ of open subsets of X such that $F_n \subset U_n$ and $\bigcap_{n=1}^{\infty} Cl_X(U_n) = \emptyset$. Put $V_1 = \bigcup [X - Cl_X(U_n)] \times W_{n+1}$, then V_1 is an open subset of $X \times M$ containing $X \times C$. On the other hand, it is easy to see that $Cl_{X \times M}(V_1) \cap G = \emptyset$, and therefore $V_2 = (X \times M) - Cl_{X \times M}(V_1)$ is an open subset of $X \times M$ containing G such that $V_1 \cap V_2 = \emptyset$. Thus, the necessity of the condition is proved. Conversely, let $\{F_n\}$ be a descending chain of closed subsets of X with empty intersection. Let p be a point of M and let $\{W_n\}$ be a countable family of open neighborhoods of p such that $\bigcap_{n=1}^{\infty} W_n = p$ and $Cl_M(W_n) \subset W_{n-1}$ for each n. Put $G = \bigcup_{n=1}^{\infty} F_n \times (M - W_n)$, then G is a closed subset of $X \times M$ such that $G \cap (X \times \{p\}) = \emptyset$. Let V_1, V_2 be open subsets of $X \times M$ such that $V_1 \supset G$, $V_2 \supset (X \times C)$ and $V_1 \cap V_2 = \emptyset$. Put $U_n = X - Pr_X[[X \times (M - W_n)] \cap [(X \times M) - V_1]]$, then $\{U_n\}$ is a descending chain of open subsets of X, and $U_n \supset F_n$ for each n. Let us put further $H_n = X - Pr_X[[X \times (M - Cl_M(W_n)] \cap V_2]$, then we have $H_n \supset Cl_X(U_n)$ for each n and $\bigcap_{i=1}^{\infty} H_n = \emptyset$. Consequently, we see that $\{U_n\}$ is a descending chain of open subsets of X such that

$U_n \supset F_n$ for each n and $\bigcap_{n=1}^{\infty} Cl_X(U_n)=\emptyset$. It follows that X is countably paracompact. The proof is completed.

Theorem 3.10.*) *Let M be any compact metrizable space containing infinitely many points. Then, X is countably compact if and only if the projection Pr_M of $X\times M$ onto M is a closed mapping. (In other words, X is countably compact if and only any two disjoint closed subsets $X\times C$ and F of $X\times M$ are separated by open sets of the form $X\times U$ and $X\times V$.)*

Proof. Suppose that X is not countably compact, then there is a countable covering $\{U_n\}$ of X such that $\bigcup_{k\leq n} U_k \not\supset X$ for each n. Let p be a point of M and let $\{W_n\}$ be a countable family of open neighborhoods of p such that $\bigcap_{n=1}^{\infty} W_n=p$ and $Cl_M(W_n)\subset W_{n-1}$ for each n. Put $F=\bigcup_{n=1}^{\infty}[X-\bigcup_{k\leq n}U_k]\times[M-W_n]$, then F is a closed subset of $X\times M$ such that $(X\times\{p\})\cap F=\emptyset$. Evidently $p\in Cl_M(Pr_M[F])$ and it follows that Pr_M is not closed. Conversely, if the projection Pr_M is not closed, then there is a closed subset F of $X\times M$ such that $Pr_M(F)$ is not closed. Let p be a point of $Cl_M(Pr_M(F)) - Pr_M(F)$, and let $\{W_n\}$ be a countable family of open neighborhoods of p such that $\bigcap_{n=1}^{\infty} Cl_M(W_n)=p$. Put $U_n=X-Pr_X[(X\times Cl_M(W_n))\cap F]$, then U_n is an open subset of X, since Pr_X is a closed mapping. On the other hand, it is easy to see that $\{U_n\}$ is a covering of X. Obviously, $(X\times Cl_M(W_n))\cap F\neq\phi$ for each n, and therefore no finite subfamily of $\{U_n\}$ can cover X. It follows that X is not countably compact. The proof is completed.

Theorem 3.11. *Let Z be any compact space containing infinitely many points. Then, X is pseudo-compact if and only if $Pr_Z[\boldsymbol{Z}(F)]$ is closed for each $F\in C(X\times Z)$, where $\boldsymbol{Z}(F)$ denotes the zero-set of F. (Pr_Z denotes the projection of $X\times Z$ onto Z.) In other words, X is pseudo-compact if and only if $\boldsymbol{Z}(F)$ and a closed subset $X\times C$ can be separated by open subsets of the form $X\times U$ and $X\times V$, whenever they are disjoint.*

*) A slightly stronger result than this is valid. Cf. [10].

Proof. In [28], the author proved that the following conditions on the product $X\times Y$ of pseudo-compact spaces (containing infinitely many points) are equivalent.
(1) Both X and Y are pseudo-compact and $Pr_X[\boldsymbol{Z}(F)]$ is a closed subset of X for each $F\in C(X\times Y)$,
(2) Both X and Y are pseudo-compact and $Pr_Y[\boldsymbol{Z}(F)]$ is a closed subset of Y for each $F\in C(X\times Y)$,
(3) $\beta X\times\beta Y=\beta(X\times Y)$.
(4) $X\times Y$ is pseudo-compact.
(The equivalence of (3) and (4) is due to Henriksen and Isbell [12] and Glicksberg [10].) The necessity of the condition follows immediately from this facts: Since Z is compact, Pr_X is a closed mapping and hence $X\times Z$ is pseudo-compact. Therefore $Pr_Z[\boldsymbol{Z}(F)]$ is a clased subset of Z for each $F\in C(X\times Z)$. To prove the sufficiency of the condition, let us note that if Z is a compact space containing infinitely many points, then there is a continuous function $f\in C(Z)$ whose zero-set is not open (C. f. [28].) Suppose that X is not pseudo-compact and let $h(x)$ be an unbounded continuous function on X. Define a continuous function $F\in C(X\times Z)$ by letting $F(x, p)=|h(x)|\cdot|f(p)|-1$, where $f(p)$ is a continuous function on Z whose zero-set is not open. Then, $Pr_Z[\boldsymbol{Z}(f)]$ is not closed as may easily be seen.

Theorem 3.12. *X is compact if and only if the projection Pr_Z of $X\times Z$ onto Z is a closed mapping for any compact space E.*

We omit the proof, which is easy.

In his paper [5], C. H. Dowker proved that the product of a normal, countably paracompact space and a compact metrizable space is normal (and countably paracompact). On the other hand, Theorem 3.9 shows that if the product of a space X with some compact metrizable space is normal, then X is (normal and) countably paracompact. Therefore, we have the following characterization of normal and countably paracompact spaces.

Theorem 3.13. *Let M be any compact metrizable space containing infinitely many points. Then, X is normal and countably* *paracompact if and only if* $X\times M$ *is normal.*

§ 4. Comments.

By virtue of Theorem 3.1, we see that the normality of $X\times BX$, where BX is any compactification of X, implies the paracompactness of X. Therefore, Michael's problem [19] can be reduced to ask whether the product of a paracompact space with any metrizable space is normal or not, as the following theorem shows.

Theorem 4.1. *The following conditions on a space* X *are equivalent.*

(1) $X\times Y$ *is normal for any paracompact space* Y.

(2) $X\times Y$ *is paracompact for any paracompact space* Y.

Proof. Let BX and BY denote any compactification of X and Y respectively. Assume that (1) is true, and consider the product $(X\times Y)\times(BX\times BY)=X\times(Y\times BX\times BY)$. Since $Y\times BX\times BY$ is paracompact, it follows that $(X\times Y)\times(BX\times BY)$ is normal. On the other hand, it is evident that $BX\times BY$ is a compactification of $X\times Y$. Therefore, $X\times Y$ is paracompact by virtue of Theorem 3.1. The implication (2)$\Rightarrow$(1) is evident.

We now notice again that a paracompact space is characterized by the property that

(A) $X\times Z$ *is normal for any compact space* Z.

While, a normal and countably paracompact space X is characterized (Theorem 3.13) by the property that

(B) $X\times M$ *is normal for any compact metrizable space* M.

In this point of view, it may be stated that the essential importance of Michael's problem lies in the following problem.

Problem 1. *What is the space* X *satisfying the following condition?*

(C) $X\times Y$ *is normal for any paracompact space* Y.

Now, let us call a space X satisfying condition (C) a π-space, then we can see from Theorem 4.1 that every π-space is paracompact and that $X \times Y$ is a π-space if and only if both X and Y are π-space. Thus, we see that the property (C) is a productive property. In [19], E. Michael proved that every σ-compact space is a π-space.

Another problem concerning Michael's problem is as follows:

Problem 2. *What is the space X satisfying the following condition?*

(D) *$X \times Y$ is normal for any metrizable space Y.*

An answer to the Problem 2 has also been given by E. Michael. [19]. He proved that a paracompact, perfectly normal space X satifies condition (D). Recently, Z. Frolik [8] has proved that a paracompact, topologically complete (in the sense of E. Čech [3]) space satisfies condition (D).

Likewise, several problems may be considered. Among them, the following seems to be interesting.

Problem 3. *What is the space X satisfying the following condition?*

(E) *$X \times Y$ is normal for any second countable space Y.*

Let E be any dense subspace of X and let βE be the Stone-Čech compactification of E. Since BX is a compactification of E, BX is the image of βE under a (unique) continuous mapping by virtue of Theorem 1.1. It follows that $X \times BX$ is the image of $X \times \beta E$ under a closed continuous mapping. Since the closed continuous image of a normal space is normal, we can see that the normality of $X \times \beta E$ implies the paracompactness of X. However, it is not known to the author whether the normality of $X \times BE$, where BE is any compactification of E, implies the paracompactness of X. "*Does the normality of $X \times BE$ implies the normality of $X \times \beta E$?*" This is closely related to the following problem presented by K. Nagami (c. f. [23].)

Problem 4. *Let f be a closed continuous mapping of X onto Y such that $f^{-1}(y)$ is compact for each $y \in Y$ and that the image of any proper closed subset of X is a proper closed subset of Y. Is* *it trus that X is normal whenever Y is normal?*

It is obvious that the mapping of $X\times\beta E$ onto $X\times BE$ satisfies the above condition.

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