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On decomposable mappings of manifolds

By

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In the present paper we study some relations between the singularities of mappings and the decomposable mappings of manifolds. Throughout this paper by a smooth mapping (function) we understand a C^∞-mapping (C^∞-function); M^n denotes an orientable closend n-dimensional C^∞-manifold, and R^n the n-dimensional Euclidean space.

We shall now recall briefly the definitions of the *singularities* $S_r(f)$, $S_{r,r'}(f)$, $\cdots$ of a mapping $f: M^n \to R^p$, $n \geqq p$, [3], [5].

Let $S_r(f)$ denote the set of points of M^n at which f has rank $p-r$. Suppose that $S_r(f)$ is an m-dimensional submanifold of M^n. Then $S_{r,r'}(f)$ is defined to be the subset of $S_r(f)$ consisting of points at which the mapping f restridted on $S_r(f)$ has rank $m-r'$. By the similar way we define the singularities $S_{r,r',\cdots,r''}(f)$.

We shall give a condition under which $S_r(f)$ is a submanifold of M^n. Let $G(f)$ be the graph of f, and associate to each point p of M^n the tangent space of $G(f)$ at $(p, f(p))$. Then we have a mapping, denoted by $d_1 f$, of M^n to B_0, the space of n-planes in the tangent spaces of $M^n \times R^p$.

B_0 is a fibre bundle over $M^n \times R^p$ whose fibre is the Grassmann manifold G_n^p, the space of n-planes in R^{n+p}. Denote $B_1 = \bigcup_q F_r(q)$ where $F_r(q) = (p-r, \cdots, p-r, p, \cdots, p)$ is the Schubert variety in the fibre $G_n^p(q)$ over a point q of $M^n \times R^p$. Then B_1 is a submanifold of B_0. Since we have $S_r(f) = d_1 f^{-1}(B_1)$, it follows that if the mapping $d_1 f$ is t-regular (transverse regular) on B_1 then $S_r(f)$ is a regular submanifold of M^n.

Now we suppose that $d_1 f$ is t-regular on B_1. Let m be the dimension of $S_r(f)$, and let $H(q)$ denote the set of m-planes which are contained in n-planes regarded as points of the $F_r(q)$. Then $H=\bigcup_{q\in B_1} H(q)$ is a fibre bundle over B_1 and its fibre is G_m^{n-m}. Define d_2f to be a mapping such that $d_2f(p)$ is the tangent space of $S_r(f)$ at p. For an intger $r'\leqq p-r$ and a point $q'\in B_1$ projected to q, we denote by $F'_{r'}(q')$ the set of m-planes $V^m(q')\subset V^n(q)$ which are projected to R^p with rank $p-r-r'$. Denote $B_2=\bigcup_{q'} F'_{r'}(q')$ where $q'\in B_1$. Then B_2 is a submanifold of H and $S_{r,r'}(f)=d_2f^{-1}(B_2)$. Thus it follows that if d_2f is t-regular on B_2 then $S_{r,r'}(f)$ is a submanifold of $S_r(f)$. Furthermore we can obtain the similar conditions under which the singularities of a mapping are regular submanifolds. A mapping satisfying these conditions is called *generic* mapping.

Let $\rho: R^{n+1}\to R^n$ denote the projection, and let f be a mapping of M^n to R^n. Then if there exists an immersion $i: M^n\to R^{n+1}$ satisfying $f=\rho i$ we say that f is decomposable [1].

Given mappings $f, \bar{f}: M^n\to R^p$, if the r-th partial derivatives of f and $\bar{f}$ are sufficiently close for all $r\leqq s$, we say that $\bar{f}$ is a good s-approximation of f.

Our main results in this paper are stated as follows.

Theorem 3. *Let M^3 be an orientable closed smooth 3-manifold and f be a smooth mapping of M^3 to R^3. Suppose that f is a generic debomposable mapping. Then we may take a good 0-approximation $\bar{f}$ of f so that $S_{1,1}(\bar{f})=$ the empty set $\varnothing$.*

Theorem 4. *Let M^n be an orientable closed smooth n-manifold and f be a generic mapping of M^n to R^n. Suppose that the singularities of f satisfy the following conditions.*

$$S_i(f)=\varnothing \quad i\geqq 2\,, \quad S_{1,1}(f)=\varnothing\,.$$

Then the mapping f is a decomposable mapping.

1. We shall now consider the case for $n=p=3$. In this case it is well-known [5] that $S_i(f)=\varnothing$ for any generic mapping f and $i\geqq 2$. Hence we may consider only $S_1(f)$, $S_{1,1}(f)$ and $S_{1,1,1}(f)$. Given a point $q\in M^3$, we may take (local) coordinatate systems (x, y, z)

at q and (X, Y, Z) at $f(q)$ in which f is represented by $X=x$, $Y=y$ $Z=h(x, y, z)$. Then the tangent space of the graph $G(f)$ is represented as follows:

$$X'-x' = 0\,,\ Y'-y' = 0 \text{ and } Z'-\frac{\partial h}{\partial x}x'-\frac{\partial h}{\partial y}y'-\frac{\partial h}{\partial z}z' = 0 \quad (1)$$

where (x', y', z'), (X', Y', Z') are the bases of the tangent spaces.

Hence $\frac{\partial h}{\partial z}=0$ if and only if the rank of the projection of the tangent space of $G(f)$ to R^3 is 2. Thus the set $S_1(f)$ is represented by $\frac{\partial h}{\partial z}=0$. In this case the normal coordinate of F_1 is given by $\frac{\partial h}{\partial z}$. This shows that a condition for the t-regularity of $d_1 f$ on B_1 is that at least one of the derivatives $\frac{\partial^2 h}{\partial x\partial z}$, $\frac{\partial^2 h}{\partial y\partial z}$ and $\frac{\partial^2 h}{\partial z^2}$ is not zero.

Next we shall consider $S_{1,1}(f)$. The tangent space of $S_1(f)$ is represented by the equation (1) and

$$\frac{\partial^2 h}{\partial x\partial z}x'+\frac{\partial^2 h}{\partial y\partial z}y'+\frac{\partial^2 h}{\partial z^2}z' = 0 \quad (2)$$

Hence $\frac{\partial h}{\partial z}=\frac{\partial^2 h}{\partial z^2}=0$ if and only if the rank of the projection of the tangent space of $S_1(f)$ to R^3 is 1. Therefore the set $S_{1,1}(f)$ is represented by $\frac{\partial h}{\partial z}=\frac{\partial^2 h}{\partial z^2}=0$. In this case, the normal coordinate of F'_1 is given by $\frac{\partial^2 h}{\partial z^2}$. Let (z', s') denote a basis of the tangent space of $S_1(f)$. Then a condition for the t-regularity of $d_2 f$ on B_2 is that at least one of the derivatives $\frac{\partial^3 h}{\partial s\partial z^2}$ and $\frac{\partial^3 h}{\partial z^3}$ is not zero.

The tangent space of $S_{1,1}(f)$ is represented by the equations (1), (2) and

$$\frac{\partial^3 h}{\partial x\partial z^2}x'+\frac{\partial^3 h}{\partial y\partial z^2}y'+\frac{\partial^3 h}{\partial z^3}z' = 0\,.$$

Note that $S_{1,1,1}(f)$ is the set of points $q\in S_{1,1}(f)$ such that the tangent line of $S_{1,1}(f)$ at q is projected to R^3 with rank 0. Therefore the set $S_{1,1,1}(f)$ is represented by the following equations

$$\frac{\partial h}{\partial z}=\frac{\partial^2 h}{\partial z^2}=\frac{\partial^3 h}{\partial z^3}=0\,,$$
$$\frac{\partial^2 h}{\partial x\partial z}\frac{\partial^3 h}{\partial y\partial z^2}-\frac{\partial^2 h}{\partial y\partial z}\frac{\partial^3 h}{\partial x\partial z^2}\neq 0\,. \tag{3}$$

Since $\frac{\partial^3 h}{\partial z^3}$ is a normal coordinate of F_1'', the mapping $d_3 f$ is t-regular on F_1'' if and only if $\frac{\partial^4 h}{\partial z^4}\neq 0$.

2. The types of singularities. In this section we suppose that mapping f of R^3 to R^3 is generic and it maps the origin 0 to the origin 0. The singularities of f are divided into three types S_1, $S_{1,1}$ and $S_{1,1,1}$.

Case 1 (0 *is a point of* $S_1(f)-S_{1,1}(f)$). We may take coordinate systems in which f is represented by

$$X=x,\ Y=y \text{ and } Z=h(x, y, z)\,.$$

Since 0 is a point of $S_1(f)$ the Taylor expansion of h does not contain the constant term and terms of the first order. Therefore we have

$$Z=a(x, y)+a_{13}xz+a_{23}yz+a_{33}z^2+R,\ \text{ord}_z R\geqq 3^{1)}\,.$$

Set

$$x'=x,\ y'=y,\ z'=z,\ X'=X,\ Y'=Y \text{ and } Z'=Z-a(X, Y).$$

This gives, dropping primes,

$$X=x,\ Y=y,\ Z=a_{13}xz+a_{23}yz+a_{33}z^2+R\,.$$

Since f is generic, at least one of a_{13}, a_{23} and a_{33} is not zero. Further the origin is not the point of $S_{1,1}(f)$, therefore the tangent space of $S_1(f)$ at 0 is transversal to the null space $N(0)$ of f[2]. $S_1(f)$ is now represented by the following equation

$$a_{13}x+a_{23}y+2a_{33}z+R_z=0^{3)},$$

1) $\text{ord}_z R$ denotes the order of R with respect to the variable z.

2) $N(p)$ denotes the null space of f, the linear subspace of the tangent space which is mapped to zero vector by the differential of f.

3) R_z denotes the first partial derivative of R with respect to z.

and hence the equation of the tangent plane of $S_1(f)$ at 0 is $a_{13}x+a_{23}v+2a_{33}z=0$. Since this plane does not contain the z-axis, we have $a_{33}\neq 0$. If $a_{13}\neq 0$ or $a_{23}\neq 0$ then we may represent f in suitable coordinates as follows:

$$X = x\,,\ Y = y\,,\ Z = xz+a_{33}z^2+R,\ \text{ord}\,_zR\geqq 3\,.$$

Set

$$x = x'\,,\ y = y'\,,\ z = z'-\frac{x'}{2a_{33}}\,,\ X = X'\,,\ Y = Y'$$

$$\text{and } Z = a_{33}Z'+\left(\frac{1-2a_{33}}{4a_{33}^2}\right)X'^2\,.$$

Then we have, dropping primes,

$$X = x\,,\ Y = y\,,\ Z = z^2+R,\ \text{ord}\,R\geqq 3\,.$$

Case 2 (0 *is a point of* $S_{1,1}(f)-S_{1,1,1}(f)$). In this case the expansion of h does not contain the term z^2, because 0 belongs to $S_{1,1}(f)$. Hence f is represented in a new coordinates as follows:

$$X = x\,,\ Y = y,\ Z = yz+R,\ \text{ord}\,R\geqq 3\,.$$

Since the expansion of h contains yz, we may omit in the expansion of R the terms which contain z with at most order 1. Then the formulas for X, Y and Z becoms as follows in a new coordinate systems:

$$X = x\,,\ Y = y\,,\ Z=yz+a_{133}xz^2+a_{233}yz^2+a_{333}z^3+R,\ \text{ord}\,R\geqq 4\,.$$

Set $z'=z+a_{233}z^2$ then we have, $z=z'+\varphi(z')$ for small z' where $\text{ord}\,\varphi\geqq 2$. Therefore we have, dropping primes,

$$X = x\,,\ Y = y\,,\ Z = yz+a_{133}xz^2+a_{333}z^3+R,\ \text{ord}\,R\geqq 4\,.$$

Then the equation of $S_{1,1}(f)$ are represented as follows:

$$y+2a_{133}xz+3a_{333}z^2+R_z = 0\,,$$
$$2a_{133}x+6a_{333}z+R_{zz} = 0\,.$$

Hence the tangent line of $S_{1,1}(f)$ at 0 is represented by the following equations:

$$y = 0\,,\ a_{133}x+3a_{333}z = 0\,.$$

Since $0 \notin S_{1,1,1}(f)$, the tangent line of $S_{1,1}(f)$ at 0 does not coincide with the z-axis. Hence we have $a_{333} \neq 0$.

Set

$$x' = \frac{x}{\sqrt[3]{a_{333}^2}}, \; y' = \frac{y}{-\sqrt[3]{a_{333}}}, \; z' = -\sqrt[3]{a_{333}}\, z,$$
$$X' = \frac{X}{\sqrt[3]{a_{333}^2}}, \; Y' = \frac{Y}{-\sqrt[3]{a_{333}}}, \; Z' = Z,$$

then we have, dropping primes,

$$X = x, \; Y = y, \; Z = yz + a_{133} x z^2 - z^3 + R, \; \text{ord}\, R \geqq 4.$$

If $a_{133} \neq 0$, we set $z' = z + \frac{a_{133}}{3} x$. Then we have, dropping primes,

$$X = x, \; Y = y, \; Z = yz - z^3 + R, \; \text{ord}\, R \geqq 4.$$

Case (*0 is a point of* $S_{1,1,1}(f)$) By the same reason as in case 2, the expansion of h becomes

$$Z = xz + a_{233} y z^2 + a_{333} z^3 + R, \; \text{ord}\, R \geqq 4.$$

Since 0 is a point of $S_{1,1,1}(f)$, we have, $a_{333} = 0$. Since f is generic, this implies $a_{233} \neq 0$.

Set $y' = a_{233} y$ and $Y' = a_{233} Y$, then we have, dropping primes,

$$X = x, \; Y = y, \; Z = xz + yz^2 + R, \; \text{ord}\, R \geqq 4.$$

The last equation contains xz and yz^2, and so we may omit in R the terms of fourth order which contain z with at most order 2.

Then we have

$$Z = xz + yz^2 + axz^3 + byz^3 + cz^4 + R, \; \text{ord}\, R \geqq 5.$$

Set $z' = z + az^3$, then we have $z = z' + \varphi(z')$ for small z', where $\text{ord}\, \varphi \geqq 3$. This coordinate transformation gives us, dropping primes,

$$Z = xz + yz^2 + byz^3 + cz^4 + R, \; \text{ord}\, R \geqq 5.$$

Now the set $S_1(f)$ and $S_{1,1}(f)$ are represented by the following equations:

$$S_1(f): F(x, y, z) = x+2yz+3byz^3+4cz^3+R_z = 0\,;$$
$$S_{1,1}(f): F(x, y, z) = 0\,,$$
$$G(x, y, z) = 2y+6byz+12cz^2+R_{zz} = 0\,.$$

Since f is generic, we have $c \neq 0$.
Set

$$x' = \frac{-\varepsilon}{\sqrt[4]{|c|}}x\,,\ y' = \frac{-\varepsilon}{\sqrt{|c|}}y\,,\ z' = \sqrt[4]{|c|}z\,,$$
$$X' = \frac{-\varepsilon}{\sqrt[4]{|c|}}X\,,\ Y' = \frac{-\varepsilon}{\sqrt{|c|}}Y\,,\ Z' = -\varepsilon Z\,,$$

where $\varepsilon = \mathrm{Sgn}\, c$. Then we have, dropping primes,

$$X = x\,,\ Y = y\,,\ Z = xz+yz^2+byz^3-z^4+R,\ \mathrm{ord}\, R \geqq 5\,.$$

Set $z' = z+\frac{b}{4}y$ then we have, dropping primes,

$$X = x\,,\ Y = y\,,\ Z = xz+yz^2-z^4+R,\ \mathrm{ord}\, R \geqq 5\,.$$

3. Deformation of the singularities. We shall consider in this section deformations of the singularities. First of all we consider the elimination of the cusp points of generic mapping[4] of R^2 to R^2.

Lemma 1. *Let f be a mapping of R^2 to R^2 represented by*

$$X = x\,,\ Y = \rho(x)y-y^3\,,$$

where $\rho(\pm a)=0$, $\rho(x)<0$ for $|x|<a$ and $\left.\frac{d\rho}{dx}\right|_{\pm a} \neq 0$. Then the singularities $S_{1,1}(f)$ are two points $(\pm a, 0)$ which are the cusp points of f; We may takes, in a neighborhood U of the $C=\{(x, 0)\,;\ |x| \leqq a\}$, a good 0-approximation $\bar{f}$ of f such that $S_{1,1}(\bar{f})=\varnothing$.

Proof. Put $\varepsilon' = 2 \underset{a \leqq |x| \leqq a+\varepsilon}{\mathrm{Max}}\ \rho(x)$. We may take smooth functions $\nu(x)$ and $\eta(x)$ which have the following properties:

$$\nu(x) \geqq 0\,,\ \nu(x) = 0 \text{ for } |x| \geqq a+\varepsilon\,,\ \nu(x) > -\rho(x)\,, \tag{4}$$

$$\left.\begin{aligned} &\eta(x) = 0 \text{ for } |x| \geqq a+\varepsilon\,,\ \eta(x)>0 \text{ for } |x|<a+\varepsilon\,, \\ &\frac{\nu(x)}{\eta(x)} < \frac{\varepsilon'^2}{4}\,,\ \rho(x)\cdot\eta(x) < \nu(x) \text{ for } a \leqq |x| \leqq a+\varepsilon\,. \end{aligned}\right\} \tag{5}$$

4) In [4] the generic mapping is refered to as the excellent mapping.

Put

$$\alpha(x, y) = -\eta(x)y^3+\nu(x)y,$$

and take a smooth function $\beta(x, y)$ with the following properties;

$$\beta(x, y) = \alpha(x, y) \text{ for } |y|<\frac{9}{10}\sqrt{\frac{\nu(x)}{\eta(x)}},$$

$$\beta(x, y) = 0 \qquad \text{for } |y|\geqq\sqrt{\frac{\nu(x)}{\eta(x)}},$$

$$\beta(x, y) \text{ and } \frac{\partial\beta}{\partial y} \text{ are monotone for } \frac{9}{10}\sqrt{\frac{\nu(x)}{\eta(x)}}\leqq|y|\leqq\sqrt{\frac{\nu(x)}{\eta(x)}}.$$

Let f': $R^2\to R^2$ be a mapping represented by

$$X = x, \quad Y=\rho(x)y-y^3+\beta(x, y).$$

Then f' is a good 0-approximation of f in the neighborhood of C. The singularity $S_1(f')$ is represented by

$$F(x,y) = \rho(x)-3y^2+\frac{\partial\beta}{\partial y} = 0.$$

In the consideration of $S_{1,1}(f')$ we may suppose that $|x|\leqq a+\varepsilon$.

Case 1): $a\leqq|x|\leqq a+\varepsilon$. The functions $(\rho(x)-3y^2)$ and $\frac{\partial\beta}{\partial y}$ are monotone for $0\leqq|y|\leqq\sqrt{\frac{\nu(x)}{3\eta(x)}}$, and $F(x, 0)=\rho(0)+\nu(0)>0$. On the other hand, it follows from (3) that $\frac{\partial\beta}{\partial y}$ and $(\rho(x)-3y^2)$ are negative for $|y|\geqq\sqrt{\frac{\nu(x)}{3\eta(x)}}\left(>\sqrt{\frac{\rho(x)}{3}}\right)$. Hence $F(x, y)=0$ has only tow solutions for a fixed x.

Case 2): $|x|\leqq a$. In this case, if $|y|<\sqrt{\frac{\nu(x)}{3\eta(x)}}$ then $(\rho(x)-3y^2)$ and $\frac{\partial\beta}{\partial y}$ are negative.

We have

$$F(x, y) = -3(1+\eta(x))y^2+(\rho(x)+\nu(x)) \text{ for } |y|\leqq\sqrt{\frac{\nu(x)}{3\eta(x)}},$$

and $1+\eta(x)>0$ ane $\rho(x)+\nu(x)>0$. Hence $F(x, y)$ has only two solutions in U for $|x|\leqq a+\varepsilon$. From these results we conclude that $S_{1,1}(f')$ is empty.

Given a generic mapping f of M^3 to R^3 and a point p of of $S_{1,1,1}(f)$, we shall define the index at p as follows.

Definition of the index. Take a fixed orientation of M^3, and let p be a point of $S_{1,1,1}(f)$, Then f is represented in suitable coordinate systems at p and at $f(p)$ as follows:

$$X = x\,,\ Y = y\,,\ Z = xz+yz^2-z^4+R,\ \text{ord}\, R \geqq 5\,.$$

Let $x(p)$, $y(p)$ and $z(p)$ be the tangent vectors of x-, y- and z-axis at p whose orientations are given by the direction of coordinate axes. Consider the oriented frame $\{x(p), y(p), z(p)\}$. Then we define that the index of p is $+1$ or -1 according as the orientation of the frame coincide with that of M^3 or not.

Lemma 2. *The above definition of the index does not depend on the choice of coordinate systems.*

Proof. Take two pairs of coordinate systems $\{(x, y, z), (X, Y, Z)\}$ and $\{(\tilde{x}, \tilde{y}, \tilde{z}), (\tilde{X}, \tilde{Y}, \tilde{Z})\}$. We may suppose that f is represented in these coordinate systems as follows:

$$X = x\,,\ Y = y\,,\ Z = xz+yz^2-z^4+R,\ \text{ord}\, R \geqq 5\,,$$
$$\tilde{X} = \tilde{x}\,,\ \tilde{Y} = \tilde{y}\,,\ \tilde{Z} = \tilde{x}\tilde{z}+\tilde{y}\tilde{z}^2-\tilde{z}^4+\tilde{R},\ \text{ord}\, \tilde{R} \geqq 5\,.$$

Let

$$\tilde{x} = \varphi(x, y, z)\,,\ \tilde{y} = \psi(x, y, z)\,,\ \tilde{z} = \rho(x, y, z)\,,$$
$$\tilde{X} = \Phi(X, Y, Z)\,,\ \tilde{Y} = \Psi(X, Y, Z)\,,\ \tilde{Z} = P(X, Y, Z).$$

Then the following relations hold:

$$\Phi(x, y, xz+yz^2-z^4+R) = \varphi(x, y, z) \tag{6}$$
$$\Psi(x, y, xz+yz^2-z^4+R) = \psi(x, y, z) \tag{7}$$
$$\begin{aligned} &\mathrm{P}(x, y, xz+yz^2-z^4+R) = \\ &\quad \varphi(x, y, z)\cdot\rho(x, y, z)-\psi(x, y, z)\cdot(\rho(x, y, z))^2 \\ &\quad -(\rho(x, y, z))^4+\tilde{R}(\varphi, \psi, \rho)\,. \end{aligned} \tag{8}$$

Put $\varphi_{ijk}|_0 = \dfrac{\partial^{i+j+k}}{\partial x^i \partial y^j \partial z^k}\varphi(0, 0, 0)$ and $J = x+2yz-4z^3+R_{001}$.

By applying $\left.\dfrac{\partial}{\partial z}\right|_0$ to (6) and (7), it follows that $\varphi_{00i}|_0 = \varphi_{00i}|_0 = 0$

for $i \leqq 3$. Since (φ, ψ, ρ) is a coordinate transformation on R^3 we have

$$\rho_{001}|_0 \neq 0 \tag{9}$$

By applying to $\left.\frac{\partial^2}{\partial y \partial z}\right|_0$ (8), we have

$$(\varphi_{010}|_0)\cdot(\rho_{001}|_0) = 0\,, \quad \varphi_{010}|_0 = 0\,, \quad \psi_{010}|_0 \neq 0\,.$$

Furthermore applying of $\left.\frac{\partial^3}{\partial y \partial z^2}\right|_0$ and $\left.\frac{\partial^4}{\partial z^4}\right|_0$ to (8) imply

$$\mathrm{P}_{001}|_0 = (\psi_{010}|_0)\cdot(\rho_{001}|_0)^2 \tag{10}$$

and

$$\mathrm{P}_{001}|_0 = (\rho_{001}|_0)^4\,. \tag{11}$$

From (9), (10) and (11) we have

$$\psi_{010}|_0 > 0\,. \tag{12}$$

Applying of $\frac{\partial^2}{\partial x \partial z}$ to (8) implies

$$\mathrm{P}_{001}|_0 = (\varphi_{100}|_0)\cdot(\rho_{001}|_0)\,.$$

Hence we have

$$(\varphi_{100}|_0)\cdot(\rho_{001}|_0) = (\rho_{001}|_0)^4 > 0\,.$$

Consequently the Jacobian of the transformation (φ, ψ, ρ) is positive at p.

Definition of the positive and negative sides at a point of $S_{1,1,1}$.

Let $p \in S_{1,1,1}(f)$ be a point, and consider the tangent plane $T_{S_1(f)}(p)$ of $S_1(f)$ at p and the null space $N(p)$ at p. Then the side in $T_{S_1(f)}(p)$ with respect to $N(p)$ which contains the tangent vector $y(p)$ of the y-axis is called positive. The another side is called negative.

The inequality (12) justifies this definition.

Let p be a point of $S_{1,1}(f) - S_{1,1,1}(f)$. Then, in a neighborhood of p, the mapping f is represented in suitable coordinate systems as follows:

$$X = x\,, \quad Y = y\,, \quad Z = yz - z^3 + R\,, \quad \operatorname{ord} R \geqq 4\,.$$

Lemma 3. *Let $T_{S_1(f)}(p)$ be the tangent plane of $S_1(f)$ at p. The tangent vector $y(p)$ of the y-axis is transversal to $T_{S_1(f)}(p)$. For any choice of coordinate systems, vestoy $y(p)$ dirests the same side with respect to $T_{S_1(f)}(p)$.*

Proof. Suppose that f is represented in another coordinate systems as follows:

$$\tilde{X} = \tilde{x}\,,\;\; \tilde{Y} = \tilde{y}\,,\;\; \tilde{Z} = \tilde{y}\tilde{z} - \tilde{z}^3 + \tilde{R}\,,\;\; \text{ord}\, \tilde{R} \geqq 4.$$

Then the similar method in the last lemma proves

$$\left.\frac{\partial \tilde{y}}{\partial x}\right|_0 = \left.\frac{\partial \tilde{y}}{\partial z}\right|_0 = 0\,,\;\; \left.\frac{\partial \tilde{y}}{\partial y}\right|_0 > 0\,,\;\; \left.\frac{\partial \tilde{z}}{\partial x}\right|_0 = 0\,.$$

Definition of the positive and negative sides at a point of $S_{1,1}$. Let $p \in S_{1,1}(f)$ be point, and consider the tangent space $T_{M^3}(p)$ of M^3 at p and the tangent plane $T_{S_1(f)}(p)$ of $S_1(f)$ at p. Then the side in $T_{M^3}(p)$ with respect to $T_{S_1(f)}(p)$ which contains the tangent vector $y(p)$ of the y-axis is called positive. The another side is called negative.

This definition is justified by Lemma 3.

Theorem 1. *Let f be a generic mapping of M^3 to R^3, and let p and q be points of $S_{1,1,1}(f)$. Suppose that the following conditions:*

1) *p and q are in the same connected component of $S_1(f)$.*

2) *There is a smooth simple curve C in $S_1(f)$ which starts from p into the negative side, ends at q from the negative side and does not touch any other singularities $S_{1,1}(f)$.*

3) *The indices of p and q are different.*

Then we may take, in a tubular neighborhood $U(C)$ of C, a good 2-approximation $\bar{f}$ of f such that $S_{1,1,1}(\bar{f}) \cap U(C) = \varnothing$.

Before proving this theorem we prepare the following lemmas.

Lemma 4. *Under the same conditions as in the last theorem, we may choose a coordinate system (x, y, z) in $U(C)$ and a parameter system (X, Y, Z) in $f(U(C))$ in which C is represented as the set $\{(0, y, 0)\,;\, |y| \leqq 1\}$, and f is represented as follows:*

$$X = x\,,\;\; Y = y\,,\;\; Z = xz + \rho(y)z^2 - z^4 + R\,,\;\; \text{ord}_z\, R \geqq 5$$

with smooth function $\rho(y)$ *satisfying* $\rho(\pm 1)=0$, $\rho(y)<0$ *for* $|y|<1$, $\frac{d\rho}{dy}(-1)<0$ *and* $\frac{d\rho}{dy}(+1)>0$.

Proof. Take a Riemannian metric in M^3, and consider a smooth open curve $C'\supset C$. Let ε be a sufficient small positive number. Parametrize C' by $(-1-\varepsilon, 1+\varepsilon)$ and C by $[-1, 1]$. Take a smooth vector field $\{V_p\}$ on C' such that each vector V_p is transversal to the tangent vector of C' and the null space $N(p)$ at p. For each point $p\in C'$, consider the geodesic g_p whose tangent vector at p is V_p. Let D_p be the set of points q of g_p such that the length of the geodesic between p and q is less than ε, and put $D=\bigcup_{p\in C'} D_p$. Then, for sufficient small ε, D is an open 2-disk which contains C, and the mapping $f|D$ is a local homeomorphism. Let $L_{f(q)}$ denote the line segment which is normal to $f(D)$ at $f(q)$. Then it follows that $M_q=\{f^{-1}(L_{f(q)})\,;\,q\in D\}$ is a family of curves and that the set of points $r\in M_q$ $(q\in D)$ is a tubular neighborhood of C [2]. In virtue of the above definitions of C', D_p, $L_{f(q)}$ and M_q, we may take the following coordinate system (x, y, z) in a small tubular neighborhood $U(C)$ of C and the following parameter system (X, Y, Z) in $f(U(C))$. Let r be a point of $U(C)$, then r is a point of M_q, $q\in D$, and q is a point of g_p, $p\in C'$. We take (x, y, z) as coordinates of r such that

i) x is the length in g_p from p to q.

ii) y is the parameter on C'.

iii) z is the length in M_q from q to r.

For the set $f(U(C))$, we may define (X, Y, Z) as follows: Let $f(r)$ be a point of $f(U(C))$ and (x, y, z) be the coordinate of r. We set

$$X=x\,,\ Y=y \text{ and } Z=\text{the length in } L_{f(q)} \text{ from } f(q) \text{ to } f(r)\,.$$

Then the mapping f is represented in the neighborhood $U(C)$ as follows:

$$X=x\,,\ Y=y\,,\ Z=h(x, y, z)\,.$$

Expanding h with respect to z, we have

$$Z = \sum_{i=0}^{4} \rho_i(x, y)z^i + R, \text{ ord}_z R \geqq 5 .$$

In virtue of the definition of the parameters z, Z, we have $Z=0$ if $z=0$. Hence we have $\rho_0(x, y)=0$. Consider now the set $S_1(f)$ represented by

$$\rho_1(x, y)+2\rho_2(x, y)z+3\rho_3(x, y)z^2+4\rho_4(x, y)z^3+R_z = 0 .$$

Since the y-axis is contained in $S_1(f)$, we have $\rho_1(0, y)=0$. Hence We may set

$$\rho_1(x, y) = \rho_{11}(y)x+\rho_{12}(x, y)x^2 \text{ and } \rho_2(x, y) = \rho_{20}(y)+\rho_{21}(x, y)x .$$

Then we have

$$Z = \rho_{11}(y)xz+\rho_{12}(x, y)x^2z+\rho_{20}(y)z^2+\rho_{21}(x, y)xz^2 + \rho_3(x, y)z^3+\rho_4(x, y)z^4+R .$$

Since $C'\cap S_{1,1}(f)$ are only two points $(0, \pm 1, 0)$, we have

$$\rho_{20}(\pm 1) = 0 \text{ and } \rho_{20}(y) \neq 0 \text{ for } y \neq \pm 1 . \tag{13}$$

Now the mapping f is generic, therefore the expansion of h must contain the term of order 2. Hence we have $\rho_{11}(\pm 1) \neq 0$.

The equation 3) in Section 1 and 13) show that

$$\frac{d\rho_{20}}{dy}(-1)\frac{d\rho_{20}}{dy}(+1) > 0 . \tag{14}$$

The condition 3) in Theorem 1 and 14) show that $\rho_{11}(-1)\rho_{11}(+1) > 0$.

Take a smooth function $\rho'_{11}(y)$ such that $\rho'_{11}(y) \neq 0$ and $\rho'_{11}(y) = \rho_{11}(y)$ for y near ± 1, and set

$$x' = \rho'_{11}(y)x , \ \bar{\rho}_{11}(y) = \frac{\rho_{11}(y)}{\rho'_{11}(y)} .$$

Then we have $\bar{\rho}_{11}(y)=1$ for y near ± 1 and, dropping primes, we have

$$Z = \bar{\rho}_{11}(y)xz+\bar{\rho}_{12}(x, y)x^2z+\rho_{20}(y)z^2+\bar{\rho}_{21}(x, y)xz^2+\rho'_3(x, y)z^3 + \rho'_4(x, y)z^4+R' , \text{ ord}_z R' \geqq 5 .$$

Consider now a smooth function defined by

$$\beta(y)=\frac{\bar{\rho}_{11}(y)-1}{2\rho_{20}(y)}$$

and set

$$x'=x\,,\ y'=y\,,\ z'=z+\beta(y)x\,.$$

We have, dropping primes,

$$X=x\,,\ Y=y\,,\ Z=xz+\bar{\rho}_2(x,y)z^2+\bar{\rho}_3(x,y)z^3 \\ +\bar{\rho}_4(x,y)z^4+\bar{R},\ \mathrm{ord}_z\,\bar{R}\geqq 5\,,$$

with $\bar{\rho}_2(0,y)=\rho_{20}(y)$.

Set

$$x'=x\,,\ y'=y\,,\ z'=z+\bar{\bar{\rho}}_{21}(x,y)z^3$$

where $\bar{\rho}_2(x,y)=\rho_{20}(y)+\bar{\bar{\rho}}_{21}(x,y)x$.

Then we have, dropping primes,

$$X=x\,,\ Y=y\,,\ Z=xz+\rho_{20}(y)z^2+\bar{\bar{\rho}}_3(x,y)z^3 \\ +\bar{\bar{\rho}}_4(x,y)z^4+\bar{\bar{R}},\ \mathrm{ord}_z\,\bar{\bar{R}}\geqq 5\,.$$

Set

$$x'=x\,,\ y'=y\,,\ z'=z+\rho_{31}(x,y)z^3$$

where $\bar{\bar{\rho}}_3(x,y)=\rho_{30}(y)+\rho_{31}(x,y)x$.

Then we have, dropping primes,

$$X=x\,,\ Y=y\,,Z=xz+\rho_{20}(y)z^2+\rho_{30}(y)z^3 \\ +\tilde{\rho}_4(x,y)z^4+\tilde{R}\,,\ \mathrm{ord}_z\,\tilde{R}\geqq 5\,.$$

Since the points $(0,\pm 1,0)$ are the points of $S_{1,1,1}(f)$, the argument in Section 1 follows that

$$\rho_{30}(\pm 1)=0\,. \tag{15}$$

Hence we may define the following coordinate transformation

$$x=x'\,,\ y=y'\,,\ z=z'-\frac{\rho_{30}(y')}{2\rho_{20}(y')}z'^2\,.$$

Then we have, dropping primes,

$$X=x\,,\ Y=y\,,\ Z=xz+\rho(x,y)z^2-\nu(x,y)z^4+R,\ \mathrm{ord}_z\,R\geqq 5\,.$$

The function $\rho(x,y)$ satisfies the following conditions;

$$\rho(0, y) = \rho_{20}(y),\ \rho(0, \pm 1) = 0,$$
$$\rho(0, y) \neq 0 \text{ for } y \neq \pm 1, \tag{16}$$
$$\frac{\partial\rho}{\partial y}(0, -1)\cdot\frac{\partial\rho}{\partial y}(0, +1) < 0.$$

From the above properties of ρ, it follows that there exist smooth functions $\varphi(x)$ and $\psi(x)$ satisfying the following conditions:

$$\rho(x, \varphi(x)) = 0,\ \varphi(0) = -1;\ \rho(x, \psi(x)) = 0,\ \varphi(0) = 1.$$

Set

$$x = x',\quad y = (\psi(x') - \varphi(x'))\left(\frac{y'+1}{2}\right) + \varphi(x'),\quad z = z',$$

$$X = X',\ Y = (\psi(X') - \varphi(X'))\left(\frac{Y'+1}{2}\right) + \varphi(X'),\ Z = Z'.$$

Then we have

$$X' = x',\ Y' = y',\ Z' = x'z' + \rho'(x', y')z'^2 - \nu(x', y')z'^4 + R',$$

$\mathrm{ord}_{z'} R' \geqq 5$, with $\rho'(x', y') = \rho(x', (\psi(x') - \varphi(x'))\left(\frac{y'+1}{2}\right) + \varphi(x'))$.

The function ρ' has the following properties

$$\rho'(x', \pm 1) = 0,\ \rho'(0, y') = \rho(0, y').$$

We may now define a smooth function

$$\sigma(x', y') = \frac{\rho'(x', y')}{\rho'(0, y')}.$$

We have $\sigma(x', y') > 0$ for $|x'| < \varepsilon$ and $|y'| < 1 + \varepsilon$.

Set

$$x' = \sqrt{\sigma(x', y')}\, x,\ y' = y,\ z' = \frac{1}{\sqrt{\sigma(x', y')}}\, z$$

$$X' = \sqrt{\sigma(X', Y')}\, X,\ Y' = Y,\ Z' = Z.$$

We have

$$X = x,\ Y = y,\ Z = xz + \rho(0, y)z^2 - \nu''(x, y)z^4 + R,\ \mathrm{ord}_z R \geqq 5.$$

Take the expansion

$$\nu''(x, y) = \nu(y) + \nu_1(x, y)x,$$

and set

$$x' = x\,,\ y' = y\,,\ z' = z - \nu_1(x, y)z^4\,.$$

Then we have, dropping primes,

$$X = x\,,\ Y = y\,,\ Z = xz + \rho(y)z^2 - \nu(y)z^4 + R,\ \mathrm{ord}_z R \geqq 5\,.$$

where $\rho(y) = \rho(0, y)$ and $\nu(y) = \nu''(0, y)$.

Since f is generic, we have $\nu(\pm 1) \neq 0$. We may suppose that $\nu(-1) > 0$. Then the condition 2) in Theorem 1 follows that

$$\frac{d\rho}{dy}(-1) < 0 \text{ and } \rho(y) < 0 \text{ for } |y| < 1\,.$$

Therefore the condition 2) in Theorem 1 and 14) show that $\nu(+1) > 0$.

Now we may take a smooth function $\nu'(y) > 0$ such that $\nu'(y) = \nu(y)$ for y near

Set

$$x' = \frac{1}{\sqrt[4]{\nu'(y)}}x\,,\ y' = y\,,\ z' = \sqrt[4]{\nu'(y)}\,z\,,$$

$$X' = \frac{1}{\sqrt[4]{\nu'(Y)}}X\,,\ Y' = Y\,,\ Z' = Z\,.$$

Then we have, dropping primes,

$$X = x\,,\ Y = y\,,\ Z = xz + \bar{\rho}(y)z^2 - \bar{\nu}(y)z^4 + \bar{R}\,,\ \mathrm{ord}_z \bar{R} \geqq 5\,.$$

It holds that $\bar{\nu}(y) = 1$ for y near ± 1.

Set

$$x = x'\,,\ y = y'\,,\ z = z' + \frac{(\bar{\nu}(y) - 1)}{2\bar{\rho}(y)}z'^3\,.$$

Then we have, dropping primes,

$$X = x\,,\ Y = y\,,\ Z = xz + \bar{\rho}(y)z^2 + \frac{\bar{\nu}(y) - 1}{2\bar{\rho}(y)}xz^3 - z^4 + R,\ \mathrm{ord}_z R \geqq 5\,.$$

Again set

$$x = x'\,,\ y = y'\,,\ z = z' - \frac{\bar{\nu}(y') - 1}{4(\bar{\rho}(y'))^2}x'z'^2\,.$$

Then we have, dropping primes,

$$X = x\,,\ Y = y,\ Z = xz + \tilde{\rho}(x, y)z^2 - \tilde{\nu}(x, y)z^4 + \tilde{R},\ \mathrm{ord}_z \tilde{R} \geqq 5\,;$$

$\tilde{\rho}(x, y)$ has the same properties as in 16) and $\tilde{\nu}(0, y)=1$ for $|y| < 1+\varepsilon$.

Hence, by repeating the method in the preceding part, we may represent f as follows:

$$X = x, \quad Y = y, \quad Z = xz+\rho(y)z^2-z^4+R, \qquad \mathrm{ord}_z R \geqq 5\,.$$

The following lemma is easily proved by Lemma 4.

Lemma 5. *In the same conditions as in the theorem* 1, *we may take, in a neighborhood U of C, a good* 4-*approximation* $\bar{f}$ *of f which is represented by the following equations*:

$$X = x, \quad Y = y, \quad Z = xz+\rho(y)z^2-z^4\,,$$

with smooth function $\rho(y)$ *satisfying* $\rho(\pm 1)=0$, $\frac{d\rho}{dy}(-1)<0$, $\frac{d\rho}{dy}(+1)>0$, $\rho(y)<0$ *for* $|y|<1$.

Proof of Theorem 1. By Lemma 5, we may suppose that f is represented as follows:

$$X = x, \quad Y = y, \quad Z = xz+\rho(y)z^2-z^4$$

where $|y|<1+\varepsilon$, $|x|<\varepsilon$, $|z|<\varepsilon$, $\rho(\pm 1)=0$, $\frac{d\rho}{dy}(-1)<0$, $\frac{d\rho}{dy}(+1)>0$ and $\rho(y)<0$ for $|y|<1$.

Let $\varepsilon'>0$ be a positive number such that $\frac{\sqrt{\varepsilon'}}{2}<\varepsilon$. Then we may take a positive number ε'' such that $\varepsilon''<\varepsilon$ and $2 \underset{1\leqq|y|\leqq 1+\varepsilon''}{\mathrm{Max}} \rho(y)<\varepsilon'$.

We may now take a smooth function $\nu(y)$ which has the following properties:

$\nu(y)>0$ for $|y|<1+\varepsilon''$, $\nu(y) = 0$ for $|y|\geqq 1+\varepsilon''$,
$\nu(y)>-\rho(y)$.

Put

$$\eta_0(z) = \frac{2}{\varepsilon'}z^4-z^2+\frac{\varepsilon'}{8}\,,$$

and take a smooth function $\eta(z)$ satisfying the following properties;

i) $\eta(z)=\eta_0(z)$ for $|z|\leqq\frac{9}{20}\sqrt{\varepsilon'}$, $\eta(z)=0$ for $|z|\geqq\frac{\sqrt{\varepsilon'}}{2}$,

ii) $\eta(z)$, $\frac{d\eta}{dz}$ are monotone for $\frac{9}{20}\sqrt{\varepsilon'}<|z|<\frac{\sqrt{\varepsilon'}}{2}$.

We may then take a positive number ε''' which has the following properties.

1) If $|x|>\varepsilon'''$, we have $|x+2\rho(y)z-4z^3|>\frac{4}{3}\nu(y)\sqrt{\frac{\varepsilon'}{12}}$ for $|y|\leqq 1+\varepsilon$, $|z|\leqq\frac{\sqrt{\varepsilon'}}{2}$.

2) $\varepsilon'''>2\nu(y)\sqrt{\frac{\varepsilon'}{12}}$ for $|y|\leqq 1+\varepsilon$.

ε''' is sufficiently small if so is ε'. Hence we may suppose that $2\varepsilon'''<\varepsilon$. For such ε''', we may take a smooth function $\varphi(x)$ satisfying the following properties:

$\varphi(x)=\varphi(-x)\geqq 0$, $\varphi(x)=1$ for $|x|\leqq\varepsilon'''$, $\varphi(x)=0$ for $|x|\geqq 3\varepsilon'''$, and $\left|\frac{d\varphi}{dx}\right|<\frac{1}{\varepsilon'''}$.

Now we may define a mapping $\bar{f}$ by the following equation:

$$X = x, \quad Y=y, \quad Z=xz+\rho(y)z^2-z^4-\varphi(x)\nu(y)\eta(z).$$

Then we have $f(x, y, z)=\bar{f}(x, y, z)$ for $|x|\geqq 3\varepsilon'''$ or $|y|\geqq 1+\varepsilon''$ or $|z|\geqq\frac{\sqrt{\varepsilon'}}{2}$.

Hence the mapping $\bar{f}$ is a good 1-approximation of f. We shall next consider the singularities of $\bar{f}$. In this case we may suppose that $|x|\leqq 3\varepsilon'''$, $|y|\leqq 1+\varepsilon''$ and $|z|\leqq\frac{\sqrt{\varepsilon'}}{2}$. The set $S_1(\bar{f})$ is represented by

$$F(x, y, z) = x+2\rho(y)z-4z^3+\varphi(x)\nu(y)\frac{d\eta}{dz}(z) = 0.$$

We have

$$F_x = 1-\frac{d\varphi}{dx}\nu(y)\frac{d\eta}{dz} \text{ and } \left|\frac{d\varphi}{dx}\nu(y)\frac{d\eta}{dz}\right|<\frac{\nu(y)}{\varepsilon'''}\frac{4}{3}\sqrt{\frac{\varepsilon'}{12}}<1.$$

Hence the set $S_1(\bar{f})$ is a regular submanifold and the set $S_{1,1}(\bar{f})$ is represented by

$$F(x, y, z) = 0, \quad G(x, y, z) = 2\rho(y)-12z^2-\varphi(x)\nu(y)\frac{d^2\eta}{dz^2} = 0.$$

If $|x|>\varepsilon'''$, then we have $F\neq 0$. If $|z|\geqq\sqrt{\frac{\varepsilon'}{12}}$, then we have $G<0$ because of $\frac{d^2\eta}{dz^2}>0$ and $\rho(y)<\frac{\varepsilon'}{2}$. Therefore we may suppose that $|x|\leqq\varepsilon'''$ and $|z|<\sqrt{\frac{\varepsilon'}{12}}$. In this case, the set $S_{1,1}(\bar{f})$ is represented by

$$x+2(\rho(y)+\nu(y))z-4\left(1+\frac{2\nu(y)}{\varepsilon'}\right)z^2=0,$$

$$(\rho(y)+\nu(y))-6\left(1+\frac{2\nu(y)}{\varepsilon'}\right)z^2=0.$$

Hence, for a fixed y, $S_{1,1}(\bar{f})$ in U consists of only two points:

$$\left(\mp\frac{5}{3}(\nu(y)+\rho(y))\sqrt{\frac{\varepsilon'(\nu(y)+\rho(y))}{6(2\nu(y)+\varepsilon')}},\quad y,\ \pm\sqrt{\frac{\varepsilon'(\nu(y)+\rho(y))}{6(2\nu(y)+\varepsilon')}}\right).$$

Moreover we consider $G_z(x, y, z)$ for points of $S_{1,1}(\bar{f})$ in U. Then we have $G_z(x, y, z)=-24\left(1+\frac{2\nu(y)}{\varepsilon'}\right)z$ because of $|x|<\varepsilon'''$ and $|z|<\sqrt{\frac{\varepsilon'}{12}}$. Hence we have $G_z(x, y, z)\neq 0$ for points of $S_{1,1}(\bar{f})$ in U. Since $S_{1,1,1}(\bar{f})$ is represented by $F=G=G_z=0$, $S_{1,1,1}(\bar{f})\cap U=\varnothing$.

Lemma 6. *Let C be a circle or a simple arc in $S_{1,1}(f)$. Suppose that C contains no point of $S_{1,1,1}(f)$. Then the mapping f is represented in a neighborhood of C as follows:*

$$X=x,\quad Y=y,\quad Z=yz-z^3+R,\qquad \mathrm{ord}_z R\geqq 4,$$

where C is represented by $y=z=0$, and x, X are real numbers mod 1 or real numbers in [0, 1] *according as C is a circle or a simple arc.*

Proof. Consider a Riemannian metric tn M^3. Then we may take a vector field $\{V_p\}$ on C such that each V_p is the normal vector of $S_1(f)$ at p. For each $p\in C$, consider the geodesic g_p whose tangent vector at p is V_p. Let D_p be the set of points q of the geodesic g_p such that the length of the geodesic between p and q is less than ε and put $D=\bigcup_{p\in C}D_p$. Then, for small ε', D

is homeomorphic to $C\times I$ where I is a interval. Now the mapping $f|D$ is a local homeomorphism. Let $L_{f(q)}$ denote the line segment which is normal to $f(D)$ at $f(q)$. Then $\{M_q=f^{-1}(L_{f(q)})\,;\,q\in D\}$ is a family of smooth curves. Consequence, as in the proof of Lemma 4, we may take paramenter systems (x, y, z) of $U(C)$ and (X, Y, Z) of $f(U(C))$ in which f is represented by

$$X = x, \quad Y = y, \quad Z = h(x, y, z)\,.$$

Expand h with respect to z:

$$h(x, y, z) = \sum_{i=0}^{3} a_i(x, y)\,z^i + R\,, \qquad \mathrm{ord}_z\, R \geqq 4\,.$$

Then, in the above choice of parameters, we have $a_0(x, y)=0$. The set $S_1(f)$ is represented by

$$a_1(x, y)+2a_2(x, y)+3a_3(x, y)\,z^2+R_z = 0\,.$$

Since C is contained in $S_1(f)$, we have $a_1(x, y)=0$. Hence we may put $a_1(x, y)=a_{11}(x, y)y$. We have$a_{11}(x, 0)\neq 0$ since f is generic.

Set

$$x' = x,\ y' = a_{11}(x, y)y,\ z' = z, \quad X' = X,\ Y' = a_{11}(X, Y)Y,\ Z' = Z\,.$$

Then we have, dropping primes,

$$X = x, \quad Y = y, \quad Z = yz+a_2(x, y)\,z^2+a_3(x, y)\,z^3+R\,.$$

Now $S_{1,1}(f)$ is represented by the following equations:

$$y+2a_2(x, y)\,z+3a_3(x, y)\,z^2+R_z = 0\,,$$
$$2a_2(x, y)+6a_3(x, y)\,z+R_{zz} = 0\,.$$

Since C is contained in $S_{1,1}(f)$, we have $a_2(x,0)=0$. Hence we may put $a_2(x, y)=a_{21}(x, y)y$.

Set

$$x' = x, \quad y' = y, \quad Z' = z+a_{21}(x, y)\,z^2\,.$$

Then we have, dropping primes,

$$X = x, \quad Y = y, \quad Z = yz+a_3(x, y)\,z^3+R, \qquad \mathrm{ord}_z\, R \geqq 4\,.$$

Since f is generic, we have $a_3(x, 0)\neq 0$.

Set

$$x' = x, \quad y' = \frac{-y}{\sqrt[3]{a_3(x, y)}}, \quad z' = -\sqrt[3]{a_3(x, y)}\, z\,,$$

$$X' = X, \quad Y' = \frac{-Y}{\sqrt[3]{a_3(xY)}}, \quad Z' = Z\,.$$

Then we have, dropping primes,

$$X = x, \quad Y = y, \quad Z = yz - z^3 + R, \qquad \mathrm{ord}_z R \geqq 4\,.$$

Then following lemma is easily obtained from the last lemma.

Lemma 7. *Suppose that C is a circle or a simple arc in $S_{1,1}(f)$ and that $C \cap S_{1,1,1}(f) = \varnothing$. Then the mapping f has, in a tubular neighborhood of C, a good 3-approximation $\bar{f}$ represented by*

$$X = x, \quad Y = y, \quad Z = yz - z^3\,.$$

Lemma 8. *Let p be a point of $S_{1,1}(f) - S_{1,1,1}(f)$, then we may take, in a neighborhood U of p, a good 2-approximation $\bar{f}$ of f such that $S_{1,1}(\bar{f}) \cap U$ is a simple curve and contains two points q, q' of $S_{1,1,1}(\bar{f})$. The indices of q and q' are the same, and the positive sides of q and q' are oposite with respect to $S_{1,1}(\bar{f})$. It is possible to take the indices of q and q' as positive or negative.*

Proof. By the last lemma, we may suppose that f is represented in a neighborhood U as follows:

$$X = x, \quad Y = y, \quad Z = yz - z^3\,.$$

Consider a function $\alpha(z) = \frac{-1}{\varepsilon^2} z^5 + 2z^3 - \varepsilon^2 z$ for sufficiently small $\varepsilon > 0$. We may then take a smooth function $\beta(z)$ satisfying:

i) $\beta(z) = \alpha(z)$ for $|z| \leqq \frac{9}{10}\varepsilon$,

ii) $\beta(z) = 0$ for $|z| \geqq \varepsilon$,

iii) $\beta(z)$, $\beta'(z) = \frac{d\beta}{dz}$ are monotone for $\frac{9}{10}\varepsilon \leqq |z| \leqq \varepsilon$.

Take a smooth function $\sigma(x)$ such that

$$\sigma(x) = \sigma(-x), \quad 0 \leqq \sigma(x) \leqq 1\,,$$

$$\sigma(x) = 1 \quad \text{for} \quad |x| \leqq \frac{\varepsilon}{4}, \quad \sigma(x) = 0 \quad \text{for} \quad |x| \geqq \frac{\varepsilon}{2}\,.$$

Consider now the mapping $\bar{f}$ represented by

$$X = x, \quad Y = y, \quad Z = yz - z^3 + \sigma(x)\sigma(y)\beta(z+x).$$

Then we have $\sigma(x)\sigma(y)\beta(z+x)=0$ for $(x, y, z) \notin U$ where $U=\{(x, y, z)\,;\ |x|\leqq\varepsilon,\ |y|\leqq\varepsilon,\ |z|\leqq 2\varepsilon\}$. Therefore $\bar{f}$ is a good 2-approximation of f in the neighborhood U of y. Now we shall consider the singularities of $\bar{f}$ in U. $S_1(\bar{f})$ is represented by the following equation

$$F(x, y, z) = y - 3z^2 + \sigma(x)\sigma(y)\beta'(z+x) = 0.$$

Since $|\beta'(z+x)|\leqq\varepsilon^2$, we have $|F(.x, y, z)|>0$ for $|y|>\frac{\varepsilon}{4}$. Hence we may suppose that $|y|\leqq\frac{\varepsilon}{4}$ and the equation of $S_1(\bar{f})$ is

$$y - 3z^2 + \sigma(x)\beta'(z+z) = 0.$$

Thus the set $S_1(\bar{f})$ is siotopic in U to $S_1(f)$. Consider $S_{1,1}(\bar{f})$ which is represented by the following equations:

$$y - 3z^2 + \sigma(x)\beta'(z+x) = 0,$$
$$-6z + \sigma(x)\beta''(z+x) = 0.$$

Set $z'=z+x$ and $g(x,z')=-6(z'-x)+\sigma(x)\beta''(z')$.

Case 1: $\frac{\varepsilon}{4}<x<\frac{\varepsilon}{2}$. We have $\beta''(z')\geqq 0$ for $z'\leqq-\sqrt{\frac{3}{5}}\varepsilon$. Hence we have $g(x, z')>0$. For $-\sqrt{\frac{3}{5}}\varepsilon<z'\leqq 0$, we have $g(x, z')$ $=-6(z'-x)+\sigma(x)\left(\frac{-20}{\varepsilon^2}z'^3+12z'\right)\geqq\frac{6}{4}\varepsilon-\frac{4}{10}\varepsilon>0$. For $\sqrt{\frac{3}{5}}\varepsilon\leqq z'$, we have $g(x, z')<0$ because $z'-x>0$ and $\beta''(z')\leqq 0$. For $0<z'$ $\leqq\sqrt{\frac{3}{5}}\varepsilon$, we have

$$g(x, z') = 6(2\sigma(x)-1)z' - \frac{2\sigma(x)}{\varepsilon^2}z'^3 + 6x.$$

It $2\sigma(x)-1\leqq 0$, then $g(x, z')$ is monotone decreasing, $g(x, 0)>0$ and $g(x)\sqrt{\frac{3}{5}}\varepsilon<0$.

If $2\sigma(x)-1>0$, then $g(x, z')>0$ for $z'=\sqrt{\frac{2\sigma(x)-1}{10\sigma(x)}}\varepsilon$.

Hence $g(x, z')=0$ has only one solution for $x>\frac{\varepsilon}{4}$.

Case 2: $-\frac{\varepsilon}{2}<x<-\frac{\varepsilon}{4}$. By the same argument in case 1, we have that $g(x, z')=0$ has only one solution for $x<-\frac{\varepsilon}{4}$.

Thus the set $S_{1,1}(\bar{f})$, for $\frac{\varepsilon}{4}<|x|$, is a simple curve.

Case 3: $|x|<\frac{\varepsilon}{4}$. The set $S_{1,1}(f)$ is represented by

$$y = 3z^2-\beta'(z+x)\,,$$
$$-6z+\beta''(z+x) = 0\,.$$

If $|z+x|\geqq\frac{9}{10}\varepsilon$, we have $(z+x)z>0$. By the definition of β, we have $\beta''(z+x)\cdot(z+x)<0$. Hence we have $-6z+\beta''(z+x)\neq 0$.

If $|z+x|<\frac{9}{10}\varepsilon$, the set $S_{1,1}(\bar{f})$ is represented by

$$y=3z^2+\frac{5}{\varepsilon^2}(z+x)^4-6(z+x)^2+\varepsilon^2\,,$$

$$3z+6x-\frac{10}{\varepsilon^2}(z+x)^3 = 0\,.$$

Hence the set $S_{1,1,1}(\bar{f})$ is represented by the above equations together with

$$1-\frac{10}{\varepsilon^2}(z+x)^2 = 0\,.$$

Thus $S_{1,1,1}(\bar{f})$ consists of two points $\{q, q'\} = \left\{\left(\pm\frac{2\varepsilon}{3\sqrt{10}}, \frac{77}{60}\varepsilon^2, \pm\frac{5\varepsilon}{3\sqrt{10}}\right)\right\}$. This proves the first part.

Expanding $yz-z^3+\beta(x+z)$ at q or q', we have

$$Z' = a+\varphi(x', y')+(y'+\psi(x'))z'+(3x'+\rho(x'))z'^2+bx'z'^3$$
$$+\left(\pm\frac{5}{\sqrt{10}\varepsilon}+\nu(x')\right)z'^4+R', \qquad \mathrm{ord}_{z'} R'\geqq 5\,,$$

with ord $\psi\geqq 1$, ord $\rho\geqq 2$, ord $\nu\geqq 1$.

Consequently we have :

1) The indicds of q and q' are the same.

2) The positive side at q is opposite to the positive side at q' with respect to $S_{1,1}(\bar{f})$

This proves the second part.

Consider mapping $\bar{\bar{f}}$ given by

$$X = x, \quad Y = y, \quad Z = yz - z^3 + \sigma(x)\sigma(y)\beta(z-x).$$

Then the above argument shows that the singularities of $\bar{\bar{f}}$ have the same properies as of f except that the indices of $\bar{\bar{f}}$ are opposite to those of f. This proves the last part.

4. Topological consideration. We suppose that $f: M^3 \to R^3$ is a generic decomposable mapping, and M^3 is an orientable closed smooth manifold. By definition there exist a locally homeomorphic mapping i and a projection π of R^4 to R^3 such that $\pi i = f$.

We may take a vector field $\{V_p\}$ on R^4 such that these vectors are projected to the null vector by $d\pi$. Since i is an immersion, the differential of i is an into-isomorshism from the tangent space of M^3 to that of R^4. Let p be a point of $S_1(f)$ then $di(T_p)$ contains the vector $V_{i(p)}$ where di is the differential of i and T_p is the tangent space of M^3 at p. Define now $\tilde{V}_p = (di)^{-1}(V_{i(p)})$, then $\{\tilde{V}_p\}$ is a smooth vector field on $S_1(f)$ which is contained in the null space $N(p)$. This vector field is called the null vector field.

Lemma 9. *Let f be a generic mapping. Then the connected components of $S_1(f)$ are orientable closed 2-manifolds.*

Proof. Since f is generic, $S_1(f)$ is a closed submsnifiold of M^3. Since the local degree of f can be defined at points of $M^3 - S_1(f)$, it follows that the normal bundle of $S_1(f)$ is trivial. Hence the lemma is proved.

Lemma 10. *Let f be a generic mapping and D be a connected component of $S_1(f)$. Then we may take, in a neighborhood U of D, a good 2-approximation $\bar{f}$ of f such that the singularity $S_{1,1}(\bar{f}) \cap U$ is a connected set.*

Proof. Let $E_1, \cdots, E_i$ be the singularities $S_{1,1}(f)$ in D, and p be a point of E_1. We may take E_j and a point $q \in E_j$ such that the points p and q are connected by a curve on D without touching any other point of $S_{1,1}(f)$. By Lemma 8, we may take in neighborhoods of p and q a deformation f' of f such that the indices of the points of $S_{1,1,1}(f')$ near p and q are different. Then, by Theorem 1 there exists a deformation f'' of f' in $U(C)$ so that E_1 and E_j are connected in the singularities $S_1(f'')$. By making such deformations successively, we obtain the lemma.

Remark. The decomposabllity of mapping is invariant under deformations if their first partial derivatives are close enough. The deformations in Section 3 are such deformations. Hence we may suppose that if the mapping f in the last lemma is decomposable then so is $\bar{f}$.

Lemma 11. *Suppose that the mapping f is a generic decomposable mapping. Let D a connected component of $S_1(f)$. If $S_{1,1}(f) \cap D=E$ is a connected set then E divides D into two connected parts.*

Proof. Let p be a point of E. Then we may suppose that the singularities of f is represented in a neighborhood of p as follows:

$$S_1(f): y-3z^2 = 0, \quad S_{1,1}(f): y = z = 0.$$

If E does not divide D, we may take a simple closed curve C in $S_1(f)$ so that C is the intersection of $S_1(f)$ and $x=0$ in a neighborhood of p and so that C intersects with E at a single point p. Take an orientation in C. Let T_r be the tangent vector of C at r and N_r be the normal vector of C in the tangent plane of $S_1(f)$ at r. If r is a point of $S_1(f)-S_{1,1}(f)$, then $\{T_r, N_r, \tilde{V}\}$ is a non-degenerate frame.

Take points $p'=\left(0, \varepsilon, \sqrt{\frac{\varepsilon}{3}}\right)$, $p''=\left(0, \varepsilon, -\sqrt{\frac{\varepsilon}{3}}\right)$ on C for small $\varepsilon>0$, and consider these frames at p' and p''. We may suppose teat $\tilde{V}_p=\left(\frac{\partial}{\partial z}\right)_{p'}$, $\tilde{V}_{p''}=\left(\frac{\partial}{\partial z}\right)_{p''}$, $N_{p'}=\left(\frac{\partial}{\partial x}\right)_{p'}$ and $N_{p''}=\left(\frac{\partial}{\partial x}\right)_{p''}$. Since the y-component of $T_{p'}$, and $T_{p''}$ are opposite, the frames $\{T_{p'}, N_{p'}, \tilde{V}_{p'}\}$ and $\{T_{p''}, N_{p''}, \tilde{V}_{p''}\}$ have opposite orientations. This contradicts to the orientability of M^3. This completes the proof.

Lemma 12. *Suppose that the mapping f is a generic mapping. Let p and q be points of $S_{1,1,1}(f)$ which are contained in a connected component of $S_{1,1}(f)$. Suppose that there is no point of $S_{1,1,1}(f)$ between p and q. Then the following two cases occur.*

1) *The positive sides at p and q are the same side with respect to $S_{1,1}(f)$ in $S_1(f)$, and the indices of p and q are different.*

2) *The positive sides at p and q are opposite with respect to $S_{1,1}(f)$ in $S_1(f)$, and the indices of p and q are the same.*

Proof. Let C be an open oriented subarc of $S_{1,1}(f)$ between p and q. There exist coordinate systems at p and q under which f is represented in the form in the sence of Section 1. Take in a tubular neighborhood of C a Riemannian metric which induces Euclidean metric determined by the coordinate systems at p and q. Let $v_1(p)$, $v_2(p)$ and $v_3(p)$ denote respectively the tangent vectors of x-, y- and z-axis in the coordinate system at p. For q, use the same notation.

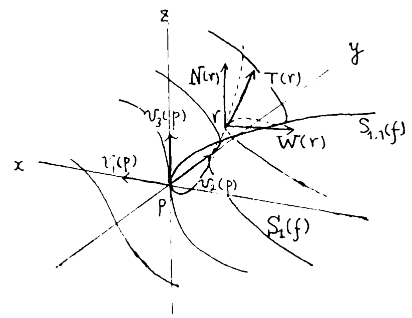
Let s be a point of C. Let $T(s)$ denote the tangent vector of $S_{1,1}(f)$ at s, $N(s)$ the null space at s, and $W(s)$ the normal vector to $S_1(f)$. The orientation of $W(s)$ is determined by the direction from the negative side to the positive.

Let r be a point of C near p, and give $N(r)$ the orientation determined by the diredtion of z-axis in the coordinate system at p. Then the plane $(W(r), N(r))$ converges to the plane $(v_1(p), v_3(p))$ if r converges to p. Hence we can compare the orientation of $\{W(r), N(r)\}$ with that of $\{v_1(p), v_3(p)\}$. We divide two cases according as the z-component of the coordinate of r is negative or positive.

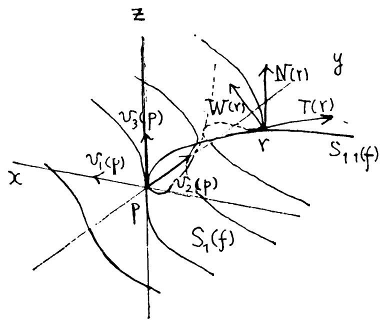
Negative case: In this case, the directions of $N(r)$ and of $v_2(p)$ are the same in $S_1(f)$ with respect to $S_{1,1}(f)$, and the orientations of $\{W(r), N(r)\}$, $\{v_1(p), v_3(p)\}$ are opposite.

Positive case: In this case the directions of $N(r)$ and of $v_2(p)$ are opposite in $S_1(f)$ with respect to $S_{1,1}(f)$ and the orientations of $\{W(r), N(r)\}$ and of $\{v_1(p), v_3(p)\}$ are the same.

For r near p, $N(r)$ is already oriented. These determine naturally the orientation of $N(s)$ for any $s\in C$. Then $\{\{W(s), T(s),$



Negative Case



Positive Case

$N(s)\}$, $s \in C\}$ is a continous family of non-degenerate frames. We may suppose that the direction of z-axis in the coordinate system at q is the same to that of $N(t)$ for a point t near q. Compare now the orientations of $\{v_1(p), v_2(p), v_3(p)\}$ and of $\{v_1(q), v_2(q), v_3(q)\}$. It occurs two cases according as the directions of $v_2(p)$ and $v_2(q)$ are the same or not in $S_1(f)$ with respect to $S_{1,1}(f)$.

Case I (*The directions are the same*). In this case, if the directions of $v_2(p)$ and of $N(r)$ are the same with respect to $S_{1,1}(f)$ then the directions of $v_2(q)$ and of $N(t)$ are the same with respect to $S_{1,1}(f)$. Hence the above negative cases arises for (p, r) and (t, q). Therefore it follows that the orientations of $\{v_1(p), v_2(p), v_3(p)\}$ and of $\{W(r), T(r), N(r)\}$ are opposite, and that the orientations of $\{W(t), T(t), N(t)\}$ and of $\{v_1(q), v_2(q), v_3(z)\}$ are the same.

If the directions of $v_2(p)$ and of $N(r)$ are opposite in $S_1(f)$ with respect to $S_{1,1}(f)$, then the directions of $v_2(q)$ and of $N(t)$ are opposite in $S_1(f)$ with respect to $S_{1,1}(f)$. Thus the above positive case arises for (p, r) and (t, q).

Hence we have that the orientations of $\{v_1(p), v_2(p), v_3(p)\}$ and of $\{W(r), T(r), N(r)\}$ are the same, and that the orientations of $\{W(t), T(t), N(t)\}$ and of $\{v_1(q), v_2(q), v_3(q)\}$ are opposite.

As a consequence of the argument above it follows that the indices of p and of q are different.

Case II (*The directions are opposite*). In this case, the similar consideration shows that the indices of p and of q are the same.

Corollary. *Let f be a generic mapping of M^3 tn R^3, where M^3*

is an orientable closed smooth manifold. Let C denote a connected component of $S_{1,1}(f)$. *Then the number of points of* $S_{1,1,1}(f)$ *in C is even.*

Theorem 2. *Let* M^3 *be an orientable closed smooth 3-manifold and f be a mapping of* M^3 *to* R^3. *Suppose that f is a generic decomposable mapping. Then we may take a good 2-approximation* $\bar{f}$ *of f so that* $S_{1,1,1}(\bar{f})$ *is empty and* $S_{1,1}(\bar{f})$ *are boundaries of domains of* $S_1(\bar{f})$.

Proof. By Lemma 10, we may suppose that the part E of $S_{1,1}(f)$ which is contained in a connected component D of $S_1(f)$ is connected. Thus, by Lemma 11, E divides D into two domains. Now let p and q be points of $S_{1,1,1}(f)\cap E$ between which there is no point of $S_{1,1,1}(f)$.

Case 1 (*The indices of p and q are different*). In this case, the positive sides at p and at q are the same side with respect to E. Now we may consider the curve C running from p to q whose interior is contained in $D-E$ and which starts from p into the negative side and which ends to q from the negative side. Then, by Theorem 1, we may eliminate p and q from $S_{1,1,1}$.

Case 2 (*The indices of p and q are the same*). In this case, the positive sides at p and at q are opposite side with respect to E. Let r be a point of E between p and q. By Lemma 8, we may take in a small neighborhood of r a good 2-approximation f' of f so that there exist, between p and q, two new point r' and r'' of $S_{1,1,1}(f')$ whose indices are different from those of p and q. Then applying the same method as in case 1 to (p, r') and (q, r), it follows that we may eliminate p and q from $S_{1,1,1}$.

The above argument shows that there exists an approximation $\bar{f}$ of f such that $S_{1,1,1}(\bar{f})$ is empty.

It is easily shown that each connected component of $S_{1,1}(\bar{f})$ is the boundary of a 2-disk or of a domain in $S_1(\bar{f})$.

Lemma 13. *Let E' be a smooth circle in* $M^3-S_1(f)$, *and suppose that E' is the boundary of an orientable smooth 2-manifold D' in* $M^3-S_1(f)$. *Then we may take in a neighborhood U(D') of D' a* *good* 0-*approximation* f' *of* f *so that the mapping* f' *is a generic mapping and* $S_{1,1}(f')\cap U(D')=E'$ *and* $S_{1,1,1}(f')\cap U(D')=\emptyset$.

Proof. For a given D', there exist sets D_1, D_2 such that $D_1\subsetneqq D'\subsetneqq D_2$, $D_2\cap S_1(f)=\emptyset$ and D_2-D_1 is diffeomorphic to $E'\times[-1, 1]$. Then we may take a neighborhood $U(D')$ of D' which is diffeomorphic to $D_2\times[-1, 1]$ and which is contained in $M^3-S_1(f)$.

Take a smooth function $\rho(x, t)$ having the following properties:

1) $\rho(x, 0)=x$, for $|x|\leqq 1$,

2) $\frac{\partial\rho}{\partial x}(x, t)>0$ for $0\leqq t<\frac{1}{2}$, $\rho(x, t)=x$ for $1\geqq|x|\geqq\frac{2}{3}$,

3) $\rho(x, t)=(4t-1)x^3+(-2t+1)x$, for $\frac{1}{2}\leqq t\leqq 1$, $|x|\leqq\frac{1}{3}$,

$\rho(x, t)=x$ for $\frac{1}{2}\leqq t\leqq 1$, $1\geqq|x|\geqq\frac{2}{3}$,

$\frac{\partial\rho}{\partial x}(x, t)>0$ for $\frac{1}{2}\leqq t\leqq 1$, $1\geqq|x|>\frac{1}{3}$.

We may take a smooth function $\nu(p)$ on D_2 such that

$$\nu(p)\geqq 0 \text{ for } p\in D_2,\quad \nu(p)=1 \text{ for } p\in D_1,$$

$$\nu(p)=\frac{1}{2} \text{ if and only if } p\in E',$$

$$\nu(p)=0 \text{ for } p\in\partial D_2.$$

Then we have a smooth mapping h of $U(D')$ to $U(D')$ defined by

$$h(p, x)=(p, \rho(x, \nu(p)))$$

where $p\in D_2$, $x\in[-1, 1]$.

Since the mapping h is the identity on the boundary of $U(D')$, h has an extension $h': M^3\to M^3$ so that $h'|M^3-U(D')=$ the identity.

It is now easily shown that the mapping $f'=fh'$ satisfies the conditions of the lemma.

Theorem 3. *Let* M^3 *be an orientable closed smooth manifold, and* f *be a generic decomposable mapping of* M^3 *to* R^3. *Then we may take a good* 0-*approximation* $\bar{f}$ *of* f *so that* $S_{1,1}(\bar{f})\neq\emptyset$.

Proof. By Theorem 2, we may suppose that $S_{1,1,1}(f)$ is empty and any circle of $S_{1,1}(f)$ is the boundary of a domain of $S_1(f)$. Denote by E one of the components of $S_{1,1}(f)$. By Lemma 7, we may take a coordinate system (x, y, z) in a neighborhood of E and the parameter system (X, Y, Z) in R^3 so that a good 3-approximation f' of f is represented by

$$X = x, \quad Y = y, \quad Z = yz - z^3 .$$

Let E' be the set of points $(x, -\varepsilon, 0)$ where $\varepsilon > 0$ is sufficiently small. Then E' satisfies the conditions of Lemma 13, and hence we may take a good 0-approximation f'' of f' which is represented by the following equations in a tubular neighborhood of E:

$$X = x, \quad Q = y, \quad Z = \rho(y)z - z^3 ,$$

where $\rho(y)$ is a smooth function which has the following conditions:

$$\rho(y) = 0 \quad \text{for} \quad y = 0, \ = \varepsilon ,$$
$$\rho(y) < 0 \quad \text{for} \quad 0 < y < \varepsilon ,$$
$$\frac{d\rho}{dy}(0) < 0, \quad \frac{d\rho}{dy}(\varepsilon) > 0 .$$

Applying Lemma 1 to each section: x = constant, we obtain a good 0-approximation f'' of f' so that E is eliminated from the singularities $S_{1,1}$. By this methods we may obtain a good 0-approximation $\bar{f}$ which satisfies the condition in the theorem.

Theorem 4. *Let M^n be an orientable closed smooth n-manifold and f be a mapping of M^n to R^n. Suppose that the singularities of f satisfy the following conditions:*

$$S_i(f) = \varnothing \quad (i \geqq 2), \quad S_{1,1}(f) = \varnothing .$$

Then the mapping f is decomposable mapping.

Proof. By the condition $S_i(f) = \varnothing (i \geqq 2)$, $S_1(f)$ is an $(n-1)$-dimensional smooth submanifold of M^n. Since $S_{1,1}(f) = \varnothing$, it follows that $f|S_1(f)$ is a local homeomorphism. The null space $N(p)$ is transversal to $S_1(f)$ because of $S_{1,1}(f) = \varnothing$. Since M^n is orientable, we may define the local degree of f at points of $M^n - S_1(f)$. Hence the normal bundle of $S_1(f)$ is trivial, and we may take an orientation in $N(p)$ so that $\{N(p) ; p \in S_1(f)\}$ is a transversal vector field. Denote by L_p the geodesic segment whose tangent vector at p is $N(p)$ Then $U=\bigcup_{p\in S_1(f)} L_p$ is a neighborhood of $S_1(f)$. Now we may take a smooth function $g(q)$ on U such that the derivative of $g(q)$ with respect to the vector $N(p)$ is not zero. For example, we may take as $g(q)$ the length of L_p from p to q. Then the function $g(q)$ can be extended to a smooth function g on M^n. Denote $h(p)=(f(p), g(p))$, then h is a smooth mapping of M^n to R^{n+1} which is a local homeomorphism.

Thus the theorem is proved.

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