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On the a priori estimate for solutions of the Cauchy problem for some non-linear wave equations

By

Masaya YAMAGUTI

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For the global Cauchy problem of wave equation, the existence of an a priori estimate of the solution is very useful as we have shown recently in another report [1] [2] for one special type of the non-linear wave equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + f(u)\frac{\partial u}{\partial t} + g(u)$$

under relatively weak conditions.

Here, we note that a priori estimate is also obtained for wave equation of a little different type with more than one space dimension which is identical to the equation treated by Konrad Jorgens [3] in the case of 3 dimension and without damping term.

At first we shall treat the case in which the space dimension is 2. Our Cauchy problem is the following: Find the solution of the equation

$$(1) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial t^2} + f\left(\frac{\partial u}{\partial t}\right) + g(u)$$

satisfying the following initial conditions.

$$(2) \qquad \begin{cases} u(x, y, 0) = u_0(x, y) \\ u_t(x, y, 0) = u_1(x, y) \end{cases}$$

where $u_0(x, y)$ belongs to C^3 and $u_1(x, y)$ belongs to C^2.

Here we do not solve this problem, but we obtain an a priori estimate for the solution of this problem assuming $u_0(x, y)$ and

$u_1(x, y)$ have compact carriers[1] under conditions for functions f and g:

Conditions

i) $f(u)$ and $g(u)$ are continuously differentiable in $-\infty < u < +\infty$

ii) $sgn\, u f(u) \geq 0$ and $G(u) = \int_0^u g(u)du \geq 0$ for $|u| > M$.

iii) $f'(u) \geq -k$ (k is one positive constant)

$|g'(u)| \leq$ Polynomial of $|u|$

Before we proceed to write our results, we define two generalized energies $E_0(t)$ and $E_1(t)$ for our solutions $u(x, y, t)$ of (1) and (2).

$$(4) \quad E_0(t) = \iint \left[G(u) + \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2 \right] dxdy$$

$$(5) \quad E_1(t) = \frac{1}{2}\iint \left[\left(\frac{\partial^2 u}{\partial t \partial x}\right)^2 + \left(\frac{\partial^2 u}{\partial t \partial y}\right)^2 + \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 u}{\partial y \partial x}\right) + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 \right] dxdy$$

and we see

$$(6) \quad E_0(0) = \iint \left[G(u_0) + \frac{1}{2}(u_1)^2 + \frac{1}{2}\left(\frac{\partial u_0}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial u_0}{\partial y}\right)^2 \right] dxdy$$

$$(7) \quad E_1(0) = \iint \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_1}{\partial y}\right)^2 + \left(\frac{\partial^2 u_0}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 u_0}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 u_0}{\partial y}\right)^2 \right] dxdy$$

where always integrals are taken in whole x, y plane which is possible, because, $u_1(x, y)$ and $u_0(x, y)$ have compact carriers and $u(x, y, t)$ also.

Now we estimate the energy $E_0(t)$ of the solution by the initial energy. First we transform (1) and (2) into a system of equations.

$$p = \frac{\partial u}{\partial x}, \qquad q = \frac{\partial u}{\partial y}$$

$$(8) \quad \begin{cases} \dfrac{\partial u}{\partial t} = v \\[2mm] \dfrac{\partial v}{\partial t} = \dfrac{\partial p}{\partial x} + \dfrac{\partial q}{\partial y} - f(v) - g(u) \\[2mm] \dfrac{\partial p}{\partial t} = \dfrac{\partial v}{\partial x} \\[2mm] \dfrac{\partial q}{\partial t} = \dfrac{\partial v}{\partial y} \end{cases}$$

$$(9)\qquad \begin{cases} v(x, y, 0) = u_1(x, y) \\ u(x, y, 0) = u_0(x, y) \\ p(x, y, 0) = \dfrac{\partial u_0}{\partial x}(x, y) \\ q(x, y, 0) = \dfrac{\partial u_0}{\partial y}(x, y) \end{cases}$$

and

$$(10)\qquad E_0(t) = \iint\left[G(u)+\frac{v^2}{2}+\frac{p^2}{2}+\frac{q^2}{2}\right]dxdy$$

Differentiating (10) with respect to t and Considering (8), we have

$$(11)\qquad \begin{aligned} \frac{dE_0}{dt} &= \iint\left[g(u)v+v\frac{\partial v}{\partial t}+p\frac{\partial p}{\partial t}+q\frac{\partial q}{\partial t}\right]dxdy \\ &= \iint\left[g(u)v+v\frac{\partial p}{\partial x}+v\frac{\partial q}{\partial y}+p\frac{\partial u}{\partial x}+q\frac{\partial v}{\partial y}-f(v)v-g(u)v\right]dxdy \\ &= \iint[-vf(v)]\,dxdy \leq \iint_{|v|\leq M}[vf(v)]\,dxdy \\ &\leq L\iint\left[\frac{v^2}{2}+\frac{p^2}{2}+\frac{q^2}{2}+G(u)\right]dxdy+L_0 l\,, \end{aligned}$$

$$(12)\qquad \frac{dE_0(t)}{dt} \leq LE_0(t)+L_0 l \qquad 0\leq t\leq h\,,$$

where l is the area of the carrier of $u_0(x, y)$ multiplied by $2h$.

$$(13)\qquad E_0(t)\leq e^{Lh}E(0)+e^{Lh}L_0 lh = e^{Lh}(E_0(0)+L_0 lh)$$

Next, we proceed to estimate $E_1(t)$. We write

$$E_1(t) = \frac{1}{2}\iint[v_x^2+v_y^2+p_x^2+p_y^2+q_x^2+q_y^2]\,dxdy\,.$$

Differentiating (8) by x and y, we have:

$$(14)\qquad \begin{cases} \dfrac{\partial v_x}{\partial t} = \dfrac{\partial p_x}{\partial x}+\dfrac{\partial q_x}{\partial y}-f'(v)v_x-g'(u)p \\ \dfrac{\partial p_x}{\partial t} = \dfrac{\partial v_x}{\partial x} \\ \dfrac{\partial q_x}{\partial t} = \dfrac{\partial v_x}{\partial y} \end{cases}$$

$$\left\{\begin{array}{l}\dfrac{\partial v_y}{\partial t}=\dfrac{\partial p_y}{\partial x}+\dfrac{\partial q_y}{\partial y}-f'(v)v_y-g'(u)q\\[2mm] \dfrac{\partial p_y}{\partial t}=\dfrac{\partial v_y}{\partial x}\\[2mm] \dfrac{\partial q_y}{\partial t}=\dfrac{\partial v_y}{\partial y}\end{array}\right.$$

And we obtain,

$$(15)\quad \frac{dE_1}{dt}=\iint\left[v_x\frac{\partial v_x}{\partial t}+v_y\frac{\partial v_y}{\partial t}+p_x\frac{\partial p_x}{\partial t}+q_x\frac{\partial q_x}{\partial t}+p_y\frac{\partial p_y}{\partial t}+q_y\frac{\partial q_y}{\partial t}\right]dxdy$$
$$=\iint[-f'(v)v_x^2-g'(u)pv_x-f'(v)v_y^2-g'(u)qv_y]dxdy$$
$$=\iint[k(v_x^2+v_y^2)-g'(u)pv_x-g'(u)qv_y]dxdy\,.$$

We consider the integral:

$$I_1=\iint|g'(u)pv_x|\,dxdy\,.$$

By the condition (iii)

$$I_1\leq\iint|u|^\alpha|p|\,|v_x|\,dxdy\leq\sqrt{\iint|u|^{2\alpha}p^2dxdy}\sqrt{\iint v_x^2dxdy}$$
$$\leq\left(\iint|u|^{4\alpha}dxdy\right)^{1/4}\cdot\left(\iint p^4dxdy\right)^{1/4}\left(\iint v_x^2dxdy\right)^{1/2}$$
$$\leq c\left[\iint(p^2+q^2)dxdy\right]^{\alpha/2}\cdot E_1(t)$$
$$\leq cE_0(t)^{\alpha/2}E_1(t)\,,$$
$$[\log E_1(t)]_0^t\leq c\int_0^t E_0(t)^{\alpha/2}dt+k\,,$$
$$E_1(t)\leq E_1(0)e^{c\int_0^t E_0(\tau)^{\alpha/2}d\tau}\leq E_1(0)e^{c\int_0^h E_0(\tau)^{\alpha/2}d\tau+k}\,.$$

Then we obtain by the Sobolev's lemma

$$|u(x,y,t)\leq C|E_0(t)+E_1(t)|$$
$$\leq C\{(E_0(0)+L_0lh)e^{Lh}+E_1(0)e^{c(E_0(0)+L_0lh)^{\alpha/2}e^{L\alpha h}h+k}\}\,.$$

This is our desired results.

We proceed to show that similar results can be obtained for the case of 3 space dimension under more stringent condition. We replace condition iii) by,

$$\text{iii)}\quad f'(u)\geq -k\,,\qquad |g'(u)|\leq c|u|^2$$

Our equation is the following:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{\partial^2 u}{\partial t^2}+f\left(\frac{\partial u}{\partial t}\right)+g(u) \tag{16}$$

We can estimate the energy $E_0(t)$ by the same argument of the preceding case, where $E_0(t)$ is defined.

$$E_0(t)=\iiint\left\{G(u)+\frac{1}{2}(v^2+p_1^2+p_2^2+p_3^2)\right\}dy \tag{17}$$

where $v=\frac{\partial u}{\partial t}$, $p=\frac{\partial u}{\partial x}$, $p_2=\frac{\partial u}{\partial y}$, $p_3=\frac{\partial u}{\partial z}$. Then we have

$$E_0(t)\leq e^{Lh}\{E_0(0)+M_0 l\}\qquad 0\leq t\leq h\,. \tag{18}$$

$E_1(t)$ is the integral:

$$\frac{1}{2}\iint\sum_{x,y,z}\{v_x^2+p_{1x}^2+p_{2x}^2+p_{3x}^2\}\,dV\,.$$

We obtain by the condition iii).

$$\begin{aligned}\frac{dE_1(t)}{dt}&\leq -\sum\iiint f'(v)v_x^2+\sum\iiint 3u^2p_1v_x dV\,.\\ &\leq k\sum\iiint v_x^2+\sum\iiint 3u^2|p_1v_x|\,dV\,.\end{aligned}$$

We treat the last term by the similar inequality as we have used in (15):

$$\begin{aligned}&\left|\iiint u^2p_1v_x dV.\right|\\ &\leq\left[\iiint u^4p_1^2dV\right]^{1/2}\left[\iiint v_x^2dV\right]^{1/2}\\ &\leq\left[\left(\iiint u^6dV\right)^{2/3}\left(\iiint p_1^6dV\right)^{1/3}\right]^{1/2}\left[\iiint v_x^2dV\right]^{1/2}\\ &\leq\left(\iiint u^6dV\right)^{1/3}\left(\iiint p_1^6dV\right)^{1/6}\left[\iiint v_x^2dV\right]^{1/2}\\ &\leq\left[\iiint(p_1^2+p_2^2+p_3^2)dV\right]\left[\iiint(p_{1x}^2+p_{1y}^2+p_{1x}^2)dV\right]^{1/2}\left[\iiint v_x^2dV\right]^{1/2}\\ &\leq E_0(t)E_1(t)\end{aligned}$$

Just similarly we can estimate other terms. It is easy to see that the maximum norm of $U(x, y, z, t)$ for $0 \leq t \leq h$ is majorized by $E_0(0)$ and $E_1(0)$[2].

NOTES

1) We assume also that the solution $u(x, y, t)$ and its derivatives of 3rd order with respect to x, y and t are square integrable in xy space for all t. By the Sobolev's work [4], we can find always this solution for sufficiently small t, for our Cauchy data.

2) We could not find the bound for $\frac{\partial u}{\partial t}$ by $E_0(0)$ and $E_1(0)$, therefore the existence of a global solution of the Cauchy problem is not proved for the equation (1) and (16). But if $f\left(\frac{\partial u}{\partial t}\right)$ is linear for $\frac{\partial u}{\partial t}$, we can easily prove the global existence of the solution of the Cauchy problem.

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