J. Math. Kyoto Univ.
2-2 (1963) 193-207.

On homotopy groups of S^3-bundles over spheres

By

Hirosi TODA

(Received Jan. 14, 1963)

§ 1. Statement of results

We shall consider the p-primary components of the homotopy groups of a cell complex

$$B(p) = S^3 \cup e^{2p+1} \cup e^{2p+4}$$

having the cohomology ring ($\mathcal{P}^1 = Sq^2$ if $p=2$) mod p

(1.1) $$H^*(B(p), Z_p) = \Lambda(u, \mathcal{P}^1 u), \qquad u \in H^3(B(p), Z_p).$$

The existence of such a complex $B(p)$ is provided by an S^3-bundle over a $(2p+1)$-sphere S^{2p+1} with a characteristic class $\alpha_1 \in \pi_{2p}(S^3)$ of a non-trivial mod p Hopf invariant [12].

Denote by X_p the 3-connective fibre space over $B(p)$ Then

(1.2) $$\pi_i(X_p) \approx \pi_i(B(p)) \qquad for \quad i > 3$$

and we have

Theorem 1. *$H^*(X_p, Z_p) = \Lambda(a, \mathcal{P}^p a) \otimes Z_p[b]$, where $a \in H^{2p+1}(X_p, Z_p)$ and the relation $\Delta b = \mathcal{P}^p a$ holds ($\Delta = Sq^1$ and $\mathcal{P}^p = Sq^4$ if $p=2$).*

Denote by $\mathcal{C}$ the class of the finite abelian groups without p-torsion, then by use of Serre's $\mathcal{C}$-theory [9], it follows from the theorem the following

Corollary. *There is a mapping $g: S^{2p+1} \rightarrow B(p)$ which induces $\mathcal{C}$-isomorphisms $g_*: \pi_i(S^{2p+1}) \rightarrow \pi_i(B(p))$ for $3 \leq i \leq 2p^2 - 1$.*

As a space of paths in the mapping-cylinder of g, we have a space Y_p which is a fibre of a fibering equivalent to g and also which is the total space of a fibering $\pi: Y_p \to S^{2p+1}$ of a fibre $\Omega(B(p))$. Then we have an exact sequence

$$(1.3) \qquad \cdots \to \pi_{i+1}(B(p)) \to \pi_i(Y_p) \xrightarrow{\pi_*} \pi_i(S^{2p+1}) \xrightarrow{g_*} \pi_i(B(p)) \to \cdots .$$

Let $f: S^n \to S^n$, $n=2p^2-1$, be a mapping of degree p and let $Z_f = S^n \cup S^n \times (0, 1]$ be the mapping-cylinder of f. By shrinking $S_1^n = S^n \times (1)$ to a point, we have a mapping-cone $C_f = Z_f / S_1^n$ of f. Let $p: Z_f \to C_f$ be the shrinking map.

Theorem 2. *There exists a mapping h of C_f into Y_p satisfying the following conditions. The composition $h\circ p$ induces $\mathcal{C}$-isomorphisms $(h\circ p)_*: \pi_i(Z_f, S_1^{2p^2-1}) \to \pi_i(Y_p)$ for $3 \leqq i \leqq 2p^3-2$. A mapping-cone of $\pi\circ h$ is a cell complex $S^{2p+1} \cup e^{2p^2} \cup e^{2p^2+1}$ with non-trivial Δ and $\mathcal{P}^p$, and the restriction $\pi\circ h | S^{2p^2-1}$ represents an element of order p in $\pi_{2p^2-1}(S^{2p+1}) \overset{\mathcal{C}}{\approx} Z_p$.*

Denote by ${}_p\pi_i(B(p))$ the p-primary component of $\pi_i(B(p))$, then the explicit value of it is given as follows.

Theorem 3. ${}_p\pi_{2p+2i(p-1)}(B(p)) \approx Z_p$ *for* $1 \leqq i < 2p$ *and* $i \neq p$,
${}_p\pi_{2p+2p(p-1)}(B(p)) \approx Z_{p^2}$,
${}_p\pi_{2p+2(p+j)(p-1)-1}(B(p)) \approx Z_p$ *for* $2 \leqq j < p$,
${}_p\pi_k(B(p)) = 0$ *otherwise for* $k < 2p + 4p(p-1) - 3$.

These results can be applied to compute the homotopy groups of Lie groups by use of the following $\mathcal{C}$-isomorphisms:

$$(1.4) \quad \pi_i(SU(p+1)) \overset{\mathcal{C}}{\approx} \pi_i(S^5) \oplus \pi_i(S^7) \oplus \cdots \oplus \pi_i(S^{2p-1}) \oplus \pi_i(B(p)),$$

$$(1.5) \quad \pi_i\left(Sp\left(\frac{p+1}{2}\right)\right) \overset{\mathcal{C}}{\approx} \pi_i(SO(p+2)) \overset{\mathcal{C}}{\approx} \pi_i(S^7) \oplus \pi_i(S^{11}) \oplus \cdots \oplus \pi_i(S^{2p-3}) \oplus \pi_i(B(p)) \qquad \text{for odd } p,$$

$$(1.6) \quad \pi_i(G_2) \overset{\mathcal{C}}{\approx} \pi_i(B(5)) \quad \text{for } p = 5.$$

§ 2. Proof of Theorem 1

We have two fiberings:

$$p : X_p \to B(p) \qquad \text{with fibre } K(Z, 2)$$

and
$$p' : B'(p) \to K(Z, 3) \qquad \text{with fibre } X_p,$$

where $K(Z, n)$ denotes Eilenberg-MacLane space of type (Z, n) and $B'(p)$ has the same homotopy type as $B(p)$.

Let $(E_r^{s,t})$ be the cohomological spectral sequence with the coefficient Z_p [7] associated with the first fibering, then

$$E_2^* \cong H^*(B(p), Z_p) \otimes H^*(Z, 2; Z_p) \approx \Lambda(u, \mathcal{P}^1 u) \otimes Z_p[v],$$
$v \in H^2(Z, 2; Z_p)$.

By concerning the dimensions of the elements of $\Lambda(u, \mathcal{P}^1 u)$, we have that the coboundary d_r is trivial except for $r=3$, $2p+1$, $2p+4$. Thus $E_2^* = E_3^*$, $E_4^* = E_{2p+1}^*$, $E_{2p+2}^* = E_{2p+4}^*$ and $E_{2p+5}^* = E_\infty^*$.

Since X_p is a 3-connective fibering, the generator v can be chosen such that $d_3(1 \otimes v) = u \otimes 1$. Then $d_3(x \otimes v^n) = n(xu \otimes v^{n-1})$ for $x \in \Lambda(u, \mathcal{P}^1 u)$. Hence we have the following isomorphism, by means of the cup-product,

$$\Lambda(\mathcal{P}^1 u \otimes 1, u \otimes v^{p-1}) \otimes Z_p[1 \otimes v^p] \cong H(E_3^*) = E_4^* = E_{2p+1}^*.$$

Since the transgression commutes with the operation $\mathcal{P}^1$ and since $\mathcal{P}^1 v = v^p$, we have $d_{2p+1}(1 \otimes v^p) = \mathcal{P}^1 u \otimes 1$ and $d_{2p+1}(u \otimes v^{p-1}) \in E_{2p+1}^{2p+4,-2} = 0$. Thus $d_{2p+1}(1 \otimes v^{mp}) = m(\mathcal{P}^1 u \otimes v^{(m-1)p})$ and $d_{2p+1}(u \otimes v^{mp-1}) = (m-1)(u \cdot \mathcal{P}^1 u \otimes v^{(m-1)p-1})$. It follows that

$$\Lambda(u \otimes v^{p-1}, \mathcal{P}^1 u \otimes v^{(p-1)p}) \otimes Z_p[1 \otimes v^{p^2}] \cong H(E_{2p+1}^*) = E_{2p+4}^*.$$

Finally, the triviality of d_{2p+4} is easily seen, and $E_\infty^* = E_{2p+4}^*$ is a graded ring associated with $H^*(X_p, Z_p)$. Thus we have obtained

$$H^*(X_p, Z_p) = \Lambda(a, c) \otimes Z_p[b], \tag{2.1}$$

where a, c and b correspond to $u \otimes v^{p-1}$, $\mathcal{P}^1 u \otimes v^{(p-1)p}$ and $1 \otimes v^{p^2}$, respectively.

Next consider the spectral sequence $(E_r^{s,t})$ associated with the second fibering $p' : B'(p) \to K(Z, 3)$. $E_2^* \cong H^*(Z, 3; Z_p) \otimes H^*(X_p, Z_p)$.

By Cartan's results [3], $H^*(Z, 3; Z_p)=\Lambda(u, \mathcal{P}^1u, \mathcal{P}^p\mathcal{P}^1u, \cdots)\otimes Z_p[\Delta\mathcal{P}^1u, \Delta\mathcal{P}^p\mathcal{P}^1u, \cdots]$ for odd p and $H^*(Z, 3; Z_2)=Z_2[u, Sq^2u, Sq^4Sq^2u, \cdots]$, where u is the fundamental class.

It is easy to see that $d_r(1\otimes a)=0$ for $r<2p+2$. Then $E_{2p+2}^{0,2p+1}\neq 0$. Since $H^{2p+2}(B(p), Z_p)=0$, $E_{2p+3}^{2p+2,0}=E_\infty^{2p+2,0}=0$. The element $\Delta\mathcal{P}^1u\otimes 1$ is not a d_r-image for $r<2p+2$. Thus it has to be a d_{2p+2}-image. By changing the coefficient of a, if it is necessary, we have that

$$d_{2p+2}(1\otimes a) = \Delta\mathcal{P}^1u\otimes 1 \quad (= Sq^3u\otimes 1 = u^2\otimes 1 \quad \text{for } p=2).$$

By Adem's relation [1], [4], $\mathcal{P}^p(\Delta\mathcal{P}^1u)=\Delta\mathcal{P}^p\mathcal{P}^1u$ for odd p and $Sq^4Sq^3u=Sq^5Sq^2u=(Sq^2u)^2$. Then $\mathcal{P}^pa$ is transgressive and

$$d_{2p^2+2}(1\otimes\mathcal{P}^pa) = \Delta\mathcal{P}^p\mathcal{P}^1u\otimes 1 \quad (d_{10}(1\otimes Sq^4a) = (Sq^2u)^2\otimes 1).$$

The element $\Delta\mathcal{P}^p\mathcal{P}^1u\otimes 1$ is not a d_r-image for $r<2p^2+2$. This shows that $\mathcal{P}^pa\neq 0$ and we can replace c by $\mathcal{P}^pa$ in (2. 1).

It is checked directly that $d_r(1\otimes b)=0$ for $r\leqq 2p+2$. Then it is verified that $E_2^*=E_{2p+2}^*$ and that

$$E_{2p+3}^* = \Lambda(u, \mathcal{P}^1u, \mathcal{P}^p\mathcal{P}^1u, \cdots)\otimes Z_p[\Delta\mathcal{P}^p\mathcal{P}^1u, \cdots]\otimes\Lambda(c)\otimes Z_p[b], \quad (p:\text{odd})$$

$$E_7^* = \Lambda(u)\otimes Z_2[Sq^2u, Sq^4Sq^2u, \cdots]\otimes\Lambda(c)\otimes Z_2[b] \quad (p=2).$$

$\mathcal{P}^p\mathcal{P}^1u$ is not a d_r-image for $r<2p^2+1$, but it is a d_r-image for $r=2p^2+1$ since $H^r(B(p), Z_p)=E_\infty^{r,0}=E_{r+1}^{r,0}=0$ for $r=2p^2+1$.

By changing the coefficient of b, if it is necessary, we have that

$$d_{2p^2+1}(1\otimes b) = \mathcal{P}^p\mathcal{P}^1u\otimes 1 \quad (= Sq^4Sq^2u\otimes 1 \quad \text{for } p=2).$$

Since the Bockstein operation Δ commutes with the transgression, we have

(2. 2) $\qquad \Delta b = c = \mathcal{P}^pa \quad (Sq^1b = c = Sq^4a \quad \textit{for } p=2),$

where the elements a, b, c are different only in coefficients $\neq 0$ from those in (2. 1).

Consequently we have proved Theorem 1.

§ 3. Proof of Theorem 2

The space X_p is a homology $(2p+1)$-sphere mod p, by Theorem 1, for dimensions $<2p^2$ and 3-connected. By Serre's $\mathcal{C}$-theory, $\pi_i(S^{2p+1})$ is $\mathcal{C}$-isomorphic to $\pi_i(X_p)$ for $i<2p^2-1$, by a homomorphism g'_* induced by a representative $g': S^{2p+1}\to X_p$ of an element of $\pi_{2p+1}(X_p)$ not divisible by p.

Then Corollary to Theorem 1 is proved by taking g as the composition of g' and the 3-connective fibering: $X_p\to B(p)$.

In order to prove Theosem 2, we may replace Y_p by a 2-connective fibre space Y'_p over Y_p, whence $B(p)$ in (1.3) may be replaced by X_p.

The space Y'_p is given as follows. Let $Z_{g'}=X_p\cup S^{2p+1}\times(0, 1]$ be the mapping cylinder of g'. Then Y'_p is the set of paths: $(I, 0, 1)\to(Z_{g'}, S^{2p+1}, *)$. The paths: $(I, 0, 1)\to(Z_{g'}, S^{2p+1}, Z_{g'})$ form a fibre space over $Z_{g'}$ with a fibre Y'_p. Consider a spectral sequence (E_r^*) associated with this fibering, then $E_2^*\approx H^*(X_p, Z_p)\otimes H^*(Y'_p, Z_p)$ and $E_\infty^*\approx H^*(S^{2p+1}, Z_p)$. We shall prove the following lemma

(3.1). *There exists an element w of $H^{2p^2-1}(Y'_p, Z_p)$ such that $H^*(Y'_p, Z_p)$ is isomorphic to $\Lambda(w)\otimes Z_p[\Delta w]$ for dimensions less than $2p^3$.*

By a simple computation of the spectral sequence, we have that b and $\Delta b=\mathcal{P}^p a$ are transgression images of w and Δw, i.e., $d_n(1\otimes w)=b\otimes 1$ and $d_{n+1}(1\otimes\Delta w)=\mathcal{P}^p a\otimes 1$, $n=2p^2$, for suitable choice of w. Construct a formal spectral sequence $('E_r^*)$ with the above d_n, d_{n+1} and $'E_2^*=H^*(X_p, Z_p)\otimes(\Lambda(w)\otimes Z_p[\Delta w])$. The spectral sequence is well-defined for dimensions less than $2p^3$ and the final term is $'E_\infty^*=\Lambda(a\otimes 1)$. Comparing $'E_r^*$ with E_r^*, it follows that (3.1) is true (*cf.* [16]).

By generalized Hurewicz theorem in $\mathcal{C}$-theory, $\pi_{2p^2-1}(Y'_p)$ is $\mathcal{C}$-isomorphic to Z_p and there exists a mapping

$$h': S^{2p^2-1}\to Y'_p$$

such that $h'^*: H^{2p^2-1}(Y'_p, Z_p)\approx H^{2p^2-1}(S^{2p^2-1}, Z_p)$ and the composition $h'\circ f$ is homotopic to zero.

Let S be a space consists of pairs (l, s) of paths $l: I \to Y'_p$ and points s of S^{2p^2-1} such that $l(1)=h'(s)$. S is a fibre spave over Y'_p with the projection π_0 given by $\pi_0(l, s)=h'(s)=l(1)$. By setting $i(s)=(l_s, s)$, $l_s(I)=h'(s)$, we have an injection i of S^{2p^2-1} into S which is a homotopy equivalence. Then

$$h' = \pi_0 \circ i .$$

Let $F=\pi_0^{-1}(*)$ be a fibre. Since $h'\circ f$ is homotopic to zero, then the injection i is extended to

$$k : Z_f \to S , \qquad k \,|\, S^{2p^2-1} = i ,$$

such that $k(S_1^{2p^2-1}) \subset F$. There exists uniquely a mapping h_0 such that the diagram

$$\begin{array}{ccc} (Z_f,\ S_1^{2p^2-1}) & \xrightarrow{k} & (S, F) \\ \downarrow p & & \downarrow \pi_0 \\ (C_f,\ *) & \xrightarrow{h_0} & (Y'_p,\ *) \end{array}$$

is commutative. h_0 is an extension of h'.

We shall prove

(3. 2). *The restriction* $k_0=k\,|\,S_1^{2p^2-1}: S_1^{2p^2-1} \to F$ *induces isomorphisms* $H^i(F, Z_p) \approx H^i(S_1^{2p^2-1}, Z_p)$ *for* $i<2p^3-1$.

Consider a spectral sequence (E_r^*) associated with the fibering $\pi_0 : S \to Y'_p$, then $E_2^* \approx H^*(Y'_p, Z_p) \otimes H^*(F, Z_p)$ and $E_\infty^* \approx H^*(S, Z_p) \approx H^*(S^{2p^2-1}, Z_p)$.

Let $n=2p^2-1$. First we have easily that $H^i(F, Z_p)=E_2^{0,i}=0$ for $i<n$. Since π_0^* is equivalent to h'^*, we have that $E_2^{n,0} \approx H^n(Y'_p, Z_p) \approx Z_p$ is mapped isomorphically onto $E_\infty^{n,0} \cong H^n(S, Z_p)$. Then it follows that $H^n(F, Z_p)$ $(\approx E_2^{0,n})$ is isomorphic to Z_p and generated by an element x such that $d_{n+1}(1 \otimes x)=\Delta w \otimes 1$. Thus $d_{n+1}((\Delta w)^k \otimes x)=(\Delta w)^{k+1} \otimes 1$ and $d_{n+1}(w \cdot (\Delta w)^k \otimes x)=w \cdot (\Delta w)^{k+1} \otimes 1$. This shows that $E_{n+2}^{t,s}=E_r^{t,s}=0$ for $r>n+2$, $s \leqq n$ and $n<t+s<2p^3$. Let $y \in H^i(F, Z_p)$ be a non-zero element of minimum $i>n$. If $i<2p^3-1$, then it is easily seen that $d_r(1 \otimes y)=0$ for all $r \geqq 2$, and thus $E_{\infty}^{0,i} \neq 0$. But this contradicts to $H^i(S, Z_p)=0$. We have obtained $H^i(F, Z_p)=0$ for $n<i<2p^3-1$.

Now, it is sufficient to prove that $k_0^*: H^n(F, Z_p) \rightarrow H^n(S_1^n, Z_p)$, $n=2p^2-1$, is an isomorphism. $h_0^*: H^n(Y_p', Z_p) \rightarrow H^n(C_f, Z_p)$ is equivalent to $h'^*: H^n(Y_p', Z_p) \rightarrow H^n(S^n, Z_p)$ and it is an isomorphism. By the naturality of Δ, it follows that $h_0^*: H^{n+1}(Y_p', Z_p) \approx H^{n+1}(C_f, Z_p)$. Also we have isomorphisms $p^*: H^i(C_f, Z_p) \approx H^i(Z_f, S_1^n; Z_p)$ and $\pi_0^*: H^i(Y_p', Z_p) \approx H^i(S, F; Z_p)$ for $i=n, n+1$. Then, by the commutativity of the previous diagram, we have isomorphisms $k^*: H^i(S, F; Z_p) \approx H^i(Z_f, S_1^n; Z_p)$ for $i=n, n+1$. Since $k: Z_f \rightarrow S$ is a homotopy equivalence, we have $H^*(S, Z_p) \approx H^*(Z_f, Z_p)$. By applying the five lemma, we have that $k_0^*: H^n(F, Z_p) \rightarrow H^n(S_1^n, Z_p)$ is an isomorphism onto. This completes the proof of (3. 2).

By generalized J.H.C. Whitehead's theorem in $\mathcal{C}$-theory, it follows from (3. 2) that $k_{0*}: \pi_i(S_1^n) \rightarrow \pi_i(F)$ is a $\mathcal{C}$-isomorphism for $i<2p^3-2$ and a $\mathcal{C}$-onto for $i \leqq 2p^3-2$. Since k is a homotopy equivalence, $k_*: \pi_i(Z_f) \approx \pi_i(S)$ for all i. By the five lemma, we have

(3. 3) $(h_0 \circ p)_* = \pi_{0*} \circ k_*: \pi_i(Z_f, S_1^{2p^2-1}) \rightarrow \pi_i(S, F) \approx \pi_i(Y_p')$ *is a $\mathcal{C}$-isomorphism onto for* $i \leqq 2p^3-2$.

Let $h: C_f \rightarrow Y_p$ be the composition of h_0 and the 2-connective fibering of Y_p' onto Y_p. Then the first assertion of Theorem 2 is proved.

The composition $\pi \circ h$ in Theorem 2 coincides with the composition of $h_0: C_f \rightarrow Y_p'$ and a fibering $\pi': Y_2' \rightarrow S^{2p+1}$ given by $\pi'(l)=l(0)$, $l \in Y_p'$. Let $W=S^{2p+1} \cup e^{2p^2} \cup e^{2p^2+1}$ be a mapping cone of $\pi \circ h$. Since the image of each point of C_f under h_0 is a path $l: (I, 0, 1) \rightarrow (Z_{g'}, S^{2p+1}, *)$, h_0 defines a mapping

$$H: W \rightarrow Z_{g'}$$

such that $H|S^{2p+1}$ is the identity and that H induces a mapping of paths $\Omega(H): \Omega(W, S^{2p+1}) \rightarrow Y_p'$ with $\Omega(H)|C_f = h_0$, where $\Omega(W, S^{2p+1}) = \{l: (I, 0, 1) \rightarrow (W, S^{2p+1}, *)\}$ and each point x of C_f is identified with a path $x \times [0, 1]$ in W.

Then it is verified that, for dimensions less than $2p^2+2p-2$, the mappings h_0, $\Omega(H)$ and H induces isomorphisms of the cohomology groups mod p. Since X_p is a deformation retract of $Z_{g'}$, it follows from Theorem 1 that $\Delta \neq 0$ and $\mathcal{P}^p \neq 0$ in W. This proves the second assertion of Theorem 2.

Let $\beta \in \pi_{2p^2-1}(S^{2p+1})$ be the class of the restriction $\pi \circ h | S^{2p^2-1}$. β is the class of the attaching map of e^{2p^2}. Since e^{2p^2+1} is attached to e^{2p^2} by a mapping of degree p, then $p\beta = 0$.

Assume that p is odd and $\beta = 0$. Then W is homotopy equivalent to a complex $W' = (S^{2p+1} \vee S^{2p^2}) \cup e^{2p^2+1}$. Then $\mathcal{P}^p \neq 0$ in $W'/S^{2p^2} = S^{2p^2} = S^{2p+1} \cup e^{2p^2+1}$. But this contradicts to the non-existence of non-trivial mod p Hopf invariant in $\pi_{2p^2+1}(S^{2p+1})$ [12]. Thus $\beta \neq 0$ for odd prime p and the last assertion of Theorem 2 is proved for odd p.

The last assertion of Theorem 2 for $p=2$ will be proved in the next section

§4. $B(2)$

In this section, we consider the case $p=2$.

We first consider $SU(3)$ which is one of $B(2)$, since the characteristic class for the bundle $p: SU(3) \to S^5$ is the generator η_3 of $\pi_4(S^3) \approx Z_2$.

We shall compute the following result.

(4. 1)

i =	4	5	6	7	8	9	10
$\pi_i(SU(3)) \approx$	0	Z	Z_6	0	Z_{12}	0	Z_6.

This follows from the exact sequence

$$\cdots \to \pi_{i+1}(S^5) \xrightarrow{\partial} \pi_i(S^3) \xrightarrow{i_*} \pi_i(SU(3)) \xrightarrow{p_*} \pi_i(S^5) \to \cdots$$

of the bundle and the following results (cf. [15]),

i =	4	5	6	7	8	9	10
$\pi_{i+1}(S^5) \approx$	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2
$\pi_i(S^3) \approx$	Z_2	Z_2	Z_{12}	Z_2	Z_2	0	Z_3,

where ∂ satisfies the relation $\partial(E\alpha) = \eta_3 \circ \alpha$ for $\alpha \in \pi_i(S^4)$. It is sufficient to show that $\partial: \pi_{i+1}(S^5) \to \pi_i(S^3)$ is not trivial for

$i=4, 5, 6, 7, 8$. In the notations of [15], we have non-trivial ∂-images: $\partial(\iota_5)=\eta_3$, $\partial(\eta_5)=\eta_3^2$, $\partial(\eta_5^2)=\eta_3^3=2\nu'$, $\partial(\nu_5)=\eta_3\circ\nu_4=\nu'\circ\eta_6$, and $\partial(\nu_5\circ\eta_8)=\eta_3\circ\nu_4\circ\eta_7=\nu'\circ\eta_6^2$. Thus (4.1) is computed.

Next we prove

(4.2). *The homotopy groups of $B(2)$ and $SU(3)$ are $\mathcal{C}$-isomorphic to each other.*

Consider 5-skeleton $S^3\cup e^5$ of $B(2)$ which has non-trivial Sq^2. The homotopy type of $S^3\cup e^5$ is characterized by Sq^2. Thus any $B(2)$ has the same homotopy type of a complex

$$(S^3\cup e^5)\cup_\gamma e^8,$$

in which e^8 is attached to a representative of a class γ of $\pi_7(S^3\cup e^5)$.

Since $\pi_7(SU(3))=0$ by (4.1), then the injection of $S^3\cup e^5$ into $SU(3)$ can be extended over a mapping $f: B(2)\to SU(3)$ which induces isomorphisms of homology groups of dimensions less than 8. By considering the ring structure mod 2 for $B(2)$ and $SU(3)$, it follows that f induces isomorphisms of the cohomology groups mod 2 and thus $\mathcal{C}$-isomorphisms of the homotopy groups.

Consider the exact sequence (1.3), in particular,

$$\pi_7(Y_2)\xrightarrow{\pi_*}\pi_7(S^5)\xrightarrow{g_*}\pi_7(B(2)).$$

g_* is trivial since $\pi_7(S^5)\approx Z_2$ and the 2-component of $\pi_7(B(2))$ vanishes by (4.1) and (4.2). Thus π_* is onto. It follows from the first assertion of Theorem 2 that the last assertion of Theorem 2 is true for $p=2$.

§5. Some results in unstable homotopy groups of spheres

In this section we assume that p is an odd prime. First we recall the following results from Theorem 8.3 of [13].

(5.1) *Let m be sufficiently large integer, then*

$${}_p\pi_{2m+2i(p-1)}(S^{2m+1})\approx Z_p \qquad \text{for} \quad 1\leqq i\leqq 2p-1 \text{ and } i\neq p,$$
$${}_p\pi_{2n+2p(p-1)}(S^{2m+1})\approx Z_{p^2},$$
$${}_p\pi_{2m+2p(p-1)-1}(S^{2m+1})\approx Z_p,$$

$$_p\pi_{2m+2(p+1)(p-1)-2}(S^{2m+1}) \approx Z_p$$

and $\quad _p\pi_{2m+1+k}(S^{2m+1}) = 0 \qquad$ *otherwise for* $k < 4p(p-1)-4$.

In the exact sequence

(5.2) $\cdots \to \pi_{i+1}(\Omega^2(S^{2m+1}), S^{2m-1}) \to \pi_i(S^{2m-1}) \xrightarrow{E^2} \pi_{i+2}(S^{2m+1}) \to \pi_i(\Omega^2(S^{2m+1}), S^{2m-1}) \to \cdots$,

we have the following $\mathcal{C}$-isomorphism, by (8.7)′ of [11],

(5.3) $\pi_i(\Omega^2(S^{2m+1}), S^{2m-1}) \overset{\mathcal{C}}{\approx} \pi_{i+1}(Z_f, S^{2pm-1})$ *for* $i < 2p^2m-3$, *where* Z_f *is the mapping-cylinder of a mapping* $f: S^{2pm-1} \to S^{2pm-1}$ *of degree* p,

If $i < 2mp-2$, then the groups in (5.3) are finite without p-torsions. Thus $E^2: \pi_i(S^{2m+1}) \to \pi_{i+2}(S^{2m+3})$ are $\mathcal{C}$-isomorphisms onto for $i < 2(m+1)p-3$, and we have

(5.1)′ (5.1) *is true for* $2n+1 > (k+2)/(p-1)$.

For $m=p$, we have

(5.4) $\quad _p\pi_{2p+2i(p-1)}(S^{2p+1}) \approx Z_p \qquad$ *for* $i = 1, 2, \cdots, p-1$,

$\quad _p\pi_{2p^2-1}(S^{2p+1}) \approx Z_p$,

$\quad _p\pi_{2p^2}(S^{2p+1}) \approx Z_{p^2}$,

$\quad _p\pi_{2p+2p^2-4}(S^{2p+1}) \approx Z_p$

and $\quad _p\pi_{2p+1+k}(S^{2p+1}) = 0 \qquad$ *otherwise for* $k < 2p^2-4$.

Furthermore, we shall prove

(5.5) $\quad _p\pi_{2p+2i(p-1)}(S^{2p+1}) \approx Z_p \qquad$ *for* $i = p+1, p+2, \cdots, 2p-1$,

$\quad _p\pi_{2p+2i(p-1)-1}(S^{2p+1}) \approx Z_p \qquad$ *for* $i = p+1, p+2, \cdots, 2p-1$,

and $\quad _p\pi_{2p+1+k}(S^{2p+1}) = 0 \quad$ *otherwise for* $2p^2-4 \leqq k < 4p(p-1)-4$.

More generally, we shall prove the following (5.6) by decreasing induction on j.

(5.6) $\quad _p\pi_{2p+2j+2i(p-1)}(S^{2p+2j+1}) \approx Z_p$ *for* $p+1 \leqq i \leqq 2p-1$ *and* $0 \leqq j$,

$\quad _p\pi_{2p+2j+2i(p-1)-1}(S^{2p+2j+1}) \approx Z_p$ *for* $p+1 \leqq i \leqq 2p-1$ *and* $0 \leqq j < i-p$,

and $\quad _p\pi_{2p+2j+1+k}(S^{2p+2j+1}) = 0 \quad$ *otherwise for* $2p^2-4 \leqq k < 4p(p-1)-4$ *and* $j \geqq 0$.

(5.6) is true for sufficiently large j, for example $j \geqq p$, by (5.1)'. By (5.3), (5.1)' and by (5.2), we have the following exact sequence.

$$\cdots \to 0 \to {}_p\pi_{2p+2(j-1)+2i(p-1)}(S^{2p+2j-1}) \xrightarrow{E^2} {}_p\pi_{2p+2j+2i(p-1)}(S^{2p+2j+1}) \to$$
$$Z_p \to {}_p\pi_{2p+2(j-1)+2i(p-1)-1}(S^{2p+2j-1}) \xrightarrow{E^2} {}_p\pi_{2p+2j+2i(p-1)-1}(S^{2p+2j+1}) \to Z_p \to$$
$${}_p\pi_{2p+2(j-1)+2i(p-1)-2}(S^{2p+2j-1}) \xrightarrow{E^2} {}_p\pi_{2p+2j+2i(p-1)-2}(S^{2p+2j+1}) \to 0 \to \cdots \to 0 \to$$
$${}_p\pi_{2p+2(j-1)+2(p+j)(p-1)}(S^{2p+2j-1}) \xrightarrow{E^2} {}_p\pi_{2p+2j+2(p+j)(p-1)}(S^{2p+2j+1}) \to Z_p \to$$
$${}_p\pi_{2p+2(j-1)+2(p+j)(p-1)-1}(S^{2p+2j-1}) \xrightarrow{E^2} {}_p\pi_{2p+2j+2(p+j)(p-1)-1}(S^{2p+2j+1}) \to 0,$$
$$(2p+j > i > p+j,\ j > 0).$$

We know [14] that there exists an element $\alpha_i \in \pi_{2i(p-1)+2}(S^3)$ of order p for each integer $i > 0$ such that $E^j\alpha_i \neq 0$ for all $j \geqq 0$. It follows that $E^2: {}_p\pi_{2p+2(j-1)+2i(p-1)}(S^{2p+2j-1}) \to {}_p\pi_{2p+2j+2i(p-1)}(S^{2p+2j+1})$ is not trivial. Then, by the above exact sequence, we have that the assertion of (5.6) for $j > 0$ implies the assertion of (5.6) for $j-1$. Thus (5.6) and (5.5) are proved.

§6. Proof of Theorem 3

For the case $p=2$, Theorem 3 is proved by (4.1) and (4.2).

In the following, we assume that p is an odd prime. By Theorem 2 and (5.1)', we have that $\pi_i(Y_p)$ is finite for $3 \leqq i \leqq 2p^3-2$ and

(6.1) ${}_p\pi_{2p+2i(p-1)-1}(Y_p) \approx Z_p$ *for* $i = p, p+1, \cdots, 2p-1$,
${}_p\pi_{2p+2i(p-1)-2}(Y_p) \approx Z_p$ *for* $i = p+1, p+2, \cdots, 2p-1$,
and ${}_p\pi_k(Y_p) = 0$ *otherwise for* $k < 2p+4p(p-1)-3$.

Apply the results (6.1), (5.4) and (5.5) to the exact sequence (1.3), then we see that Theorem 3 is a consequence of the following lemma

(6.2). *The homomorphisms* $\pi_*: \pi_k(Y_p) \to \pi_k(S^{2p+1})$ *for* $k=2p+2i(p-1)-1$, $i=p, p+1, \cdots, 2p-1$ *and for* $k=2p+2p^2-4$ *are isomorphisms of the* p*-components.*

i): *The case* $k=2p+2p(p-1)-1=2p^2-1$. In this case, a generator of ${}_p\pi_k(Y_p)$ is represented by $h|S^k$. By the last assertion of Theorem 2, we have that (6.2) is true for this case.

ii): *The case* $k=2p+2p^2-4$. In this case, the image of π_* contains the composition $\beta\circ\alpha$ of the class $\beta\in\pi_{2p^2-1}(S^{2p+1})$ of $\pi\circ h|S^{2p^2-1}$ and a generator α of ${}_p\pi_k(S^{2p^2-1})\approx Z_p$. In the *stable range*, we know in [13] that the composition $E^\infty(\beta\circ\alpha)=E^\infty(\beta)\circ E^\infty(\alpha)$ is not zero, Thus π_* is not trivial for p-components and (6.2) is true for this case.

iii): *The cases* $k=2p+2(p+j)(p-1)-1$ *and* $j=1,2,\cdots,p-1$.

Let $K=S^{2p^2-4}\cup e^{2p^2-3}$ be the mapping-cone of a mapping of degree p. We may assume that C_f is a three fold iterated suspension E^3K of K. Then $\pi\circ h$ defines a mapping $\Omega^3(\pi\circ h):K\to\Omega^3(S^{2p+1})$. Set $Q=\Omega(\Omega^2(S^{2p+1}),\ S^{2p-1})$, then the homomorphism $\pi_{i+2}(S^{2p+1})\to\pi_i(\Omega^2(S^{2p+1}),\ S^{2p-1})$ in (5.2) is equivalent to a homomorphism i_*: $\pi_{i-1}(\Omega^3(S^{2p+1}))\to\pi_{i-1}(Q)$ induced by the natural injection i.

Since the class of $\pi\circ h|S^{2p^2-1}$ is an E^2-image, $\Omega^3(\pi\circ h)|S^{2p^2-4}$ is homotopic to zero. Thus $\Omega^3(\pi\circ h)$ is factorized to $K\to S^{2p^2-3}\to Q$.

Next we have

(6.3). *$H^*(Q, Z_p)$ is spanned by* 1, w *and* Δw *for dimensions less than* $4p^2-5$, $w\in H^{2p^2-3}(Q, Z_p)$.

This follows from the results on $H_*(\Omega^2(S^{2p+1}), Z_p)$ in [6].

Then $\pi_{2p-3}(Q)$ is $\mathcal{C}$-isomorphic to Z_p. Thus $\Omega^3(\pi\circ h)$ is homotopic to the composition of a mapping $q:K\to EK$ and a mapping $g:EK\to Q$ such that $q(S^{2p^2-4})=*$ and $q^*:H^n(EK, Z_p)\approx H^n(K, Z_p)$ for $n=2p^2-3$. We prove

(6.4). *g induces isomorphism of cohomology groups mod p and thus $\mathcal{C}$-isomorphisms of homotopy groups for dimensions less than* $4p^2-6$.

It is sufficient to prove that $g|S^{2p^2-3}$ is not homotopic to zero. Assume that $g|S^{2p^2-3}$ is homotopic to zero. Then $\Omega^3(\pi\circ h)$ is homotopic to zero in Q. It follows that $\Omega^2(\pi\circ h):EK\to\Omega^2(S^{2p+1})$ is homotopic to a mapping into S^{2p-1}. Let $L=S^{2p-1}\cup e^{2p^2-2}\cup e^{2p^2-1}$ be the mapping-cone of the last mapping. Then the mapping-cone $S^{2p+1}\cup e^{2p^2}\cup e^{2p^2+1}$ of $\pi\circ h$ in Theorem 2 is homotopy equivalent to

E^2L. Then $\mathcal{P}^p \neq 0$ in E^2L and thus $\mathcal{P}^p \neq 0$ in L. But $\mathcal{P}^p H^{2p-1}(\ , Z_p)=0$ in general. We have a contradiction, hence $g|S^{2p^2-2}$ is not homotopic to zero and (6.4) is proved.

Now consider an element γ of $\pi_{k-3}(K)$ such that, by shrinking S^{2p^2-4} to a point, γ is carried to a generator of ${}_p\pi_{k-3}(S^{2p^2-3})$. Then $q_*(\gamma) \neq 0$. By (6.4), $\Omega^3(\pi \circ h)_*(\gamma)=g_* q_*(\gamma) \neq 0$ in $\pi_{k-3}(Q)$. Then $\Omega^3(\pi \circ h)_* (\gamma) \neq 0$ in $\pi_{k-3}(\Omega^3(S^{2p+1}))$. It follows that $(\pi \circ h)_* E^3\gamma \neq 0$ in $\pi_k(S^{2p+1})$. Thus π_* in (6.2) is not trivial for the case iii) and it is an isomorphism of the p-components.

Consequenty, Theorem 3 has been proved.

§ 7. Remarks on homotopy groups of Lie groups

Since $\pi_{2n}(S^{2k+1})$ is finite and has no p-torsion if $k<n<p$, it follows from the exact sequence for the bundle $SU(k+1) \rightarrow S^{2k+1} = SU(k+1)/SU(k)$ that $\pi_{2n}(SU(k+1))$ is finite and has no p-torsion.

From the exactness of the sequence $\pi_{2n+1}(SU(n+1)) \xrightarrow{\pi_*} \pi_{2n+1}(S^{2n+1}) \rightarrow \pi_{2n}(SU(n))$, we have that if $p<n$ then there exists a mapping $f_n : S^{2n+1} \rightarrow SU(n+1)$ such that the mapping degree of the composition $\pi \circ f_n : S^{2n+1} \rightarrow S^{2n+1}$ is prime to p. The multiplication in $SU(n+1)$ and the mappings $f_1, f_2, \cdots, f_n$ define a mapping

$$f : S^3 \times S^5 \times \cdots \times S^{2n+1} \rightarrow SU(n+1).$$

Then it is verified that f induces isomorphisms of the cohomology groups mod p and thus $\mathcal{C}$-isomorphisms

(7.1) $f^* : \pi_i(S^3) \oplus \pi_i(S^5) \oplus \cdots \oplus \pi_i(S^{2n+1}) \rightarrow \pi_i(SU(n+1))$ *for all* i *and for* $n<p$.

We have also that $\pi_{2p}(SU(p))$ is finite and the injection homomorphism : $\pi_{2p}(SU(2)) \rightarrow \pi_{2p}(SU(p))$ is an onto map of the p-components. This injection homomorphism is equivalent to the projection homomorphism : $\pi_{2p+1}(B_{SU(2)}) \rightarrow \pi_{2p+1}(B_{SU(p)})$. Let $g : S^{2p+1} \rightarrow B_{SU(p)}$ be a mapping which induces the $SU(p)$-bundle : $SU(p+1) \rightarrow S^{2p+1}$. Then there exists a mapping $q : S^{2p+1} \rightarrow S^{2p+1}$ of the degree prime to p such that the composition $g \circ q$ is homotopic to a mapping into $B_{SU(2)}$. Let $\bar{q} : X \rightarrow SU(p+1)$ be a bundle map induced by q. Then X is equivalent to a $SU(p)$-bundle, whose group of structure can be reduced into $SU(2)$. Thus there exists a $SU(2)$-bundle $B(p)$ over S^{2p+1} such that the diagram

$$\begin{array}{ccc} B(p) & \xrightarrow{g'} & SU(p+1) \\ \downarrow & & \downarrow \pi \\ S^{2p+1} & \xrightarrow{q} & S^{2p+1} \end{array}$$

is commutative, for a mapping g'. By use of g' and $f_2, \cdots, f_{p-1}$, construct a mapping

$$f' : S^5 \times \cdots \times S^{2p-1} \times B(p) \to SU(p+1)$$

as above, then f' induces isomorphisms of the cohomology groups mod p and thus $\mathcal{C}$-isomorphisms

$$(1.4) \qquad f'_* : \pi_i(S^5) \oplus \cdots \oplus \pi_i(S^{2p-1}) \oplus \pi_i(B(p)) \to \pi_i(SU(p+1))$$

for all i. By [2], $\mathcal{P}^1 \neq 0$ in $SU(p+1)$. Thus $B(p)$ satisfies (1.1).

Similarly, we have mappings

$$f \; : \; S^3 \times S^7 \times \cdots \times S^{4n-1} \to Sp(n)$$

and $$f' : \; S^7 \times \cdots \times S^{2p-3} \times B(p) \to Sp\left(\frac{p+1}{2}\right) \qquad (p : \text{odd})$$

which induce $\mathcal{C}$-isomorphisms

$$(7.2) \qquad f_* : \; \pi_i(S^3) \oplus \pi_i(S^7) \oplus \cdots \oplus \pi_i(S^{4n-1}) \to \pi_i(Sp(n))$$

for all i and for $p \geqq 2n$ and

$$(1.5) \quad f'_* : \; \pi_i(S^7) \oplus \cdots \oplus \pi_i(S^{2p-3}) \oplus \pi_i(B(p)) \to \pi_i\left(Sp\left(\frac{p+1}{2}\right)\right)$$

for all i and for odd p.

By [5], we have $\mathcal{C}$-isomorphisms

$$(1.5)' \qquad \pi_i(Spin(n+2)) \overset{\mathcal{C}}{\approx} \pi_i(SO(n+2)) \overset{\mathcal{C}}{\approx} \pi_i\left(Sp\left(\frac{n+1}{2}\right)\right)$$

for odd n, odd p and for all i.

There is a G_2-bundle: $Spin(7) \to S^7$ with a characteristic class of order 2. Then we have $\mathcal{C}$-isomorphisms

$$\pi_i(G_2) \oplus \pi_i(S^7) \overset{\mathcal{C}}{\approx} \pi_i(Spin(7)) \overset{\mathcal{C}}{\approx} \pi_i(S^7) \oplus \pi_i(B(5))$$

for all i and for $p=5$. It follows

(1. 6) $$\pi_i(G_2) \overset{\mathcal{C}}{\approx} \pi_i(B(5)) \qquad (p = 5).$$

REFERENCES

[1] J. Adem, *The relations on Steenrod powers of cohomology classes*, Algebraic geometry and topology, Princeton.
[2] A. Borel and J.-P. Serre, *Groupes de Lie et puissances réduites de Steenrod*, Amer. J. Math., 75 (1953), 409-448.
[3] H. Cartan, Séminaire de topologie, E. N. S., 1954-55.
[4] H. Cartan, *Sur l'itérations des opérations de Steenrod*, Comm. Math. Helv., 29 (1955), 40-58.
[5] B. Harris, *On the homotopy groups of the classical groups*, Ann. of Math., 74 (1961), 407-413.
[6] J. C. Moore, *Some applications of homology theory to homotopy problems*, Ann. of Math., 58 (1953), 325-350.
[7] J.-P. Serre, *Homologie singulière des espaces fibrés*, Ann. of Math., 54 (1951), 425-505.
[8] J.-P. Serre, *Cohomologie modulo 2 des complexes d'Eilenberg-MacLane*, Comm. Math. Helv., 27 (1953), 198-231.
[9] J.-P. Serre, *Groupes d'homotopie et classes des groupes abéliens*, Ann. of Math., 58 (1953), 258-294.
[10] N. E. Steenrod, *Topology of fibre bundles*, Princeton 1951.
[11] H. Toda, *On double suspension E^2*, J. Inst. Poly. Osaka City Univ., 7 (1956), 103-145.
[12] H. Toda, *p-primary components of homotopy groups II, mod p Hopf invariant*, Mem. Coll. Sci. Kyoto Univ., 31 (1958), 143-160.
[13] H. Toda, *p-primary components of homotopy groups IV, Compositions and toric constructions*, Mem. Coll. Sci. Kyoto Univ., 32 (1959), 297-332.
[14] H. Toda, *On unstable homotopy groups of spheres and classical groups*, Proc. Nat. Acad. Sci. U. S. A., 46 (1960), 1102-1105.
[15] H. Toda, *Composition methods in homotopy groups of spheres*, Princeton 1962.
[16] E. C. Zeeman, *A proof of the Comparision theorem for spectral sequences*, Proc. Camb. Phil. Soc., 53 (1957), 57-62.