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On $\mathfrak{p}$-equations and normal extensions of finite $\mathfrak{p}$-type I

To Yasuo Akizuki on his 60th Birthday

By

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§1. **Introduction.** Let p be a prime number and Δ be a field of characteristic p. Let Δ' be the separable closure of Δ and G_Δ be the galois group of Δ'/Δ. We mean by a *Witt vector* with coefficients in Δ' an infinite ordered set $(\alpha_0, \alpha_1, \alpha_2, \cdots)$ of elements α_ν $(\nu=0, 1, 2, \cdots)$ in Δ'. Putting $\mathbf{0}=(0, 0, \cdots)$, $\mathbf{1}=(1, 0, 0, \cdots)$, $\boldsymbol{p}=(0, 1, 0, \cdots)$ and $\boldsymbol{p}^\nu=(\overbrace{0, \cdots, 0}^{\nu}, 1, 0, \cdots)$, we write $\sum_{\nu=0}^{\infty}\alpha_\nu \boldsymbol{p}^\nu$ instead of $(\alpha_0, \alpha_1, \alpha_2, \cdots)$. *E.* Witt introduced the sum, the difference and the product of two Witt vectors $\sum_{\nu=0}^{\infty}\beta_\nu \boldsymbol{p}^\nu$ and $\sum_{\nu=0}^{\infty}\beta_\nu \boldsymbol{p}^\nu$ by means of a system of infinite polynomials $\Phi_{\underset{\cdot}{\pm},\nu}(X_0, \cdots, X_{\nu-1}, Y_0, \cdots, Y_{\nu-1})$ with coefficients in the prime field $GF(p)$ as follows: $(\sum_{\nu=0}^{\infty}\alpha_\nu \boldsymbol{p}^\nu)\underset{\cdot}{\pm}(\sum_{\nu=0}^{\infty}\beta_\nu \boldsymbol{p}^\nu)=\sum_{\nu=0}^{\infty}\gamma_{\underset{\cdot}{\pm},\nu}\boldsymbol{p}^\nu$,

$$(1)\qquad \gamma_{\pm,\nu} = \alpha_\nu \pm \beta_\nu + \Phi_{\pm,\nu}(\alpha_0, \cdots, \alpha_{\nu-1};\ \beta_0, \cdots, \alpha_{\nu-1}),$$

$$(2)\qquad \gamma_{\cdot,\nu} = \alpha_0\beta_\nu + \alpha_\nu\beta_0 + \Phi_{\cdot,\nu}(\alpha_0, \cdots, \alpha_{\nu-1};\ \beta_0, \cdots, \beta_{\nu-1}).$$ 1)

By mean of these operations all the Witt vectors with coefficients in Δ' forms a commutative integral domain $W_{\Delta'}$. We call $W_{\Delta'}$ the *ring of Witt vectors with coefficients in* Δ'. The ring W_Δ of Witt vectors with coefficients in Δ is naturally embedded in $W_{\Delta'}$. Since the ring $\boldsymbol{Z}_p$ of *p-adic integers* is canonically isomor-

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1) See [1] p.p. 126-128.

phic to the ring of Witt vectors with coefficients in *the prime field* $GF(p)$, we may consider $\boldsymbol{Z}_p$ as a subring of $\boldsymbol{W}_{\Delta'}$. We denote by $\boldsymbol{K}_{\Delta'}$ (resp. $\boldsymbol{K}_\Delta$) the quatient field of $\boldsymbol{W}_{\Delta'}$ (resp. $\boldsymbol{W}_\Delta$), then we may consider $K_{\Delta'}$ as the field of *p-series* $\{\sum_{\nu=-n}^{\infty}\alpha_\nu \boldsymbol{p}^\nu \mid \alpha\in\Delta'\}$ with finite negative terms. The field $\boldsymbol{Q}_p$ *of p-adic numbers* is also regarded as a subfield of $\boldsymbol{K}_{\Delta'}$.

We shall identify the galois group of $\boldsymbol{K}_{\Delta'}/\boldsymbol{K}_\Delta$ with the galois group $\boldsymbol{G}_\Delta$ of Δ'/Δ in the following mean: $(\sum_{\nu=-n}^{\infty}\alpha_\nu \boldsymbol{p}^\nu)^\sigma=\sum_{\nu=-n}^{\infty}\alpha_\nu^\sigma \boldsymbol{p}^\nu$ $(\sigma\in\boldsymbol{G}_\Delta)$, and consider $\boldsymbol{K}_{\Delta'}$ (resp. $\boldsymbol{W}_{\Delta'}$) as a $\boldsymbol{Q}_p[\boldsymbol{G}_\Delta]$-module (resp. $\boldsymbol{Z}_p[\boldsymbol{G}_\Delta]$-module), where we mean by galois automorphisms the continuous automorphisms in $\boldsymbol{p}$-adic topology. We denote by $\mathfrak{p}$ the meromorphism of $\boldsymbol{K}_{\Delta'}$ defined by

$$(3)\qquad (\sum_{\nu=-n}^{\infty}\alpha_\nu \boldsymbol{p}^\nu)^{\mathfrak{p}} = \sum_{\nu=-n}^{\infty}\alpha_\nu^p \boldsymbol{p}^\nu$$

and mean by a $\mathfrak{p}$-*equation with coefficients in* $\boldsymbol{K}_\Delta$ (*resp.* $\boldsymbol{W}_\Delta$) an equation $\sum_{\nu=0}^{n}\boldsymbol{a}_\nu \boldsymbol{X}^{\mathfrak{p}^\nu}=0$ with coefficients α_ν in $\boldsymbol{K}_\Delta$ (resp. $\boldsymbol{W}_\Delta$). The solutions in $\boldsymbol{K}_{\Delta'}$ of a non-zero $\mathfrak{p}$-equation $\boldsymbol{f}(\boldsymbol{X})=0$ with coefficients in $\boldsymbol{K}_\Delta$ form a $\boldsymbol{Q}_p$-finite-dimensional $\boldsymbol{Q}_p[\boldsymbol{G}_\Delta]$-submodule $\boldsymbol{V}_f$ in $\boldsymbol{K}_{\Delta'}$ and conversely each $\boldsymbol{Q}_p$-finite-dimensional $\boldsymbol{Q}_p[\boldsymbol{G}_\Delta]$-submodule $\boldsymbol{V}$ in $\boldsymbol{K}_{\Delta'}$ is uniquely expressed as the module of solutions $\boldsymbol{V}_\varphi$ of a $\mathfrak{p}$-*equation* $\varphi(\boldsymbol{X})=0$ such that 1° the coefficients belong to W_Δ, 2° the coefficient of the highest term is 1, 3° the coefficient of $\boldsymbol{X}$ (the lowest term) is not congruent to zero modulo $\boldsymbol{p}\boldsymbol{W}_\Delta$. The correspondence between $\boldsymbol{Q}_p$-finite-dimensional $\boldsymbol{Q}_p[\boldsymbol{G}_\Delta]$-submodules in $\boldsymbol{K}_{\Delta'}$ and $\mathfrak{p}$-equations satisfying the conditions 1°, 2°, 3° is one-to-one (Theorem 1). For a $\boldsymbol{Q}_p$-finite-dimensional $\boldsymbol{Q}_p[\boldsymbol{G}_\Delta]$-submodule $\boldsymbol{V}$ in $\boldsymbol{K}_{\Delta'}$ we denote by $(\sum_{\nu=0}^{\infty}\xi_{n\nu}\boldsymbol{p}^\nu,\cdots,\sum_{\nu=0}^{\infty}\xi_{n\nu}\boldsymbol{p}^\nu)$ a $\boldsymbol{Z}_p$-base of the intersection $\boldsymbol{V}\cap\boldsymbol{W}_{\Delta'}$, by $\boldsymbol{\Gamma}_V=\{\boldsymbol{M}_V(\sigma)\in GL(n,\boldsymbol{Z}_p)\mid\sigma\in\boldsymbol{G}_\Delta\}$ the respresentation of $\boldsymbol{G}_\Delta$ by mean of the base and by $\boldsymbol{\Gamma}_V(\boldsymbol{p}^\nu)$ the subgroup $\{\boldsymbol{M}\in\boldsymbol{\Gamma}_V\mid\boldsymbol{M}$ identity mod $\boldsymbol{p}^\nu\}$. then the galois groups of the normal extensions $\boldsymbol{K}_\Delta(\xi_{1,0},\cdots,\xi_{n,0},\sum_{l=0}^{1}\xi_1\boldsymbol{p}^l,\cdots,\sum_{l=0}^{1}\xi_{nl}\boldsymbol{p}^l,\cdots,\sum_{l=0}^{\nu-1}\xi_{1l}\boldsymbol{p}^l,\cdots,$

$\sum_{l=0}^{\nu-1} \xi_{nl} p^l)/K_\Delta$ and $\Delta(\xi_{10}, \cdots, \xi_{n0}, \cdots, \xi_{1,\nu-1}, \cdots, \xi_{n,\nu-1})/\Delta$ are canonically isomorphic to $\Gamma_V/\Gamma_V(p^\nu)$. We put $K_\Delta(V) = \bigcup_{\nu=1}^{\infty} K_\Delta(\xi_{10}, \cdots, \xi_{n0}, \sum_{l=0}^{1} \xi_{1l} p^l, \cdots, \sum_{l=0}^{1} \xi_{nl} p^l, \cdots, \sum_{l=0}^{\nu-1} \xi_{1l} p^l, \cdots, \sum_{l=0}^{\nu-1} \xi_l p^l) \cup K_\Delta(\sum_{l=1}^{\infty} \xi_{1l} p^l, \cdots, \sum_{l=0}^{\infty} \xi_{nl} p^l)$, $\Delta(V) = \bigcup_{\nu=1}^{\infty} \Delta(\xi_{1,0}, \cdots, \xi_{n,0}, \cdots, \xi_{l,\nu}, \cdots, \xi_{n,\nu})$ and call $K_\Delta(V)/K_\Delta$ and $\Delta(V)/\Delta$ normal extensions of finite $\mathfrak{p}$-type.

If a $K_\Delta[G_\Delta]$-module $\mathfrak{p}$ has a K_Δ-base $(\xi_1, \cdots, \xi_n)$ such that the coefficients of the representation $\{M(\sigma) \mid \sigma \in G_\Delta\}$ defined by $(\xi_1^\sigma, \cdots, \xi_n^\sigma) = (\xi_1, \cdots, \xi_n) M(\sigma)$ $(\sigma \in G_\Delta)$ belong to a finite algebraic extension of Q_p, we call $\mathfrak{V}$ a *$K_\Delta[G_\Delta]$-module of finite $\mathfrak{p}$-type.* We shall determine the structure of the $K_\Delta[G_\Delta]$-submodule of $K_{\Delta'}$ which is the union of all semi-simple $K_\Delta[G_\Delta]$-modules of finite $\mathfrak{p}$-type in $K_{\Delta'}$. The results (Theorem 3) is a partial generalization of the existence theorem of normal base for a finite normal extension.[2]

§ 2. $\mathfrak{p}$-Wronskians.

As an analogy in theory of differential equation we shall define Wronskian and give a criterion of linearly independency over Q_p. We means by *the $\mathfrak{p}$-Wronskian* of a system $(\xi_1, \cdots, \xi_n)$ of quantities $\xi_1, \cdots, \xi_n$ the determinant

$$W_\mathfrak{p}(\xi_1, \cdots, \xi_n) = \begin{vmatrix} \xi_1 & , \cdots , & \xi_n \\ \xi_1^\mathfrak{p} & , \cdots , & \xi_n^\mathfrak{p} \\ \vdots & & \\ \xi_1^{\mathfrak{p}^{n-1}} & , \cdots , & \xi_n^{\mathfrak{p}^{n-1}} \end{vmatrix}.$$

Proposition 1. *Let $\xi_1, \cdots, \xi_n$ be elements in $K_{\Delta'}$. Then $\xi_1, \cdots, \xi_n$ are linearly independent over Q_p if and only if $W_\mathfrak{p}(\xi_1, \cdots, \xi_n) \neq 0$*[3].

Proof. From the definition of $\mathfrak{p}$ it follows that an element in $K_{\Delta'}$ is fixed by $\mathfrak{p}$ if and only if it belongs to Q_p. This shows that if $\xi_1, \cdots, \xi_n$ are linearly dependent over Q_p the $\mathfrak{p}$-Wronskian $W_\mathfrak{p}(\xi_1, \cdots, \xi_n)$ is zero. We shall prove the converse by the induction

2) In Part II we shall treat the analogy for $\mathfrak{p}$-equations of Riemann's-problem for linear differential equations and shall determine the structure of the union of all simi-simple $K_\Delta[G_\Delta]$-submodules of finite $\mathfrak{p}$-type in $K_{\Delta'}$.

3) Replacing $\mathfrak{p}$ by $\mathfrak{p}^r$ and Q_p by the unramified extension of degree r over Q_p, we have the same result for $\mathfrak{p}^r$-Wronskian as $\mathfrak{p}$-Wronskians.

on n. Assume the result for $n-1$ and $\xi_1 \neq 0$. Suppose $W_{\mathfrak{p}}(\xi_1, \cdots, \xi_n)=0$. Then it follows

$$W_{\mathfrak{p}}(\xi_1, \cdots, \xi_n) = \xi_1^{1+\mathfrak{p}+\cdots+\mathfrak{p}^{n-1}} \begin{vmatrix} 1, & \xi_2\xi_1^{-1} & , \cdots, \xi_n\xi_1^{-1} \\ 1, & (\xi_2\xi_1^{-1})^{\mathfrak{p}} & , \cdots, (\xi_n\xi_1^{-1})^{\mathfrak{p}} \\ 1, & (\xi_2\xi_1^{-1})^{\mathfrak{p}^{n-1}} & , \cdots, (\xi_n\xi_1^{-1})^{\mathfrak{p}^{n-1}} \end{vmatrix}$$

$$= \xi_1^{1+\mathfrak{p}+\cdots+\mathfrak{p}^{n-1}} \begin{vmatrix} 1, & \xi_2\xi_1^{-1} & , \cdots, \xi_n\xi_1^{-1} \\ 0, & (\xi_2\xi_1^{-1})^{\mathfrak{p}}-\xi_2\xi_1^{-1} & , \cdots, (\xi_n\xi_1^{-1})^{\mathfrak{p}}-\xi_n\xi_1^{-1} \\ 0, & (\xi_2\xi_1^{-1})^{\mathfrak{p}}-\xi_2\xi_1^{-1})^{\mathfrak{p}} & , \cdots, (\xi_n\xi_1^{-1})^{\mathfrak{p}}-\xi_n\xi_1^{-1})^{\mathfrak{p}} \\ \vdots & & \\ 0, & ((\xi_2\xi_1^{-1})^{\mathfrak{p}}-\xi_2\xi_1^{-1})^{\mathfrak{p}^{n-2}} & , \cdots, ((\xi_n\xi_1^{-1})^{\mathfrak{p}}-\xi_n\xi_1^{-1})^{\mathfrak{p}^{n-2}} \end{vmatrix}$$

$$= 0.$$

Hence, by virtue of the assumption of the induction, there are elements $a_2, \cdots, a_n$ of Q_p which are not all zero such that $\sum_{i=2}^{n} a_i((\xi_i\xi_1^{-1})^{\mathfrak{p}}-\xi_i\xi_1^{-1})=0$, and thus $(\sum_{i=2}^{n} a_i\xi_i\xi_1^{-1})^{\mathfrak{p}} = \sum_{i=2}^{n} a_i\xi_i\xi_1^{-1}$. This shows that $\sum_{i=2}^{n} a_i\xi_i\xi_1^{-1}$ equals to element, say $-a_1$, in Q_p. Namely these we $a_1, \cdots, a_n$ in Q_p which are not all zero $\sum_{i=1}^{n} a_i\xi_i=0$. For $n=1$ the result is obviously true, hence we complete the proof of Proposition 1.

We mean by the p-Wronskian of a system $(\xi_1, \cdots, \xi_n)$ of elements $\xi_1, \cdots, \xi_n$ the determinant:

$$W_p(\xi_1, \cdots, \xi_n) = \begin{vmatrix} \xi_1 & , \cdots, \xi_n \\ \xi_1^p & , \cdots, \xi_n^p \\ \xi_1^{p^{n-1}} & , \cdots, \xi_n^{p^{n-1}} \end{vmatrix}.$$

Then by replacing $\mathfrak{p}$ by p we have the following the analogious results as Proposition 1 by the completely same reason.

Proposition 1′. *Let $\xi_1, \cdots, \xi_n$ be elements in Δ'. Then $\xi_1, \cdots, \xi_n$ are linearly independent over the prime field $GF(p)$ if and only if $W_p(\xi_1, \cdots, \xi_n) \neq 0$.*

§ 3. Non-commutative $\mathfrak{p}$-polynomials and $Q_p[G_\Delta]$-submodules in $K_{\Delta'}$.

We denote by $K_\Delta\langle t\rangle$ (resp. $W_\Delta\langle t\rangle$) the ring of non-commutative polynomials in t with coefficients in K_Δ (resp. W_Δ) with the law of multiplication: $ta = a^{\mathfrak{p}}t$, $t^\mu t^\nu = t^{\mu+\nu}$ ($a \in K_\Delta$; $\mu, \nu > 0$). We call elements in $K_\Delta\langle t\rangle$ non-commutative $\mathfrak{p}$-polynomials with coefficients in K_Δ and mean by the rank of a non-commutative $\mathfrak{p}$-polynomials $\mathfrak{p}$-polynomial f the highest degree in t in f. We denote by rank f the rank of f. Each element $f = \sum_{\nu=0}^{n} a_\nu t^\nu$ in $K_\nu\langle t\rangle$ acts on $K_{\Delta'}$ in the following way: $f(\xi) = (\sum_{\nu=0}^{n} a_\nu t)(\xi) = \sum_{\nu=0}^{n} a\xi^{\mathfrak{p}^\nu}$. For each $\mathfrak{p}$-equation $f(X) = \sum_{\nu=0}^{n} a_\nu X^{\mathfrak{p}^\nu} = 0$ we mean by f the non-commutative $\mathfrak{p}$-polynormial $\sum_{\nu=0}^{n} a_\nu t^\nu$.

Lemma 1. *Let V be a Q_p-finite-dimensional Q_p-vector subsace in $K_{\Delta'}$ and $(\xi_1, \cdots, \xi_n)$ be a Z_p-base of the intersection $V \cap W_{\Delta'}$ regarded as a Z_p-module. Then $W_{\mathfrak{p}}(\xi_1, \cdots, \xi_n)$ is a unit in $W_{\Delta'}$*[4].

Proof. Assume $\xi_1, \cdots, \xi_r$ are linearly independent modulo $p(W_{\Delta'} \cap V)$ and $\xi_{r+1}, \cdots, \xi_n$ are linearly dependent on $\xi_1, \cdots, \xi_n$ modulo $p(W_{\Delta'} \cap V)$. Obviousely $1 \leq r \leq n$ Suppose for a monent $r \lneqq n$. Then there exist elements $a_1, \cdots, a_r, b_1, \cdots, b_n$ in Z_p such that $a\xi_1 + \cdots + a_r\xi_r - \xi_n = p(b_1\xi_1 + \cdots + b_r\xi_n)$. Since $\xi_1, \cdots, \xi_n$ are linearly independent over Z_p, we have $a_1 = pb_1, \cdots, a_r = pb_r$, $b_{r+1} = \cdots = b_{n-1} = 0$ and $1 + pb_n = 0$. This is contradiction, because $1 \not\equiv 0 \bmod p$. This shows $\xi_1, \cdots, \xi_n$ are linearly independent modulo $p(W_{\Delta'} \cap V)$. Since $p(W_\Delta \cap V) = pW_{\Delta'} \cap V$ and $W_{\Delta'}/pW_{\Delta'}$ is canonically isomorphic to Δ', by virtue of Proposition 1′ we have $W_{\mathfrak{p}}(\xi_1, \cdots, \xi_n) \not\equiv 0 \bmod pW_{\Delta'}$. This proves Lemma 1.

Proposition 2. *Let V be a Q_p-finite-dimensional $Q_p[G_\Delta]$-module in $K_{\Delta'}$ and $(\xi_1, \cdots, \xi_n)$ be a Q_p-base of V. Put $f_V(X) = (-1)^n W_{\mathfrak{p}}(\xi_1, \cdots, \xi_n)^{-1} W_{\mathfrak{p}}(X, \xi_1, \cdots, \xi_n)$. Then the non-commutative $\mathfrak{p}$-polynomial f_V*[5] *is an element in $W_\Delta\langle t\rangle$ with the properties* 1° f_V *does not depend on the choic of Q_p-base,* 2° *the highest coefficient equals* 1, 3° *the consiant term is $(-1)^n W_{\mathfrak{p}}\xi_1, \cdots, \xi_n)^{\mathfrak{p}-1}$ and $\not\equiv 0$ mod pW_Δ*[6].

Proof. First we shall prove the independence of f_V on the

4), 6) The situation is the same as 3).

5) f_V is the non-commutative $\mathfrak{p}$-polynomial associated with the $\mathfrak{p}$-equation $f_V(X) = 0$.

choice of the Q_p-base of V. Let A be any non-singular $n\times n$ matrix with coefficients in Q_p and put $(\eta_1, \cdots, \eta_n)=(\xi_1, \cdots, \xi_n)A$. Then it follows

$$\begin{aligned}&(-1)^n W_{\mathfrak{p}}(\eta_1, \cdots, \eta_n)^{-1} W_{\mathfrak{p}}(X, \eta_1, \cdots, \eta_n)\\ &= (-1)^n |A^{-1}| W_{\mathfrak{p}}(\xi_1, \cdots, \xi_n)^{-1} W_{\mathfrak{p}}(X, \xi_1, \cdots, \xi_n)\left|\begin{pmatrix}1 & 0\\ 0 & A\end{pmatrix}\right|\\ &= (-1)^n W_{\mathfrak{p}}(\xi_1, \cdots, \xi_n)^{-1} W_{\mathfrak{p}}(X, \xi_1, \cdots, \xi_n)\end{aligned}$$

This proves the independence of f_V on the choice of the Q_p-base. Since for every σ in G_Δ $(\xi_1^\sigma, \cdots, \xi_n^\sigma)$ is also a Q_p-base of V and K_Δ is the subfield of $K_{\Delta'}$ consisting of all the elements fixed by every element in G_Δ, we can conclude that the coefficients in f_V belong to K_Δ. From the definition of f_V the highest coefficient in f_V equals to 1. Let $(\zeta_1, \cdots, \zeta_n)$ be a Z_p-base of the intersection $V\cap W_{\Delta'}$. Then by virtue of Lemma 1 we have $W_{\mathfrak{p}}(\zeta_1, \cdots, \zeta_n)\not\equiv 0$ mod $pW_{\Delta'}$. Since the coefficient of X^n in $W_{\mathfrak{p}}(X, \zeta_1, \cdots, \zeta_n)$ is $W_{\mathfrak{p}}(\zeta^{\mathfrak{p}}, \cdots, \zeta_n^{\mathfrak{p}})=W_{\mathfrak{p}}(\zeta_1, \cdots, \zeta_n)^{\mathfrak{p}}$, this shows that the constant term in f_V is $(-1)^n$ $W_{\mathfrak{p}}(\zeta_1, \cdots, \zeta_n)^{\mathfrak{p}-1}$ and is not congruent to zero modulo pW_Δ. On the other hand, since $\zeta_1, \cdots, \zeta_n\in W_{\Delta'}$, the coefficients in $W_{\mathfrak{p}}(X, \zeta_1, \cdots, \zeta_n)$ with respect to X are elements in $W_{\Delta'}$. Therefore we can conclude f_V belongs to $W_\Delta\langle t\rangle$, because $W_{\mathfrak{p}}(\zeta_1, \cdots, \zeta_n)$ is a unit in $W_{\Delta'}$ and f_V belongs to $K_\Delta\langle t\rangle$.

For any element f $(\neq 0)$ in $K_\Delta\langle t\rangle$ we mean by V_f the subset in $K_{\Delta'}$ consisting of all the solutions ξ of the $\mathfrak{p}$-equation $f(X)=0$. Then we have

Proposition 3. (i) *V_f is a $Q_p[G_\Delta]$-submodule in $K_{\Delta'}$ such that $\dim_{Q_p} V_f\leqq \operatorname{rank} f$.* (ii) *$V=V_{f_V}$.* (iii) *If V' is a $Q_p[G_\Delta]$-submodule of V_f, then there exists g in $K_\Delta\langle t\rangle$ such that $f=gf_{V'}$.*

Proof. Since $(a\xi+b\eta)^{\mathfrak{p}}=a\xi^{\mathfrak{p}}+b\eta^{\mathfrak{p}}$ for a, b in Q_p and ξ, η in $K_{\Delta'}$, we have $f(a\xi+b\eta)=af(\xi)+bf(\eta)$ for $a, b\in Q_p$. This shows V_f is a Q_p-module. On the other hand all the coefficients in f belong K_Δ and $\mathfrak{p}$ commutes with every element $\sigma\in G_\Delta$, hence ξ^σ $(\sigma\in G_\Delta)$ belongs to V_f if and only if $\xi\in V_f$. This means V_f is a $Q_p[G_\Delta]$-module. Let $\xi_1, \cdots, \xi_m$ be linearly independent elements in V_f over Q_p. Then by virtue of Proposition 1 we have $W_{\mathfrak{p}}(\xi_1, \cdots, \xi_m)\neq 0$. On the other hand, if we write $f=\sum_{\nu=0}^{n} a_\nu p^\nu$, we have

$$(\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_n)\begin{pmatrix} \xi_1 & ,\cdots, & \xi_m \\ \xi_1^{\mathfrak{p}} & ,\cdots, & \xi_m^{\mathfrak{p}} \\ \xi_1^{\mathfrak{p}n} & ,\cdots, & \xi_m^{\mathfrak{p}n} \end{pmatrix} = (0, 0, \cdots, 0)\,.$$

This shows $m \leqq n$, and thus (i) has been proved. From Proposition 2 it follows rank $\boldsymbol{f}_V = \dim_{Q_p} V$, hence by virtue of (i) we have $\dim_{Q_p} V_{f_V} <$ rand $\boldsymbol{f}_V = \dim_{Q_p} V$. On the other hand $V \subset V_{f_V}$, hence $V = V_{f_V}$. From (i) and (ii) we know that $\boldsymbol{f}_{V'}$ is the element $\boldsymbol{h}$ in $\boldsymbol{K}_\Delta\langle \boldsymbol{t}\rangle$ with the smallest rank such that $V_h \supset V'$. We can choose $\boldsymbol{g}$ and θ in $\boldsymbol{K}_\Delta\langle \boldsymbol{t}\rangle$ such that $\boldsymbol{f} = \boldsymbol{g}\boldsymbol{f}_{V'} + \theta$ and rank $\theta <$ rank $\boldsymbol{f}_{V'}$, because rank $\boldsymbol{f}_{V'} \leqq$ rank $\boldsymbol{f}$. Since $V_f \supset V'$ and $V_{gf_{V'}} \supset V'$, we have $V_\theta \supset V'$. Thus rank $\theta \geqq \dim_{Q_p} V_\theta > \dim_{Q_p} V' = \boldsymbol{f}_{V'}$. Therefore, if $\theta \neq 0$, this is a contradiction. This proves $\theta = 0$.

We shall now show the reverse of Proposition 2.

Proposition 4. *Let* $\boldsymbol{f}$ *be an element of* $\boldsymbol{W}_\Delta\langle \boldsymbol{t}\rangle$ *such that the highest coefficient is* 1 *and the constant term is not congruence to zero modulo* $p\boldsymbol{W}_\Delta$. *Then* $\dim_{Q_p} V_f = \operatorname{rank} \boldsymbol{f}$ *and* $f = f_{V_f}$.

Proof. Let n be the rank of $\boldsymbol{f}$ and put $\boldsymbol{f} = \sum_{\nu=0}^{n} \boldsymbol{a}_\nu \boldsymbol{t}^\nu$. Let $x_0, x_1, x_2, \cdots$ be indeterminates and put $\boldsymbol{X} = \sum_{\nu=0}^{\infty} x_\nu \boldsymbol{p}^\nu$, $\boldsymbol{f}(\boldsymbol{X}) = \sum_{\nu=0}^{\infty} \varphi_\nu(x_0, x_1, \cdots, x_\nu)\boldsymbol{p}^\nu$. It is sufficient to show that the number of solutions of $\varphi_0(x_0) = 0$, $\varphi_1(x_0, x) = 0, \cdots, \varphi_{\nu-1}(x_0, x_1, \cdots, x_{\nu-1}) = 0$ in Δ' is exactly $p^{\nu n}$. Since $\boldsymbol{a}_n = 1$ and $\boldsymbol{a}_0 \not\equiv 0 \bmod p\boldsymbol{W}_\Delta$, by virtue of (1), (2), (3) we have $\frac{\partial}{\partial x_\mu}\varphi_\mu(x_0, \cdots, x_\mu) \not\equiv 0$, and thus $\varphi_\mu(\xi_0, \cdots, \xi_{\mu-1}, x_\mu) = 0$ has no multiple root for given value $\xi_0, \cdots, \xi_{\mu-1}$ in Δ'. On the other hand the degree of $\varphi_\mu(\xi_0, \cdots, \xi_{\mu-1}, x_\mu)$ in x_μ is p^n, hence we conclude the number of solutions of $\varphi_0(x_0) = 0$, $\varphi_1(x_0, x_1) = 0, \cdots, \varphi_{\nu-1}(x_0, \cdots, x_{\nu-1}) = 0$ in Δ' is exactly $p^{\nu n}$. This proves Proposition 4.

We now sum up the results in Proposition 2, 3, 4.

Theorem 1. *The correspondence* $V \leftrightarrow \boldsymbol{f}_V(X) = 0$ *gives a bijective map between the set of* $\boldsymbol{Q}_p$*-finite-dimensional* $\boldsymbol{Q}_p[G_\Delta]$*-submodules in* $\boldsymbol{K}_{\Delta'}$ *and the set of* $\mathfrak{p}$*-equations with the propertics* 1° *the coefficients belong to* $\boldsymbol{W}_\Delta$, 2° *the highest coefficient is* 1, 3° *the coefficient of* $\boldsymbol{X}$

is not congruent to zero modulo $\boldsymbol{p}W_{\Delta}$. *By this correspondence* $\boldsymbol{Q}_p[G_{\Delta}]$-*modules correscpond to irreducible* $\mathfrak{p}$-*equaiions and conversely.*

§ 4. $K_{\Delta}[G_{\Delta}]$-modules of finite $\mathfrak{p}$-type in $K_{\Delta'}$.

Definition. *If a* $\boldsymbol{K}_{\Delta}[G_{\Delta}]$-*module* $\mathfrak{V}$ *in* $\boldsymbol{K}_{\Delta'}$ *has a* $\boldsymbol{K}_{\Delta}$-*base* $(\xi_1, \cdots, \xi_n)$ *such that the coefficients of the representation* $\{M(\sigma) \mid \sigma \in G_{\Delta}\}$ *defined by* $(\xi_1^{\sigma}, \cdots, \xi_n^{\sigma}) = (\xi_1, \cdots, \xi_n) M(\sigma)$ *belong to a finite algebraic extension of* $\boldsymbol{Q}_p$, *then* we *call* $\mathfrak{V}$ *a* $\boldsymbol{K}_{\Delta}[G_{\Delta}]$-*module of finiie* $\mathfrak{p}$-*type.*

In the present paragraph we shall be concerned with $\boldsymbol{K}_{\Delta}[G_{\Delta}]$-modules of finite $\mathfrak{p}$-type in $\boldsymbol{K}_{\Delta'}$, especially semi-simple $\boldsymbol{K}_{\Delta}[G_{\Delta}]$-modules of finite $\mathfrak{p}$-type in $\boldsymbol{K}_{\Delta'}$.

Lemma 1. *If* $\mathfrak{V}$ *is a* $\boldsymbol{K}_{\Delta}[G_{\Delta}]$-*module of finite* $\mathfrak{p}$-*type in* $\boldsymbol{K}_{\Delta'}$, *then there exists a* $\boldsymbol{Q}_p$-*finite-dimensional* $\boldsymbol{Q}_p[G_{\Delta}]$-*module* V *in* $\boldsymbol{K}_{\Delta'}$ *such that* $\mathfrak{V} = \boldsymbol{K}_{\Delta} V$. *If* $\mathfrak{V}$ *is simple, we can choose a simple* $\boldsymbol{Q}_p[G_{\Delta}]$-*module as* V.

Proof. Let $\mathfrak{V}$ be a $\boldsymbol{K}_{\Delta}[G_{\Delta}]$-module in $\boldsymbol{K}_{\Delta'}$ with a K_{Δ}-base $(\eta_1, \cdots, \eta_n)$ such that the field Λ generated by the coefficients of the representation $\{M(\sigma) \mid (\xi_1^{\sigma}, \cdots, \xi_n^{\sigma}) = (\xi_1, \cdots, \xi_n) M(\sigma),\ \sigma \in G_{\Delta}\}$ is a finite algebraic extension of $\boldsymbol{Q}_p$. Let $(\beta_1, \cdots, \beta_r)$ be a $\boldsymbol{Q}_p$-base of Λ and put $\eta_{ij} = \beta_i \xi_j$ $(1 \leqq i \leqq r;\ 1 \leqq j \leqq n)$. Then we have a $\boldsymbol{Q}_p[G]$-module $V = \boldsymbol{Q}_p \eta_{11} + \cdots + \boldsymbol{Q}_p \eta_{rn} = \Lambda \xi_1 + \cdots + \Lambda \xi_n$ in $\boldsymbol{K}_{\Delta'}$ such that $\boldsymbol{K}V = \mathfrak{V}$. Assume $\mathfrak{V}$ is simple. Then the enveloping algebra of $\{M(\sigma) \mid \sigma \in G\}$ over Λ is a simple Λ-algebra. Hence V is a direct sum $V_1 \oplus \cdots \oplus V_r$ of simple $\boldsymbol{Q}_p[G]$-submodules. Since $\boldsymbol{K}_{\Delta} V = \mathfrak{V}$ and $\mathfrak{V}$ is simple there exists as V_i such that $\boldsymbol{K}_{\Delta} V_i = \mathfrak{V}$.

Theorem 2. *Let* $\mathfrak{V}$ *be a* $\boldsymbol{K}_{\Delta}[G_{\Delta}]$-*module of finite* $\mathfrak{p}$-*type in* $\boldsymbol{K}_{\Delta'}$ *and* Λ *be a suffield of* $\boldsymbol{K}_{\Delta}$ *in which the coefficients of a representation* $\{M(\sigma) \mid (\xi_1^{\sigma}, \cdots, \xi_n^{\sigma}) = (\xi_1, \cdots, \xi_n) M(\sigma),\ \sigma \in G_{\Delta}\}$ *of* G_{Δ} *by the* $\boldsymbol{K}_{\Delta}$-*base* $(\xi_1, \cdots, \xi_n)$ *are contained. Let* r *be the degree of* Λ *over* $\boldsymbol{Q}_p$. *Then every* $\boldsymbol{K}_{\Delta}[G_{\Delta}]$-*module in* $\boldsymbol{K}_{\Delta'}$ *isomorphic to* V *is contained in the sum* $\tilde{\mathfrak{V}} = \mathfrak{V} + \boldsymbol{K}_{\Delta} \mathfrak{V}^{\mathfrak{p}^r} + \cdots + \boldsymbol{K}_{\Delta} \mathfrak{V}^{\mathfrak{p}^{r(n-1)}}$ *in* $\boldsymbol{K}_{\Delta'}$.

Proof. We notice that $\Lambda / \boldsymbol{Q}_p$ is cyclic and the galois automorphisms are induced by $\{1, \mathfrak{p}, \cdots, \mathfrak{p}^{r-1}\}$, because $\Lambda \subset \boldsymbol{K}_{\Delta}$ and $\boldsymbol{K}_{\Delta}$ is unramified for p. Let $\mathfrak{U}$ be a $K_\Delta[G_\Delta]$-module in $K_{\Delta'}$ isomorphic to $\mathfrak{V}$ and φ be the isomorphism of $\mathfrak{V}$ onto $\mathfrak{U}$. Then, putting $M(\sigma)=(m_{ij}(\sigma))$ $(\sigma\in G_\Delta)$, we have $(\varphi(\xi^1)^\sigma,\cdots,\varphi(\xi_n)^\sigma)=(\varphi(\xi_1^\sigma),\cdots,\varphi(\xi_n^\sigma))=(\varphi(\sum_{l=1}^n m_{l1}(\sigma)\xi_l),\cdots,\varphi(\sum_{l=1}^n m_{ln}(\sigma)\xi_l))=(\sum_{l=1}^n m_{l1}(\sigma)\varphi(\xi_l),\cdots,\sum_{l=1}^n m_{ln}(\sigma)\varphi(\xi_l))=(\varphi(\xi_1),\cdots,\varphi(\xi_n))M(\sigma)$. Replacing $\mathfrak{p}$ by $\mathfrak{p}^r$ in Proposition 1, by the same reason as for $\mathfrak{p}$, we have

$$W_{\mathfrak{p}^r}(\xi_1,\cdots,\xi_n)=\begin{vmatrix}\xi_1 & ,\cdots, & \xi_n\\ \xi_1^{\mathfrak{p}^r} & ,\cdots, & \xi_n^{\mathfrak{p}^r}\\ \vdots & & \\ \xi_1^{\mathfrak{p}^{r(n-1)}} & ,\cdots, & \xi_n^{\mathfrak{p}^{r(n-1)}}\end{vmatrix}\neq 0$$

Hence putting

$$\begin{pmatrix}\varphi(\xi_1) & ,\cdots, & \varphi(\xi_n)\\ \varphi(\xi_1)^{\mathfrak{p}^r} & ,\cdots, & \varphi(\xi_n)^{\mathfrak{p}^r}\\ \vdots & & \vdots\\ \varphi(\xi_1)^{\mathfrak{p}^{r(n-1)}} & ,\cdots, & \varphi(\xi_n)^{\mathfrak{p}^{r(n-1)}}\end{pmatrix}\begin{pmatrix}\xi_1 & ,\cdots, & \xi_n\\ \xi_1^{\mathfrak{p}^r} & ,\cdots, & \xi_n^{\mathfrak{p}^r}\\ \vdots & & \\ \xi_1^{\mathfrak{p}^{r(n-1)}} & ,\cdots, & \xi_n^{\mathfrak{p}^{r(n-1)}}\end{pmatrix}^{-1}=\begin{pmatrix}a_{11}, & \cdots, & a_{1n}\\ \vdots & & \vdots\\ a_{n1}, & \cdots, & a_{nn}\end{pmatrix},$$

we get a matrix with coefficients $a_{ij}(1\leqq i,j\leqq n)$ in K_Δ. Since $\varphi(\xi_i)=\sum_{l=1}^n a_{il}\xi^{\mathfrak{p}^{r(l-1)}}$ $(1\leqq i\leqq m)$ with $a_{ij}\in K_\Delta$ and $\varphi(\xi_1),\cdots,\varphi(\xi_n)$ generate $\mathfrak{U}$ over K_Δ, we conclude $K_\Delta\mathfrak{V}+K_\Delta\mathfrak{V}^{\mathfrak{p}^r}+\cdots+K_\Delta\mathfrak{V}^{\mathfrak{p}^{r(n-1)}}\supset\mathfrak{U}$.

We shall now calculate the multiplicity of simple $K_\Delta[G_\Delta]$-module in the union $K_{\Delta',s}$ of semi-simple $K_\Delta[G_\Delta]$-modules of finite $\mathfrak{p}$-type in $K_{\Delta'}$.

Theorem 3. *Let $\mathfrak{V}$ be a simple $K_\Delta[G_\Delta]$-module of finite $\mathfrak{p}$-type in $K_{\Delta'}$ and $\{M(\sigma)\mid\sigma\in G_\Delta\}$ be a representation of G_Δ by a K_Δ-base of $\mathfrak{V}$ such that the coefficients in $\{M(\sigma)\mid\sigma\in G_\Delta\}$ belong to a finite algebraic extension Λ of Q_p in K_Δ. If the envelopeng algebra of $\{M(\sigma)\mid\sigma\in G_\Delta\}$ over Λ is a full matrix ring of degree d_0 over a division ring and r is the degree of Λ over Q_p, then the sum $\tilde{\mathfrak{V}}=\mathfrak{V}+K_\Delta\mathfrak{V}^{\mathfrak{p}^r}+\cdots+K_\Delta\mathfrak{V}^{\mathfrak{p}^{r(d_0-1)}}$ in $K_{\Delta'}$ is a direct sum such that every $K_\Delta[G_\Delta]$-module in $K_{\Delta'}$ isomorphic to $\mathfrak{V}$ is contained in $\tilde{\mathfrak{V}}$. Namely the multiplicity of $\mathfrak{V}$ in the union $K_{\Delta',s}$ of semi-simple $K_\Delta[G_\Delta]$-moules of finit $\mathfrak{p}$ type is d_0.*

Proof. Let $(\xi_p,\cdots,\xi_n)$ be a K_Δ-base of $\mathfrak{V}$ such that $(\xi_1^\sigma,\cdots,\xi_n^\sigma)=(\xi_1\cdots,\xi_n)M(\sigma)$ $(\sigma\in G_\Delta)$ and put $V=\Lambda\xi_1+\cdots+\Lambda\xi_n$. Since Λ is algebraic subfield in K_Δ of degree r over Q_p, Λ/Q_p is a cyclic extension and the galois automorphisms are induced by $\{1, \mathfrak{p}, \cdots, \mathfrak{p}^{r-1}\}$. Since V is a simple $\Lambda[G_\Delta]$-module, $V^{\mathfrak{p}^{r\nu}}$ $(\nu=1, 2, \cdots)$ are also simple $\Lambda[G_\Delta]$-modules isomorphic to V, and thus $K_\Delta V^{\mathfrak{p}^{r\nu}}$ $(\nu=1, 2, \cdots)$ are simple $K_\Delta[G_\Delta]$-modules isomorphic to $\mathfrak{V}=K_\Delta V$. This shows that the sum $\mathfrak{V}=K_\Delta V+\cdots+K_\Delta\mathfrak{V}^{\mathfrak{p}^{(n-1)r}}$ in $K_{\Delta'}$ is a direct sum $K_\Delta V\oplus K_\Delta V^{\mathfrak{p}^r}\oplus\cdots\oplus K_\Delta V^{\mathfrak{p}^{(t-1)r}}$ with a positive integer t. The purpose of the proof is to show $t=d_0$. Let A_Λ be the enveloping algebra of $\{M(\sigma)\,|\,\sigma\in G_\Delta\}$ over Λ and D be the division algebra of A_Λ. Then $[A_\Lambda : D]=d_0^2$. Let Ω be the center of A_Λ and T be the minimal extension of Ω such that $D\otimes_\Omega T$ splits. Then we have $[A_\Lambda : \Lambda]=d_0^2[T:\Omega]^2[\Omega:\Lambda]$ and $T\cap K_\Delta=\Lambda$. We put $d=d_0[T:\Omega]$. We introduce the endomorphism $\mathfrak{q}$ of $T\otimes_\Lambda K_{\Delta'}$ by by $(\alpha\otimes\xi)^{\mathfrak{q}}=\alpha\otimes\xi^{\mathfrak{p}^r}$ $(\alpha\in T,\ \xi\in K_{\Delta'})$. Since Λ is the subfield of $K_{\Delta'}$ consisting of all the elements fixed by $\mathfrak{p}^r$, the endomorphism $\mathfrak{q}$ is well defined. There exists an absolutely simple $T[G_\Delta]$-module U in $T\otimes_\Lambda V$, because $T\otimes_\Lambda A_\Lambda$ is a full matrix algebra over T. We choose a T-base $(\eta_1, \cdots, \eta_d)$ of U. Then, since

$$\begin{vmatrix} \eta_1 & , \cdots , & \eta_d \\ \eta_1^{\mathfrak{q}} & , \cdots , & \eta_d^{\mathfrak{q}} \\ \vdots & & \\ \eta_1^{\mathfrak{q}^{d-1}} & , \cdots , & \eta_d^{\mathfrak{q}^{d-1}} \end{vmatrix} \neq 0 ,$$

putting

$$f_U(X) = (-1)^d \begin{vmatrix} X, \eta_1 & , \cdots , & \eta_d \\ X^{\mathfrak{q}}, \eta_1^{\mathfrak{q}} & , \cdots , & \eta_d^{\mathfrak{q}} \\ \vdots & & \\ X^{\mathfrak{q}^d}, \eta_1^{\mathfrak{q}^d} & , \cdots , & \eta_d^{\mathfrak{q}^d} \end{vmatrix} \begin{vmatrix} \eta_1 & , \cdots , & \eta_d \\ \eta_1^{\mathfrak{q}} & , \cdots , & \eta_d^{\mathfrak{q}} \\ \vdots & & \\ \eta_1^{\mathfrak{q}^{d-1}} & , \cdots , & \eta_d^{\mathfrak{q}^{d-1}} \end{vmatrix}^{-1},$$

we know that $f_U(X)=0$ is an irreducible $\mathfrak{q}$-equation[7] with coefficients in $T\otimes_\Lambda K_\Delta$ and U coincides with the $T[G_\Delta]$-module of of solutions of $f_U(X)=0$ in $T\otimes_\Lambda K_{\Delta'}$. Next we write the (i, i)-th unit $(1\leq i\leq d)$ in the full matrix ring $T\otimes_\Lambda A_\Lambda$ as follows $\sum_{l=1}^{t}\gamma_{il}N(\sigma_l)$[8] $(1\leq i\leq d)$ with γ_{il} in T and σ_l in G_Δ. Assume $\sum^{d}\sum^{n}\lambda_{ij}\eta_1^{\mathfrak{q}^{j-1}}=0$

7) The situation is the same as 2).

8) $\{N(\sigma)\,|\,\sigma\in G_\Delta\}$ is the representation by the base $(\eta_1, \cdots, \eta_d)$.

$i=1\ j=1$

with $\boldsymbol{\lambda}_{ij}$ in $\boldsymbol{T}\otimes_{\Lambda}\boldsymbol{K}_{\varDelta}$. Then, since $\sum_{l=1}^{t}\boldsymbol{\gamma}_{il}\boldsymbol{\eta}_{j}^{\sigma_l}=\boldsymbol{\eta}_i\delta_{ij}$ $(1\leqq i\leqq d)$ and σ_l $(1\leqq l\leqq t)$ commute with $\mathfrak{q}$, we have $\sum_{l=1}^{t}\boldsymbol{\gamma}_{il}(\sum_{h,k}\boldsymbol{\lambda}_{hk}\boldsymbol{\eta}_h^{\mathfrak{q}^{k-1}})^{\sigma_l}=\sum_{k=1}^{n}\boldsymbol{\lambda}_{ik}\boldsymbol{\eta}_i^{\mathfrak{q}^{k-1}}=0$ $(1\leqq i\leqq d)$. On the other hand by virtue of the irreducibility of the $\mathfrak{q}$-equation $\boldsymbol{f}_U(X)=0$ we know that $\boldsymbol{\eta}_i$, $\boldsymbol{\eta}_i^{\mathfrak{q}}, \cdots, \boldsymbol{\eta}_i^{\mathfrak{q}^{d-1}}$ are linearly independent over $\boldsymbol{T}\otimes_{\Lambda}\boldsymbol{K}_{\varDelta}$ and $\boldsymbol{\eta}_i^{\mathfrak{q}^d}$ is a linear combination of $\boldsymbol{\eta}_i, \boldsymbol{\eta}_i^{\mathfrak{q}}, \cdots, \boldsymbol{\eta}_i^{\mathfrak{q}^{d-1}}$ with coefficients in $\boldsymbol{T}\otimes_{\Lambda}\boldsymbol{K}_{\varDelta}$. Therefore we can conclude that $(\boldsymbol{\eta}_1, \cdots, \boldsymbol{\eta}_d, \boldsymbol{\eta}_1^{\mathfrak{q}}, \cdots, \boldsymbol{\eta}_d^{\mathfrak{q}}, \cdots, \boldsymbol{\eta}_1^{\mathfrak{q}^{d-1}}, \cdots, \boldsymbol{\eta}_d^{\mathfrak{q}^{d-1}})$ is a $(\boldsymbol{T}\otimes_{\Lambda}\boldsymbol{K}_{\varDelta})$-base of $(\boldsymbol{T}\otimes_{\Lambda}\boldsymbol{K}_{\varDelta})[G_{\varDelta}]$-module $\tilde{\mathfrak{U}}=(\boldsymbol{T}\otimes_{\Lambda}\boldsymbol{K}_{\varDelta})\boldsymbol{U}+(\boldsymbol{T}\otimes_{\Lambda}\boldsymbol{K}_{\varDelta})\boldsymbol{U}^{\mathfrak{q}}+\cdots+(\boldsymbol{T}\otimes_{\Lambda}\boldsymbol{K}_{\varDelta})\boldsymbol{U}^{\mathfrak{q}^{n-1}}$, and thus $\tilde{\mathfrak{U}}=(\boldsymbol{T}\otimes_{\Lambda}\boldsymbol{K}_{\varDelta})\boldsymbol{U}\oplus(\boldsymbol{T}\otimes_{\Lambda}\boldsymbol{K}_{\varDelta})\boldsymbol{U}^{\mathfrak{q}}\oplus\cdots\oplus(\boldsymbol{T}\otimes_{\Lambda}\boldsymbol{K}_{\varDelta})\boldsymbol{U}^{\mathfrak{q}^{d-1}}$. By virtue of Theorem 2 every $[\boldsymbol{T}\otimes_{\Lambda}\boldsymbol{K}_{\varDelta}](G_{\varDelta})$-module in $\boldsymbol{T}\otimes_{\Lambda}\boldsymbol{K}_{\varDelta'}$ isomorphic to $(\boldsymbol{T}\otimes_{\Lambda}\boldsymbol{K}_{\varDelta})\boldsymbol{U}$ is contained in $\mathfrak{U}$. We shall return to the culculation of t. Since Ω is the center of $\boldsymbol{A}_{\Lambda}$, $\boldsymbol{\Omega}\otimes_{\Lambda}\boldsymbol{V}$ is isomorphic to the direct sum $\boldsymbol{V}_1+\boldsymbol{V}_2+\cdots+\boldsymbol{V}_{\omega}$ $(\omega=[\boldsymbol{\Omega}:\Lambda])$ of mutually inequivalent $G_{\varDelta}$-modules $\boldsymbol{V}_1, \cdots, \boldsymbol{V}_{\omega}$ such that $\boldsymbol{V}_1$ is a simple $\Omega[G_{\varDelta}]$-module and other $\boldsymbol{V}_i$ are conjugate of $\boldsymbol{V}_1$ over Λ. Moreover $\boldsymbol{T}\otimes_{\boldsymbol{\Omega}}\boldsymbol{A}_{\Lambda}$ is the full matrix ring over $\boldsymbol{T}$, $\boldsymbol{T}\otimes_{\boldsymbol{\Omega}}\boldsymbol{V}_1$ is the $[\boldsymbol{T}:\boldsymbol{\Omega}]$-times direct sum of an absolutely simple $\boldsymbol{T}[G_{\varDelta}]$-module $\boldsymbol{U}$. This shows that $(\boldsymbol{T}\otimes_{\Omega}\boldsymbol{K}_{\varDelta})(\boldsymbol{V}_1^{\mathfrak{q}}+\cdots+\boldsymbol{V}_1^{\mathfrak{q}^{n-1}})=\tilde{\mathfrak{U}}$ and $d^2=\dim_{\boldsymbol{T}_{\Lambda}\otimes\boldsymbol{K}_{\varDelta}}\tilde{\mathfrak{U}}=i[\boldsymbol{T}:\boldsymbol{\Omega}]\dim_{\boldsymbol{T}}\boldsymbol{U}$. Since $d=d_0[\boldsymbol{T}:\Omega]$ and $d=\dim_{\boldsymbol{T}}\boldsymbol{U}$, we conclude $t=d_0$. This completes the proof of Theorem 3.

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