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# On the analytic semiexact differentials on an open Riemann surface

By

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**Introduction.** In the present paper, we treat the problem on the $A$-periods of square integrable analytic semiexact differentials on an arbitrary Riemann surface.

Virtanen [1] treated this theory of $A$-periods for the parabolic Riemann surface, where Riemann's bilinear relation plays a fundamental role. Kusunoki [3] proved a bilinear relation and remarked that this theory can be generalized to a Riemann surface belonging to the class $O_{HD}$. We generalized the argument of Virtanen by making use of the distinguished harmonic differentials to an arbitray open Riemann surface and obtained a condition which is necessary and sufficient for the existence of an square integrable analytic semiexact differential with given $A$-periods. This result was obtained also by Kusunoki, independently, by means of the canonical differential introduced in [4] (cf. Kusunoki [5]). It follows from this result that there exists an analytic semiexact differential with a finite number of analytic singularities which is square integrable outside of each neighborhood of singularities and has vanishing $A$-periods. Also we give a metrical criterion for the unique determination by the $A$-periods. For this purpose, we define a conformal invariant which generalize the notion of Pfluger's analytic modulus. Then we give a sufficient condition for the unique determination as an application of this invariant. Further we show, for a special choice of canonical homology basis, an inequality that connects this invariant and the harmonic modulus.

Finally we treat with a few subclasses of analytic semiexact differentials.

## §1. Square integrable analytic semiexact differentials

**1.** We denote by $\Gamma_h$ the class of all square integrable harmonic differentials which are defined on the Riemann surface $R$ and also denote by $\Gamma_{hse}(\Gamma_{ase})$ the class of semiexact harmonic (analytic) differentials in $\Gamma_h$. Further we denote by $\Gamma_{he}$ the class of exact harmonic differentials in $\Gamma_h$ and by $\Gamma_{ho}(\Gamma_{hm})$ the orthogonal complement in $\Gamma_h$ of $\Gamma_{he}^*(\Gamma_{hse}^*)$, where $\Gamma_{he}^*(\Gamma_{hse}^*)$ is the class of differentials whose conjugates are in $\Gamma_{he}(\Gamma_{hse})$.

Given an exhaustion $\{F_n\}$ of $R$ by regular regions, then there exists a canonical homology basis $A_1, B_1, \cdots, A_{p(n)}, A_{p(n)}, \cdots$ such that $A_1, B_1, \cdots, A_{p(n)}, B_{p(n)}$ form a canonical homology basis modulo $\partial F_n$. Now let $c$ be a cycle, then there exists a regular *distinguished* harmonic differential $\sigma(c)$ so that $\int_c \omega = (\omega, \sigma(c)^*)$ for $\omega \in \Gamma_{hse} \cap \Gamma_{hse}^*$. Such a $\sigma(c)$ is real and belongs to $\Gamma_{ho} \cap \Gamma_{hse}^*$ and $\sigma(c) = \omega_{hm} + \omega_{eo}$ outside of a compact set, where $\omega_{hm} \in \Gamma_{hm}$, $\omega_{eo} \in \Gamma_{eo} \cap \Gamma^1$. If $c$ and $c'$ are two cycles, then $(\sigma(c'), \sigma(c)*)$ is equal to the intersection number $c' \times c$ of $c'$ and $c$ (for the definition and the existence, see Ahlfors-Sario [6]). Particularly, for cycles $A_i, B_i$ in a canonical homology basis, $\sigma(A_i)$ and $\sigma(B_i)$ have the following periods;

$$\int_{Aj} \sigma(A_i) = 0 \quad , \quad \int_{Bj} \sigma(A_i) = \delta_{ij} \ ,$$
$$\int_{Aj} \sigma(B_i) = -\delta_{ij} \ , \quad \int_{Bj} \sigma(B_i) = 0 \ .$$

LEMMA[*)]. *If $\varphi$ is a regular distinguished harmonic differential and $\sigma$ belongs to $\Gamma_{hse} \cap \Gamma_{hse}^*$, then holds the bilinear relation*

$$(1.1) \qquad (\varphi, \sigma^*) = \sum_n \left( \int_{A_n} \varphi \int_{B_n} \bar{\sigma} - \int_{A_n} \bar{\sigma} \int_{B_n} \varphi \right).$$

Proof. Since $\varphi$ is a distinguished differential, it has a finite number of non-vanishing $A$ and $B$-periods. Let the $A_n$-periods and the $B_n$-periods of $\varphi$ be denoted respectively by $x_n$ and $y_n$,

*) The bilinear relation for the canonical differential $\varphi$ in [4] and $\sigma \in \Gamma_{hse}^* \cap \Gamma_{hse}$ is shown by Kusunoki (unpublished).

then belongs

$$\tau = \varphi - \sum_n (y_n \tilde{\sigma}(A_n) - x_n \tilde{\sigma}(B_n))$$

to $\Gamma_{he} \cap \Gamma^*_{hse}$ and is the distinguished differential. Hence $\tau = \tau_{hm} + \tau_{eo} + df$, where $\tau_{hm} \in \Gamma_{hm}$, $\tau_{eo} \in \Gamma_{eo} \cap \Gamma^1$ and $f$ is constant in each complementary component of a sufficiently large regular region $\Omega$. Since $\Gamma_{hm} \perp \Gamma^*_{hse}$ and $\Gamma_{eo} \perp \Gamma_h$, we have

$$(1.2) \qquad (\tau_{hm}, \sigma^*) = 0 \quad \text{and} \quad (\tau_{eo}, \sigma^*) = 0\,.$$

By Green's formula, we obtain also

$$(1.3) \qquad (df, \sigma^*) = (df, \sigma^*)_\Omega = -\int_{\partial\Omega} f\bar{\sigma} = 0\,.$$

Consequently, by (1.2) and (1.3) we obtain the bilinear relation

$$(\varphi, \sigma^*) = \sum_n (y_n(\tilde{\sigma}(A_n), \sigma^*) - x_n(\tilde{\sigma}(B_n), \sigma^*))$$
$$= \sum_n \left( \int_{A_n} \varphi \int_{B_n} \bar{\sigma} - \int_{A_n} \bar{\sigma} \int_{B_n} \varphi \right),$$

i.e. the above-mentioned result.

In case when all $A$-periods of a differential $\omega$ happen to be null, $\omega$ is said to be *A-exact*. Then if $\varphi$ is a regular distinguished $A$-exact harmonic differential and $\varphi^*$ has vanishing $A$-period along the cycle $A$ conjugate to the cycle $B$ such that $\varphi$ has non-vanishing $B$-period, then it follows from lemma that $\|\varphi\|^2 = (\varphi, \varphi) = -(\varphi, \varphi^{**}) = 0$, that is, $\varphi = 0$. Thus, particularly, we have the following

COROLLARY. *A-exact analytic semiexact differential whose real part is a regular distinguished differential vanishes identically.*

**2.** We consider the analytic differentials $\omega(A_n) = \tilde{\sigma}(A_n) + i\tilde{\sigma}(A_n)^*$ $(n = 1, 2, \cdots)$. Let $\alpha_{nm} = \int_{A_m} \tilde{\sigma}(A_n)^*$ and $\beta_{nm} = \int_{B_m} \tilde{\sigma}(A_n)^*$, then $\alpha_{nm}$ and $\beta_{nm}$ are all real and $\omega(A_n)$ has the following periods schema:

| $A$-periods | $B$-periods |
|---|---|
| $i\alpha_{n1}$ | $i\beta_{n1}$ |
| $i\alpha_{n2}$ | $i\beta_{n2}$ |
| $\vdots$ | $\vdots$ |
| $i\alpha_{nn-1}$ | $i\beta_{nn-1}$ |
| $i\alpha_{nn}$ | $1 + i\beta_{nn}$ |
| $i\alpha_{nn+1}$ | $i\beta_{nn+1}$ |
| $\vdots$ | $\vdots$ |

Since $\tilde{\sigma}(A_n)$ and $\tilde{\sigma}(A_m)$ are real differentials, we have

$$\alpha_{nm} = (\tilde{\sigma}(A_n)^*, \quad \tilde{\sigma}(A_m)^*) = (\tilde{\sigma}(A_m)^*, \quad \tilde{\sigma}(A_n)^*) = \alpha_{mn}.$$

Also, for a finite number of real numbers $x_1, \cdots, x_k$ such that at least one of $x_i$ $(i=1, \cdots, k)$ differs from zero, we obtain

$$\sum_{i,j=1}^{k} \alpha_{ij} x_i x_j = \| x_1 \tilde{\sigma}(A_1) + \cdots + x_k \tilde{\sigma}(A_k) \|^2 > 0 .$$

Thus we know that the period matrix $\| \alpha_{nm} \|$ $(n, m=1, \cdots, k)$ is *symmetric and positive definite.* Therefore, if any positive numbers $a_{kk}$ $(k=1, 2, \cdots)$ are given, we can determine a linear combination $\varphi_k = \sum_{m=1}^{k} x_m \omega(A_m)$ with real coefficients $x_m$ such that $\varphi_k$ has the following $A_n$-periods :

$$\int_{A_n} \varphi_k = \sum_{m=1}^{k} x_m \int_{A_n} \omega(A_m) = i \sum_{m=1}^{k} x_m \alpha_{mn} = \begin{cases} 0 & (n < k) \\ i\, a_{kk} & (n = k) . \end{cases}$$

Since $x_m$ are real, the real part of $\varphi_k$ is also a distinguished harmonic differential. Let $\int_{A_n} \varphi_k = i\, a_{kn}$ and $\int_{B_n} \varphi_k = b_{kn} + i c_{kn}$, where $a_{kn}$, $b_{kn}$ and $c_{kn}$ are all real and $a_{kn}=0$ $(k>n)$, $a_{kk}>0$ and $b_{kn}=0$ $(k<n)$, then $\varphi_k$ has the following period schema :

| $A$-periods | $B$-periods |
|---|---|
| $0$ | $b_{k1} + i c_{k1}$ |
| $\vdots$ | $\vdots$ |
| $0$ | $b_{kk-1} + i c_{kk-1}$ |
| $i\, a_{kk}$ | $b_{kk} + i c_{kk}$ |
| $i\, a_{kk+1}$ | $i c_{kk+1}$ |
| $\vdots$ | $\vdots$ . |

We will show that *the system* $\{\varphi_k\}$ $(k=1, 2, \cdots)$ *constitutes an orthogonal system.* To see this, we put $\tau_k$ as the real part of $\varphi_k$, then

$$\begin{aligned} (\varphi_k, \varphi_h) &= (\tau_k + i \tau_k^*, \ \tau_h + i \tau_h^*) \\ &= (\tau_k, \tau_h) + i(\tau_k^*, \tau_h) - i(\tau_k, \tau_h^*) + (\tau_k^*, \tau_h^*) \\ &= 2(\tau_k, \tau_h) - 2i(\tau_k, \tau_h^*) . \end{aligned}$$

Since $\tau_k$ and $\tau_h$ are real distinguished harmonic differentials, by the application of the bilinear relation (1. 1), we have

$$(\tau_k, \tau_h) = -\sum_n \left( \int_{A_n} \tau_k \int_{B_n} \tau_h^* - \int_{A_n} \tau_h^* \int_{B_n} \tau_k \right) = \sum_n \int_{A_n} \tau_h^* \int_{B_n} \tau_k$$
$$= \begin{cases} 0 & (h > k) \\ a_{kk} b_{kk} & (h = k) \end{cases}$$

and

$$(\tau_k, \tau_h^*) = 0 .$$

By this follows the orthogonal relation:

$$(\varphi_k, \varphi_h) = 0 \qquad (k \neq h)$$
$$(\varphi_k, \varphi_k) = 2 a_{kk} b_{kk} .$$

Let us normalize $\varphi_k$ and denote by $\psi_k$ the normalized differential, that is, $\psi_k = \varphi_k / \sqrt{2 a_{kk} b_{kk}}$. By the general theory of square integrable analytic differentials on a Riemann surface (Nevanlinna [9]), it follows that *if* $\{c_k\}$ *be a sequence of complex numbers for which the series* $\sum_{k=1}^{\infty} |c_k|^2$ *converges, then*

$$\omega = \sum_{k=1}^{\infty} c_k \psi_k$$

*converges and represents an element of* $\Gamma_{ase}$ *and*

$$\|\omega\|^2 = \sum_{k=1}^{\infty} |c_k|^2 \quad \textit{and} \quad (\omega, \psi_k) = c_k .$$

We point out that if $c_k$ be real, the real part of $\omega$ belongs to $\Gamma_{ho} \cap \Gamma_{hse}^*$, but is in general not a distinguished differential.

**3.** THEOREM I. *Let* $\{a_n + i b_n\}$ *be a sequence of given complex numbers. A necessary and sufficient condition for the existence of an analytic differential* $\omega \in \Gamma_{ase}$ *having* $a_n + i b_n$ *as its* $A_n$*-periods is the convergence of the series*

$$\sum_{k=1}^{\infty} \left| \sum_{n=1}^{k} (a_n + i b_n) l_{kn} \right|^2 , \tag{1.4}$$

*where* $l_{kn}$ *is the real part of* $B_n$*-periods of* $\psi_k$, *that is,*

$$l_{kn} = Re \int_{B_n} \psi_k = \frac{b_{kn}}{\sqrt{2 a_{kk} b_{kk}}} \qquad (k \geqq n) , \tag{1.5}$$
$$l_{kn} = 0 \qquad (k < n) .$$

*Such an analytic differential* $\omega$ *can be represented by*

$$(1.6)\qquad \omega=\sum_{k=1}^{\infty}(\omega,\psi_k)\psi_k=-2i\sum_{k=1}^{\infty}\Big[\sum_{n=1}^{k}(a_n+ib_n)l_{kn}\Big]\psi_k\,.$$

Proof. *Necessity.* Suppose that there exists an analytic differential $\omega$ satisfying the condition. Let $\tau$ be the real part of $\omega$, then $\tau$ and $\tau^*$ belong to $\Gamma_{hse}\cap\Gamma_{hse}^*$. Also let $\rho_k$ be the real part of $\psi_k$, then, by the help of the bilinear relation (1.1), we obtain the relations

$$(\rho_k,\tau)=-\sum_n\left(\int_{A_n}\rho_k\int_{B_n}\tau^*-\int_{A_n}\tau^*\int_{B_n}\rho_k\right)=\sum_{n=1}^{k}b_nl_{kn}\,,$$

$$(\rho_k,\tau^*)=\sum_n\left(\int_{A_n}\rho_k\int_{B_n}\tau-\int_{A_n}\tau\int_{B_n}\rho_k\right)=-\sum_{n=1}^{k}a_nl_{kn}\,.$$

Hence

$$\begin{aligned}(1.7)\qquad (\omega,\psi_k)&=2(\tau,\rho_k)-2i(\tau,\rho_k^*)\\&=2\Big(\sum_{n=1}^{k}b_nl_{kn}-i\sum_{n=1}^{k}a_nl_{kn}\Big)=-2i\sum_{n=1}^{k}(a_n+ib_n)l_{kn}\,.\end{aligned}$$

We put $c_k=-2i\sum_{n=1}^{k}(a_n+ib_n)l_{kn}$ and make the linear comblinations $\omega_m=\sum_{n=1}^{m}c_k\psi_k$. Then by Bessel's inequality we obtain

$$4\sum_{k=1}^{m}\Big|\sum_{n=1}^{k}(a_n+ib_n)l_{kn}\Big|^2=\sum_{k=1}^{m}|c_k|^2=\|\omega_m\|^2\leqq\|\omega\|^2\,.$$

Therefore, $$\sum_{k=1}^{\infty}\Big|\sum_{n=1}^{k}(a_n+ib_n)l_{kn}\Big|^2<\infty\,.$$

*Sufficiency.* Suppose that the series (1.4) converges, then

$$\omega=-2i\sum_{k=1}^{\infty}\Big[\sum_{n=1}^{k}(a_n+ib_n)l_{kn}\Big]\psi_k$$

represents an element of $\Gamma_{ase}$. To show that the $A_n$-periods of $\omega$ are $a_n+ib_n$, we put

$$\int_{A_n}\omega=p_n+iq_n\,,$$

then, by making use of the bilinear relation, we have (see (1.7))

$$(\omega,\psi_k)=-2i\sum_{n=1}^{k}(p_n+iq_n)l_{kn}\qquad(k=1,2,\cdots)\,.$$

On the other hand, the Fourier coefficients of $\omega$ are

$$(\omega,\psi_k)=-2i\sum_{n=1}^{k}(a_n+ib_n)l_{kn}\qquad(k=1,2,\cdots)\,.$$

Hence

$$\sum_{n=1}^{k}(p_n+iq_n)l_{kn}=\sum_{n=1}^{k}(a_n+ib_n)l_{kn} \qquad (k=1, 2, \cdots).$$

Since $l_{kk}=\dfrac{b_{kk}}{\sqrt{2a_{kk}b_{kk}}}\neq 0$, we have successively

$$p_n+iq_n=a_n+ib \qquad (n=1, 2, \cdots).$$

Thus the desired result is obtained.

If the series $\sum_{k=1}^{\infty}|\sum_{n=1}^{k}(a_n+ib_n)l_{kn}|^2$ converges, then $\sum_{k=1}^{\infty}|\sum_{n=1}^{k}a_n l_{kn}|^2$ and $\sum_{k=1}^{\infty}|\sum_{n=1}^{k}b_n l_{kn}|^2$ converge, hence we obtain the following

COROLLARY. *If there exists an analytic differential* $\omega\in\Gamma_{ase}$ *whose* $A_n$*-periods are* $a_n+ib_n$, *then there exist two analytic differentials* $\omega_1$ *and* $\omega_2$ *in* $\Gamma_{ase}$ *such that* $A_n$*-periods of* $\omega_1$ *and* $\omega_2$ *are* $a_n$ *and* $ib_n$, *respectively.*

Such two differentials $\omega_1$ and $\omega_2$ are represented, respectively, by

$$\omega_1=-2i\sum_{k=1}^{\infty}(\sum_{n=1}^{k}a_n l_{kn})\psi_k$$

and

$$\omega_2=2\sum_{k=1}^{\infty}(\sum_{n=1}^{k}b_n l_{kn})\psi_k.$$

We notice therefore that, since the real part of $\psi_k$ is distinguished harmonic differential and belongs to $\Gamma_{ho}\cap\Gamma^*_{hse}$, the imaginary part of $-2i\sum_{k=1}^{\infty}(\sum_{n=1}^{k}a_n l_{kn})\psi_k$ belongs to $\Gamma_{ho}\cap\Gamma^*_{hse}$ ; on the other hand, the real part of $2\sum_{k=1}^{\infty}(\sum_{n=1}^{k}b_n l_{kn})\psi_k$ belongs to $\Gamma_{ho}\cap\Gamma^*_{hse}$. If the genus of $R$ is finite and the real part (imaginary part) of $\omega_2(\omega_1)$ is restricted to be distinguished, by the corollary of the lemma in **1** $\omega_2(\omega_1)$ is represented by the above series (finite series) uniquely. But in general, $\omega\in\Gamma_{ase}$ is not determined uniquely by its $A$-periods on an arbitrary Riemann surface.

The analytic differential $-(\sigma(B_n)+i\sigma(B_n)^*)$ has $\delta_{mn}-i\beta_{mn}$ as its $A_m$-periods. Thus, by the help of the corollary of the theorem I, we have the

COROLLARY (Virtanen [2])[1]. *There exists an analytic differential* $\omega(A_n)\in\Gamma_{ase}$ *such that*

$$\int_{A_n}\omega(A)=1\,,\quad \int_{A_m}\omega(A_n)=0\qquad(m\neq n)\,.$$

*Such a differential* $\omega(A_n)$ *is represented by the series*

$$-2i\sum_{k=n}^{\infty}l_{kn}\psi_k\,.$$

**4.** Let $\tilde{\tau}$ be a distinguished harmonic differential with a finite number of harmonic singularities (residue sum 0) without $A$ and $B$-periods. We put $\tilde{\varphi}=\frac{1}{2}(\tilde{\tau}+i\tilde{\tau}^*)$ and $\tilde{\psi}=\frac{1}{2}(\bar{\tilde{\tau}}+i\bar{\tilde{\tau}}^*)$, then $\tilde{\varphi}$ is a square integrable analytic differential outside of each neighborhood of poles and is semiexact on $R$ with suitable slits joinning simple poles. On the other hand, $\tilde{\psi}$ belongs to $\Gamma_{ase}$ (Ahlfors-Sario [6]). $\tilde{\varphi}+\tilde{\psi}$ has purely imaginary $A$, $B$-periods and $\tilde{\varphi}-\tilde{\psi}$ has real $A$, $B$-periods. Since $2\tilde{\psi}=\tilde{\varphi}+\tilde{\psi}-(\tilde{\varphi}-\tilde{\psi})$ belongs to $\Gamma_{ase}$, by the above corollary we can find an analytic differential $\sigma\in\Gamma_{ase}$ such that $\sigma$ has the same $A$-periods as the imaginary part of $A$-periods of $2\tilde{\psi}$. Then $\tilde{\varphi}+\tilde{\psi}-\sigma$ is $A$-exact and has the given analytic singularities (residue sum 0). Thus we have

THEOREM II. *There exists an A-exact analytic differential* $\omega$ *with given a finite number of analytic singularities (residue sum 0) such that* $\omega$ *is square integrable outside of neighborhoods of singularities and semiexact on R with suitable slits joinning simple poles.*

Since the real part of $\tilde{\varphi}+\tilde{\psi}$ is distinguished, if the genus of $R$ is finite, then the differential constructed in theorem II exists uniquely whenever its real part is restricted to be distinguished.

## § 2. A condition for uniqueness.

**1.** Let $\omega$ be an element of $\Gamma_{ase}$ and denote by $a_n+ib_n$ its $A_n$-periods, then it follows from theorem I that $\omega^0=-2i\sum_{k=1}^{\infty}\sum_{n=1}^{k}(a_n+ib_n)l_{kn}\psi_k=\sum_{k=1}^{\infty}(\omega,\psi_k)\psi_k$ has the same $A$-periods as $\omega$. Hence

1) K. Oikawa has constructed also such a normal differential $\omega(A_n)$ by Accola's method (unpublished).

$\omega-\omega^0=\tau$ is an $A$-exact analytic semiexact differential. We denote by $\Gamma^A_{ase}$ the class of all $A$-exact differentials which belong to $\Gamma_{ase}$, then $\Gamma^A_{ase}=\{0\}$ is equivalent to the completeness of orthonormal system $\{\psi_k\}$ in $\Gamma_{ase}$. Since $\Gamma_{ae}\subset\Gamma^A_{ase}$, if the class $\Gamma^A_{ase}$ is an empty class on an Riemann surface $R$, then $R$ must belong to the class $0_{AD}$. In other paper (Kobori-Sainouchi [10]) a sufficient condition for $\Gamma^A_{ase}=\{0\}$ was given. We are going, in the following, to give a more general metrical criterion.

At first, we suppose that the exhaustion $\{F_n\}$ $(n=1, 2, \cdots)$ is canonical, that is, each contour $\alpha_n^{(i)}$ $(i=1, 2, \cdots, m(n))$ of $\partial F_n$ is a dividing cycle. Then, for any two harmonic semiexact differentials $\omega$ and $\sigma$, holds the generalized Green's formula

$$(2.1)\qquad (\omega, \sigma^*)_{F_n}=\sum_{k=1}^{p(n)}\left(\int_{A_k}\omega\int_{B_k}\bar{\sigma}-\int_{A_k}\bar{\sigma}\int_{B_k}\omega\right)-\int_{\partial F_n}u\bar{\sigma},$$

where $u(p)$ is a function defined separately on each contour $\alpha_n^{(i)}$ of $\partial F_n$: $u(p)=\int_{p_i}^{p}\omega$ $(p, p_i\in\alpha_n^{(i)})$ (cf. [10] or Accola [11]).

Since $(\omega, \omega^*)_{F_n}=(\omega, -i\omega)_{F_n}=i\|\omega\|^2_{F_n}$ for an analytic differential $\omega$, if $\omega$ belongs to $\Gamma^A_{ase}$, then by (2.1), we get

$$i\|\omega\|^2_{F_n}=\sum_{k=1}^{p(n)}\left(\int_{A_k}\omega\int_{B_k}\bar{\omega}-\int_{A_k}\bar{\omega}\int_{B_k}\omega\right)-\int_{\partial F_n}u\bar{\omega}=-\int_{\partial F_n}u\bar{\omega}.$$

Thus $i\int_{\partial F_n}u\bar{\omega}=\|\omega\|^2_{F_n}>0$ $(\omega\neq 0)$. We consider the class $\Gamma^A_{ase}(\bar{F}_n-F_1)$ of analytic semiexact differential without $A$-periods on $\bar{F}_n-F_1$. If $\bar{F}_n-F_1$ is not a connected set, $\Gamma^A_{ase}(\bar{F}_n-F_1)$ represents the class of element $\omega$ such that the restriction of $\omega$ to each component of $\bar{F}_n-F_1$ is $A$-exact analytic semiexact differential. We define a quantity $K_{1n}$ as follows:

$$K_{1n}=K_{1n}(\bar{F}_n-F_1)=\inf_{\omega}\frac{\int_{\partial F_n}u\bar{\omega}}{\int_{\partial F_1}u\bar{\omega}},$$

where $\omega$ varies over $\Gamma^A_{ase}(\bar{F}_n-F_1)$ such that $i\int_{\partial F_1}u\bar{\omega}>0$. We call it the *generalized analytic modulus* associated with $\bar{F}_n-F_1$. We remark that $K_{1n}$ depends, in general, upon the choice of canonical homology basis, but is a conformal invariant for a suitably fixed homology basis. Since $||\omega||^2_{F_n-F_1}=i\int_{\partial F_n}u\bar{\omega}-i\int_{\partial F_1}u\bar{\omega}>0$, we obtain

$$K_{1n}\geqq 1\,.$$

If we restrict $\omega$ to the class $\Gamma_{ae}(\bar{F}_n-F_1)$ of analytic exact differentials, the corresponding invariant becomes the Pfluger's analytic modulus $k_{1n}$ (Pfluger [12]). When the genus of each component of $\bar{F}_n-F_1$ is zero, $\Gamma^A_{ase}(\bar{F}_n-F_1)=\Gamma_{ae}(\bar{F}_n-F_1)$, hence we have $K_{1n}=k_{1n}$. On the other hand, if the genus of at least one component of $\bar{F}_n-F_1$ is positive, there are $A$-exact analytic semiexact differentials having non-vanishing $B$-periods on its component (cf. Ahlfors [8] theorem 4 or Behnke und Stein [13]), hence $\Gamma_{ae}(\bar{F}_n-F_1)\subsetneqq\Gamma^A_{ase}(\bar{F}_n-F_1)$ and so we have in general

$$k_{1n}\geqq K_{1n}\,.$$

THEOREM III. *If* $\lim_{n\to\infty}K_{1n}=\infty$, *then any element of* $\Gamma_{ase}$ *without A-periods is identically zero.*

Proof. Suppose that $\omega$ is an $A$-exact analytic semiexact differential and is not identically zero on $R$, then

$$||\omega||^2_{F_n}=i\int_{\partial F_n}u\bar{\omega}\,.$$

The definition of $K_{1n}$ implies

$$||\omega||^2_{F_n}\geqq K_{1n}\,||\omega_1||^2_{F_1}\,.$$

Since $||\omega||^2_{F_1}>0$ and $\lim_{n\to\infty}K_{1n}=\infty$, $||\omega||^2$ cannot be finite. Therefore, we obtain the above mentioned result.

Thus we know that, if $\lim_{n\to\infty}K_{1n}=\infty$, the square integrable analytic semiexact differential is determined uniquely by its $A$-periods. Let us denote by $K_n$ the generalized analytic modulus associated with $\bar{F}_{n+1}-F_n$, then easily we get

$$K_{1n}\geqq K_1K_2\cdots K_{n-1}\,. \tag{2.2}$$

Thus we have

COROLLARY. *If* $\prod_{n=1}^{\infty}K_n=\infty$, *then any element of* $\Gamma_{ase}$ *without A-periods is identically zero.*

More generally, we shall consider the class $\Gamma_{ase}^{A_1,\cdots,A_N}$ of square integrable analytic semiexact differential whose $A$-periods vanish except for a finite number of $A_i$-periods ($i=1,2,\cdots,N$). Let $A_i$ ($i=1,\cdots,N$) $\subset F_{n_0}$, then we can conclude that *if* $\lim_{n\to\infty} K_{n_0 n}=\infty$, *then any element* $\omega$ *of* $\Gamma_{ase}^{A_1,\cdots,A_N}$ *such that* $i\int_{\partial F_{n_0}} u\bar{\omega}>0$ *is identically zero.* Because, by (2.1) we have

$$\|\omega\|_{F_n}^2 = 2\sum_{k=1}^{N} \mathcal{Im}\left\{\int_{A_k}\omega\int_{B_k}\bar{\omega}\right\}+i\int_{\partial F_n} u\bar{\omega} \qquad (n>n_0),$$

hence

$$\|\omega\|_{F_n}^2-2\sum_{k=1}^{N} \mathcal{Im}\left\{\int_{A_k}\omega\int_{B_k}\bar{\omega}\right\} = i\int_{\partial F_n} u\bar{\omega} \geqq K_{n_0 n}\cdot\left(i\int_{\partial F_{n_0}} u\bar{\omega}\right).$$

From this we obtain the desired rerult.

**2.** Let $\omega$ be an element of $\Gamma_{hse}(\bar{F}_n)\frown\Gamma_{hse}^*(\bar{F}_n)$, then by (2.1) we get

(2.3)
$$\|\omega\|_{F_n}^2 = -(\omega,\omega^{**})_{F_n} = -\sum_{k=1}^{p(n)}\left(\int_{A_k}\omega\int_{B_k}\bar{\omega}^*-\int_{A_k}\bar{\omega}^*\int_{B_k}\omega\right)+\int_{\partial F_n} u\bar{\omega}^*.$$

Now we consider the class $\Gamma_{he}(\bar{F}_n-F_1)\frown\Gamma_{hse}^*(\bar{F}_n-F_1)$, then under the use of (2.3) we can define a conformal invariant $h_{1n}$ as follows:

$$h_{1n} = \inf_{\omega}\frac{\int_{\partial F_n} u\bar{\omega}^*}{\int_{\partial F_1} u\bar{\omega}^*},$$

where $\omega$ varies over the above class and $\int_{\partial F_1} u\bar{\omega}^*>0$ [2].

Then $h_{1n}\geqq 1$ and by the same way as we did in the proof of the theorem III we have the following result: *If* $\lim_{n\to\infty} h_{1n}=\infty$, *then* $\Gamma_{he}\frown\Gamma_{hse}^*=\{0\}$, *that is, R belongs to the class* $0_{KD}$.

Let us denote by $\Gamma_{hse}^A(\bar{F}_n-F_1)\frown\Gamma_{hse}^{*A}(\bar{F}_n-F_1)$ the class of $\omega\in\Gamma_{hse}(\bar{F}_n-F_1)\frown\Gamma_{hse}^*(\bar{F}_n-F_1)$ such that both $\omega$ and $\omega^*$ have vanishing $A$-periods. We note that since $\Gamma_{hse}^A(\bar{F}_n-F_1)\frown\Gamma_{hse}^{*A}(\bar{F}_n-F_1)\supset\Gamma_{ase}^A(\bar{F}_n-F_1)$, we have

2) When $\omega$ does not be restricted to an element of $\Gamma_{hse}^*$, it may happen that the corresponding modulus is always equal to 1 (cf. [12]).

$$\inf_{\omega} \frac{\int_{\partial F_n} u\bar{\omega}^*}{\int_{\partial F_1} u\bar{\omega}^*} \leqq K_{1n},$$

where $\omega$ varies over the above class and $\int_{\partial F_1} u\bar{\omega}^* > 0$. Analogously, we have

$$\inf_{\omega} \frac{\int_{\partial F_n} u\bar{\omega}^*}{\int_{\partial F_1} u\bar{\omega}^*} \leqq k_{1n},$$

where $\omega$ varies over the class $\Gamma_{he}(\bar{F}_n - F_1) \cap \Gamma^*_{he}(\bar{F}_n - F_1)$ and $\int_{\partial F_1} u\bar{\omega}^* > 0$. It is clear that $k_{1n} \geqq h_{1n}$, but since $\Gamma^A_{hse}(\bar{F}_n - F_1) \supset \Gamma_{he}(\bar{F}_n - F_1)$ and $\Gamma^*_{hse}(\bar{F}_n - F_1) \supset \Gamma^{*A}_{hse}(\bar{F}_n - F_1)$, we can not, in general, determine the inequality of $K_{1n}$ and $h_{1n}$.

**3.** In the case that the exhaustion $\{F_n\}$ is not necessarily canonical, the restriction of a semiexact differential to the regular region $F_n$ is not in general semiexact on $F_n$. But if we choose a canonical homology basis with respect to an exhaustion $\{F_n\}$ of $R$ such that the cycles on $\partial F_n$ are *weakly* homologous to a linear combination of $A$-cycles only (Ahlfors [10]), then a semiexact differential without $A$-periods becomes also semiexact on $F_n$. Because any dividing cycle in $F_n$ is homologous to a linear combination of cycles on $\partial F_n$. We point out that in this case theorem III is valid also.

The following relation of the Pfluger's analytic modulus $k_n$ and harmonic modulus $\mu_n$ associated with $\bar{F}_{n+1} - F_n$ is well known:

$$k_n \geqq \mu_n^{4\pi},$$

where the equality $k_n = \mu_n^{4\pi}$ holds for a doubly connected region. We shall show that *the inequality* $K_n \geqq \mu_n^{4\pi}$ *holds also for a special choice of canonical homology basis.* To see this, we denote by $v(p)$ the harmonic function which is 0 on $\partial F_n$, 1 on $\partial F_{n+1}$. Let $\gamma(t_j)$ be a set of finite number of level curves; $v(p) = t_j$ $(0 = t_1 < t_2 < \cdots < t_{\nu-1} < t_\nu = 1)$ such that at least one critical point of $v(p)$ is contained in $\gamma(t_j)$ $(j \neq 1, \nu)$. We add the relatively compact region

$\Omega_j$ bounded by $\gamma(t_j)$ $(j=2, \cdots, \nu-1)$ to the exhaustion $\{F_n\}$ and introduce the above mentioned homology basis with respect to the exhaustion $\{F_1, F_2, \cdots, F_n, \Omega_2, \cdots, \Omega_{\nu-1}, F_{n+1}, \cdots\}$, then the region bounded by $\gamma(t)$ $(t_j \leq t < t_{j+1})$ has the same canonical homology basis as that of $\Omega_j$ (cf. Ahlfors [7], Hilfssatz 5). Then, for $\omega \in \Gamma^A_{ase}(\bar{F}_{n+1}-F_n)$, both $\omega$ and $\omega^*$ have the vanishing periods along each component of the level curve $\gamma(t)$ $(0 \leqslant t \leqslant 1)$. We use $v+iv^* = x+iy=z$ $(0 \leqslant v \leqslant 1,\ 0 \leqslant v^* \leqslant d = D(v))$ as the local variable and set $\omega = A(z)dz$. We write

$$m(t) = i\int_{v=t} u\bar{\omega} = \int_0^d u\bar{A}\,dy = \int_0^d u\bar{u}'\,dy\,,$$

where $u(iy) = \int_0^{iy} A\,d(iy)$ and the differentiation is with respect to the local variable. Then $m(t)$ is determined uniquely by $\omega \in \Gamma^A_{ase}(\bar{F}_{n+1} - F_n)$, because $u$ is determined except for an additive constant on each component of $\gamma(t)$, but $\omega$ has vanishing periods along each component of $\gamma(t)$. Therefore we can apply the same argument as that in [6] (pp. 231-232) and we have

$$K_n \geqq \mu_n^{4\pi}\,, \tag{2.4}$$

where the equality holds for a doubly connected region.

Let us denote by $K_n^{(j)}(\mu_n^{(j)})$ the generalized analytic modulus (harmonic modulus) of the component $F_n^{(j)}$ of $\bar{F}_{n+1} - F_n$. Then, by the same way as that in the case of the proof of $k_n = \min_j k_n^{(j)}$ (cf. [6] p. 233) we can conclude that

$$K_n = \min_j K_n^{(j)}\,. \tag{2.5}$$

Now let $\{F_n\}$ be an exhaustion of $R$ by regular regions and for each $n$ we make a set of level curves $\gamma(t_j)$ defined as above. The regions bounded by those level curves construct an exhaustion. We introduce an above mentioned canonical homology basis with respect to this exhaustion. Then by (2.2), (2.4) and (2.5) we know that the following result holds.

THEOREM IV. *If* $\prod_{n=1}^{\infty} (\min_j \mu_n^{(j)}) = \infty$, *then there exists an exhaustion and a corresponding canonical homology basis such that* $\Gamma^A_{ase} = \{0\}$.

Thus, on such a surface, for such a canonical homology basis, $\omega$ ($\in \Gamma_{ase}$) is determined uniquely by its $A$-periods. In [10] we have proved this theorem by the other way.

### § 3. A special subclass of $\Gamma_{ase}$.

**1.** We saw that if $\omega$ be an element of $\Gamma_{ase}$, then $\omega-\omega^0=\tau$ belongs to $\Gamma^A_{ase}$, where $\omega^0=\sum_{k=1}^{\infty}(\omega, \psi_k)\psi_k=-2i\sum_{k=1}^{\infty}\sum_{n=1}^{k}(a_n+ib_n)l_{kn}\psi_k$. Since $\tau$ is orthogonal to elements of the system $\{\psi_k\}$, hence $(\omega^0, \tau)=0$. Thus we get

$$||\omega-\omega^0||^2=(\omega-\omega^0, \tau)=(\omega, \tau)=(\omega, \omega)-(\omega, \omega^0)=||\omega||^2-||\omega^0||^2,$$

hence $\omega^0$ has the smallest norm in the class of $\sigma\in\Gamma_{ase}$ such that $\sigma$ has the same $A$-periods as $\omega$. After Virtanen [1] we call such a $\omega^0$ *normal differential* and denote by $\Gamma^0_{ase}$ the class of normal differential. $\Gamma^0_{ase}$ is the subspace spanned by $\{\psi_k\}$ and we have an orthogonal decomposition

$$\Gamma_{ase}=\Gamma^0_{ase}\dotplus\Gamma^A_{ase}.$$

If we use $\{\bar{\sigma}(B_n)\}$ instead of using $\{\bar{\sigma}(A_n)\}$, we have an orthonormal system $\{\varphi_k\}$ in $\Gamma_{ase}$ by the same way as we did in § 1. Thus we have also an orthogonal decomposition

$$\Gamma_{ase}=\Gamma'_{ase}\dotplus\Gamma^B_{ase},$$

where $\Gamma'_{ase}$ is the subspace spanned by $\{\varphi_k\}$ and $\Gamma^B_{ase}$ the class of $B$-exact $\omega\in\Gamma_{ase}$.

**2.** Let $\omega$ be an element of $\Gamma_{ase}$ and $\{\omega_n\}$ be a system of the elementary differentials of 1st kind, that is,

$$\int_{A_n}\omega_n=1, \quad \int_{A_m}\omega_n=0 \qquad (m\neq n);$$

further if the condition

$$\sum_{n=1}^{\infty}||\omega_n||\left|\int_{A_n}\omega\right|<\infty \tag{3.1}$$

be satisfied, then

$$\|\sum_{n=N}^{M}\omega_n\int_{A_n}\omega\|^2=\sum_{n=N}^{M}\sum_{m=N}^{M}\left(\omega_n\int_{A_n}\omega,\ \omega_m\int_{A_m}\omega\right)$$
$$\leqq\sum_{n=N}^{M}\sum_{m=N}^{M}\left|\int_{A_n}\omega\int_{A_m}\omega\right||(\omega_n,\omega_m)|\leqq\sum_{n=N}^{M}\sum_{m=N}^{M}\left|\int_{A_n}\omega\right|\left|\int_{A_m}\omega\right|\|\omega_n\|\|\omega_m\|$$
$$\leqq\left(\sum_{n=N}^{M}\|\omega_n\|\left|\int_{A_n}\omega\right|\right)^2.$$

It follows from (3.1) that the series

$$\sum_{n=1}^{\infty}\omega_n\int_{A_n}\omega \tag{3.2}$$

converges and has the same $A$-periods as $\omega$. Thus, when the surface $R$ satisfies the condition for uniqueness and moreover $\omega$ satisfies (3.1), then $\omega$ is expressed by the series (3.2) uniquely. On an arbitrary Riemann surface, if $\omega$ and $\{\omega_n\}$ be restricted to the class $\Gamma^0_{ase}$, we denote them by $\omega^0$ and $\{\omega^0_n\}$, then the corresponding series $\sum_{n=1}^{\infty}\omega^0_n\int_{A_n}w^0$ belongs to $\Gamma^0_{ase}$. Therefore we have

THEOREM V. *Let $\omega^0$ be an element of $\Gamma^0_{ase}$ and $\{\omega^0_n\}$ be the system of elementary differentials $\in\Gamma^0_{ase}$, then, if the series*

$$\sum_{n=1}^{\infty}\|\omega^0_n\|\left|\int_{A_n}\omega^0\right|$$

*converges, $\omega^0$ is represented by $\sum_{n=1}^{\infty}\omega^0_n\int_{A_n}\omega^0$ uniquely.*

**3.** Finally, following the method in [1], we establish a bilinear relation for two elements of $\Gamma^0_{ase}$. For this purpose, we show that if

$$\sum_{n=1}^{\infty}\|\omega^0_n\|\left|\int_{A_n}\omega\right|<\infty \qquad (\omega\in\Gamma_{ase}), \tag{3.3}$$

then

$$\sum_{k=1}^{\infty}\sum_{n=1}^{k}\left(l_{kn}\cdot\left|\int_{A_n}\omega\right|\right)^2<\infty, \tag{3.4}$$

where $l_{kn}=Re\int_{B_n}\psi_k$ (cf. (1.5)).

Since $\|\omega^0_n\|^2=\sum_{k=1}^{\infty}l^2_{kn}$ and $l_{kn}=0$ $(k<n)$, this follows from

$$\infty > \left(\sum_{n=1}^{\infty} \|\omega_n^0\| \cdot \left|\int_{A_n} \omega\right|\right)^2 = \left(\sum_{n=1}^{\infty} |a_n+ib_n| \cdot \sqrt{\sum_{k=1}^{\infty} l_{kn}^2}\right)^2$$

$$= \sum_{n,m=1}^{\infty} |a_n+ib_n| \cdot |a_m+ib_m| \cdot \sqrt{(\sum_{k=1}^{\infty} l_{kn}^2)(\sum_{k=1}^{\infty} l_{km}^2)}$$

$$\geqq \sum_{n,m=1}^{\infty} |a_n+ib_n| \cdot |a_m+ib_m| \cdot \sum_{k=1}^{\infty} |l_{kn} l_{km}|$$

$$\geqq \sum_{k=1}^{\infty} \sum_{n,m=1}^{\infty} |a_n+ib_n|\, |a_m+ib_m|\, |l_{kn} l_{km}|$$

$$= \sum_{k=1}^{\infty} (\sum_{n=1}^{k} |a_n+ib_n|\, |l_{kn}|) \cdot (\sum_{m=1}^{k} |a_m+ib_m|\, |l_{km}|)$$

$$= \sum_{k=1}^{\infty} (\sum_{n=1}^{k} |l_{kn} \int_{A_n} \omega|)^2 .$$

Now let $\sigma_i^0$ $(i=1, 2)$ be two elements of $\Gamma_{ase}^0$ and satisfy (3.3), then by (3.4)

$$\sum_{k=1}^{\infty} \sum_{n=1}^{k} \left| l_{kn} \int_{A_n} \sigma_i^0 \right|^2 < \infty \qquad (i=1, 2) . \tag{3.5}$$

Moreover, we suppose that $\sigma_i^0$ $(i=1, 2)$ have pure imaginary $A$-periods. Set $\int_{A_n} \sigma_i^0 = i b_n^{(i)}$ $(i=1, 2)$, then by (1.6)

$$\sigma_i^0 = \sum_{k=1}^{\infty} (\sigma_i^0, \psi_k)\psi_k = 2 \sum_{k=1}^{\infty} \sum_{n=1}^{k} b_n^{(i)} l_{kn} \psi_k . \tag{3.6}$$

Hence the inner product of $\sigma_1^0$ and $\sigma_2^0$ is

$$(\sigma_1^0, \sigma_2^0) = \sum_{k=1}^{\infty} (\sigma_1^0, \psi_k)\overline{(\sigma_2^0, \psi_k)} \tag{3.7}$$
$$= 4 \sum_{k=1}^{\infty} [(\sum_{n=1}^{k} b_n^{(1)} l_{kn}) \cdot (\sum_{m=1}^{k} b_m^{(2)} l_{km})] .$$

Since $l_{kn}=0$ $(k<n)$ and

$$\sum_{k=1}^{\infty}(\sum_{n=1}^{k}|b_n^{(1)} l_{kn}| \cdot \sum_{m=1}^{k}|b_m^{(2)} l_{km}|) \leqq [\sum_{k=1}^{\infty}(\sum_{n=1}^{k}|l_{kn} b_n^{(1)}|)^2 \cdot \sum_{k=1}^{\infty}(\sum_{m=1}^{k}|l_{km} b_m^{(2)}|)^2]^{1/2} < \infty ,$$

we get

$$4 \sum_{k=1}^{\infty} (\sum_{n=1}^{k} b_n^{(1)} l_{kn}) \cdot (\sum_{m=1}^{k} b_m^{(2)} l_{km}) = 4 \sum_{n=1}^{\infty} b_n^{(1)} \cdot \sum_{k=1}^{\infty} l_{kn} (\sum_{m=1}^{\infty} b_m^{(2)} l_{km}) .$$

On the other hand, by (3.6) and (1.5)

$$Re \int_{B_n} \sigma_2^0 = \sum_{k=1}^{\infty} (\sigma_2^0, \psi_k) Re \int_{B_n} \psi_k$$
$$= 2 \sum_{k=1}^{\infty} (\sum_{m=1}^{k} b_m^{(2)} l_{km}) l_{kn} .$$

Hence (3. 7) is equal to

$$2\sum_{n=1}^{\infty} b_n^{(1)} Re\int_{B_n} \sigma_2^0 = 2\sum_{n=1}^{\infty} \mathfrak{Im}\int_{A_n} \sigma_1^0 \cdot Re\int_{B_n} \sigma_2^0 .$$

Summing up the above result, we have

THEOREM VI. *If both* $\sigma_1^0$ *and* $\sigma_2^0$ *have the pure imaginary A-periods and satisfy* (3. 3), *then holds the bilinear relation*

$$(\sigma_1^0, \sigma_2^0) = 2\sum_{n=1}^{\infty} \mathfrak{Im}\int_{A_n} \sigma_1^0 \cdot Re\int_{B_n} \sigma_2^0 .$$

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