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# Canonical conformal mappings of open Riemann surfaces

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## Introduction

Koebe was the first who proved strictly the existence of conformal mappings of an arbitrary planar Riemann surface onto slit regions. Concerning an open Riemann surface $R$ of positive genus $g$, Kusunoki [9] and Nehari [13] showed that for any $g+1$ points $P_0$, $P_1,\cdots,P_g$ on $R$ there exists a conformal mapping of $R$ with possible poles at $P_0,P_1,\cdots,P_g$ onto a covering surface of the extended plane which is at most $(g+1)$-sheeted and bounded by parallel slits, only if the boundary of $R$ consists of a finite number of closed Jordan courves. Under the same condition on the boundary, Mori [12] showed the existence of conformal mappings of $R$ onto covering surfaces of the extended plane which are $(g+1)$-sheeted and bounded by slits along parallel segments. As for circular and radial slit mappings, Kusunoki [10] and Nehari [13] established an analogous theorem under the same condition on the boundary. But the condition on the boundary is very restrictive, and our intention in this paper is to remove the restriction.

In §1 we shall consider some definitions and properties which are necessary for our conclusions. Here we also show that a Riemann surface of genus $g>1$ can be considered as an at most $g$-sheeted covering surface of the extended plane.

In §2 we shall treat parallel slit mappings. If $f$ is a function on $R$ which maps this onto a covering surface of the extended plane with $q$-sheets and parallel slits as a boundary, a subset $E$

of the extended plane, formed by points $w$ such that there exist at most $q-1$ points on $R$ at which $f(P)=w$ (counted with their multiplicities), consists of parallel segments. Therefore we shall call a conformal mapping $f$ of $R$ onto a covering surface of the extended plane a *parallel slit mapping* if every component of the set $E$ is either a point or a segment parallel to a fixed line. In other words, let $\{R_n\}$ be a canonical exhaustion of $R$ and $\{G_n'\}$ the sequence of the projections of $f(R-\bar{R}_n)$, where every $\bar{R}_n$ denotes the closure of $R_n$, then the set $E$ coincides with the intersection $\bigcap_{n=1}^{\infty}\bar{G}_n'$. In this connection, we mean in the following by *projection of the boundary of* $f(R)$ the intersection $\bigcap_{n=1}^{\infty}\bar{G}_n'$. We shall show, for an arbitrary open Riemann surface of finite genus $g$, that there exist parallel slit mappings onto at most $(g+1)$-sheeted covering surfaces of the extended plane, and the total area of the projection of the boundary of an image under each of these mapping is zero.

Furthermore, we shall prove, if a number of the boundary components of $R$ is at most countable, that the class of all parallel slit mappings of $R$ coincides with the class of single-valued integrals of canonical differntials, except for rotations around the origin on the plane.

With an analogous definition for circular and radial slit mappings, we shall show, in §3, the existence of mappings of this kind and the fact that the logarithmic area of the projection of the boundary of an image under anyone of these mapping is zero.

## §1. Preliminaries

1. *The canonical semiexact differentials.* At first we recall the definition and some properties of the canonicl semiexact differentials on an arbitrary open Riemann surface $R$, which were introduced by Kusunoki [9].

Let $B$ be a canonical regular region on $R$ (Ahlfors and Sario [2] p. 26, p. 80 and p. 117), whose complement consists of $n$ disjoiut non-compact domains $G_1$, $G_2,\cdots,G_n$ with relative boundaries $\Gamma_1$, $\Gamma_2,\cdots,\Gamma_n$, respectively, then a harmonic function $u$ on $R$ is called a *canonical potential associated with B*, if on each domain $G_j$ $(j=1,2,\cdots,n)$ it is a normalized potential, except a possible real constant, which is a single-valued harmonic function on $G_j$ satisfying the normalization condition

$$(1) \qquad u(P)=\int_{\Gamma_j} u(Q)d\omega_j(Q,P), \quad P\in G_j \qquad (j=1,2,\cdots,n),$$

where $\omega_j$ stands for the harmonic measure of an arc $\widehat{Q_jQ}$ on $\Gamma_j$ with a fixed $Q_j$, with respect to $G_j$ $(j=1.\,2,\cdots,n)$ (Nevanlinna [14] pp. 320–333). It may have a finite number of additive periods and singularities in $B$.

Let $T=T_0+T_1$ be the real vector space of canonical potentials associated with canonical regular regions, where the subspace $T_0$ consists of those single-valued and regular on $R$. $T_0$ is a subspace of the Hilbert space consisting of single-valued harmonic functions $u$ with finite norms $||u||=\sqrt{D(u)}$. Here, two elements of $T_0$ are identified if the difference is a constant. Let $\bar{T}_0$ be the completion of $T_0$ by this metric and let $\tilde{T}=\bar{T}_0+T_1$. We call any element of $\tilde{T}$ a *canonical potential* on $R$. The Abelian differentials $\varphi$ such that $Re\int\varphi$ are, except constants, canonical potentials are called *canonical differentials* provided that the sums of the residues vanish.

A differential on $R$ is said to be *semiexact* if it has no periods along every dividing cycle on $R$.

Let us denote by $\mathfrak{K}$ the class of canonical semiexact differentials (or integrals) on $R$, and by $\mathfrak{K}_0$ the class of single-valued integrals (functions) of class $\mathfrak{K}$.

Many properties with respect to the differentials of class $\mathfrak{K}$ and the functions of class $\mathfrak{K}_0$ were found by Kusunoki [9], among which the following are important for the present research.

*For any canonical potential u associated with a canonical regular regions B bounded by* $\Gamma_1, \Gamma_2,\cdots,\Gamma_n$, *and differential* $dv+idv^*$ *which is square integrable over* $R-B$ *and* $\int_{\Gamma_j} dv^*=0$ $(j=1,2,\cdots,n)$, *we have*

$$(2) \qquad D_{R-B}(u, v) = -\int_{\partial B} u\,dv^*$$

*where $\partial B$ denotes the boundary of $B$ described in the positive sense with respect to $B$.*

*Let $u \in \tilde{T}$ and let $v$ be a harmonic function such that $dv$ is square integrable outside of a compact set $K$ and such that $\int_\Gamma dv^* = 0$ for every dividing curve $\Gamma \subset R-K$, then for any exhaustion $\{R_n\}$ of $R$ we have*

$$(3) \qquad \lim_{n\to\infty} \int_{\partial R_n} u\,dv^* = 0\,.$$

*A differential of class $\mathfrak{K}$, whose integral has single-valued real part and regular on $R$, is identically zero on $R$* (Uniquness tneorem).

*For a canonical homology basis $\{A_k, B_k\}$ modulo dividing cycles on $R$, there exist differentials $\varphi_{A_k}$, $\varphi_{B_k}$ $(k=1, 2, \cdots)$ of class $\mathfrak{K}$ of the first kind such that $Re\,\varphi_{A_k}$, $Re\,\varphi_{B_k}$ have only non-vanishing periods $+1$ and $-1$ along $B_k$ and $A_k$ respectively. Let $P_0$ and $Q_0$ be given points on $R$, then there exist differentials $\psi^{(r)}_{P_0}$, $\tilde{\psi}^{(r)}_{P_0}$ of class $\mathfrak{K}$ of the second kind whose integrals have single-valued real parts, and singularities $1/z^r$ and $i/z^r$ $(r \geqq 1)$ at $P_0$ respectively. Also there exist differentials $\phi_{P_0Q_0}$, $\tilde{\phi}_{P_0Q_0}$ of class $\mathfrak{K}$ of the third kind whose integrals have single-valued real parts except an arc $\widehat{P_0Q_0}$, and logaritmic singularities at $P_0$ and $Q_0$ with residues $-1$, $-i$ (at $P_0$) and $+1$, $+i$ (at $Q_0$) respectively.*

Moreover it is known that *these differentials $\varphi_{A_k}$, $\varphi_{B_k}$ $\psi^{(r)}_{P_0}$ and $\tilde{\psi}^{(r)}_{P_0}$ are represented as the limit differentials in terms of norm convergence,*

$$(4) \qquad \varphi_{\substack{A_{kn}\\B_{kn}}} \to \varphi_{\substack{A_{k}\\B_{k}}}\,, \qquad \begin{matrix} \psi^{(r)}_{P_0n} \to \psi^{(r)}_{P_0} \\ \tilde{\psi}^{(r)}_{P_0n} \to \tilde{\psi}^{(r)}_{P_0} \end{matrix} \qquad (n \to \infty)\,,$$

*where $\varphi_{A_{kn}}$, $\psi^{(r)}_{P_0n}$ etc. are the corresponding differentials on canonical regular regions $R_n$ $(n=1, 2, \cdots)$ which constitute an exhaustion of $R$.*

We call an exhaustion of $R$ consisting of canonical regular regions a *canonical exhaustion* of $R$ (Ahlfors and Sario [2] p. 80).

2. *A representation of $(g+1)$-valent functions of class $\mathfrak{K}_0$.* Now we restrict $R$ to be of finite genus $g>0$, unless otherwise stated. For an arbitrary divisor $\delta$ of finite degree $d[\delta]$ on $R$, the following Riemann-Roch's theorem was established by Kusunoki [9]:

$$A[\delta^{-1}]-B[\delta] = 2(d[\delta]-g+1), \tag{5}$$

where $A[\delta^{-1}]$ denotes the number of linearly independent (in the real sense) functions of class $\mathfrak{K}_0$ which are multiples of $\delta^{-1}$, and $B[\delta]$ the number of linearly independent differentials of class $\mathfrak{K}$ which are multiples of $\delta$.

In case of a divisor consisting of a single point $P^{g+1}$, we see that there exist functions of class $\mathfrak{K}_0$ with single poles of order at most $g+1$ at $P$. *The set of points such that $B[P^g]=0$ is dense in $R$ and there exists a function of class $\mathfrak{K}_0$ with a single pole of order just $g+1$ at any point of the subset* (Mori [12]).

Take a point $P_0$ such that $B[P_0^g]=0$. Such a point is a point at which the function on a parametric disk

$$V(z) =$$
$$\begin{vmatrix} Re\, h_1(z) & Im\, h_1(z) & Re\, h_1'(z) & Im\, h_1'(z) & \cdots Re\, h_1^{(g-1)}(z) & Im\, h_1^{(g-1)}(z) \\ Re\, h_2(z) & Im\, h_2(z) & Re\, h_2'(z) & Im\, h_2'(z) & \cdots Re\, h_2^{(g-1)}(z) & Im\, h_2^{(g-1)}(z) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ Re\, h_{2g}(z) & Im\, h_{2g}(z) & Re\, h_{2g}'(z) & Im\, h_{2g}'(z) & \cdots Re\, h_{2g}^{(g-1)}(z) & Im\, h_{2g}^{(g-1)}(z) \end{vmatrix} \tag{6}$$

does not vanish, where $z$ is a local parameter at $P_0$ and $h_k(z)dz = \varphi_{A_k}$ and $h_{g+k}(z)dz=\varphi_{B_k}$ $(k=1,2,\cdots,g)$ (Mori [12] and Springer [20] p. 272).

Let $\{R_n\}$ be a canonical exhaustion of $R$ and let $V_n(z)$ be the corresponding function on a parametric disk about $P_0$ on each $R_n$, then we have $V_n(z)\to V(z)$ as $n\to\infty$ because of (4). Hence we have $V_n(z)\neq 0$ for sufficiently large $n$. Therefore there exists a function $\tilde{f}_n$ of class $\mathfrak{K}_0$ on $R_n$ which has a single pole of order $g+1$ at $P_0$ for such large $n$. From now on we consider $R_n$ only for such large $n$.

The function $\tilde{f}$ of class $\mathfrak{K}_0$ on $R$ which has a single pole of order $g+1$ at $P_0$ has an expression

$$\tilde{f} = \sum_{r=1}^{g+1} d_r \int \psi_{P_0}^{(r)} + \sum_{r=1}^{g+1} \tilde{d}_r \int \tilde{\psi}_{P_0}^{(r)}$$

by the uniqueness theorem, where $(d_1, d_2, \cdots, d_{g+1}, \tilde{d}_1, \tilde{d}_2, \cdots, \tilde{d}_{g+1})$ is a solution of the following system of equations

$$(7) \qquad \sum_{r=1}^{g+1} d_r \int_{\substack{A_k\\B_k}} \psi_{P_0}^{(r)} + \sum_{r=1}^{g+1} \tilde{d}_r \int_{\substack{A_k\\B_k}} \tilde{\psi}_{P_0}^{(r)} = 0\,, \qquad (k = 1, 2, \cdots, g)\,.$$

By dint of the relation (5), we can see that the system (7) has two linearly independent solutions. While $\tilde{f}_n$ has an expression

$$\tilde{f}_n = \sum_{r=1}^{g+1} d_r^{(n)} \int \psi_{P_0 n}^{(r)} + \sum_{r=1}^{g+1} \tilde{d}_r^{(n)} \int \tilde{\psi}_{P_0 n}^{(r)}$$

where $(d_1^{(n)}, d_2^{(n)}, \cdots, d_{g+1}^{(n)}, \tilde{d}_1^{(n)}, \tilde{d}_2^{(n)}, \cdots, \tilde{d}_{g+1}^{(n)})$ is a solution of the following system of equations

$$(8) \qquad \sum_{r=1}^{g+1} d_r^{(n)} \int_{\substack{A_k\\B_k}} \psi_{P_0 n}^{(r)} + \sum_{r=1}^{g+1} \tilde{d}_r^{(n)} \int_{\substack{A_k\\B_k}} \tilde{\psi}_{P_0 n}^{(r)} = 0\,, \qquad (k = 1, 2, \cdots, g)\,.$$

Therefore we see easily, by (4), that

$$(9) \qquad d_r^{(n)} \to d_r, \quad \tilde{d}_r^{(n)} \to \tilde{d}_r \quad (n \to \infty) \qquad (r = 1, 2, \cdots, g+1)\,.$$

Let $Q_0$ be any point on $R$ which is different from $P_0$, and put

$$(10) \qquad f(P) = \tilde{f}(P) - \tilde{f}(Q_0) = u + iu^*$$

and

$$(11) \qquad f_n(P) = \tilde{f}_n(P) - \tilde{f}_n(Q_0) = u_n + iu_n^*\,.$$

Then we have by (4) and (9)

$$(12) \qquad f_n(P) \to f(P)\,, \quad (n \to \infty)\,.$$

The convergence is uniform on every compact set contained in $R - P_0$.

Thus we have shown the following result.

**Lemma 1.** *Let $R$ be an arbitrary open Riemann surface of finite genus $g>0$, $P_0$ a point on $R$ such that $B[P_0^g]=0$, and let $f$ be a function of class $\mathfrak{R}_0$ on $R$ with a single pole of order $g+1$ at*

$P_0$. *Then, for a canonical exhausion* $\{R_n\}$ *of* $R$, *there exists a function* $f_n$ *of class* $\mathfrak{R}_0$ *on each* $R_n$ *with a single pole of order* $g+1$ *at* $P_0$, *for sufficiently large* $n$, *such that the sequence* $\{f_n\}$ *converges to the function* $f$.

3. *A boundary component and its weakness.* For an arbitrary open Riemann surface $R$, we consider an infinite suquence $\{G_n\}$ of subregions of $R$ such that:

1) the relative boundary of each $G_n$ is compact,

2) $G_n \supset G_{n+1}$ $(n=1, 2, \cdots)$, and

3) $\bigcap_{n=1}^{\infty} \bar{G}_n = \phi$, where $\bar{G}_n$ denotes the closure of $G_n$.

The sequence $\{G_n\}$ is said to be a *defining sequence* of a boundary component $\gamma$ of $R$ (Stoilow [21] p. 85).

If $f$ is a conformal mapping of $R$ onto $R'$, the sequence $\{f(G_n)\}$ of the images of $G_n$ under $f$ is a defining sequence of a boundary component $\gamma'$ of $R'$. We shall say that $\gamma'$ *corresponds to* $\gamma$ *under* $f$ in this sence.

For a canonical exhaustion $\{R_n\}$ of $R$, each set $R-\bar{R}_n$ is said to be a *boundary neighborhood*, and a component of $R-\bar{R}_n$ whose boundary contains $\gamma$ is said to be a *neighborhood of* $\gamma$. We define a property to be a *$\gamma$-property*, after Jurchescu [7], if a Riemann surface $R$ carrying $\gamma$ has the property, then every Riemann surface $R'$ which admits a conformal mapping of a neighborhood of a boundary component $\gamma'$ of $R'$ onto a neighborhood of $\gamma$ has the same property.

Let $P_0$ be a fixed point on $R$, and let $K: |z| \leqq 1$ be a fixed parametric disk on $R$ such that $z=0$ corresponds to $P_0$. Consider a canonical exhaustion $\{R_n\}$ of $R$ with $P_0 \in R_1$, and let us denote by $\beta_n$ the boundary of each $R_n$ and by $\gamma_n$ the component of $\beta_n$ which separates $\gamma$ from $P_0$.

Consider the class $\{t\}_\gamma$ of single-valued functions on $R$ which satisfy the following conditions:

1) each $t$ is harmonic on $R-P_0$ and

$$t = \log|z| + h(z)$$

in $K$, where $h(z)$ is a harmonic function with $h(0)=0$,

2) $$\int_{\gamma_n} dt^* = 2\pi \quad \text{and} \quad \int_{\beta_{nj} \neq \gamma_n} dt^* = 0\,, \qquad (n = 1, 2, \cdots)$$

where $\gamma_n$ and $\beta_{nj}$ are described in the positive sense with respect to $R_n$.

Further we consider the corresponding class $\{t\}_{\gamma_n}$ on $R_n$, then there exists a function $t_n$ in this class which is uniquely determined by the conditions:

$$t_n = k_n \text{ on } \gamma_n \quad \text{and} \quad t_n = k_{nj} \text{ on } \beta_{nj} \neq \gamma_n\,,$$

where $k_n$ and $k_{nj}$ are certain real constants.

Then, *there is a principal function $t_\gamma$ furnishing the condition*

$$min \int_\beta t\, dt^* = \int_\beta t_\gamma\, dt^*_\gamma\,,$$

*where $\int_\beta t\, dt^*$ means the limit of integrals taken along $\beta_n$. For any $t \in \{t\}_\gamma$, the deviation of $\int_\beta t\, dt^*$ from the minimum is*

$$\int_\beta t\, dt^* - \int_\beta t_\gamma\, dt^*_\gamma = D_R(t - t_\gamma)\,.$$

*Moreover*

$$k_n \leqq k_{n+1} \quad \textit{and} \quad \int_\beta t_\gamma\, dt^*_\gamma = 2\pi \lim_{n \to \infty} k_n = 2\pi k_\gamma\,.$$

The function $t_\gamma$ is called the *capacity function* of $R$ for $\gamma$, and the quantity $c_\gamma = e^{-k_\gamma}$ is called the *capacity* of $\gamma$ with respect to $K$ (Sario [18] and Savage [19]).

The condition $c_\gamma = 0$ is independent of a local parameter at $P_0$ and of $P_0$. A boundary component $\gamma$ is said to be *weak* if $c_\gamma = 0$. *The weakness of $\gamma$ is a $\gamma$-property* (Jurchescu [7]), *and a boundary component of a planar Riemann surface is weak if and only if its image under any univalent conformal mapping of the surface is a point* (Savage [19]). The boundary of a Riemann surface whose boundary components are all weak is called *absolutely disconnected* (Sario [18] and Savage [19]).

4. *A property of the weak boundary components.* Let $S$ be a subregion of any Riemann surface and let $\rho$ be a conformal metric on $S$. We define $\rho$-length of any cycle $c$ on $S$ by the following lower Darboux integral

$$l(\rho ; c) = \underline{\int_c} \rho(z)|dz| .$$

If $\rho$ is a measurable conformal metric on $S$, we define $\rho$-area of $S$ by the following Lebesgue integral

$$A(\rho ; S) = \int_S \rho^2(z)\, d\sigma_z ,$$

where $\sigma_z$ is the Lebesgue measure on a parametric disk $K_z$ on $S$. A measurable conformal metric $\rho$ defined on $S$ is said to be *A-bounded* on $S$ if $A(\rho ; S) < \infty$ (Jurchescu [7]).

We use the following result obtained by Jurchescu [7].

*A boundary component $\gamma$ of a Riemann surface is weak if and only if, for any neighborhood $S$ of $\gamma$ and for any A-bounded conformal metric $\rho$ on $S$, there exists a dividing cycle separating $\gamma$ from the relative boundary of $S$ with an arbitrarily small $\rho$-length.*

Then it is readily seen:

**Lemma 2.** *Suppose that $R$ is an arbitrary open Riemann surface of finite genus and $f$ is a conformal mapping of $R$ onto a covering surface of the extended plane which has at most a finite number of sheets. Then, $f(P)$ has a limit as $P$ tends to any weak boundary component of $R$.*

It is known that if $f(P)$ is a bounded analytic function on an end of a Riemann surface of class $O_G$, it has a limit as $P$ approaches an ideal boundary component (Heins [6]).

Proof. A projection of a boundary component $\tilde{\gamma}$ of $f(R)$ means the intersection $\bigcap_{n=1}^{\infty} \bar{G}'_n$, where $G'_n$ are the projections of subregions $\tilde{G}_n$ of $f(R)$ which constitute a defining sequence of $\tilde{\gamma}$. Then $f(P)$ has a limit when $P$ tends to $\gamma$, if and only if the projection of $\tilde{\gamma}$ which is the image of $\gamma$ under $f$ is a single point. We shall prove that if the projection $\gamma'$ of $\tilde{\gamma}$ is a continuum, then $\gamma$ is not weak.

Let us denote by $K_{w}$ the disk $|w-w_0|\leqq 1$ with $w_0\in\gamma'$. There exists a disk $K\subset K_{w}\cap S'$, where $S'$ is the projection of a neighborhood of $\tilde{\gamma}$. In $K_{w}$ we consider a rectangle $H=aba'b'$ such that its side $a$ is completely in the interior of $K$ and its neighboring sides $b$, $b'$ have common points with $\gamma'$. Let $\tilde{K}$ be a set on $f(R)$ which lies over $K$, and $\tilde{H}$ a set on $f(R)$ which lies over $H$, and take a neighborhood $\tilde{S}$ of $\tilde{\gamma}$ so that $\tilde{S}\subset f(R)-\tilde{K}$.

We define a conformal metric $\rho$ on $\tilde{S}$ by putting $\rho(w-w_0)=1$ on $\tilde{H}\cap\tilde{S}$, except for branch points of $f(R)$ with respect to the extended plane, and $\rho=0$ otherwise. For any point $\tilde{P}_1$ on $\tilde{H}\cap\tilde{S}$ which lies over $w_1$ and is not a branch point of $f(R)$, $w-w_1$ is a local parameter at $\tilde{P}_1$ and we have $\rho(w-w_1)=\rho(w-w_0)\left|\dfrac{d(w-w_0)}{d(w-w_1)}\right|$ $=1$, and for any other local parameter $z$ at $\tilde{P}_1$ we have $\rho(z)$ $=\rho(w-w_1)\left|\dfrac{d(w-w_1)}{dz}\right|$. Clearly $\rho$ is $A$-bounded and satisfies $l(\rho\,;c)$ $\geqq l_0>0$, where $l_0$ is the length of $a$ in $K$ and $c$ is any dividing cycle separating $\tilde{\gamma}$ from the relative boundary of $\tilde{S}$. Hence $\tilde{\gamma}$ is not weak by the result quoted above, and $\gamma$ is not weak, because the weakness is a $\gamma$-property.

5. *A canonical continuation of a Riemann surface.* We say that a Riemann surface $R^*$ is a *continuation* of a Riemann surface $R$ if $R^*$ has an open subset $\tilde{R}$ which is conformally equivalent to $R$, and if $R^*$ is compact it is called a *compact continuation* (Bochner [4] and Radó [15]). If $R$ is an open Riemann surface of finite genus, there exist always compact continuations of the same genus (Bochner [4] and Mori [11]). Moreover, if $\tilde{R}$ is dense in $R^*$, it is called a *dense continuation* (Heins [5]), and if $R^*-\tilde{R}$ contains interior points, the continuation is said to be *essential* (Sario [17]).

We form a special continuation $R^*$ of $R$ of genus $g$ as follows. We get a planar surface by cutting $R$ along $g$ closed analytic

Jordan curves $\Gamma_j$ $(j=1, 2,\cdots,g)$, which do not intersect each other and the union of which does not divide $R$. We map $R'$ conformally onto a domain $D$ on the plane, which is bounded by $2g$ closed analytic Jordan curves $C_j$, $C'_j$ $(j=1, 2,\cdots,g)$ and a bounded closed set $\beta'$, so that $C_j$ and $C'_j$ correspond to $\Gamma_j$ $(j=1, 2,\cdots,g)$ and $\beta'$ corresponds to the ideal boundary of $R$ respectively. Let us denote by $D'$ the domain on the extended plane bounded by $\beta'$, then $D'$ can be mapped conformally onto a horizontal slit region $D''$ bounded by $\beta^*$ (Ahlfors and Sario [2] p. 177). Let $C^*_j$ and $C'^*_j$ be the images of $C_j$ and $C'_j$, and $D^*$ be the image of $D$ under that mapping respectively. Since there exists analytic correspondence between $C^*_j$ and $C'^*_j$, $D^*\cup\beta^*$ can be regarded as a closed Riemann surface $R^*$ of genus $g$ by identifying the corresponding points on $C^*_j$ and $C'^*_j$ $(j=1, 2,\cdots, g)$. $R$ is conformally equivalent to $\tilde{R}=R^*-\beta^*$.

On this continuation $R^*$ of $R$, we gain a realization $\beta^*$ of the ideal boundary $\beta$ of $R$. A weak boundary component corresponds to a point on $R^*$, while a boundary component which is not weak corresponds to a point or a slit, which is seen by the fact that the weakness of a boundary component is a $\gamma$-property and a weak boundary component of a plane region corresponds to a point under any univalent conformal mapping of the region. We shall call this kind of continuation $R^*$ of $R$ a *canonical continuation* of $R$, that is a continuation which satisfies the following conditions:

1) $R^*$ is a compact continuation,
2) it is of the same genus as $R$,
3) $\tilde{R}$ is dense in $R^*$, and
4) every component of the realization of the ideal boundary of $R$ on $R^*$ is either a point or an analytic curve.

Thus we have shown

**Lemma 3.** *If $R$ is an arbitrary open Riemann surface of finite genus, then there exist canonical continuations of $R$.*

Next we are going to prove:

**Lemma 4.** *Let $R$ be an arbitrary open Riemann surface of finite genus whose boundary is not absolutely disconnected, and let*

*$\gamma$ be a boundary component of $R$ which is not weak, then there exists a canonical continuation $R^*$ of $R$ on which the realization $\gamma^*$ of $\gamma$ is an analytic curve.*

Proof. Let $R_1^*$ be a canonical continuation of $R$, and suppose that the realization $\gamma_1^*$ of $\gamma$ on $R_1^*$ is a point. Then there must be a neighborhood $U_1^*$ of $\gamma_1^*$ and a conformal mapping of $U_1^*$ onto a domain $D$ of the plane, such that the boundary component $\gamma'$ of $D$ which corresponds to $\gamma_1^*$ under that mapping is a continuum. Let $\Gamma_1^*$ be a closed analytic Jordan curve which separates the relative boundary of $U_1^*$ from $\gamma_1^*$, and $S_1^*$ be the subregion on $R_1^*$ which is bounded by $\Gamma_1^*$ and does not contain $\gamma_1^*$. We map a complementary domain of $\gamma'$ with respect to the extended plane onto a horizontal slit region, and let $\Gamma^*$ be the resulting image of $\Gamma_1^*$ on the plane and $V^*$ be the subregion on the plane which is bounded by $\Gamma^*$ and contains the image of $\gamma'$. Then, there is an analytic correspondence between $\Gamma_1^*$ and $\Gamma^*$, and by identifying the corresponding points on $\Gamma_1^*$ and $\Gamma^*$ we get a canonical continuation $R^*=\bar{S}_1^*\cup\bar{V}^*$ of $R$ on which the realization of $\gamma$ is an analytic curve.

By use of a canonical continuation, we can easily get:

**Theorem 1.** *Let $R$ be an arbitrary open Riemann surface of finite genus $g>1$, then $R$ is conformally equivalent to an at most $g$-sheeted covering surface of the extended plane which is bounded by a set consisting of analytic curves and totally disconnected set.*

Proof. We form a canonical continuation $R^*$ of $R$. On the compact Riemann surface $R^*$ of genus $g>1$, there are $n$ Weierstrass points where $n=2g+2$ if $R^*$ is hyperelliptic and $2(g+1)<n\leq(g-1)g(g+1)$ if $R^*$ is not hyperelliptic (Behnke und Sommer [3] pp. 573-577). Then there exists a conformal mapping of $R^*$, with a single pole at a Weierstrass point, onto an at most $g$-sheeted covering surface of the extended plane. Under that mapping, the realization of the ideal boundary of $R$ on $R^*$ corresponds to a set consisning of analytic curves ond totally disconnected set.

## § 2. Parallel slit mappings

In this section we treat the functions of class $\mathfrak{K}_0$, and show that they are parallel slit mappings of $R$ onto covering surfaces of the extended plane, that is, every component of the projection of the boundary of an image $f(R)$ under any function $f$ of this class is either a point or a segment which is parallel to the imaginary axis.

If $\{R_n\}$ is a canonical exhaustion of $R$, with the definition of functions of class $\mathfrak{K}_0$ and by the way of the construction of differentials of class $\mathfrak{K}$ (Kusunoki [9] pp. 243–248), a function $f(P)$ of class $\mathfrak{K}_0$ is the limit function of a sequence of functions $\{f_n(P)\}$, each of which is not necessarily of single-valued. But the real part $u_n(P)$ of each $f_n(P)$ is single-valued and has the expression

$$u_n(P) = \int_{\beta_{nj}} \{u_n(Q)-k_{nj}\}\, d\omega_{nj}(Q\,, P)+k_{nj}\,, \quad P\in G_{nj} \tag{13}$$

on each component $G_{nj}$ of the complement of $R_n$, where $\beta_{nj}$ is the relative boundary of $G_{nj}$, $\omega_{nj}(Q, P)$ is the harmonic measure of an arc $\widehat{Q_{nj}Q}$ on $\beta_{nj}$ with fixed $Q_{nj}$, with respect to $G_{nj}$, and $k_{nj}$ is a real constant $(j=1, 2, \cdots, \nu_n)$. Therefore the property of the harmonic measure $\omega_{nj}(Q, P)$ plays an important role.

At first we shall see a property of a harmonic measure of an ideal boundary.

6. *A harmonic measure of an ideal boundary.* Let $R$ be an arbitrary open Riemann surface, $\{R_n\}$ a canonical exhaustion of $R$, and let $R_0$ be a regular region with the boundary $\alpha$. The limit function

$$\omega(P, \alpha) = \lim_{n\to\infty} \omega_n(P, \alpha) \tag{14}$$

of harmonic measures $\omega_n(P, \alpha)$ of $\alpha$ with respect to $R_n-R_0$, is a harmonic function on $R-R_0$ with $0\leqq\omega(P, \alpha)\leqq 1$. We form the difference

$$\omega(P, \beta) = 1-\omega(P, \alpha) \tag{15}$$

and define it as a *harmonic measure of the ideal boundary* $\beta$ with respect to $R-R_0$ (Nevanlinna [14] p. 317).

**Lemma 5.** *Let $R$ be an open Riemann surface of finite genus whose boundary is not absolutely disconnected, and let $\gamma$ be an ideal boundary component of $R$ such that the realization $\gamma^*$ of $\gamma$ on a canonical continuation $R^*$ of $R$ is an analytic curve. Then we have*

$$\omega(P, \beta^*) = 1 \qquad \text{for} \quad P \in \gamma^*,$$

*where $\beta^*$ denotes the realization of the ideal boundary of $R$ on $R^*$.*

Proof. By (15), it is sufficient to prove that

$$\omega(P, \alpha) = 0 \qquad \text{for} \quad P \in \gamma^* .$$

Let $z$ be a local parameter at $P$ by which $\gamma^*$ is made to correspond to the real axis and a part of $\gamma^*$ which corresponds to the positive part of the real axis is not empty. Choose a real number $\lambda$ so small that $z=\lambda$ is on $\gamma^*$, and let us denote the disk $|z|\leqq\lambda$ by $K_\lambda$. Take a parameter $t=\log z$ and consider the harmonic measure $\omega^*(t)$ of the image of the circle $|z|=\lambda$, $0\leqq\arg z\leqq 2\pi$, with respect to the half plane $Re\, t\leqq\log\lambda$. The function $\omega^*(\log z)$ is harmonic for $0<|z|\leqq\lambda$ and single-valued if we take the branch $0<\arg z<2\pi$. Clearly we have

$$\omega^*(\log z) \leqq \frac{2}{\pi}\operatorname{Tan}^{-1}\pi\left(\log\frac{\lambda}{|z|}\right)^{-1},$$

where $\operatorname{Tan}^{-1}$ denotes the principal value of arctangent.

We take $n$ so large that $(R_n-R_0)\cap K_\lambda \neq \phi$, and compare the harmonic measure $\omega_n(P(z), \alpha)$ with $\omega^*(\log z)$. We obtain the estimate

$$\omega_n(P(z), \alpha) < \omega^*(\log z) ,$$

which is easily seen by using the maximum principle on each component of $(R_n-R_0)\cap K_\lambda$. Then we have

$$\begin{aligned}\lim_{n\to\infty}\omega_n(P(z), \alpha) &\leqq \omega^*(\log z)\\ &= \frac{2}{\pi}\operatorname{Tan}^{-1}\pi\left(\log\frac{\lambda}{|z|}\right)^{-1}.\end{aligned}$$

This inequality shows that $\omega(P(z), \alpha)\rightarrow 0$ as $z\rightarrow 0$.

Moreover, this fact is independent of a canonical continuation of $R$ on which $\gamma$ is realized, beacuse there is a conformal mapping between $\tilde{R}_1$ and $\tilde{R}_2$ which are the subregions of any two continuations of $R$ and conformally equivalent to $R$. Thus we have obtained by Lemma 4 and 5:

**Theorem 2.** *If $R$ is an open Rieman surface of finite genus whose boundary is not absolutely disconnected, the harmonic measure $\omega(P, \beta)$ of the ideal boundary $\beta$ of $R$ tends to 1 as $P$ approaches $\gamma$, where $\gamma$ is any component of $\beta$ which is not weak.*

The Green's function $g(P, P_0)$ of $R$ with a logarithmic singularity at $P_0$ is expressed as the limit function

$$g(P, P_0)=\lim_{n\rightarrow\infty} g_n(P, P_0)\,,$$

where $g_n(P, P_0)$ denotes the Green's function of each $R_n$ (Nevanlinna [14] p. 316). We can find a number $m$ so large that a region $G$ which consists of such points $P$ that satisfy $g(P, P_0)>m$ is a regular region. Hence by making use of the function $m\omega^*(\log z)$, we can prove quite analogously as we did to establish Theorem 2:

**Corollary.** *If $R$ is an open Riemann surface of finite genus whose boundary is not absolutely disconnected, then the Green's function $g(P, P_0)$ tends to 0 as $P$ approaches any boundary component which is not weak.*

7. *Area of the projection of the boundary of $f(R)$.* We shall now prove

**Lemma 6.** *Let $R$ be an arbitrary open Riemann surface of finite genus $g>0$, $f$ a function of class $\mathfrak{K}_0$ with $q$ poles (counted with their multiplicities), and let $E$ be a subset of the plane consisted of all points $w$ such that there exist at most $q-1$ points on $R$ at which $f(P)=w$ (counted with their multiplicities). Then $E$ is a closed set of the area zero.*

Proof. For any point $w\in CE$, where $CE$ denotes the complement of $E$ with respect to the extended plane, there exist exactly

$q$ points $P_1, P_2, \cdots, P_q$ on $R$ at which $f(P_j)=w$ (counted with their multiplicities). We take neighborhoods $U_j(P_j)$ of $P_j$ $(j=1,2,\cdots,q)$, then the intersection of the projections of the images $f(U_j(P_j))$ of $q$ neighborhoods is a neighborhood of $w$ contained in $CE$. Therefore the set $E$ is closed.

In order to see that the area of $E$ is zero, we take a canonical exhaustion $\{R_n\}$ of $R$. Let us denote the image of the boundary $\beta_n$ of $R_n$ under $f$ on the $q$-sheeted covering surface of the extended plane by $\tilde{\beta}_n$, and its projection on the plane by $\beta'_n$. Let $E_n^{(k)}$ $(0 \leqq k \leqq q-1)$ be a set consisting of all points $w$ such that there exist at most $k$ points on $R_n$ at which $f(P)=w$ (counted with their multiplicies). Then we have $E_n^{(k)} \subset E_n^{(k+1)}$ $(0 \leqq k \leqq q-2)$, and

$$E_n^{(q-1)} \supset E_{n+1}^{(q-1)}, \quad E_n^{(q-1)} \supset E \qquad \text{for all } n. \tag{16}$$

Now we consider about $\partial E_n^{(k)}$ $(0 \leqq k \leqq q-1)$ for any $n$. Clearly $\partial E_n^{(k)}$ is piecewise analytic and all $\partial E_n^{(k)}$ $(0 \leqq k \leqq q-1)$ amount to $\beta'_n$. If a point $P$ moves along $\beta_n$ in the positive direction with respect to $R_n$, the image $f(P)$ moves along $\tilde{\beta}_n$ in the positive direction with respect to $f(R_n)$, and the projection $w$ moves along $\beta'_n$ in the positive direction with respect to $CE_n^{(k)}$. For, suppose that there exists a piece $\alpha^{(k)} \subset E_n^{(k)}$ along which $w$ moves in the opposite direction, that is in the positive direction with respect to $E_n^{(k)}$. We choose a point $w_0 \in \alpha^{(k)}$ and a neighborhood $U$ of $w_0$ so that it is divided by $\alpha^{(k)}$ into two disjoint sets and it does not contain any branch point of $f^{-1}$. Let $\tilde{\alpha}^{(k)}$ be a piece of $\tilde{\beta}_n$ which lies over $\alpha^{(k)}$ and let $\tilde{U}$ be a connected set over $U$ on the same sheet on which $\tilde{\alpha}^{(k)}$ passes through, then $\tilde{\alpha}^{(k)}$ divides $f(R_n) \cap \tilde{U}$ and $f(CR_n) \cap \tilde{U}$, where $CR_n$ denotes the complement of $R_n$ with respect to $R$. $f(R_n) \cap \tilde{U}$ must be over $E_n^{(k)} \cap U$ because of the assumption that $w$ moves along $\alpha^{(k)}$ in the positive direction with respect to $E_n^{(k)}$. On the other hand, each point of $E_n^{(k)} \cap U$ has at most $k$ inverse images on $R_n$ by $f$ and each point of $CE_n^{(k)} \cap U$ has at least $k+1$ inverse images on $R_n$ by $f$, which is absurd, unless the projection of $f(P)$ passes along $\alpha^{(k)}$ in the positive direction with respect to $CE_n^{(k)}$ at least two times, and it has the same effect as to pass along $\alpha^{(k)}$ in this direction.

Therefore

$$-\int_{\beta_n} u\,du^* = \sum_{k=0}^{q-1} \int_{\partial E_n^{(k)}} u\,du^* = \sum_{k=0}^{q-1} M(E_n^{(k)}) \geqq M(E_n^{(q-1)}), \qquad (n = 1, 2, \cdots)$$

where $f=u+iu^*$ and $M(E_n^{(k)})$ denotes the area of $E_n^{(k)}$. Then we have by (3) and (16)

$$\lim_{n\to\infty} M(E_n^{(q-1)}) = 0,$$

and we attain to our conclusion $M(E)=0$.

6. *A parallel slit mapping with a single pole.* Let $P_0$ be a point on $R$ such that $B[P_0^g]=0$, then there exist two linearly independent functions of class $\mathfrak{K}_0$ with single poles of order $g+1$ at $P_0$. Let $f$ be one of these functions, then $f$ is a conformal mapping of $R$ onto a $(g+1)$-sheeted covering surface of the extended plane.

Let $\tilde{\gamma}$ be a boundary component of $f(R)$ whose projection is not a point, then the boundary component $\gamma$ of $R$ which corresponds to $\tilde{\gamma}$ under $f$ is not weak by Lemma 2, and there is a canonical continuation $R^*$ of $R$ on which $\gamma$ is realized as an analytic curve by Lemma 4. We identify $R$ and $\tilde{R}$ which is the subregion on $R^*$ conformally equivalent to $R$, and we retain the notation $\gamma$ for the realization of $\gamma$ on $R^*$.

Let $\{R_n\}$ be a canonical exhaustion of $R$ with $P_0 \in R_1$, then the sequence $\{G_n^{(\gamma)}\}$ of the components of $R-\bar{R}_n$ $(n=1,2,\cdots)$, whose relative boundaries $\gamma_n$ separate $\gamma$ from $P_0$ is a defining sequence of $\gamma$. The function $f(P)=u(P)+iu^*(P)$ is the limit function of a sequence of functions $\{f_n(P)\}$, where each $f_n(P)$ has the same singularity as $f(P)$ and the real part $u_n(P)$ of $f_n(P)$ is single-valued and has the expression (13) on each component of $CR_n$. We consider the behavior of $u_n(P)$ on $G_n$. We have

$$u_n(P) = \int_{\gamma_n} \{u_n(Q)-k_n^{(\gamma)}\}\, d\omega_n^{(\gamma)}(Q, P) + k_n^{(\gamma)}, \quad P \in G_n^{(\gamma)},$$

and we can show that $\omega_n^{(\gamma)}(Q, P)\equiv 0$ for $P\in\gamma$ and $u_n(P)$ is a constant $k_n^{(\gamma)}$ along $\gamma$ in a quite analogous way as we did in Lemma 5.

We obtain a sequence of real numbers $\{k_n^{(\gamma)}\}$, and we know that $\lim_{n\to\infty} k_n^{(\gamma)}$ exists and is finite, and it is equal to the value of $u(P)$ along $\gamma$ as follows. We can select a subsequence $\{k_{n_\nu}^{(\gamma)}\}$ for which $\lim_{\nu\to\infty} k_{n_\nu}^{(\gamma)}$ exists. The corresponding subsequence $\{f_{n_\nu}(P)\}$ of $\{f_n(P)\}$ also converges to the function $f(P)$, and we see that $u(P)$ is a constant $k^{(\gamma)}=\lim_{\nu\to\infty} k_{n_\nu}^{(\gamma)}$ along $\gamma$. The sequence $\{k_n^{(\gamma)}\}$ must also converge to $k^{(\gamma)}$. Since $k^{(\gamma)}$ is the value of $u(P)$ on $\gamma$ and the point at infinity belongs to $CE$, it must be finite.

Thus, the real part $u(P)$ of $f(P)$ is a constant along $\gamma$, and we have obtained together with Lemma 6, the following

**Theorem 3.** *Let $R$ be an arbitrary open Riemann surface of finite genus $g>0$, then there exists a parallel slit mapping of $R$ onto a $(g+1)$-sheeted covering surface of the extended plane. Moreover, the total area of the projection of the boundary of the image of $R$ under this mapping is zero.*

We can choose a point $P$ as a pole of this mapping *arbitrarily near the prescribed point*, because the set of points $P$ such that $B[P^g]=0$ is dense in $R$, and when we take a point $P$ at which $B[P^g]=0$ as a pole, there are *two linearly independent such mappings*, as we can see it by (5).

9. *Parallel slit mappings with poles of order* 1. For any $g+1$ points $P_0, P_1,\cdots,P_g$ on an open Riemann surface of finite genus $g>0$, there are functions of class $\mathfrak{K}_0$ with poles of order 1 at some of $P_0, P_1,\cdots,P_g$, which is easily seen by Riemann-Roch's theorem (5) (see Kusunoki [9] Theorem 12). Then, quite analogously as we did in the previous theorem, we can prove

**Theorem 4.** *Given an open Riemann surface $R$ of finite genus $g>0$ and $g+1$ points $P_0, P_1,\cdots,P_g$ on $R$, then there exist parallel slit mappings of $R$, with poles of order* 1 *at some of $P_0, P_1,\cdots,P_g$, onto at most $(g+1)$-sheeted covering surfaces of the extended plane. Moreover, the total area of the projection of the boundary of the image under anyone of this mapping is zero.*

10. *Another characterization of functions of class $\mathfrak{K}_0$.* For

any function $f$ of class $\mathfrak{K}_0$ with $q$ poles (counted with their multiplicities), we can show by the same reasoning as we did in the previous theorems, that $f$ is a parallel slit mapping of $R$ onto a $q$-sheeted covering surface of the extended plane.

Conversely, if a number of the boundary components of $R$ is at most countable, and if $F$ is a parallel slit mapping of $R$ such that the projection of the boundary of the image $F(R)$ does not contain the point at infinity, we can show that $F$ can be expressed as $e^{i\theta}f$, where $f\in\mathfrak{K}_0$. Let us note that, with the definition we have chosen, a number of sheets of a covering surface of the extended plane, onto which $R$ is mapped under a parallel slit mapping is at most finite.

**Theorem 5.** *Suppose that $R$ is an open Riemann surface of finite genus and a number of boundary components of $R$ is at most countable. Then a conformal mapping $F$ of $R$ onto a covering surface of the extended plane, such that the projection of the boundary of the image $F(R)$ does not contain the point at infinity, is a parallel slit mapping if and only if it can be expressed as $e^{i\theta}f$, where $f\in\mathfrak{K}_0$.*

Proof. It suffices to prove only the necessary part of the theorem. Let $F$ be a parallel slit mapping of $R$ onto a $q$-sheeted covering surface of the extended plane, then we can find $\theta$ such that the projection of the boundary of the image under $e^{-i\theta}F$ becomes segments parallel to the imaginary axis and totally disconnected set.

Let

$$(17)\qquad e^{-i\theta}F = v+iv^* = \sum_{r=1}^{m_j}\frac{a_{jr}}{z^r}+\sum_{k=0}^{\infty}b_{jk}z^k \qquad \text{at} \quad P_j\ (j=1,2,\cdots,n),$$

where $\sum_{j=1}^{n} m_j=q$. The real part $v$ is a constant along each boundary component of $R$. Moreover, the Dirichlet integral of $v$ taken over $R-K$, where $K$ is a compact set which contains all $P_j$ $(j=1,2,\cdots,n)$, is finite. In fact, let $\{R_n\}$ be a canonical exhaustion of $R$, and $\beta_n$ the boundary of each $R_n$. The projection $E$ of the boundary of $e^{-i\theta}F(R)$ can be covered by an open set $U$ with the arbitrarily small area, because the area of $E$ is zero by assumption on $F$ and on $R$, and the projection of the image of $\beta_n$ under $e^{-i\theta}F$ is contained in $U$ for sufficiently large $n$. Then by an analogous consideration as we did in the proof of Lemma 6, it can be readily seen that

$$\lim_{n\to\infty}\int_{\beta_n} v\,dv^* = 0\,,$$

and it follows that

$$D_{R-K}(v) = \lim_{n\to\infty}\int_{\beta_n} v\,dv^* - \int_{\partial K} v\,dv^* = -\int_{\partial K} v\,dv^*\,. \tag{18}$$

On the other hand, the projection of the boundary of $e^{-i\theta}F(R)$ and the projecton of the image of $\partial K$ under $e^{-i\theta}F$ are contained in a bounded region $D$ on the plane, and we know that

$$D_{R-K}(v) \leqq qM(D) < \infty\,. \tag{19}$$

Take the differentials $\psi_{P_j}^{(r)}$, $\tilde{\psi}_{P_j}^{(r)}$ $(j=1,2,\cdots,n\,;\,r=1,2,\cdots,m_j)$ of class $\mathfrak{K}$ of the second kind on $R$, and form a function

$$f = u+iu^* = \sum_{j=1}^{n}\sum_{r=1}^{m_j}\{(Re\,a_{jr})\int\psi_{P_j}^{(r)}+(Im\,a_{jr})\int\tilde{\psi}_{P_j}^{(r)}\}\,. \tag{20}$$

Then, $u$ is single-valued and $du^*$ is semiexact, and we get

$$\begin{aligned} D_{R_n}(u-v) &= \int_{\beta_n}(u-v)(du^*-dv^*) \\ &\leqq -\int_{\beta_n} u\,dv^* - \int_{\beta_n} u\,du^*\,, \end{aligned} \tag{21}$$

because

$$\int_{\beta_n} u\,du^* = -D_{R-R_n}(u) \leqq 0$$

by (2), and

$$\int_{\beta_n} v\,dv^* = -D_{R-R_n}(v) \leqq 0\,.$$

Since $u$ and $v$ are single-valued and constant along any boundary component of $R$ and $du^{*\prime}$ is semiexact, $\int_{\gamma} v\,du^{*}=0$ along each $\gamma$, and because a number of boundary components of $R$ is at most countable,

$$(22)\qquad |\int_{\beta_n} v\,du^{*}| = |\int_{\beta} v\,du^{*} - \int_{\beta_n} v\,du^{*}|$$
$$= D_{R-R_n}(v, u) \leqq \sqrt{D_{R-R_n}(v)\,D_{R-R_n}(u)}\,.$$

Consequently, we find by (3) and (19) that

$$D_{R_n}(u-v) \to 0\,, \qquad (n \to \infty)\,.$$

Thus, $F=e^{i\theta}f$ except for an additive constant on $R$, and $f$ is a function of class $\mathfrak{K}_0$, for it has the form (20) and it should be single-valued.

If the restriction on the boundary of $R$ is removed, in order to derive the result $F=e^{i\theta}f$, where $f\in\mathfrak{K}_0$, we must require that the projection of the boundary of $F(R)$ lies on an at most countable number of parallel lines. But it must be noticed that this is not a necessary condition, because the real part $u$ of a function of class $\mathfrak{K}_0$ well takes more than a countable number of values on the boundary of $R$.

**Theorem 6.** *Let $R$ be an arbitrary open Riemann surface of finite genus. If $F$ is a parallel slit mapping of $R$ such that the projection of the boundary of $F(R)$ lies on an at most countable number of parallel lines and does not contains the point at infinity, then $F=e^{i\theta}f$ where $f$ is a function of class $\mathfrak{K}_0$.*

Proof. Suppose that $e^{-i\theta}F=v+iv^{*}$, where $v$ is a constant along each boundary component of $R$, has the expansion (17) at the poles $P_j$ $(j=1, 2,\cdots,n)$ and construct a function $f$ as (20). By the assumption on $F$, the area of the projection of the boundary of $F(R)$ is zero, and we have (18), (19) and (21) under the same consideration as above.

In order to obtain the estimate (22), we form a partition of the ideal boundary of $R$ (Ahlfors and Sario [2] p. 87) so that each part of the partition consists of the boundary components on which $v$ is a same constant. Then we have

$$\int v\,du^* = 0\,,$$

where the integral is taken along each part of the partition, because $du^*$ is semiexact. According to the restriction of $F$, a number of parts of the partition is at most countable, and we get

$$\int_\beta v\,du^* = 0\,.$$

Then, the proof of the theorem can be carried out as before.

## § 3. Circular and radial slit mappings

11. *Circular and radial slit mappings.* After Kusunoki [9], let us denote by $\hat{\mathfrak{K}}$ the class of single-valued meromorphic functions on $R$ which can be written as $\exp\int\varphi$, where $\varphi\in\mathfrak{K}$. We denote by $\phi_{PQ}$ the differential of class $\mathfrak{K}$ of the third kind whose integral has single-valued real part except an arc $\widehat{PQ}$, and the singularities $-\log z$ at $P$ and $\log z$ at $Q$.

Then, for any given points $P_0$ and $P_1$ on $R$, $2g$ points $P_2,\cdots,P_g$, $Q_0$, $Q_1,\cdots,Q_g$ can be chosen so that the function

$$\hat{f}(P) = \exp f(P) \in \hat{\mathfrak{K}}$$

where

$$f(P) = \int^P \phi_{Q_0P_0} + \sum_{j=1}^{g}\int^P \phi_{P_jQ_j} \in \mathfrak{K}$$

becomes single-valued, whose poles are at $Q_0$, $P_1,\cdots,P_g$ and zero points are at $P_0$, $Q_1,\cdots,Q_g$ (Kusunoki [10]).

The function $\hat{f}(P)$ has a limit as $P$ approaches any boundary component which is weak, which is seen by Lemma 2. Moreover, if the projection of a boundary component $\tilde{\gamma}$ of $\hat{f}(P)$ is a continuum, the real part of the function $\hat{f}(P)$ is a constant along $\gamma$, which corresponds to $\tilde{\gamma}$ under $\hat{f}(P)$ (see the proof of Theorem 3).

We shall call a conformal mapping $\hat{f}$ of $R$ onto a covering surface of the extended plane a *circular* (*radial*) *slit mapping*, if every component of the projection of the boundary of $\hat{f}(R)$ is either a point or a circular arc (radial segment) with the center at the origin.

**Theorem 7.** *Let $R$ be an arbitrary open Riemann surface of finite genus $g$, then there exists a circular* (*radial*) *slit mapping of $R$ onto an at most $(g+1)$-sheeted covering surface of the extended plane. We can presccribe the location of one pole and one zero of the mapping function. Moreover, the logarithmic area of the projection of the boundary of the image under that mapping is zero.*

Let $\{R_n\}$ be a canonical exhaustion of $R$, and $E_n^{(k)}$ be a set on the plane which consists of points $w$ such that there exist at most $k$ points on $R_n$ at which $\hat{f}(P)=w$ for $0 \leqq k < q$, where $q$ is a number of poles of $\hat{f}$ counted with multiplicities. For any measurable set $D$ on the $w$-plane, we denote by $M_{\log}(D)$ the logarithmic area of $D$, then we have

$$M_{\log}(E_n^{(k)}) = \int_{\partial E_n^{(k)}} \log|w|\, d \arg w\,,$$

because $E_n^{(k)}$ is bounded by piecewise analytic curves. Therefore we can prove, quite analogously as Lemma 6, that the logarithmic area of the projection of the boundary of an image under a circular slit mapping is zero.

We get a radial slit mapping by exp (if), and the fact that the logarithmic area of the projection of the boundary of an image under a radial slit mapping is also zero is readily seen by the way of the proof of Lemma 6, because we have

$$\int_{\partial E_n^{(k)}} u\, du^* = - \int_{\partial E_n^{(k)}} u^* du\,.$$

Further, by a suitable choice of $Q_j$, $R$ can be mapped conformally onto an *exactly* $(g+1)$-*sheeted* covering surface of the extended plane.

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