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# Linear transformations of Finsler connections

Dedicated to Professor J. Kanitani on his 70th birthday

By

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We introduced, in a previous paper [1], a notion of a linear transformation of the tangent bundle $B$ of a differentiable manifold $M$, which was a generalization of a notion of a transformation induced from the one of $M$. A Finsler connection is defined in a certain principal bundle $Q$, the base space of which is the total space $B$.

A theory of transformations of a Finsler connection by a linear transformation will be developed under a certain special condition. The paper [1] was devoted to the study of affine linear transformations, and we intend to treat a projective one. The present paper is written as necessary preparation for it. The terminologies and signs of the paper [1] will be used in the following without too much comment.

## §1. Preliminaries

### [1] Principal bundle $Q$

In the first place, we recall the principal bundle $Q$, in which a Finsler connection is defined [1], [3].

Let $P(M, \pi, G)$ be the principal bundle of frames tangent to a differentiable manifold $M$ of $n$ dimensions. The group of structure is the full linear real group $GL(n, R)$, and an element $g$ of $G$ acts on $P$ by $p \in P \rightarrow p \cdot g$, which is called a right translation $R_g$ of $P$ by $g$. The total space $P$ is interpreted as the set of all admissible mappings $F \rightarrow B$, where $F$ is a $n$-dimensional real vector space and

$B$ is the total space of the tangent bundle $B(M, \tau, F, G)$ of the manifold $M$. Throughout the paper, we assume that $b \in B$ is a non-null tangent vector of $M$. Take a fixed base $(e_a)$, $a=1, 2, \cdots, n$, of $F$ and denote by $\rho_g$, $g \in G$, the operation of $g$ on $F$, namely, $\rho_g(f)=g_b{}^a f^b e_a$, where $g=(g_b{}^a)$, $a, b=1, 2, \cdots, n$, and $f=f^a e_a$.

The projection $\tau: B \rightarrow M$ gives an induced bundle $\tau^{-1}P = Q(B, \bar{\pi}, G)$, the total space of which is defined by $Q=\{(b, p) \mid b \in B, p \in P, \tau(b)=\pi(p)\}$. Then the projection $\bar{\pi}: Q \rightarrow B$ and the induced mapping $\eta: Q \rightarrow P$ are given by $\bar{\pi}(b, p)=b$ and $\eta(b, p)=p$. A right translation $R_g$ of $P$ by $g \in G$ is transfered into $Q$, and we have a right translation $\bar{R}_g$ of $Q$, which is defined by $\bar{R}_g(b, p)=(b, R_g(p))$. Later on, we shall use the same latter $R_g$, instead of $\bar{R}_g$, for a right translation of $Q$. By a right translation of $Q$, a fundamental vector field $F(A)$ on $Q$ corresponding to $A \in \hat{G}$ (the Lie algebra of $G$) is induced, which is determined by $F(A)_q=L_q(A)$, where $L_q: G \rightarrow Q$, $g \rightarrow R_g(q)$.

## 2 Left translations of $Q$

We introduce a mapping

$$L: G \times Q \rightarrow Q, (g, (b, p)) \rightarrow (p(g \cdot p^{-1}b), p).$$

Then, for a fixed element $g \in G$, we have a mapping $L_g: Q \rightarrow Q$, $q \rightarrow L(g, q)$, which is called a *left translation* of $Q$ by $g \in G$. It is easily seen that $L_g$ acts on $\eta^{-1}(p)$, $p \in P$, transitively, $\eta^{-1}(p)$ being called the *$\eta$-fibre* on $p \in P$. If we take the identification $i: Q \rightarrow F \times P$, used in [1, §2], the above $L_g$ is expressed simply by $(f, p) \in F \times P \rightarrow (gf, p)$.

Let $q \in Q$ be a fixed point and $R_q$ be a mapping defined by $R_q: G \rightarrow Q$, $g \rightarrow L(g, q)$. By a mapping $R_q$, we can introduce the *second fundamental vector field* $E(A)$ on $Q$ corresponding to $A \in \hat{G}$, which is defined by $E(A)_q=R_q(A)$. Since $\eta E(A)=0$ is obvious, we can say that $E(A)$ is tangent to $\eta$-fibre at any point of $Q$. Take the natural base $(\hat{g}_b{}^a)$, $\hat{g}_b{}^a=(\partial/\partial g_a{}^b)_e$, of $\hat{G}$ and put $E_b{}^a=E(\hat{g}_b{}^a)$. Then the expression

$$E_b{}^a(q) = p_b{}^i p_j^{-1a} b^j \frac{\partial}{\partial b^i}$$

is easily derived, where $(x^i, b^i, p_a{}^i)$ is the canonical coordinate of $q \in Q$ [1, §1].

### [3] Characteristic field

The notion of the *characteristic field* $\gamma$ on $Q$ [1, §1] is important for a theory of Finsler connections, which is simply a mapping $Q \to F$, $(b, p) \to p^{-1}b$. We shall find an expression of the differential of $\gamma$ for the later use. Take following mappings:

$$\xi \ : F \times P \to B,\ (f, p) \to pf\,,$$
$$\sigma_f \ : Q \to B,\ (b, p) \to pf\,,$$
$$K_f : P \to B,\ p \to pf\,.$$

Then it is clear that $\xi \circ i = \bar{\pi}$ and $\sigma_f = K_f \circ \eta$. Hence, if we take a tangent vector $X \in Q_q$ and $f = \gamma(q)$, the differential $\bar{\pi}$ is expressed by

$$\begin{aligned}\bar{\pi}(X) &= \xi \circ i(X) = \xi(\gamma(X),\ \eta(X)) = \eta(X) \cdot f + p \cdot \gamma(X) \\ &= K_f \circ \eta(X) + p \cdot \gamma(X) = \sigma_f(X) + p \cdot \gamma(X)\,.\end{aligned}$$

Consequently we obtain

$$\gamma = p^{-1}(\bar{\pi} - \sigma_f)\,, \quad q = (b, p)\,, \quad f = \gamma(q)\,, \tag{1.1}$$

which is the desired equation.

It follows from (1.1) that

$$\begin{aligned}F(A)\gamma &= d\gamma F(A) = p^{-1}(\bar{\pi}F(A) - \sigma_f F(A)) \\ &= -p^{-1}\sigma_f F(A) = -p^{-1}\sigma_f L_q(A)\,.\end{aligned}$$

Since we have $p^{-1}\sigma_f L_q(g) = g \cdot f$, $g \in G$, we obtain

$$F(A)\gamma = -A \cdot \gamma\,. \tag{1.2}$$

### [4] Mapping $C(f)$

In [1], we sketched a Finsler connection in $Q$, which was originally introduced by T. Okada [3]. In terms of a canonical coordinate, the connection is given by coefficients of connection of three kinds [1, §1], namely, $F_j{}^i(x^i, b^i)$, $F_j{}^i{}_k(x^i, b^i)$, and $C_j{}^i{}_k(x^i, b^i)$. Among them, the last $C_j{}^i{}_k$ behaves as a (1, 2)-tensor under a transformation of a canonical coordinate. By virtue of this property, we define a mapping $C$, which is given by

$$C : F \times Q \to \hat{G}, \quad (f, q) \to C_b{}^a{}_c(q) f^c \hat{g}_a{}^b\,,$$

where $C_b{}^a{}_c(q)=C_j{}^i{}_k(x^i, b^i)p_i{}^{-1a}p_b{}^jp_c{}^k$ and $(x^i, b^i, p_a{}^i)$ is a canonical coordinate of $q\in Q$. For a fixed element $f\in F$, the mapping $C(f)$: $Q\to\hat{G}$ is derived from $C$. It follows from [1, (2.9)] that

$$\omega_{(p)f}\circ j_f(f_1)=C(f_1)_{(pf,p)}, \qquad f, f_1\in F, \quad p\in P. \tag{1.3}$$

In [1, §9], we used a *$\mathring{v}$-basic vector field* $\mathring{B}^v(f)$, which was defined by a mapping $\pi_p^{-1}: b\in B\to(b, p)\in Q$ as $\mathring{B}^v(f)_q=\pi_p^{-1}\circ p\circ j_{\gamma(q)}(f)$, where $q=(b, p)$. Since $R_q(g)=\pi_p^{-1}\circ p(g\cdot\gamma(q))$, we have the relation

$$E(A)=\mathring{B}^v(A\cdot\gamma), \tag{1.4}$$

where $E(A)$ is the second fundamental vector field. $\pi E(A)_q$, $q=(b, p)$, is vertical in $B$ and is equal to $p(A\cdot\gamma)$, because $\pi R_q(g)=p(g\cdot\gamma(q))$, $g\in G$. Hence we see that $h^hE(A)_q=0$ ($h$-horizontal component), while $h^vE(A)_q=l_q\circ p(A\cdot\gamma)=B^v(A\cdot\gamma)_q$ ($v$-horizontal component), where $l_q$ indicates a lift to $q\in Q$. We shall find the vertical component of $E(A)$. The $p$-induced form $\omega_{(p)}$ [1, §2] on $F$ from the connection form $\omega$ of a Finsler connection is given by $\omega_{(p)}=\omega\circ i^{-1}\circ\chi_p$, where $\chi_p$: $F\to F\times P$, $f\to(f, p)$. It follows from $i^{-1}\circ\chi_p=\pi_p^{-1}\circ p$, that $\omega\circ i^{-1}\circ\chi_p\circ j_f(f_1)=\omega(\mathring{B}^v(f_1))_{(pf,p)}$. Therefore the equation (1.3) gives $v\mathring{B}^v(f)_q=F(C(f)_q)$ (vertical component), and hence we see that $vE(A)=F(C(A\cdot\gamma))$, by virtue of (1.4). Consequently $E(A)$ and $\mathring{B}^v(f)$ are expressed, with respect to a Finsler connection, as follows:

$$E(A)=B^v(A\cdot\gamma)+F(C(A\cdot\gamma)), \tag{1.5}$$

$$\mathring{B}^v(f)=B^v(f)+F(C(f)). \tag{1.6}$$

It, however, is remarked that $E(A)$ and $\mathring{B}^v(f)$ are defined without use of a Finsler connection.

**5 Condition of homogeneity**

In [1, §9], we discussed the complete integrability of infinitesimal affine transformation under the condition of homogeneity. This condition seems very essential for a theory of Finsler geometry [3], [4]. The definition of this condition is as follows. Let $R^+$ be the set of positive numbers, and a mapping $R^+\times F\to F$ be such that $(z, f)\to z\cdot f$ (ordinary product), $z\in R^+$, $f\in F$. Then we introduce mappings [2, p. 174]

$$h: R^{+}\times B \to B\,,\quad (z, b) \to z\cdot b = p(z\cdot p^{-1}b)\,,\quad p \in \pi^{-1}\circ\tau(b)\,,$$
$$\bar{h}: R^{+}\times Q \to Q\,,\quad (z, (b, p)) \to (z\cdot b, p)\,.$$

It is clear that $z\cdot b$ as thus defined does not depend on the choice of $p$. We denote by $h_z$ (resp. $h_b$) the mapping $B\to B$ (resp. $R^{+}\to B$) obtained from the above $h$ for a fixed $z\in R^{+}$ (resp. $b\in B$). For the another mapping $\bar{h}$, the similar signs $\bar{h}_z$ and $\bar{h}_q$ are used.

Now, the *condition of homogeneity* is that *a Finsler connection* $(\Gamma^v, \Gamma^h)$ *is invariant by every mapping* $\bar{h}_z$, that is, $\bar{h}_z\Gamma^v = \Gamma^v$ and $\bar{h}_z\Gamma^h = \Gamma^h$.

Let $X$ be a tangent vector field to $Q$. If $X$ satisfies the equation $\bar{h}_z(X) = z^r\cdot X$, then we say that $X$ is *positively homogeneous of degree* $r$ (*p.h.*$(r)$, for brevity) [4, p. 7]. The same term is used for a differential form $\alpha$ on $Q$, if $\alpha\circ\bar{h}_z = z^r\cdot\alpha$. The following proposition will be easily verified [3].

**Proposition 1.** *The condition of homogeneity is equivalent to one of the following three properties.*

1. $F(A)$, $B^v(f)$ *and* $B^h(f)$ *are p.h.*(0), (1) *and* (0) *respectively.*
2. $\omega$, $\theta^v$ *and* $\theta^h$ *are p.h.*(0), (1) *and* (0) *respectively.*
3. $F_j{}^i$, $F_j{}^i{}_k$ *and* $C_j{}^i{}_k$ *are functions of p.h.*(1), (0) *and* $(-1)$ *respectively with respect to variables* $b^i$.

## § 2. Linear transformations

A *linear transformation* $\varphi$ of the total space $B$ of the tangent bundle $B(M, \tau, G)$ is defined in [1], which is a transformation such that

1. $\varphi$ is fibre-preserving.
2. $\varphi$ is linear on each fibre.

By virtue of the first property of $\varphi$, a transformation $\underline{\varphi}$ of the base manifold $M$ is derived which satisfies the equation $\tau\circ\varphi = \underline{\varphi}\circ\tau$. $\underline{\varphi}$ is called the *projection* of $\varphi$. On the other hand, $\varphi$ gives naturally a transformation $\varphi^*$ of $P$, which is termed the *associated* transformation with $\varphi$.

A linear transformation of $P$ is by definition a transformation which commutes with every right translation. The following fact was proved in [1].

**Proposition 2.** *Any linear transformation $\varphi^*$ of $P$ is associated with a linear transformation $\varphi$ of $B$, and the relation*

$$(2.1)\qquad \varphi^*(p)\cdot f = \varphi(p\cdot f), \quad p\in P, \ f\in F,$$

*is satisfied.*

We have naturally a transformation $\bar{\varphi}$ of the total space $Q$ of the induced bundle $\tau^{-1}P$ from a linear transformation $\varphi$ of $B$, such that $\bar{\varphi}(b, p)=(\varphi(b), \varphi^*(p))$. $\bar{\varphi}$ is called the transformation *induced from* $\varphi$, or, for brevity, the linear transformation of $Q$. In the following, we shall use the same letter $\varphi$ for the induced one, in case there is no danger of confusion.

The notion of the *deviation* $\lambda: P\rightarrow G$ of a linear transformation $\varphi$ is essential in our discussion. Let $\varphi_0$ be the differential of the projection $\underline{\varphi}$. $\varphi_0$ is obviously linear and then we have the associated $\varphi_0^*$. Then the mapping $\lambda$ is defined by the equation

$$(2.2)\qquad \varphi^*(p)=\varphi_0^*(p)\cdot\lambda(p).$$

If the projection $\underline{\varphi}$ is the identity transformation of $M$, $\varphi$ is called a *rotation*. In this case, $\varphi^*$ coincides with the right translation $R_\lambda$ by the deviation $\lambda$.

We proved in [1] that a fundamental vector field $F(A)$ and the characteristic field $\gamma$ were invariant by the induced transformation $\varphi$. Another important property of $\varphi$ is that *the second fundamental vector field $E(A)$ is also invariant by $\varphi$.* In fact, we have first

$$\begin{aligned}\varphi\circ L_g(b, p) &= \varphi(p(g\cdot p^{-1}b), p) = (\varphi(p(g\cdot p^{-1}b)), \varphi^*(p))\\ &= (\varphi^*(p)(g\cdot p^{-1}b), \varphi^*(p)),\end{aligned}$$

where we made use of (2.1). On the other hand, we have

$$\begin{aligned}L_g\circ\varphi(b, p) &= L_g(\varphi(b), \varphi^*(p)) = (\varphi^*(p)(g\cdot\varphi^*(p)^{-1}\varphi(b)), \varphi^*(p))\\ &= (\varphi^*(p)(g\cdot p^{-1}b), \varphi^*(p)),\end{aligned}$$

where we made use of the invariance of $\gamma$. Thus $\varphi$ *commutes with every left translation,* from which it follows immediately that $E(A)$ is invariant by $\varphi$.

**Theorem 1.** *The necessary and sufficient condition for a transformation $\bar{\varphi}$ of $Q$ to be linear is that the following three properties are satisfied.*

1. *$\bar{\varphi}$ commutes with every right translation.*
2. *$\bar{\varphi}$ commutes with every left translation.*
3. *The characteristic field $\gamma$ is invariant by $\bar{\varphi}$.*

Proof. We define, in the first place, transformation $\varphi$ of $B$ and $\varphi^*$ of $P$ as follows:

$$\varphi(b) = \bar{\pi}\circ\bar{\varphi}(q), \quad q\in\bar{\pi}^{-1}(b),\ b\in B,$$
$$\varphi^*(p) = \eta\circ\bar{\varphi}(q), \quad q\in\eta^{-1}(p),\ p\in P.$$

It follows from the properties 1 and 2 that $\varphi(b)$ and $\varphi^*(p)$ are well defined, independent of the choice of $q$. Then $\bar{\varphi}$ is written by $\bar{\varphi}(b, p)=(\varphi(b), \varphi^*(p))$. The property 3 means that $\varphi^*(p)^{-1}\varphi(b)= p^{-1}b$, from which it follows that $\varphi(b)=\varphi^*(p)(p^{-1}b)$, that is, (2.1). Further, by means of the property 1, we see that $\varphi^*$ as thus defined commutes with every right translation of $P$. Consequently the theorem is established by virtue of Proposition 2.

## § 3. Transformation of a Finsler connection

We consider a Finsler connection $(\Gamma^v, \Gamma^h)$ in $Q$, and $B^v(f)$ and $B^h(f)$ are $v$-basic and $h$-basic vector fields respectively. We discuss behaviours of $F(A)$, $B^v(f)$ and $B^h(f)$ under a linear transformation $\varphi$. First, the following equations will be derived:

$$\begin{aligned}\varphi F(A) &= F(A), \\ \varphi B^v(f) &= F(\mu_v(f))+B^v(f), \\ \varphi B^h(f) &= F(\mu_h(f))+B^v(\mu(f))+B^h(\lambda^{-1}f),\end{aligned} \tag{3.1}$$

where $\lambda$ is the deviation of $\varphi$, and $\mu_v$, $\mu_h$ and $\mu$ will be defined in the following. It follows from (3.1) directly that the connection form $\omega$, the $v$-basic form $\theta^v$ and the $h$-basic form $\theta^h$ subject to the following transformations:

$$\begin{aligned}\omega\circ\varphi &= \omega+\mu_v(\theta^v)+\mu_h(\theta^h), \\ \theta^v\circ\varphi &= \theta^v+\mu(\theta^h), \\ \theta^h\circ\varphi &= \lambda^{-1}\theta^h.\end{aligned} \tag{3.2}$$

We shall show (3.1). The first of (3.1) is obvious by [1, Prop. 2]. Next we have, by means of [1, Prop. 3],

$$\theta^h(\varphi B^v(f)) = 0\,, \quad \theta^h(\varphi B^h(f)) = \lambda^{-1} f\,. \tag{3.3}$$

Further we show that

$$\theta^v(\varphi B^v(f)) = f\,. \tag{3.4}$$

In fact, it follows from the definition of $B^v(f)$ that

$$\bar{\pi}\circ\varphi B^v(f)_q = \varphi\circ\bar{\pi} B^v(f)_q = \varphi(pf) = \varphi^*(p)f\,,$$

where we put $q=(b, p)$. Therefore we obtain $h^v\varphi B^v(f)_q = l_{q'}(p'f)$, $q'=(b', p')=\varphi(q)$. Thus (3.4) is a consequence of the definition of the form $\theta^v$. Finally we introduce three mappings $\mu_v$, $\mu_h$ and $\mu$, which depend on the choice of $f\in F$, as follows:

$$\begin{aligned} &\mu_v(f): Q \to \hat{G}\,, \quad q \to \omega(\varphi B^v(f))_q\,, \\ &\mu_h(f): Q \to \hat{G}\,, \quad q \to \omega(\varphi B^h(f))_q\,, \\ &\mu(f)\,: Q \to F\,, \quad q \to \theta^v(\varphi B^h(f))_q\,. \end{aligned} \tag{3.5}$$

Thus (3.1) is deduced from (3.3), (3.4) and (3.5).

Above mappings $\mu_v$, $\mu_h$ and $\mu$ satisfy the equations

$$\begin{aligned} &\mu_v(g^{-1}f)\circ R_g = ad(g^{-1})\mu_v(f)\,, \\ &\mu_h(g^{-1}f)\circ R_g = ad(g^{-1})\mu_h(f)\,, \\ &\mu(g^{-1}f)\circ R_g = g^{-1}\mu(f)\,. \end{aligned} \tag{3.6}$$

We shall prove the first of (3.6). If we put $\varphi(q')=q$, we see

$$\begin{aligned} \mu_v(g^{-1}f)\circ R_g(q) &= \omega\varphi B^v(g^{-1}f)_{q'g} = \omega\varphi R_{g^{-1}} B^v(f)_{q'} \\ &= \omega R_{g^{-1}}(\varphi B^v(f))_q = ad(g^{-1})\omega(\varphi B^v(f))_q\,. \end{aligned}$$

In like manner we can show the second. By making use of $\theta^v\circ R_g = g^{-1}\theta^v$, the third will be also verified.

An induced transformation $\varphi$ is characterized by the three properties given by Theorem 1, and (3.6) is a direct result from the property 1. In the following, we discuss the behaviour of the differential of $\varphi$ arising from the properties 2 and 3.

The property 2 gives $\varphi E(A)=E(A)$. If we put $q=\varphi(q')$, it follows from (1.5) and (3.1) that

$$\varphi E(A)_{q'} = F(C(A\cdot\gamma(q'))_{q'})_q + F(\mu_v(A\cdot\gamma(q'))_q)_q + B^v(A\cdot\gamma(q'))_q$$
$$= F(C(A\cdot\gamma(q))_{q'})_q + F(\mu_v(A\cdot\gamma(q))_q)_q + B^v(A\cdot\gamma(q))q\,,$$

where we made use of the invariance of $\gamma$. Thus $\varphi E(A)=E(A)$ is expressed by

$$\mu_v(A\cdot\gamma(q))_q = C(A\cdot\gamma(q))_q - C(A\cdot\gamma(q))_{q'}\,.$$

Since $A\in\hat{G}$ is an arbitrary element, the above equation gives

$$\mu_v(f)_q = C(f)_q - C(f)_{q'}\,,\quad q=\varphi(q')\,. \tag{3.7}$$

Next, we turn to the consideration of the property 3 of Theorem 1. It follows from the second of (3.1) and $\gamma\circ\varphi=\gamma$ that

$$\gamma B^v(f)_{q'} = \gamma F(\mu_v(f))_q + \gamma B^v(f)_q\,,\quad q=\varphi(q')\,.$$

By virtue of (1.2), the first term of the right hand side is written in the form $-\mu_v(f)_q\gamma$. If we put $\gamma^a|_b f^b e_a=\gamma|(f)$ ($v$-covariant derivative), then the above equation gives

$$\mu_v(f)_q\gamma = \gamma|(f)_q - \gamma|(f)_{q'}\,.$$

This, however, is solely a consequence of (3.7), because $\gamma|(f)_q = f + C(f)_q\gamma$. In like manner, from the third of (3.1), it follows that

$$\gamma_|(f)_{q'} = \gamma_|(\lambda^{-1}f)_q + \gamma_|(\mu(f))_q - \mu_h(f)_q\gamma,\quad q=\varphi(q')\,, \tag{3.8}$$

where $\gamma_|(f)=\gamma^a{}_{|b}f^b e_a$ ($h$-covariant derivative).

Summarizing the above results, we can state that

**Theorem 2.** *The tranformation of a Finsler connection by a linear transformation $\varphi$ of $B$ is given by* (3.1) *or* (3.2), *where $\mu_v$, $\mu_h$ and $\mu$ are defined by* (3.5) *and satisfy* (3.6), (3.7) *and* (3.8).

If we take the fixed base $(e_a)$ of $F$ and $(\hat{g}_b{}^a)$ of $\hat{G}$, we may write

$$\mu_v(e_a) = \mu_{v)a}{}^b{}_c\hat{g}_b{}^c\,,\quad \mu(e_a) = \mu_a{}^b e_b\,,$$
$$\mu_h(e_a) = \mu_{h)a}{}^b{}_c\hat{g}_b{}^c\,.$$

Then (3.6) means that quantities

$$\begin{aligned} \mu_{v)j}{}^i{}_k &= \mu_{v)b}{}^a{}_c p_a{}^i p_j^{-1b} p_k^{-1c}\,,\\ \mu_{h)j}{}^i{}_k &= \mu_{h)b}{}^a{}_c p_a{}^i p_j^{-1b} p_k^{-1c}\,,\\ \mu_j{}^i &= \mu_b{}^a p_a{}^i p_j^{-1b}\,, \end{aligned} \tag{3.6'}$$

are functions of $x^i$ and $b^i$ only, where $(x^i, b^i, p_a{}^i)$ is a canonical coordinate. On the other hand, (3.7) and (3 8) are written

$$(3.7') \qquad \mu_{v)b}{}^a{}_c(q) = C_c{}^a{}_b(q) - C_c{}^a{}_b(q'),$$

$$(3.8') \qquad \gamma^a{}_{|b}(q') = \gamma^a{}_{|c}(q)(\lambda_b^{-1c}(q) + \mu_b{}^c(q)) - \mu_{h)b}{}^a{}_c(q)\gamma^c(q).$$

It is remarked here that $\gamma^a{}_{|b} = -D_b{}^a$ [1, §7], where

$$D_b{}^a = D_j{}^i p_i^{-1a} p_b{}^j, \quad D_j{}^i = F_j{}^i - b^k F_k{}^i{}_j.$$

The following fact will be immediately verified by Proposition 1 and (3.5).

**Proposition 3.** *If a Finsler connection satisfies the condition of homogeneity, then* $\mu_v$, $\mu_h$ *and* $\mu$ *are p.h.*$(-1)$, $(0)$ *and* $(1)$ *respectively.*

## §4. Transformation of quasi-connection

We introduced, in [1, §2], the quasi-$f$-connection $\Gamma_f$ in the bundle $P$ of frames of $M$ induced from a Finsler connection in $Q$ and a fixed element $f \in F$. The quasi-connection form $\omega^*_{(f)}$ is also given by [1, Theo. 1] In the following we shall find the expression of $\omega^*_{(f)} \circ \varphi^*$, corresponding to (3.2).

We have first from [1, (2.3)]

$$(4.1) \qquad \theta^v_{(p)f} \circ j_f = \text{identity}.$$

Next, if we denote by $\theta^h_{(f)}$ and $\theta^h_{(p)}$ the $f$-induced and $p$-induced forms from the $h$-basic form $\theta^h$ [1, §7], then the equations

$$(4.2) \qquad \theta^h_{(f)} = \theta, \quad \theta^h_{(p)} = 0,$$

will be obtained, where $\theta$ is the basic form on $P$ [5]. In fact, it follows from $\tau \circ \bar{\pi} \circ i^{-1} = \pi$ that

$$\theta^h_{(f)p} = \theta^h \circ i^{-1} \circ X_f = p^{-1} \circ \tau \circ \bar{\pi} \circ i^{-1} \circ X_f,$$
$$\theta^h_{(p)f} = \theta^h \circ i^{-1} \circ X_p = p^{-1} \circ \tau \circ \bar{\pi} \circ i^{-1} \circ X_p.$$

Since $\tau \circ \bar{\pi} \circ i^{-1} \circ X_f = \eta$ and $\tau \circ \bar{\pi} \circ i^{-1} \circ X_p = \text{constant}$, we obtain (4.2). Next, we shall show that

$$(4.3) \qquad \omega_{(p)} = \omega_{(p')} + \mu_v(\theta^v{}_{(p')}), \quad p = \varphi^*(p').$$

Observing that $\chi_p(f)=(f, \varphi^*(p'))=(1, \varphi^*)\circ\chi_{p'}(f)$, $f\in F$, we get

$$\omega_{(p)} = \omega\circ i^{-1}\circ\chi_p = \omega\circ i^{-1}\circ(1, \varphi^*)\circ\chi_{p'} = \omega\circ\varphi\circ i^{-1}\circ\chi_{p'},$$

and substitution of (3.2) gives

$$= (\omega+\mu_v(\theta^v)+\mu_h(\theta^h))\circ i^{-1}\circ\chi_{p'}.$$

Thus, we have (4.3) from the second of (4.2).

Now, it follows from the definition of $\omega^*_{(f)}$ that

$$\begin{aligned}\omega^*_{(f)p}\circ\varphi^* &= \omega_{(f)p}\circ\varphi^* - \omega_{(p)f}\circ j_f\circ\theta^v_{(f)p}\circ\varphi^* \\ &= (\omega\circ i^{-1}\circ\chi_f)_p\circ\varphi^* - \omega_{(p)f}\circ j_f\circ(\theta^v\circ i\circ\chi_f)_p\circ\varphi^* \\ &= (\omega\circ\varphi\circ i^{-1}\circ\chi_f)_{p'} - \omega_{(p)f}\circ j_f\circ(\theta^v\circ\varphi\circ i^{-1}\circ\chi_f)_{p'},\end{aligned}$$

where we put $p=\varphi^*(p')$. Substituting from (3.2) and making use of (4.2) and (4.3), we obtain

$$\begin{aligned}&= \omega_{(f)p'}+\mu_v(\theta^v_{(f)p'})+\mu_h(\theta_{p'})-\omega_{(p)f}\circ j_f\circ\theta^v_{(f)p'}-\omega_{(p)f}\circ j_f\circ\mu(\theta_{p'}) \\ &= \omega^*_{(f)p'}+\mu_v(\theta^v_{(f)p'}-\theta^v_{(p')}\circ j_f\circ\theta^v_{(f)p'})+\mu_h(\theta_{p'})-\omega_{(p)f}\circ j_f\circ\mu(\theta_{p'}).\end{aligned}$$

Consequently, by virtue of (4.1) and (4.3), we have finally

$$\omega^*_{(f)}\circ\varphi^* = \omega^*_{(f)}+\mu_h(\theta)-C(\mu(\theta))_{\bar\varphi\bar K_f}, \tag{4.4}$$

where we put $\bar K_f\colon P\to Q$, $p\to(pf, p)$.

As an application of (4.4), we consider the particular case where $\varphi$ is a rotation. In this case, from [1, (6.3)], we see

$$\varphi^* = R_\lambda+F(\Lambda), \tag{4.5}$$

where $\varphi^*$ is the differential and $\Lambda$ is the $\lambda$-form of a rotation [1, § 6]. Hence we have, by means of [1, (2.6)] and [1, Theo. 1],

$$\omega^*_{(f)}\circ\varphi^* = ad(\lambda^{-1})\omega^*_{(\lambda f)}+\Lambda.$$

Therefore we obtain

$$ad(\lambda^{-1})\omega^*_{(\lambda f)}-\omega^*_{(f)} = \mu_h(\theta)-C(\mu(\theta))_{\bar\varphi\bar K_f}-\Lambda. \tag{4.6}$$

This equation is the relation satisfied by $\mu_h$ and $\mu$ for the case of rotation.

Gathering these results we have

**Proposition 4.** *The transformation of a quasi-f-connection form* $\omega^*_{(f)}$ *induced from a Finsler connection by a linear transformation is given by* (4.4), *corresponding to* (3.2). *In the case of a rotation, we have* (4.6).

## § 5. Induced Finsler connections

Let $(\Gamma^v, \Gamma^h)$ be a Finsler connection in $Q$. Then a linear transformation $\varphi$ gives a new pair of distributions $\varphi(\Gamma^v, \Gamma^h)=(\bar{\Gamma}^v, \bar{\Gamma}^h)$. This new pair satisfies the condition of a Finsler connection [1, § 1], as is easily verified. We call this new connection the *induced Finsler connection* from $(\Gamma^v, \Gamma^h)$ by $\varphi$.

**Proposition 5.** *If a Finsler connection satisfies the condition of homogeneity, the same is true for the induced connection by a linear transformation.*

In order to prove this, it is enough to show that the mapping $\bar{h}_z$, as introduced in [5] of § 1, commutes a linear transformation $\varphi$. The commutability is obvious from the linearity of $\varphi$.

Thus, we can say that any linear transformation preserves the condition of homogeneity.

**Proposition 6.** *The connection form* $\bar{\omega}$, *the v-basic form* $\bar{\theta}^v$, *and etc. of the induced connection are given by*

$$(5.1)\qquad \begin{array}{llll} (1) & \bar{\omega} = \omega\circ\varphi^{-1}, & (4) & \bar{F} = F, \\ (2) & \bar{\theta}^v = \theta^v\circ\varphi^{-1}, & (5) & \bar{B}^v = \varphi B^v, \\ (3) & \bar{\theta}^h = \theta^h, & (6) & \bar{B}^h = \varphi B^h(\lambda). \end{array}$$

Proof. Since the $h$-basic form $\theta^h$ and fundamental vector fields are defined independent of a Finsler connection, the equations (3) and (4) are obvious.

$$\begin{aligned} (1): \quad & \bar{\omega}\circ R_g = \omega\circ\varphi^{-1}\circ R_g = \omega\circ R_g\circ\varphi^{-1} = ad(g^{-1})\omega\circ\varphi^{-1} = ad(g^{-1})\bar{\omega}, \\ & \bar{\omega}(F(A)) = \omega\circ\varphi^{-1}(F(A)) = \omega F(A) = A, \\ & \bar{\omega}(\bar{\Gamma}) = \omega\circ\varphi^{-1}(\varphi\Gamma) = \omega(\Gamma) = 0. \end{aligned}$$

Thus all of conditions satisfied by a connection form hold for $\bar{\omega}$ and hence we have (1).

$$(5): \quad \varphi B^v \in \bar{\Gamma}^v ,$$
$$\pi \circ \varphi B^v(f)_q = \varphi \circ \pi B^v(f)_q = \varphi(pf) = \varphi^*(p)f ,$$

where $q=(b, p)$.

$$(2): \quad \theta^v \circ \varphi^{-1}(\bar{F}) = \theta^v(F) = 0 ,$$
$$\theta^v \circ \varphi^{-1}(\bar{\Gamma}^h) = \theta^v(\Gamma^h) = 0 ,$$
$$\theta^v \circ \varphi^{-1}(\bar{B}^v(f)) = \theta^v B^v(f) = f .$$
$$(6): \quad \bar{\omega}(\varphi B^h(\lambda f)) = \omega B^h(\lambda f) = 0 ,$$
$$\bar{\theta}^v(\varphi B^h(\lambda f)) = \theta^v B^h(\lambda f) = 0 ,$$
$$\bar{\theta}^h(\varphi B^h(\lambda f)) = \theta^h \circ \varphi B^h(\lambda f) = \lambda^{-1}\theta^h B^h(\lambda f) = f .$$

Thus all of equations of (5.1) are obtained.

From (3.1), (3.2) and (5.1), we have the concrete expressions of $\bar{B}^v(f)$ and etc. as follows:

$$(5.2) \qquad \bar{B}^h(f) = F(\mu_v(f)) + B^v(f) ,$$
$$(5.3) \qquad \bar{B}^h(f) = F(\mu_h(\lambda f)) + B^v(\mu(\lambda f)) + B^h(f) ,$$
$$(5.4) \qquad \bar{\omega} = \omega - \mu_v(\theta^v) - (\mu_h - \mu_v \mu)(\lambda \theta^h) ,$$
$$(5.5) \qquad \bar{\theta}^v = \theta^v - \mu(\lambda \theta^h) .$$

By virtue of these equations, we can write down expressions of new coefficients of connection as follows:

$$(5.6) \qquad \bar{F}_j{}^i = F_j{}^i - \mu_k{}^i \lambda_j{}^k ,$$
$$(5.7) \qquad \bar{F}_j{}^i{}_k = F_j{}^i{}_k - \mu_{h)l}{}^i{}_j \lambda_k{}^l + C_j{}^i{}_l \mu_h{}^l \lambda_k{}^h ,$$
$$(5.8) \qquad \bar{C}_j{}^i{}_k = C_j{}^i{}_k - \mu_{v)k}{}^i{}_j .$$

## § 6. Various conditions

A Finsler connection as above treated is very general, even if the condition of homogeneity is imposed. T. Okada [3] introduced various conditions satisfied by a Finsler connection, in order to derive the euclidean connection due to E. Cartan. In the following we consider those conditions.

**Condition F:** *A Finsler connection is said to satisfy the condition F if* $\sigma_f \Gamma^h{}_q = H_b$ *holds,* where $q=(b, p)$, $f=\gamma(q)$, the mapping $\sigma_f$ was defined in [3] of § 1, and $H_b$ is the non-linear connection induced from the Finsler connection.

**Proposition 7.** *The condition F is equivalent to one of following equations*:

(6.1) $$\sigma_f B^h{}_q = \bar{\pi} B^h{}_q , \quad f = \gamma(q) ,$$

(6.2) $$\gamma B^h(f) = 0 .$$

Proof. (6.1) is clear. (6.2) is easily obtained from (1.1) and (6.1).

It follows from (6.2) that the classical expression of the condition $F$ in terms of coefficients of connection is

(6.3) $$D_j{}^i = F_j{}^i - b^k F_k{}^i{}_j \equiv 0 .$$

Now, if a Finsler connection satisfies the condition $F$ and the induced connection by a linear transformation $\varphi$ does so, then we say that the transformation $\varphi$ *preserves* the condition $F$. This term will be used, in the following, for other conditions.

**Proposition 8.** *The necessary and sufficient condition for a linear transformation $\varphi$ to preserve the condition F is that the equation*

(6.4) $$\mu_{h)b}{}^a{}_c \gamma^c = \gamma^a |_c \mu_b{}^c$$

*is satisfied.*

Proof. It follows from (5.3) and (1.2) that

$$\gamma(\bar{B}^h(f) - B^h(f)) = -\mu_h(\lambda f) + \gamma B^v(\mu' \lambda f)) .$$

Since the det. $(\lambda_b{}^a)$ does not vanish, we obtain (6.4) at once.

**Condition $C_1$:** *A Finsler connection is said to satisfy the condition $C_1$ if $\sigma_f \Gamma^v{}_q = 0$, $f = \gamma(q)$.*

**Proposition 9.** *The condition $C_1$ is equivalent to one of following equations*:

(6.5) $$\sigma_f B^v{}_q = 0 , \qquad f = \gamma(q) ,$$

(6.6) $$\gamma B^v(f) = f .$$

This is easily verified by means of (1.1). From (6.6) we have the classical expression of the condition $C_1$ in terms of coefficients of the connection as follows:

(6.7) $$b^k C_k{}^i{}_j = 0\,.$$

As for the preservation of the condition $C_1$, we have from (5.2) and (1.2)

**Proposition 10.** *The necessary and sufficient condition for a linear transformation $\varphi$ to preserve the condition $C_1$ is that the equation*

(6.8) $$\mu_{v)b}{}^a{}_c \gamma^c = 0$$

*is satisfied.*

To introduce an another condition, we recall the mapping $\bar{h}$, by means of which the condition of homogeneity is defined in [5] of §1. If we denote by $\hat{z}$ the tangent vector $(d/dz)_z$ to $R^+$, then a tangent vector $\bar{h}_q(\hat{z})$ is obtained. Thus we have a vector field $\bar{h}(\hat{z})$ on $Q$. This vector field is equal to the second fundamental vector field $E(\sum_a \hat{g}_a{}^a)$, because, if we take a one-parameter group $z\delta = (z\delta_a{}^b)$ of the group $G$, we see $z\delta\cdot f = z\cdot f$ for any $f \in F$. Therefore it follows from (1.5) that

(6.9) $$\bar{h}(\hat{z}) = B^v(\gamma) + F(C(\gamma))\,,$$

and hence $\bar{h}(\hat{z})$ is contained in $Q^v{}_q + V^v{}_q$, the $h$-horizontal component being equal to zero.

**Condition $C_2$:** *A Finsler connection is said to satisfy the condition $C_2$ if $\bar{h}(\hat{z})$ is v-horizontal at every point.*

From (6.9) we obtain at once

**Proposition 11.** *The condition $C_2$ means that $C(\gamma)$ vanishes, that is,*

(6.10) $$C_j{}^i{}_k b^k = 0\,.$$

The next proposition is a consequence of (5.4) and (6.9).

**Proposition 12.** *The necessary and sufficient condition for a linear transformation $\varphi$ to preserve the condition $C_2$ is that the equation*

(6.11) $$\mu_{v)c}{}^a{}_b \gamma^c = 0$$

*is satisfied.*

## § 7. Torsions and curvatures of the induced connection

We shall find torsions and curvatures of the induced connection $(\bar{\Gamma}^v, \bar{\Gamma}^h)$. To do this, we shall make use of brackets of two of $F(A)$, $B^v(f)$ and $B^h(f)$. In [1, § 1] formulas of those brackets are given in the case where $A$ and $f$ are fixed elements. However, if $A$ and $f$ are function on $Q$, those formulas become more complicated. It is well known that

$$[fX, gY] = fg[X, Y] + f\cdot X(g)\cdot Y - g\cdot Y(f)\cdot X,$$

where $X$ and $Y$ are vector fields and $f$ and $g$ are functions. Making use of this, we obtain the following expressions of brackets.

$$(7.1) \quad [F(A), F(A')] = F([A, A']) + F(F(A)A') - F(F(A')A),$$

$$(7.2) \quad [F(A), B^v(f)] = B^v(Af) + B^v(F(A)f) - F(B^v(f)A),$$

$$(7.3) \quad [F(A), B^h(f)] = B^h(Af) + B^h(F(A)f) - F(B^h(f)A),$$

$$(7.4) \quad [B^v(f), B^v(f')] = F(S^2(f, f')) + B^v(S^1(f, f')) + B^v(B^v(f)f')_{[f, f']},$$

$$(7.5) \quad [B^v(f), B^h(f')] = -F(P^2(f', f)) - B^v(P^1(f', f)) - B^h(C(f', f)) + B^h(B^v(f)f') - B^v(B^h(f')f),$$

$$(7.6) \quad [B^h(f), B^h(f')] = F(R^2(f, f')) + B^v(R^1(f, f')) + B^h(T(f, f')) + B^h(B^h(f)f')_{[f, f']},$$

where the subscript $[f, f']$ means, for an example, $W(f, f')_{[f, f']} = W(f, f') - W(f', f)$, and $S^2, S^1, P^2, P^1, C, R^2, R^1$ and $T$ are torsions and curvatures, and are written, for an example,

$$S^2(f, f') = S^2{}_{cd} f^c f'^d = S_{b\cdot cd}^{\ a} f^c f'^d \hat{g}_a{}^b,$$
$$P^1(f, f') = P^1{}_{cd} f'^c f^d = P_c{}^a{}_d f'^c f^d e_a.$$

We have also (7.1), ⋯, (7.6) (with bars) for the induced connection.

Substituting first from (5.2) and (5.3) into (7.2) and (7.3) (with bars), we have, by direct calculation

$$(7.7) \quad \begin{aligned} F(A)\mu_v(f) &= -[A, \mu_v(f)] + \mu_v(Af), \\ F(A)\mu_h(\lambda f) &= -[A, \mu_h(\lambda f)] + \mu_h(\lambda Af), \\ F(A)\mu'(\lambda f) &= -A\mu'(\lambda f) + \mu'(\lambda Af). \end{aligned}$$

We may, however, expect that those equations are automatically satisfied. In fact, by means of (3.6'), we obtain easily that

$$F_b{}^a(\mu_{v)d}{}^c{}_e) = -\delta_b{}^c\mu_{v)d}{}^a{}_e + \delta_d{}^a\mu_{v)b}{}^c{}_e + \delta_e{}^a\mu_{v)d}{}^c{}_b\,,$$

which shows that the first of (7.7) holds. In similar manner, remaining equations are verified.

Next, substituting in (7.4) (with bars) from (5.2), we obtain

$$\bar{S}^1(f, f') = S^1(f, f') + \mu_v(f)f'_{[f,f]}\,, \tag{7.8}$$

and moreover

$$\begin{aligned}\bar{S}^2(f, f') + \mu_v(\bar{S}^1(f, f')) = {}& S^2(f, f') + [\mu_v(f), \mu_v(f')] \\ & + F(\mu_v(f))\mu_v(f')_{[f,f']} + B^v(f)\mu_v(f')_{[f,f']}\,.\end{aligned}$$

This equation will be rewritten, by virtue of (7.8) and (7.7), in the form

$$\begin{aligned}\bar{S}^2(f, f') = {}& S^2(f, f') - \mu_v(S^1(f, f')) + B^v(f)\mu_v(f')_{[f,f']} \\ & - [\mu_v(f), \mu_v(f')]\,.\end{aligned} \tag{7.9}$$

It will be convenient to use $\mathring{B}^v(f)$, instead of $B^v(f)$, in (7.9) and in the following, because $\mathring{B}^v(f)$ is defined without use of a connection. We have already deduced the equation (1.6), and hence we obtain

$$B^v(f)\mu_v(f')_{[f,f']} = \mathring{B}^v(f)\mu_v(f')_{[f,f']} - F(C(f))\mu_v(f')_{[f,f']}\,,$$

and substitution of (7.7) gives

$$= \mathring{B}^v(f)\mu_v(f')_{[f,f']} + [C(f), \mu_v(f')]_{[f,f']} - \mu_v(C(f)f')_{[f,f']}\,.$$

Observing that $C(f)f'_{[f,f']} = -S^1(f, f')$ from the definition of the torsion $S^1$, we have from (7.9)

$$\begin{aligned}\bar{S}^2(f, f') = {}& S^2(f, f') + \mathring{B}^v(f)\mu_v(f')_{[f,f']} - [\mu_v(f), \mu_v(f')] \\ & + [C(f), \mu_v(f')]_{[f,f']}\,.\end{aligned} \tag{7.9$'$}$$

The similar process is applied to (7.5) and (7.6), and then we obtain

$$\bar{C}(f', f) = C(f', f) - \mu_v(f)f'\,, \tag{7.10}$$

$$\bar{P}^1(f', f) = P^1(f', f) - C(f, \mu(\lambda f')) + \mu_h(\lambda f')f - \mathring{B}^v(f)\mu(\lambda f')\,, \tag{7.11}$$

$$\begin{aligned}\bar{P}^2(f', f) = {}& P^2(f', f) - S^2(f, \mu(\lambda f')) - \mu_v(P^1(f', f)) \\ & + [C(\mu(\lambda f')), \mu_v(f)] - [C(f), \mu_h(\lambda f')] \\ & - [\mu_h(\lambda f'), \mu_v(f)] + \mu_v(\mathring{B}^v(f)\mu(\lambda f')) \\ & + \mathring{B}^v(\mu(\lambda f'))\mu_v(f) - \mathring{B}^v(f)\mu_h(\lambda f') + B^h(f')\mu_v(f)\,,\end{aligned} \tag{7.10}$$

$$(7.13)\quad \bar{T}(f, f') = T(f, f') + \mu_h(\lambda f)f'_{[f,f']} - C(f', \mu(\lambda f))_{[f,f']},$$

$$(7.14)\quad \bar{R}^1(f, f') = R^1(f, f') - P^1(f', \mu(\lambda f))_{[f,f']} - \mu(\lambda T(f, f')) - B^h(f')\mu(\lambda f)_{[f,f']} + \mathring{B}^v(\mu(\lambda f))\mu(\lambda f')_{[f,f']},$$

$$(7.15)\quad \bar{R}^2(f, f') = R^2(f, f') - P^2(f', \mu(\lambda f))_{[f,f']} + S^2(\mu(\lambda f), \mu(\lambda f')) - \mu_v(R^1(f, f')) + \mu_v(P^1(f', \mu(\lambda f))_{[f,f']} + (\mu_v\mu - \mu_h)(\lambda T(f, f')) - B^h(f')\mu_h(\lambda f)_{[f,f']} + \mu_v(B^h(f')\mu(\lambda f))_{[f,f']} - \mathring{B}^v(\mu(\lambda f'))\mu_h(\lambda f)_{[f,f']} + \mu_v\mathring{B}^v(\mu(\lambda f'))\mu(\lambda f))_{[f,f']} - [\mu_h(\lambda f), \mu_h(\lambda f')] - [C(\mu(\lambda f')), \mu_h(\lambda f)]_{[f,f']}.$$

It is obvious that (7.10) is equivalent to (5.6)

For the case of a projective transformation of an ordinary connection in the bundle $P$ of frames, it is usual that the connection is assumed to be symmetric. On the other hand, the condition of symmetry of a Finsler connection is defined as follows.

**Condition of symmetry:** *A Finsler connection is said to be symmetric if the torsion $T$ vanishes.*

From [1, (1.3)], we see that $T$ is coefficient of $h$-component of the $h$-torsion form. Since $T_j{}^i{}_k = F_j{}^i{}_k - F_k{}^i{}_j$, the above condition means that $F_j{}^i{}_k$ is symmetric with respect to subscripts. It follows from (7.13) that

**Proposition 13.** *The necessary and sufficient condition for a linear transformation $\varphi$ to preserve the condition of symmetry is that the equation*

$$(7.16)\qquad \mu_h(\lambda f)f'_{[f,f']} - C(f', \mu(\lambda f))_{[f,f']} = 0$$

*is satisfied.*

In terms of components, the equation (7.16) is written by

$$(7.16')\qquad \mu_{(h)d}{}^a{}_{[b}\lambda_{c]}{}^d - C_{[b}{}^a{}_d\mu_e{}^d\lambda_{c]}{}^e = 0.$$

## §8. Infinitesimal linear transformations

Let $\varphi_t$ be a 1-parameter quasi-group of linear transformations and let $X$ be the infinitesimal transformation of $\varphi_t$. As has already been shown, a linear transformation is characterized by the three properties of Theorem 1, and hence $X$ is such that

$$\mathfrak{L}_X F(A) = 0\,, \tag{8.1}$$

$$\mathfrak{L}_X E(A) = 0\,, \tag{8.2}$$

$$\mathfrak{L}_X \gamma = 0\,, \tag{8.3}$$

where $\mathfrak{L}_X$ indicates the Lie derivative with respect to X.

Ey making use of these equations, we shall find the expression of $X$ in terms of canonical coordinate $(x^i, b^i, p_a{}^i)$. If we take the base $(\hat{g}_a{}^b)$ of the Lie algebra $\hat{G}$ and put $F_a{}^b = F(\hat{g}_a{}^b)$ and $E_a{}^b = E(\hat{g}_a{}^b)$, then we have

$$F_a{}^b = p_a{}^i \frac{\partial}{\partial p_b{}^i}\,, \quad E_a{}^b = p_a{}^i p_j^{-1b} b^j \frac{\partial}{\partial b^i}\,, \quad \gamma = p_i^{-1a} b^i e_a\,.$$

By putting

$$X = X^i \frac{\partial}{\partial x^i} + X^{(i)} \frac{\partial}{\partial b^i} + X_a{}^i \frac{\partial}{\partial p_a{}^i}\,,$$

the equation (8.1), that is, $[X, F(A)]=0$ gives

$$\frac{\partial X^i}{\partial p_a{}^j} = \frac{\partial X^{(i)}}{\partial p_a{}^j} = 0\,, \quad \frac{\partial X_a{}^i}{\partial p_b{}^j} = \delta_a{}^b p_j^{-1c} X_c{}^i\,.$$

From (8.2) and (8.3) we deduce similarly that

$$\frac{\partial X^i}{\partial b^j} = \frac{\partial X_a{}^i}{\partial b^j} = 0\,, \quad \frac{\partial X^{(i)}}{\partial b^j} = X_a{}^i p_j^{-1a}\,, \quad X^{(i)} = X_a{}^i p_j^{-1a} b^j\,.$$

It follows from these equations that $X^i$ and $X_j{}^i = X_a{}^i p_j^{-1a}$ are functions of $x^i$ only, and $X^{(i)}$ is equal to $X_j{}^i b^j$. Therefore we have

$$X = X^i(x) \frac{\partial}{\partial x^i} + X_j{}^i(x) b^j \frac{\partial}{\partial b^i} + X_j{}^i(x) p_a{}^j \frac{\partial}{\partial p_a{}^i}\,. \tag{8.4}$$

We consider the special case where $\varphi_t$ are induced transformations from the projection $\underline{\varphi}_t$. In this case deviations $\lambda_t$ are the unit $e \in G$ and we obtain easily $X_j{}^i = \partial X^i / \partial x^j = X^i{}_{,j}$. If we use the letter $Y$, instead of $X$, then it follows from (8.4) that

$$Y = Y^i(x) \frac{\partial}{\partial x^i} + Y^i{}_{,j} b^j \frac{\partial}{\partial b^i} + Y^i{}_{,j} p_a{}^j \frac{\partial}{\partial p_a{}^i}\,. \tag{8.5}$$

On the other hand, if $\varphi_t$ are rotations, we have $X^i = 0$, because projections $\underline{\varphi}_t$ are reduced to the indentity. Since a rotation $\varphi^*$

is expressible by $(x^i, p_a{}^i) \to (x^i, \lambda_j{}^i p_a{}^j)$ [1, (3. 6)], we see that $X_j{}^i p_a{}^j = (d\lambda_j{}^i/dt)_{t=0} p_a{}^j$. Hence, if we use the letter $Z$, instead of $X$, then we obtain

$$Z = \eta_j{}^i b^j \frac{\partial}{\partial b^i} + \eta_j{}^i p_a{}^j \frac{\partial}{\partial p_a{}^i}, \tag{8.7}$$

where $\eta \in \hat{G}$ is the tangent vector of the curve $\lambda_t$ at $e \in G$.

It is obvious that a general $X$ is the sum of the induced part $Y$ and the rotation part $Z$, and hence we conclude that

**Proposition 14.** *The infinitesimal transformation $X$ of a 1-parameter quasi-group of linear transformations is written as the sum of the induced part $Y$ and the rotation part $Z$, where $Y$ and $Z$ are given by* (8. 5) *and* (8. 6) *respectively.*

## §9. The Lie derivative of a Finsler connection

Let $X$ be the infinitesimal transformation as treated in the last section. The Lie derivative $\mathfrak{L}_X \alpha$ of a form $\alpha$ on $Q$ with respect to the $X$ is defined by

$$\mathfrak{L}_X \alpha = \lim_{t \to 0} \frac{1}{t}(\alpha \circ \varphi_t - \alpha).$$

Hence, from (3. 2), we can derive directly the following formulas:

$$\begin{aligned} \mathfrak{L}_X \omega &= \nu_v(\theta^v) + \nu_h(\theta^h), \\ \mathfrak{L}_X \theta^v &= \nu(\theta^h), \\ \mathfrak{L}_X \theta^h &= -\eta \theta^h, \end{aligned} \tag{9.1}$$

where $\nu_v$ and $\nu_h$ are tangent vectors of curves $\mu_{vt}$ and $\mu_{ht}$ in $G$ at $e \in G$, and $\nu$ is the tangent vector of the curve $\mu_t$ in $F$ at the origin.

On the other hand, for a tangent vector field $U$ on $Q$, the Lie derivative $\mathfrak{L}_X U$ is defined by

$$\mathfrak{L}_X U = \lim_{t \to 0} \frac{1}{t}(U - U \circ \varphi_t).$$

Then, from (3. 1), it is easy to see that

$$\begin{aligned} &\mathfrak{L}_X F(A) = 0, \\ &\mathfrak{L}_X B^v(f) = -F(\nu_v(f)), \\ &\mathfrak{L}_X B^h(f) = -F(\nu_h(f)) - B^v(\nu(f)) + B^h(\eta f). \end{aligned} \tag{9.2}$$

We can deduce from (9.2) a system of differential equations satisfied by the vector field $X$. To do this, we remember that $\mathfrak{L}_X U=[X, U]$. Since the decomposition of $X$ with respect to a Finsler connection is written as $X=F(\omega(X))+B^v(\theta^v(X))+B^h(\theta^h(X))$, it follows from the third equation of (9.2) that

$$\begin{aligned}
&[F(\omega(X))+B^v(\theta^v(X))+B^h(\theta^h(X)),\ B^h(f)] \\
&\quad = B^h(\omega(X)f)-F(B^h(f)\omega(X))-F(P^2(f,\ \theta^v(X)))-B^v(P^1(f,\ \theta^v(X))) \\
&\qquad -B^h(C(f,\ \theta^v(X)))-B^v(B^h(f)\theta^v(X))+F(R^2(\theta^h(X),\ f)) \\
&\qquad +B^v(R^1(\theta^h(X),\ f))+B^h(T(\theta^h(X),\ f))-B^h(B^h(f)\theta^h(X)),
\end{aligned}$$

where we made use of (7.3), (7.4) and (7.5). Therefore the third equation of (9.2) is equivalent to the following:

$$(9.3)\quad \begin{aligned}
B^h(f)\omega(X) &= -P^2(f,\ \theta^v(X))+R^2(\theta^h(X),\ f)+\nu_h(f), \\
B^h(f)\theta^v(X) &= -P^1(f,\ \theta^v(X))+R^1(\theta^h(X),\ f)+\nu(f), \\
B^h(f)\theta^h(X) &= -C(f,\ \theta^v(X))+T(\theta^h(X),\ f)+\omega(X)f-\eta f.
\end{aligned}$$

In an entirely similar way we deduce from the second equation of (9.2) that

$$(9.4)\quad \begin{aligned}
B^v(f)\omega(X) &= S^2(\omega(X),\ f)+P^2(\theta^h(X),\ f)+\nu_v(f), \\
B^v(f)\theta^v(X) &= S^1(\omega(X),\ f)+P^1(\theta^h(X),\ f)+\omega(X)f, \\
B^v(f)\theta^h(X) &= C(\theta^h(X),\ f).
\end{aligned}$$

(9.3) and (9.4) are differential equations satisfied by the $X$, because, if we put $\theta^v(X)=X^a e_a$, we have $B^v(f)\theta^v(X)=X^a|_b f^b e_a$ ($v$-covariant derivative) and $B^h(f)\theta^h(X)=X^a{}_{|b} f^b e_a$ ($h$-covariant derivative).

On the other hand, the first equation of (9.2) does not give differential equations, but we obtain

$$(9.5)\quad \begin{aligned}
F(A)\omega(X)+[A,\ \omega(X)] &= 0, \\
F(A)\theta^v(X)+A\cdot\theta^v(X) &= 0, \\
F(A)\theta^h(X)+A\cdot\theta^h(X) &= 0,
\end{aligned}$$

which do not contain derivatives of components of $X$.

**Proposition 15.** *The infinitesimal transformation $X$ of a* 1-*parameter quasi-group of linear transformations has to satisfy a system of differential equations* (9.3) *and* (9.4), *and moreover a system of algebraic equations* (9.5).

We shall, finally, find Lie derivatives of torsions and curvatures of a Finsler connection. In (9.2), $A\in\hat{G}$ and $f\in F$ are fixed elements, while, if $A$ and $f$ are functions on $Q$, we can easily derive from (9.2) that

$$\begin{aligned} &\mathfrak{L}_X F(A) = F(\mathfrak{L}_X A), \\ (9.6)\quad &\mathfrak{L}_X B^v(f) = -F(\nu_v(f)) + B^v(\mathfrak{L}_X f), \\ &\mathfrak{L}_X B^h(f) = -F(\nu_h(f)) - B^v(\nu(f)) + B^h(\eta f) + B^h(\mathfrak{L}_X f). \end{aligned}$$

Next, it follows from the Jacobi identity that $\mathfrak{L}_X[U, V] = [\mathfrak{L}_X U, V] + [U, \mathfrak{L}_X V]$, where $U$ and $V$ are vector fields on $Q$.

Now, if $f$ and $f'$ are fixed elements of $F$, we obtain from (7.4) that

$$[B^v(f), B^v(f')] = F(S^2(f, f')) + B^v(S^1(f, f')).$$

By operating $\mathfrak{L}_X$ on the above equation and using (9.6), we have by direct calculation that

$$(9.7)\quad \mathfrak{L}_X S^1(f, f') = -\nu_v(f)f'_{[f,f']},$$

$$(9.8)\quad \mathfrak{L}_X S^2(f, f') = \nu_v(S^1(f, f')) + B^v(f')\nu_v(f)_{[f,f']}.$$

The similar way leads us to the following:

$$(9.9)\quad \mathfrak{L}_X C(f', f) = -\eta C(f', f) + C(\eta f', f) - B^v(f)\eta f' + \nu_v(f)f',$$

$$(9.10)\quad \mathfrak{L}_X P^1(f', f) = S^1(f, \nu(f')) + P^1(\eta f', f) + B^v(f)\nu(f') + \nu(C(f', f)) - \nu_h(f')f,$$

$$(9.11)\quad \mathfrak{L}_X P^2(f', f) = S^2(f, \nu(f')) + P^2(\eta f', f) + B^v(f)\nu_h(f') - B^h(f')\nu_v(f) + \nu_v(P^1(f', f)) + \nu_h(C(f', f)),$$

$$(9.12)\quad \mathfrak{L}_X T(f, f') = T(\eta f, f')_{[f,f']} + C(f', \nu(f))_{[f,f']} - B^h(f')\eta f_{[f,f']} - \nu_h(f)f'_{[f,f']} - \eta T(f, f'),$$

$$(9.13)\quad \mathfrak{L}_X R^1(f, f') = P^1(f', \nu(f))_{[f,f']} + R^1(\eta f, f')_{[f,f']} + B^h(f')\nu(f)_{[f,f']} + \nu(T(f, f')),$$

$$(9.14)\quad \mathfrak{L}_X R^2(f, f') = P^2(f', \nu(f))_{[f,f']} + R^2(\eta f, f')_{[f,f']} + B^h(f')\nu_h(f)_{[f,f']} + \nu_v(R^1(f, f')) + \nu_h(T(f, f')).$$

**Proposition 16.** *Lie derivatives of torsions and curvatures of a Finsler connection with respect to the infinitesimal transformation X of a* 1-*parameter quasi-group of linear transformations are given by* (9.7), $\cdots$, (9.14).

It will be convenient to write those equations in terms of a canonical coordinate, and we obtain

$$(9.7')\quad \mathfrak{L}_X S_j{}^i{}_k = -\nu_{v)[j}{}^i{}_{k]}\,,$$

$$(9.9')\quad \mathfrak{L}_X C_j{}^i{}_k = \nu_{v)k}{}^i{}_j\,,$$

$$(9.10')\quad \mathfrak{L}_X P_j{}^i{}_k = \nu_l{}^i C_j{}^l{}_k + \nu_j{}^l S_k{}^i{}_l + \eta_j{}^l P_l{}^i{}_k - \nu_{h)j}{}^i{}_k + \nu_j{}^i|_k\,,$$

$$(9.12')\quad \mathfrak{L}_X T_j{}^i{}_k = -\eta_l{}^i T_j{}^l{}_k + \eta_{[j}{}^l T_l{}^i{}_{k]} + \nu_{[j}{}^l C_{k]}{}^i{}_l - \nu_{h)[j}{}^i{}_{k]} - \eta_{[j}{}^i{}_{|k]}\,,$$

$$(9.13')\quad \mathfrak{L}_X R_j{}^j{}_k = \nu_l{}^i T_j{}^l{}_k + \nu_{[j}{}^l P_{k]}{}^j{}_l + \eta_{[j}{}^l R_l{}^i{}_{k]} + \nu_{[j|k]}^{i}\,,$$

Above equations give Lie derivatives of torsions, and the following equations do that of curvatures:

$$(9.8')\quad \mathfrak{L}_X S_j{}^i{}_{kl} = \nu_{v)m}{}^i{}_j S_k{}^m{}_l + \nu_{v)[k}{}^i{}_j|_{l]}\,,$$

$$(9.11')\quad \mathfrak{L}_X P_j{}^i{}_{kl} = \eta_k{}^m P_j{}^i{}_{ml} + \nu_k{}^m S_j{}^i{}_{ml} + \nu_{v)m}{}^j{}_j P_k{}^m{}_l + \nu_{h)m}{}^i{}_j C_k{}^m{}_l - \nu_{v)l}{}^i{}_{j|k} + \nu_{h)k}{}^j{}_j|_l\,,$$

$$(9.14')\quad \mathfrak{L}_X R_j{}^i{}_{kl} = \eta_{[k}{}^m R_j{}^i{}_{ml]} + \nu_{[k}{}^m P_j{}^i{}_{l]m} + \nu_{v)m}{}^i{}_j R_k{}^m{}_l + \nu_{h)m}{}^i{}_j T_k{}^m{}_l + \nu_{h)[k}{}^i{}_{j|l]}\,.$$

In the case where $X$ is the induced $Y$, those equations will be written somewhat simple, for the infinitesimal deviation $\eta_j{}^i$ vanishes.

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